Proceedings

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Lattice Theory Conference

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PROCEEDINGS OF THE UNIVERSITY OF HOUSTON

LATTICE THEORY CONFERENCE

March 22-24, 1973

Conference Directors

Dennison R. Brown | Jürgen Schmidt

Assistant Directors

Michael Friedberg | George Grätzer

Editors

Siemion Fajtlowicz | Klaus Kaiser

Houston, 1973

The Lattice Theory Conference, of which these are the proceedings, is an outgrowth of some conversations between Professors Jürgen Schmidt and Dennison Brown of the University of Houston, held during the late spring and summer of 1972. Desire to sponsor such an event grew from additional discussion with Professor George Grätzer of the University of Manitoba.

Funding for this undertaking was obtained primarily from the University of Houston Department of Mathematics, Professor James N. Younglove, Chairman. Additional monies were obtained by grants from the University of Houston Graduate School, Ronald F. Bunn, Dean, and from the University of Houston Office of Research, Francis B. Smith, Director. The sponsors are extremely grateful to these agencies for this opportunity.

The conference was held at the Shamrock Hilton Hotel in Houston on March 22-24, 1973. In excess of forty papers were presented, many by researchers of international repute in the areas of lattice and semilattice theory.

Special acknowledgment is due Professor Michael Friedberg, who was in charge of all hotel arrangements, and Professors Siemion Fajtlowicz and Klaus Kaiser, who edited these proceedings.

Professor Dennison R. Brown
Professor Jürgen Schmidt

PROGRAM OF THE UNIVERSITY OF HOUSTON LATTICE THEORY CONFERENCE

SHAMROCK HILTON HOTEL

Thursday, March 22, 1973

Castillian ABC Room

8:00 - 5:00 Registration

Morning Session

Chair: J. Younglove

9:00 - 10:00	R. McKenzie	Lattice Theory and equational logic
10:00 - 11:00	W. Lampe	Complete lattices as congruence lattices of infinitary algebras
11:00 - 12:00	V. Faber	On the Birkhoff problem

Afternoon Session

Chair: T. Evans

2:00 - 3:00	G. C. Rota	Topics in combinatorial lattice theory
3:00 - 3:30	G. Markowsky	Combinatorial aspects of lattice theory and the free distributive lattice
3:30 - 4:30	G. Hutchinson	On classes of lattices representable by modules
4:30 - 5:00	R. Mena	Lattices of ideals
5:00 - 6:00	C. Herrmann	Lattices of normal subgroups and word problems

UNIVERSITY OF HOUSTON LATTICE THEORY CONFERENCE

SHAMROCK HILTON HOTEL

Friday, March 23, 1973

Columbia Room

Morning Session

Chair: M. Neff

9:00 - 10:00	B. Banaschewski	The filter space of a lattice: its role in general topology
10:00 - 10:30	B. Johnson	Finite sublattices of a free lattice: a progress report
10:30 - 11:00	H. Lakser	Simple sublattices of free products of lattices
11:00 - 11:30	M. Hoft	The order-sum in classes of algebras
11:30 - 11:50	S . D. Comer	Arithmetical properties of relatively free products
11:50 - 12:10	R. Smith	Distributivity of the lattice of filters of a groupoid

Afternoon Session

Chair: D. G. Bourgin

2:00 - 3:00	K. H. Hofmann	The duality of discrete and compact semilattices
3:00 - 4:00	J. Lawson	Intrinsic topologies for lattices and semilattices

Chair: M. Friedberg

4:00 - 4:30	A. Stralka	Distributive topological lattices
4:30 - 5:00	K. Keimel	Representations of lattice ordered rings
5:00 - 5:20	K. H. Hofmann & M. W. Mislove	Dimensional stability of zero-dimensional compact semilattices
5:20 - 5:40	M. Bennett	Modular centers of additive lattices
5:40 - 6:00	J. Martinez	Structure of archimedean lattices

UNIVERSITY OF HOUSTON LATTICE THEORY CONFERENCE

SHAMROCK HILTON HOTEL

Saturday, March 24, 1973

Columbia Room

Morning Session

Chair: J. S. Mac Nerney

9:00 - 10:00	K. Baker	Inside free semilattices
10:00 - 11:00	R. Wille	Free modular lattices over partial lattices
11:00 - 11:30	G. Bruns	Orthomodular lattices
11;30 - 12:00	R. Freese	Breadth two modular lattices
12:00 - 12:30	H. Werner	Stone duality for varieties generated by quasi-primal ideals

Afternoon Session

Chair: J. Schmidt

2:00 - 2:20	L. Chawla Kansas State University	Isomorphic embedding of the lattice of all subgroups of a group
2:20 - 2:40	A. Day Lakehead University	Finitely projected algebras with application to varieties of lattices
2:40 - 3:00	C. Holmes University of Miami	Projectivities of free products of groups
3:00 - 3:20	D. Adams University of Massachusetts	Equational classes of orthomodular lattices and Foulis semigroups
3:20 - 3:40	G. Kalmbach Penn State University	Orthomodular logic
3:40 - 4:00	R. Redfield Simon Fraser University	Ordering uniform completions of partially ordered sets
4:00 - 4:20	R. Quackenbush University of Manitoba	On planar lattices
4:20 - 4:40	E. Schreiner Western Michigan University	Tight residuated mappings
4:40 - 5:00	R. Podmanabhan University of Manitoba	On lattices with unique complements

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REPRESENTATIONS OF LATTICES AS CONGRUENCE LATTICES

William A. Lampe

§1. INTRODUCTION

In [1] Birkhoff posed the problem of characterizing the lattice of all congruence relations of an algebra. It is easy to see that this lattice is a complete lattice. In [9] G. Grätzer and E. T. Schmidt showed that every algebraic lattice is isomorphic to the lattice of all congruence relations of some <u>finitary</u> algebra. The converse had been known for some time. Recently, a number of other representation theorems involving the lattice of congruence relations of an algebra have been proved. One such theorem is that every complete lattice is isomorphic to the lattice of all congruence relations of some algebra. In this paper we will survey these results and discuss the basic method used in their proofs. We will also mention some of the open problems. (No originality is claimed for the problems).

§2. TERMS AND NOTATIONS

Let α be an ordinal and A be a set. If $f : A^{\alpha} \to A$, then we say that f is an <u>α-ary operation</u> on A. $\mathfrak{A} = \langle A; F \rangle$ is an <u>algebra</u> iff F is a family of operations on the set A. We say $\mathfrak{A}$ *is of characteristic* $\mathfrak{m}$ iff $\mathfrak{m}$ is the least regular cardinal such that for any operation f of $\mathfrak{A}$ if f is α-ary then $\alpha < \mathfrak{m}$. $\mathfrak{A}$ is *finitary* iff $\mathfrak{A}$ is of characteristic $\aleph_0$. $\mathfrak{A}$ is *infinitary* if $\mathfrak{A}$ is not finitary. If $\mathbf{x} \in A^\alpha$, then the i^{th} component of $\mathbf{x}$ is denoted x_i. If Θ is an equivalence relation on A and if $\mathbf{x}, \mathbf{y} \in A^\alpha$, we write $\mathbf{x} \equiv \mathbf{y}\ (\Theta)$ iff $x_i \equiv y_i\ (\Theta)$ for every $i < \alpha$. Θ is a *congruence relation* of $\mathfrak{A}$ iff Θ is an equivalence relation on A and for any α and any α-ary operation f and any $\mathbf{x}, \mathbf{y} \in A^\alpha$ $f(\mathbf{x}) \equiv f(\mathbf{y})\ (\Theta)$ whenever $\mathbf{x} \equiv \mathbf{y}\ (\Theta)$. $\mathrm{Con}(\mathfrak{A})$ is the set of all congruence relations of $\mathfrak{A}$. $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}); \subseteq \rangle$ is the *congruence lattice* of $\mathfrak{A}$. $\mathfrak{A}$ is *simple* if $\mathbf{Con}(\mathfrak{A})$ is the two element chain. Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra, and let $B \subseteq A$. B is a *subalgebra* of $\mathfrak{A}$ iff for every α and for every α-ary operation f of $\mathfrak{A}$ and for every $\mathbf{x} \in B^\alpha$ it holds that $f(\mathbf{x}) \in B$. $\mathrm{Sub}(\mathfrak{A})$ is the set of all subalgebras of $\mathfrak{A}$. By convention $\emptyset \in \mathrm{Sub}(\mathfrak{A})$ iff $\mathfrak{A}$ has no 0-ary operations. $\mathbf{Sub}(\mathfrak{A}) = \langle \mathrm{Sub}(\mathfrak{A}); \subseteq \rangle$ is the *subalgebra lattice* of $\mathfrak{A}$. Let $\mathbf{x} \in A^\alpha$ and $\sigma : A \to A$ then $\mathbf{x}\sigma$ is the sequence $\mathbf{y} \in A^\alpha$ with $y_i = x_i\sigma$ for every $i < \alpha$. σ is an *endomorphism* iff $f(\mathbf{x}\sigma) = f(\mathbf{x})\sigma$ for every operation f and every $\mathbf{x}$. $\mathrm{End}(\mathfrak{A})$ is the set of all endomorphisms of $\mathfrak{A}$, and $\mathbf{End}(\mathfrak{A}) = \langle \mathrm{End}(\mathfrak{A}); \circ \rangle$ is the *endomorphism semigroup* of $\mathfrak{A}$. A 1-1 onto endomorphism is an *automorphism*, and $\mathbf{Aut}(\mathfrak{A}) = \langle \mathrm{Aut}(\mathfrak{A}); \circ \rangle$ denotes the *automorphism group*.

$\mathcal{L} = \langle L; \leq \rangle$ is a complete lattice iff $\mathcal{L}$ is a partially ordered set such that any $H \subseteq L$ has a join (sup, $\bigvee H$) and a meet (inf, $\bigwedge H$). Let $\mathfrak{m}$ be a regular cardinal. The element c of the complete lattice $\mathcal{L}$ is <u>m-compact</u> iff whenever $c \leq \bigvee H$ then $c \leq \bigvee H_0$ for some H_0 with $H_0 \subseteq H$ and $|H_0| < \mathfrak{m}$. The complete lattice is <u>m-algebraic</u> iff every element is the join of some set of m-compact elements. $\aleph_0$-algebraic lattices are simply called <u>algebraic lattices</u>. Clearly, any complete lattice $\mathcal{L}$ is $|L|^+$-algebraic.

$\mathcal{L}$ is a <u>partition lattice</u> iff $\mathcal{L}$ is a sublattice of the lattice of all equivalence relations on some set such that equality and the total relation are members of $\mathcal{L}$.

§3. HISTORY AND RESULTS

In [3] G. Birkhoff and O. Frink showed that the congruence lattice of a finitary algebra is an algebraic lattice. The converse appeared in 1963.

<u>Theorem 1</u>. (G. Grätzer and E. T. Schmidt [9]): If $\mathcal{L}$ is any algebraic lattice, then there is a finitary algebra $\mathfrak{A}$ such that $\mathbf{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

In [9] Grätzer and Schmidt gave the construction for an algebra $\mathfrak{A}$, all of whose operations were unary, such that $\mathbf{Con}(\mathfrak{A})$ is isomorphic to the specified lattice $\mathcal{L}$. A simpler proof appears in [16]. Other proofs appear in [4], [13], [14] and [21]. The proofs in [14] and [21] are essentially the same. The various proofs differ in detail but all use basically the same construction. The proof in [13] is due to R. N. McKenzie.

Let C be the set of compact elements of $\mathcal{L}$. The algebra in each of the proofs has $|C| \cdot \aleph_0$ elements and $|C| \cdot \aleph_0$ unary operations. A long standing problem is to show that the representation in Theorem 1 can be effected with an algebra having one binary operation (or at least finitely many finitary operations). The known results on this problem are fragmentary.

G. Birkhoff showed in [2] that any group could be isomorphic to the automorphism group of some finitary algebra (in fact a unary algebra). His proof has been extended to show that any semigroup with unit can be the endomorphism semigroup of some finitary algebra. (That such a representation could be effected using only one binary operation or two unary operations was shown in a series of papers which ended with [10]).

The "kernel" of any homomorphism is a congruence relation. This provides a mechanism thru which the endomorphism semigroup of an algebra can affect the congruence lattice. (Very little is known about the connection between $\underset{\sim}{\mathrm{End}}(\mathfrak{A})$ and $\underset{\sim}{\mathrm{Con}}(\mathfrak{A})$. See, for example [5] and [15]). There is no such obvious mechanism through which the automorphism group can affect the congruence lattice.

So it was conjectured some time ago that in general the congruence lattice and the automorphism group are "independent". More precisely, it was conjectured that if $\mathcal{L}$ is any algebraic lattice and $\mathfrak{G}$ is any group then there is a finitary algebra $\mathfrak{A}$ such that $\mathbf{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$ and $\mathbf{Aut}(\mathfrak{A})$ is isomorphic to $\mathfrak{G}$. That this conjecture is true follows from Theorem 2. In [20] E. T. Schmidt published an incorrect proof that this conjecture is true. However, the intuitive picture of the construction in Theorem 2 is in some ways similar to E. T. Schmidt's.

G. Birkhoff and O. Frink proved in [3] that any algebraic lattice was isomorphic to $\mathbf{Sub}(\mathfrak{A})$ for some finitary $\mathfrak{A}$. E. T. Schmidt gave a very nice proof in [19] that $\mathbf{Sub}(\mathfrak{A})$ and $\mathbf{Aut}(\mathfrak{A})$ are independent. This result is also a Corollary to Theorem 2. There is obviously a third corollary to Theorem 2 which gives a representation for any pair of algebraic lattices.

Theorem 2. (W. A. Lampe [18]): If $\mathfrak{G}$ is any group and $\mathcal{L}_0$ and $\mathcal{L}_1$ are any two algebraic lattices each having two or more elements, then there is a finitary algebra $\mathfrak{A}$ such that:

(i) $\mathbf{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_0$;

(ii) $\mathbf{Sub}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_1$;

(iii) $\mathbf{Aut}(\mathfrak{A})$ is isomorphic to $\mathfrak{G}$.

The $\mathfrak{A}$ in the proof of Theorem 2 actually has n-ary operations for every $n > 0$. Binary operations would have done as well, but the proof would have been a little bit longer. If C_i represents the set of compact elements of $\mathcal{L}_i$, then $\mathfrak{A}$ has $[|C_0| \cdot |C_1| \cdot \aleph_0]$ elements and operations.

In what ways can one "improve" this representation? If $\mathfrak{A}$ is a finitary algebra having at most countably many operations, then each finitely generated subalgebra is countable, and so each finitely generated subalgebra has at most countably many finitely generated subalgebras. Thus, in $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$ each compact element has at most countably many compact elements below it. (The converse was first proved by W. Hanf. It appeared in [13] and [22]). It is clear then that in general one cannot put a bound on the number of operations that the $\mathfrak{A}$ in Theorem 2 has. But if one omits conclusion (ii), then it seems likely that one could produce a representation using only finitely many finitary operations.

One must use at least one binary operation in the $\mathfrak{A}$ of Theorem 2 for two reasons. First, among other things, G. Grätzer showed in [5] that the automorphism group of a simple algebra having only unary or nullary operations was a group of order p where p = 1 or p is a prime. (A corollary of the main result of [5] is that any group is the automorphism group of some simple algebra having one binary and many unary operations. The unary operations have been eliminated by J. Ježek in a recent paper appearing in Comm. Math. Univ. Carolinae). Secondly, if $\mathfrak{A}$ is unary then the join in $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$ is just set union, and so $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$

is then a "completely" distributive lattice.

Let Θ and Φ be equivalence relations on some set, and let $\Theta \cdot \Phi$ represent the "composition" of Θ and Φ. Let $\Psi_0 = \Theta$, $\Psi_1 = \Theta \cdot \Phi$, $\Psi_2 = \Theta \cdot \Phi \cdot \Theta$, $\Psi_3 = \Theta \cdot \Phi \cdot \Theta \cdot \Phi$, etc. In the lattice of all equivalence relations on the set, $\Theta \vee \Phi = \bigcup(\Psi_i \mid i = 0, 1, \ldots)$. We say the join in a partition lattice is of type-n if for any Θ, Φ, $\Theta \vee \Phi = \Psi_n$. B. Jónsson showed in [12] that a lattice $\mathcal{L}$ is modular iff $\mathcal{L}$ is isomorphic to a partition lattice in which the join is of type-2. $\mathrm{Con}(\mathfrak{A})$ is a partition lattice but it is a special kind of partition lattice. So a natural and non-trivial question arises which is answered by Theorem 3.

Theorem 3. (G. Grätzer and W. A. Lampe [7]): If $\mathcal{L}$ is a modular algebraic lattice, then there is a finitary algebra $\mathfrak{A}$ such that $\mathrm{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$ and the join in $\mathrm{Con}(\mathfrak{A})$ is of type-2.

Incorrect proofs for the above theorem appeared in [9] and [21].

The algebra $\mathfrak{A}$ in the proof is unary and has $|C| \cdot \aleph_0$ elements and operations where C is the set of compact elements of $\mathcal{L}$. One can ask the familiar questions about the number and kind of operations required for this representation.

The new techniques of [16] were essential to the proof of Theorem 3. Incidentally, the join in $\mathrm{Con}(\mathfrak{A})$ is "automatically" of type-3 for the particular algebra $\mathfrak{A}$ in the proof of Theorem 1 given in [16]. The same is probably true for the other proofs.

By generalizing the technique in the proof of Theorem 3 we can make the algebra $\mathfrak{A}$ in Theorem 2 be such that the join in $\mathrm{Con}(\mathfrak{A})$ is of type-n and not type n-1 for any $n \geq 3$. We can also make the join in $\mathrm{Con}(\mathfrak{A})$ be of "type ω" - i.e. not of type n for any n. If $\mathfrak{G}$ is the one-element group and $\mathcal{L}_0$ is modular, we can construct an $\mathfrak{A}$ for Theorem 2 such that the join in $\mathrm{Con}(\mathfrak{A})$ is of type-2. Another problem is: what are the automorphism groups of algebras having modular congruence lattices in which the join is of type-2?

As mentioned in the introduction, we also know that

Theorem 4: If $\mathcal{L}$ is a complete lattice, then there is an algebra $\mathfrak{A}$ such that $\mathrm{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

More generally, we know

Theorem 5. (G. Grätzer and W. A. Lampe [8]): If $\mathcal{L}$ is an $\mathfrak{m}$-algebraic lattice, then there is an algebra $\mathfrak{A}$ of characteristic $\mathfrak{m}$ such that $\mathrm{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

In general, the congruence lattice of an infinitary algebra is not a partition lattice. However, we can build the $\mathfrak{A}$ for the proof of Theorem 5 in such a way that $\mathrm{Con}(\mathfrak{A})$ is a partition lattice in which the join is of type-3.

Such a result is not automatic for Theorem 5 as it was for Theorem 1. In fact, one uses a generalization of the technique for Theorem 3.

Once again the algebra has very many operations, and it's not clear one needs so many.

Consider Theorems 2 and 3 and all their previously mentioned extensions. A natural question is, "Are all the straightforward generalizations of all these theorems to $\mathfrak{m}$-algebraic lattices and algebras of characteristic $\mathfrak{m}$ true?" The answer is yes. But the proofs are not exactly straightforward generalizations of the corresponding finitary case proofs. There is also a corresponding array of open problems.

A "master" construction from which all these theorems follow will appear in [8].

§4. THE BASIC METHOD

All the above mentioned theorems are proved using constructions that have their roots in the original construction by Grätzer and Schmidt for Theorem 1. In this section we will make some remarks about this method.

To some extent, the method is derived from the proof of the Birkhoff-Frink Theorem on $\mathrm{Sub}(\mathfrak{A})$. So we will start the discussion there. But first we need to define some more terms.

Let C be some family of subsets of the set A. C is a <u>closure system</u> iff given any family $(D_i \mid i \in I)$

with $D_i \in C$ for every $i \in I$ it also holds that $\cap(D_i \mid i \in I) \in C$. For $B \subseteq A$ we define the C-closure (or simply, closure) of B by $[B]_C = \cap(D \mid D \in C, B \subseteq D)$. Since $A \in C$, $B \subseteq [B]_C \in C$. B is closed iff $B = [B]_C \in C$. The closure system C is an algebraic closure system iff C is also closed under directed unions; i.e., if the family $(D_i \mid i \in I)$ is a directed partially ordered set (under set inclusion) and each $D_i \in C$, then $\cup(D_i \mid i \in I) \in C$. In an algebraic closure system a set is closed iff it contains the closure of each of its finite subsets. For a regular cardinal $\mathfrak{m}$ one can define an $\mathfrak{m}$-algebraic closure system to be a closure system in which a set is closed iff it contains the closure of each of its subsets having less than $\mathfrak{m}$ elements.

If C is an algebraic closure system, then $\langle C; \subseteq\rangle$ is an algebraic lattice. Conversely, any algebraic lattice is isomorphic to some $\langle C; \subseteq\rangle$ where C is an algebraic closure system. Similar statements hold for $\mathfrak{m}$-algebraic lattices and $\mathfrak{m}$-algebraic closure systems.

Let C be an algebraic closure system on the set A. It is easy to describe a family F of finitary operations on A such that $C = \mathrm{Sub}(\langle A; F\rangle)$. In particular, for each finite sequence $a_0, \ldots, a_n$ of elements of A such that $a_n \in [a_0, \ldots, a_{n-1}]_C$ define an n-ary operation $f_{a_0, \ldots, a_n}$

by $f_{a_0, \ldots, a_n}(a_0, \ldots, a_{n-1}) = a_n$ and $f_{a_0, \ldots, a_n}(x_0, \ldots, x_{n-1}) = x_0$ otherwise. One takes F to be the family of all such operations.

Suppose now you have some algebraic lattice $\mathcal{L}$ that you want to represent as $\mathrm{Sub}(\mathfrak{A} \times \mathfrak{A})$. A first step is to find some algebraic closure system C on a set of the form $B \times B$ where $\mathcal{L}$ is isomorphic to $\langle C; \subseteq \rangle$. Obviously, one then should try the approach from the preceding paragraph. So for each $\langle a_0, b_o \rangle, \ldots, \langle a_n, b_n \rangle$ with $\langle a_n, b_n \rangle \in [\langle a_0, b_o \rangle, \ldots, \langle a_{j-1}, b_{n-1} \rangle]_C$ one defines an operation f on B with $f(a_0, \ldots, a_{n-1}) = a_n$ and $f(b_0, \ldots, b_{n-1}) = b_n$ and $f(x_0, \ldots, x_{n-1}) = x_0$ otherwise. Unfortunately, this doesn't work. Such an f has some unwanted side effects. In particular $f(\langle a_0, c_0 \rangle, \ldots, \langle a_{n-1}, c_{n-1} \rangle) = \langle a_n, c_0 \rangle$ and it may happen of course that $\langle a_n, c_0 \rangle \notin [\langle a_0, c_0 \rangle, \ldots, \langle a_{n-1}, c_{n-1} \rangle]_C$. So one drops the statement "$f(x_0, \ldots, x_{n-1}) = x_0$ otherwise" and leaves f undefined otherwise. One can take B together with these partly defined operations and form a "partial algebra" $\mathfrak{B}$. One can extend $\mathfrak{B}$ to the "algebra freely generated by $\mathfrak{B}$" ($\underset{\sim}{F}(\mathfrak{B})$) by filling in the "tables" for the operations as freely as possible. The subalgebras generated by subsets of $B \times B$ in $\underset{\sim}{F}(\mathfrak{B}) \times \underset{\sim}{F}(\mathfrak{B})$ are "right". But there are many new subsets that don't generate the "right" subalgebras. So add some new partial operations to take care of this. Freely generate. Repeat ad infinitum. Take the direct limit, and call it $\mathfrak{A}$. $\underset{\sim}{\text{Sub}}(\mathfrak{A} \times \mathfrak{A})$ is isomorphic to $\mathcal{L}$. (Actually one must choose the initial C so that the "diagonal" is the smallest member.) (That this works is shown in [6], essentially. See [11] also.)

Now suppose you want an $\mathfrak{A}$ so that $\underset{\sim}{\text{Con}}(\mathfrak{A})$ is isomorphic to the algebraic lattice $\mathcal{L}$. It is easy to check that $\text{Con}(\mathfrak{A})$ is always an algebraic closure system on $A \times A$. So one might look for a set B and some algebraic closure system C on $B \times B$ such that each member of C is an equivalence relation on B and such that $\langle C; \subseteq\rangle$ is isomorphic to $\mathcal{L}$. One could then hope to proceed as in the preceding paragraph. Unfortunately, transitivity rears its ugly head, and that idea doesn't work either. The following modification does work. Given $\langle a_0, b_0\rangle, \ldots, \langle a_n, b_n\rangle$ with $\langle a_n, b_n\rangle \in [\langle a_0, b_0\rangle, \ldots, \langle a_{n-1}, b_{n-1}\rangle]_C$ one defines three partial operations, say f, g, h, with

$f(a_0, \ldots, a_{n-1}) = a_n$, $f(b_0, \ldots, b_{n-1}) = g(b_0, \ldots, b_{n-1})$, $g(a_0, \ldots, a_{n-1}) = h(a_0, \ldots, a_{n-1})$ and $h(b_0, \ldots, b_{n-1}) = b_n$. Now when Θ is a congruence relation with $a_i \equiv b_i\ (\Theta)$ for $0 \leq i \leq n-1$ then under Θ we have $a_n = f(a_0, \ldots, a_{n-1})$ $\equiv f(b_0, \ldots, b_{n-1}) = g(b_0, \ldots, b_{n-1}) \equiv g(a_0, \ldots, a_{n-1})$ $= h(a_0, \ldots, a_{n-1}) \equiv h(b_0, \ldots, b_{n-1}) = b_n$. Transitivity gives us the desired result, $a_n \equiv b_n\ (\Theta)$. Now if one replaces each partial operation of the proceeding paragraph by three partial operations (as in this paragraph), and if one otherwise proceeds as in the preceeding paragraph, one then obtains an algebra $\mathfrak{A}$ with $\underset{\sim}{\text{Con}}(\mathfrak{A})$ isomorphic to $\mathcal{L}$.

Now let us go back to the $\mathfrak{B}$ and the $\underset{\sim}{F}(\mathfrak{B})$ above. Each congruence relation Θ of $\mathfrak{B}$ has an extension $\underset{\sim}{F}(\Theta)$ to a congruence of $\underset{\sim}{F}(\mathfrak{B})$. It is fairly obvious that if the ideas are going to work then one must have $\underset{\sim}{F}(\Theta \cap \Phi) = \underset{\sim}{F}(\Theta) \cap \underset{\sim}{F}(\Phi)$. Unfortunately, this fails in general. This is the technical problem that is cured by using a triple of operations in place of each "natural" operation. This problem is caused by transitivity.

So it becomes important to discover lemmas giving sufficient conditions on a partial algebra $\mathfrak{B}$ so that $\underset{\sim}{F}(\cap(\Theta_i \mid i \in I)) = \cap(\underset{\sim}{F}(\Theta_i) \mid i \in I)$. Such a lemma was implicit in [9]. It was made explicit in both [14] and [21]. But this lemma was true only if $\mathfrak{B}$ was a unary partial algebra. A lemma of this sort for arbitrary finitary partial algebras appears in [17]. This made Theorems 2 and 3 possible. (There are some other innovations required also.)

One would hope that the construction outlined above (when appropriately generalized) would work for proving Theorem 5. It does, but a new proof is required. One of the main new ingredients is a new, mildly complicated lemma giving sufficient conditions on an infinitary partial algebra $\mathfrak{B}$ so that $\underset{\sim}{F}(\cap\ \Theta_i \mid i \in I) = \cap(\underset{\sim}{F}(\Theta_i \mid i \in F)$ always holds.

The proofs of all the theorems use variations on the above construction.

The reader has probably noticed that the construction outlined above for Theorem 1 gives an algebra $\mathfrak{A}$ having n-ary operations for every $n > 0$. Yet it was stated in §3 that the algebra $\mathfrak{A}$ used in the proof had only unary operations. One can do this by starting with a C such that an equivalence relation Θ is closed iff it contains the closure of its one element subsets. If $\mathcal{L}$ is algebraic, such a C exists. As previously noted, Grätzer and Schmidt were forced to do this because their techniques were valid only for unary partial algebras.

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REPRESENTATIONS OF FINITE LATTICES AS PARTITION LATTICES ON FINITE SETS

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§ 0. A lattice is a set with two associative commutative and idempotent binary operations $\vee$ (meet) and $\wedge$ (join) satisfying

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x .$$

We put $x \leq y$ if $x \vee y = y$ and $x < y$ if $x \leq y$ and $x \neq y$. We consider here only lattices L with a least element 0_L and a greatest element 1_L. A sublattice of a lattice L is a subset X of L such that $a \in X$ and $b \in X$ imply that $a \wedge b \in X$ and $a \vee b \in X$. If 0_L and $1_L \in X$, X is called a normal sublattice.

For any set S we denote by $\Pi(S)$ the lattice of partitions on S, that is, the lattice of all equivalence relations on S with $\leq$ defined as set inclusion, relations being treated as sets of ordered pairs. Thus $1_{\Pi(S)} = S \times S$, $0_{\Pi(S)} = \{(x,x): x \in S\}$ and $a \wedge b = a \cap b$ for all $a,b \in \Pi(S)$.

A representation of a lattice L as a lattice of partitions is an isomorphism $\varphi: L \to \Pi(S)$. Then we call φ a representation of L on S. The representation φ is called normal if $\varphi(L)$ is a normal sublattice of $\Pi(S)$. For each lattice L, let $\mu(L)$ be the least cardinal μ such that L has a representation on S, where $|S| = \mu$. Whitman has shown [10] that $\mu(L) \leq \aleph_0 + |L|$. A well-known and still unsolved problem of Birkhoff [2, p. 97] is whether $\mu(L)$ is finite whenever L is finite.

§ 1. For any $x \in \Pi(S)$ and $a,b \in S$ we write $a(x)b$ for $(a,b) \in x$. Let A and B be sets such that $A \cap B = \{v\}$. Let L and M be normal sublattices of $\Pi(A)$ and $\Pi(B)$, respectively. For $x \in L$ and $y \in M$, let $x \circ y$ denote the partition of $A \cup B$ defined by $a(x \circ y)b$ if and only if $a(x)b$ or $a(y)b$ or both $a(x)v$ and $b(y)v$.

Theorem 1. The set N of all partitions of the form $x \circ y$ with $x \in L$ and $y \in M$ is a normal sublattice of $\Pi(A \cup B)$ and this lattice is isomorphic to $L \times M$.

Proof. Clearly the map $\varphi\colon L \times M \to N$ given by $\varphi(x,y) = x \circ y$ is a bijection. We need only establish for all $x,u \in L$ and $y,v \in M$ the equations

(i) $1_{\Pi(A)} \circ 1_{\Pi(B)} = 1_{\Pi(A \cup B)}$,

(ii) $0_{\Pi(A)} \circ 0_{\Pi(B)} = 0_{\Pi(A \cup B)}$,

(iii) $(x \circ y) \vee (u \circ v) = (x \vee u) \circ (y \vee v)$,

(iv) $(x \circ y) \wedge (u \circ v) = (x \wedge u) \circ (y \wedge v)$.

These equations can be proved by examining all possible special cases. In place of (iii) and (iv) it is sufficient to prove the cases

(v) $x \circ y = (x \circ 0_M) \vee (0_L \circ y) = (x \circ 1_M) \wedge (1_L \circ y)$,

$$\text{(vi)} \qquad \begin{cases} (x \circ 0_M) \vee (u \circ 0_M) = (x \vee u) \circ 0_M , \\ (0_L \circ y) \vee (0_L \circ v) = 0_L \circ (y \vee v) , \\ (x \circ 1_M) \wedge (u \circ 1_M) = (x \wedge u) \circ 1_M , \\ (1_L \circ y) \wedge (1_L \circ v) = 1_L \circ (y \wedge v) \end{cases}$$

which are obvious. We prove (iii) from (v) and (vi) as follows:

$$\begin{aligned} (x \circ y) \vee (u \circ v) &= (x \circ 0_M) \vee (0_L \circ y) \vee (u \circ 0_M) \vee (0_L \circ v) \\ &= (x \circ 0_M) \vee (u \circ 0_M) \vee (0_L \circ y) \vee (0_L \circ v) \\ &= ((x \vee u) \circ 0_M) \vee (0_L \circ (y \vee v)) \\ &= (x \vee u) \circ (y \vee v) . \end{aligned}$$

The remaining facts are established in a similar way.

Corollary 2. If L is a sublattice of the product of the lattices L_i $(i = 1,\ldots,k)$, then

$$\mu(L) \le \sum_{i=1}^{k} \mu(L_i) - k + 1 .$$

Proof. The proof follows directly from Theorem 1 by induction.

Theorem 3. If L is a subdirect product of M and P, if $\mu(M)$ and $\mu(P)$ are finite and if $(0_M, 1_P) \in L$, e.g., $L = M \times P$, then

$$\mu(L) = \mu(M) + \mu(P) - 1 .$$

Proof. For each $x \in M$ there exists a $y_x \in P$ such that $(x, y_x) \in L$. Similarly, for each $y \in P$ there exists an $x_y \in M$ such that $(x_y, y) \in L$. Thus for each $x \in M$ and $y \in P$ we have

$(0_M,y) = (0_M,1_P) \wedge (x_y,y) \in L$ and $(x,1_P) = (0_M,1_P) \vee (x,y_x) \in L$.

By Corollary 2, we know that $\mu(L) \leq \mu(M) + \mu(P) - 1$. Suppose that φ is a representation of L on a set T with $\mu(L)$ elements. Suppose that $\varphi(0_M,1_P)$ has k equivalence classes $A_1,A_2,\ldots,A_k$ of cardinalities $n_1,n_2,\ldots n_k$. Let P_{A_i} be the lattice of partitions of A_i formed by restricting the elements $\varphi(0_M,y)$ with $y \in P$ to A_i , that is, $P_{A_i} = \{\varphi(0_M,y)|_{A_i} : y \in P\}$. Let $\varphi(y) = (\varphi(0_M,y)|_{A_1},$ $\varphi(0_M,y)|_{A_2},\ldots,\varphi(0_M,y)_{A_k})$. Then φ is an isomorphism of P into $P_{A_1} \times \ldots \times P_{A_k}$ and thus Corollary 2 yields

$$\mu(P) \leq \sum_{i=1}^{k} n_i - k + 1 = \mu(L) - k + 1 .$$

On the other hand, M is isomorphic to $\{(x,1) | \; x \in M\} \subseteq L$. Thus M can be represented on $T/(\varphi(0_M,1_P))$ (T factored by the equivalence relation $\varphi(0_M,1_P)$), so $k \geq \mu(M)$. Hence

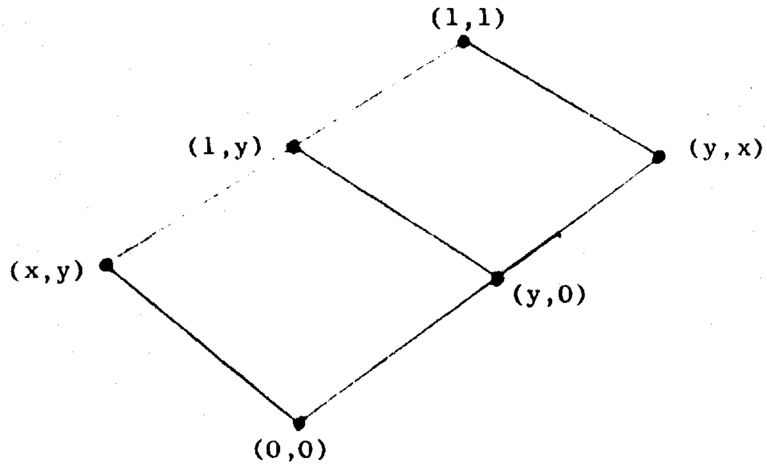
$$\mu(L) \geq \mu(P) + k - 1 \geq \mu(P) + \mu(M) - 1 .$$

Corollary 4. If $\mu(L)$ is finite and L is a sublattice of $\Pi(S)$, where $|S| = \mu(L)$, then L is a normal sublattice. Thus a minimum finite representation is a normal representation.

Proof. Since L can be represented on $S/0_L$, the fact that $\mu(L)$ is minimum implies that $0_L = 0_{\Pi(S)}$. If 1_L has equivalence classes $A_1,A_2,\ldots,A_k$, then L is isomorphic to a sublattice of the product of the L_{A_i} . Corollary 2 gives $\mu(L) \leq \sum_{i=1}^{k} |A_i| - k + 1 = \mu(L) - k + 1$, a contradiction unless $k = 1$. Thus $1_L = 1_{\Pi(S)}$.

Remark 1. By Theorem 3, the problem of finding $\mu(L)$ for all finite lattices L reduces to the determination of $\mu(L)$ for all finite directly indecomposable L's. This reduces this problem for various special classes of lattices: Dilworth [3] has shown that every finite relatively complemented lattice is a product of simple lattices. This applies also to finite geometric lattices since they can be characterized as finite relatively complemented semi-modular lattices [2; p. 89]. Birkhoff has shown that every modular geometric lattice is a product of a Boolean algebra and projective geometries [2; § 7]. Dilworth (see [2; p. 97]) has shown that every finite lattice is isomorphic to some sublattice of a finite semi-modular lattice. Hartmanis [5] has shown both that every finite lattice is isomorphic to some sublattice of the lattice of subspaces of a geometry on a finite set and that every finite lattice is isomorphic to the lattice of geometries of a finite set. Jónsson [7] has shown that every finite lattice is isomorphic to a sublattice of a finite subdirectly irreducible lattice.

Remark 2. The assumption $(0_M, 1_P) \in L$ in Theorem 3 is essential. In fact, if C_n is the n-element chain and if $L = C_3 \times C_2$, then Theorem 3 gives $\mu(L) = 4$; however, by Figure 1, L is also isomorphic to a subdirect product of $\Pi(2) \times \Pi(2)$ and $\Pi(2) \times \Pi(2)$, which would lead to $\mu(L) = 5$ if Theorem 3 applied.



<u>Figure 1</u>.

<u>Remark 3</u>. Let $L \lhd \Pi(n)$ mean that L has a normal representation on n. Theorem 1 shows that $\Pi(\ell) \times \Pi(\ell) \lhd \Pi(2\ell - 1)$. Since $\Pi(\ell) \lhd \Pi(\ell) \times \Pi(\ell)$, this suggests the question: For what ℓ and m is $\Pi(\ell) \lhd \Pi(m)$? If $\Pi(\ell) \lhd \Pi(\ell_1)$ and $\Pi(\ell) \lhd \Pi(\ell_2)$, then $\Pi(\ell) \lhd \Pi(\ell_1 + \ell_2 - 1)$. Since $\Pi(3) \lhd \Pi(4)$, we have $\Pi(3) \lhd \Pi(m)$ for all $m \geq 3$. Ralph McKenzie has proved (private communication) that $\Pi(\ell) \lhd \Pi(\ell + 1)$ does not hold for $\ell \geq 4$.

§ 2. We now examine μ for some special lattices. We recall that by a <u>complement</u> of x in a lattice L is meant an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

<u>Lemma 5</u>. If $P_1, P_2, \ldots P_k$ and Q are partitions of a set S with n elements and $P_1 \vee \ldots \vee P_k = Q$, then $\sum_{i=1}^{k} |S/P_i| \leq n(k-1) + |S/Q|$ in addition, $P_i \vee P_j = Q$ for all $i \neq j$, then $\sum_{i=1}^{k} |S/P_i| \leq \frac{k}{2}(n + |S/Q|)$.

Proof. For every $A \in S/P_i$ $(i = 1,2,\ldots,k)$ form a path through all points of A. Thus S obtains a graph structure and by $P_1 \vee \cdots \vee P_k = Q$, this graph has $\ell = |S/Q|$ connected components containing, in some order, $n_1, n_2, \ldots, n_\ell$ points. Since a connected graph with m points has at least $m-1$ edges,

$$\sum_{i=1}^{k} \sum_{A \in S/P_i} (|A| - 1) \geq \sum_{j=1}^{\ell} (n_j - 1) \; ;$$

$$\sum_{i=1}^{k} \left(\sum_{A \in S/P_i} |A| - |S/P_i| \right) \geq n - \ell \; ;$$

$$\sum_{i=1}^{k} (|S| - |S/P_i|) \geq n - |S/Q| \; ;$$

$$kn - \sum_{i=1}^{k} |S/P_i| \geq n - |S/Q| \; ;$$

$$\sum_{i=1}^{k} |S/P_i| \leq n(k-1) + |S/Q| \; .$$

Now suppose $P_i \vee P_j = Q$ for all $i \neq j$. Then by the last equation with $k = 2$, for all $i \neq j$, $|S/P_i| + |S/P_j| \leq n + |S/Q|$. Hence we have

$$\begin{aligned} (k-1) \sum_{i=1}^{k} |S/P_i| &= \sum_{i \neq j} (|S/P_i| + |S/P_j|) \\ &= \binom{k}{2} (n + |S/Q|) \; . \end{aligned}$$

The lemma follows.

Theorem 6. Consider the lattice $L(\ell,m)$ consisting of 0 and 1 and of two chains $P_1 > \dots > P_\ell$ of length ℓ and the other $Q_1 > \dots > Q_m$ of length m, such that P_i and Q_j are complementary for all i and j (see Figure 2). If $\ell > 1$, then

$$\mu(L(\ell,m)) = \ell + m - 1 + \{2\sqrt{\ell + m - 2}\} .$$

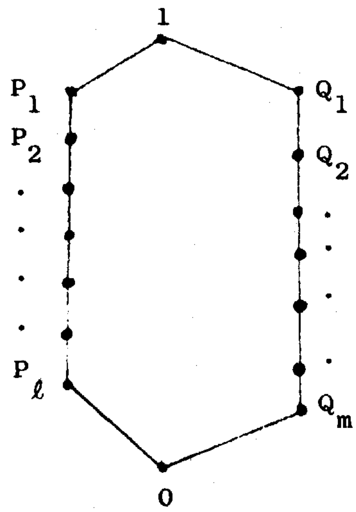


Figure 2.

Proof. Here, the symbol $\{x\}$ denotes the least integer not less than x. We suppose that $k = |P_1| \le |Q_1|$. Then $|P_\ell| \ge |P_1| + \ell - 1$ and $|Q_m| \ge |Q_1| + m - 1$. By Lemma 5, if $\mu(L) = n$, then

$$n + 1 \ge |P_\ell| + |Q_m| \ge |P_1| + \ell - 1 + |Q_1| + m - 1 .$$

Letting $x = \ell + m$, we have

$$k \le |Q_1| \le n + 3 - k - x .$$

Since $P_1 \wedge Q_1 = 0$, no class of Q_1 can have more than k elements. Thus

$$n \le k|Q_1| \le k(n + 3 - k - x) .$$

Since the maximum of the right hand side of this equation occurs when $k = \frac{1}{2}(n+3-x)$,

$$n \le \left(\frac{n+3-x}{2}\right)^2 .$$

Solving this equation, we find that

$$n \ge x - 1 + 2\sqrt{x-2} .$$

We first demonstrate a representation of $L(\ell,1)$. Let k be the first integer such that $k^2 \ge \ell + 2\sqrt{\ell - 1}$ $(k = 1 + \{\sqrt{\ell-1}\})$. Let n be the initial segment of length $\ell + \{2\sqrt{\ell-1}\}$ in the lexicographic ordering on $\mathbb{Z}_k \times \mathbb{Z}_k$. The partition P_1 on n is defined by $((x,y),(u,v)) \in P_1$ if and only if $x = u$. The partition Q_1 on n is defined by $((x,y),(u,v)) \in Q_1$ if and only if $y = v$. (Note that $\ell \ge 2$ implies that $k \ge 4$ and thus $P_1 \ne Q_1$.) The partition P_ℓ is defined by $((s,y),(u,v)) \in P_\ell$ if and only if either $x = 0 = u$ or $(x,y) = (u,v)$. The partitions P_i with $1 < i < \ell$ are formed by interpolation between P_1 and P_ℓ (separating off each of the singletons in P_ℓ one at a time from P_1). We must verify that a sufficient number of partitions can be formed in this way. Since $|P_\ell| = n - k + 1$ and $|P_1| = \{\frac{n}{k}\}$, if all possible interpolations were made, the length of the chain from P_1 to P_ℓ would be

$$p = n - k + 1 - \{\tfrac{n}{k}\} + 1 .$$

If $\{\frac{n}{k}\} \le k - 1$, we have

$$p \ge \ell + 1 + \{2\sqrt{\ell-1}\} - 2\{\sqrt{\ell-1}\} \ge \ell .$$

If $\{\frac{n}{k}\} = k$, we have

$$p = \ell + \{2\sqrt{\ell-1}\} - 2\{\sqrt{\ell-1}\} .$$

Suppose $s < \sqrt{\ell - 1} \le s + \frac{1}{2}$ for some integer s. Then $n = \ell + 2s + 1$, $k = s + 2$ and $\ell \le s^2 + s + \frac{5}{4}$. Since ℓ is an integer, $\ell \le s^2 + s + 1$ and thus

$$n \le s^2 + 3s + 2 = k(k-1) .$$

This gives $\{\frac{n}{k}\} = k - 1$, a contradiction. Thus $\{2\sqrt{\ell - 1}\} = 2\{\sqrt{\ell - 1}\}$ and hence $p = \ell$.

To complete the proof, we show that $L(\ell - 1, m + 1)$ can be represented on the same set as $L(\ell, m)$. Suppose $\ell \ge 2$ and $P_\ell \in L(\ell, m)$ has classes C_i, $1 \le i \le n$. Since $P_{\ell-1} \ge P_\ell$, we may assume that $P_{\ell-1}$ has a class containing $C_1 \cup C_2$. Since $P_{\ell-1} \wedge Q_m = 0$, for every $x \in C_1$ and $y \in C_2$, $(x,y) \notin Q_m$. Consider a shortest $P_\ell - Q_m$ path $x_1 x_2 \dots x_n$ $(n \ge 3)$ from C_1 to C_2. Then $x_1 \in C_1$ and $x_n \in C_2$ but $x_i \notin C_1 \cup C_2$, $2 \le i < n$. Thus $(x_1, x_2) \in Q_m$. Let $Q_{m+1} \le Q_m$ be the partition defined by: for all $x, y \ne x_1$, $(x,y) \in Q_{m+1}$ if and only if $(x,y) \in Q_m$; for all x, $(x, x_1) \in Q_{m+1}$ if and only if $x = x_1$. To show $P_{\ell-1} \vee Q_{m+1} = 1$, we need only show $(x_1, x_2) \in P_{\ell-1} \vee Q_{m+1}$ for then $P_{\ell-1} \vee Q_{m+1} \ge P_{\ell-1} \vee Q_m = 1$. Since the $P_\ell - Q_m$ path $x_2 \dots x_n$ does not contain x_1, it is a $P_\ell - Q_{m+1}$ path. Since $(x_n, x_1) \in P_{\ell-1}$, $x_2 \dots x_n x_1$ is a $P_{\ell-1} - Q_{m+1}$ path from x_2 to x_1.

We now consider the lattice L_n of subspaces of the geometry G_n with n points and 1 line. L_n consists of n mutually complementary elements and 0 and 1 (see Figure 3). Hartmanis [6] has shown that $\mu(L_n) \le 2p$ where p is the first prime larger than n. We shall prove $\mu(L_n) \leq p$, where p is the first prime not less than n (see Theorems 7, 8 and 9 below).

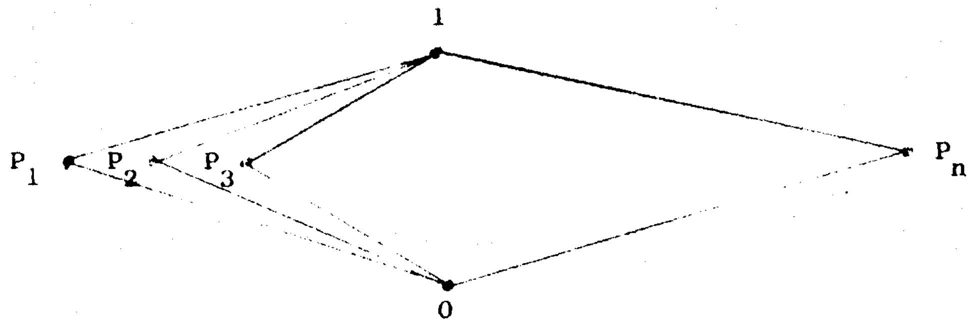


Figure 3.

Theorem 7.
$$\mu(L_n) \geq \begin{cases} n+1 \ ; & n \ \text{even} \\ n \ ; & n \ \text{odd}. \end{cases}$$

Proof. Suppose L_n can be represented as a sublattice L of the lattice of partitions of m. Each non-trivial $P \in L$ defines a set of edges $L_P = \{\{a,b\}: (a,b) \in P, \ a \neq b\}$. Since $P \wedge Q = 0$ and $P \vee Q = 1$ when $P \neq Q$, we have that $L_P \cup L_Q$ is a connected graph. Thus

(i) $$|L_P| + |L_Q| = |L_P \cup L_Q| \geq m-1,$$

(ii) $$\sum_{P \in L} |L_P| \leq \frac{1}{2} m(m-1).$$

From (i) we get

$$(n-1) \sum_{P \in L} |L_P| = \sum_{P \neq Q} (|L_P| + |L_Q|) \geq \frac{n(n-1)}{2} (m-1).$$

Hence from (ii),

$$\frac{1}{2}\, m(m-1) \ge \sum_{P \in L} |L_P| \ge \frac{1}{2}\, n(m-1)$$

which yields $m \ge n$. Equality can occur only if $|L_P| + |L_Q| = m-1$ for all non-trivial $P \neq Q \in L$, which implies that $m-1$ is even whenever $m = n = 3$. Small cases are handled by inspection.

<u>Theorem 8</u>. The following four statements are equivalent:

(i) $\mu(L_{2n-1}) = 2n-1$;

(ii) The complete graph on $2n-1$ points, K_{2n-1}, can be edge-colored with $2n-1$ colors so that the union of any two color classes is a spanning path;

(iii) K_{2n} can be edge-colored with $2n-1$ colors so that the union of any two color classes is a spanning cycle;

(iv) The symmetric group on $2n$ elements, S_{2n}, contains a set $\{I_i : i = 1,2,\ldots,2n-1\}$ of involutions such that the group generated by I_i and I_j is transitive whenever $i \neq j$.

<u>Proof</u>. (i) $\leftrightarrow$ (ii). If we assume (ii), each color class is a partition, so (i) follows easily. Suppose (i) holds. As we have seen above $|L_P \cup L_Q| = 2n-2$ for all $P \neq Q$. Since $L_P \cup L_Q$ is connected, it must be a tree. Thus $|L_P| = n-1$ and L_P contains no cycles, that is, P is a maximum matching of the points of K_{2n-1}. (ii) now follows.

(ii) $\leftrightarrow$ (iii). Suppose K_{2n} has been $(2n-1)$ edge-colored so that the union of any two color classes is a spanning cyle. Clearly $K_{2n} \setminus \{v\}$ satisfies (ii). On the other hand, if K_{2n-1} has been $(2n-1)$ edge-colored so that the union of two color classes is a spanning path, each point misses one color and, by counting, each color misses one point. $K_{2n} = K_{2n-1} \cup \{\{v,a\}: a \in K_{2n-1}\}$ is $2n-1$ edge-colored by coloring $\{v,a\}$, $a \in K_{2n-1}$, with the color missing at a. It is easy to show that this coloring satisfies (iii).

(iii) $\leftrightarrow$ (iv). Each 1-factor of K_{2n} defines an involution on $2n$ and vice versa. Since the elements of the group generated by the involutions I and J have the form $\ldots IJIJ \ldots$, the union of two 1-factors spans K_{2n} if and only if the group generated by the corresponding involutions is transitive.

Theorem 9. The statement 8 (i) holds if n (see [1] and [8]) or $2n-1$ (see [1] and [9]) is a prime.

Remark 4. B. A. Anderson (private communication) has also shown that 8 (i) holds for $n = 8$ and $n = 14$. Thus the first unknown case is $n = 18$. We would like to know a similar result to Theorem 6 about a lattice $L(\ell_1, \ell_2, \ldots, \ell_w)$ consisting of 0 and 1 and of w chains $P_{i1} > \ldots > P_{i\ell_i}$, $1 \leq i \leq w$, such that P_{ij} and $P_{i'j'}$ are complementary when $i \neq i'$. However, the method of proof used in Theorem 6 gives only $\mu(L(\ell_1, \ldots, \ell_w)) \geq f(\bar{\ell}, w)$ where $\bar{\ell} = w^{-1} \sum_{i=1}^{w} \ell_i$ and

$$f(\bar{\ell},w) = 2\bar{\ell} - 3 + 8\,\frac{w-1}{w^2} + 4\,\frac{\sqrt{w-1}}{w^2}\,\sqrt{4 + w^2(2\ell - 3)}\ .$$

Although this reduces to Theorem 6 when $w = 2$, for large values of w it is a very bad estimate since $\lim_{w\to\infty} f(\bar{\ell},w) = 2\bar{\ell} - 3$, an absurdity. Actually, proofs of this type seem to indicate that the best results for these lattices are obtained by partitions with nearly equal classes. For this reason, we mention the following theorem.

Theorem 10. L_{k+2} has a normal representation $\varphi: L_{k+2} \to \Pi(S)$, where $|S| = n^2$ such that $|S/\varphi(a)| = n$ and $|A| = n$ for each $A \in S/\varphi(a)$ whenever $a \in L_{k+2}$, $a \neq 0_{L_{k+2}}, 1_{L_{k+2}}$, if and only if there are k mutually orthogonal Latin squares of order n.

Proof. Suppose L exists. Let the partitions be $C_i = \{C_{i1}, \ldots, C_{in}\}$, $1 \leq i \leq k$, $A = \{A_1, \ldots, A_n\}$, and $B = \{B_1, \ldots, B_n\}$. We form the Latin square $L^i_{\ell m}$ as follows: let $L^i_{\ell m} = j$ if $C_{ij} \cap A_\ell \cap B_m \neq \phi$. The definition is possible since $A_\ell \cap B_m = \{x_{\ell m}\}$ for all ℓ and m, and given i, some C_{ij} must contain $x_{\ell m}$. Suppose $L^i_{\ell m} = L^i_{\ell' m} = j$. Then $C_{ij} \cap A_\ell \cap B_m \neq \phi$ and $C_{ij} \cap A_{\ell'} \cap B_m \neq \phi$, contradicting $A_\ell \cap A_{\ell'} = \phi$ unless $\ell = \ell'$. Similarly $L^i_{\ell m} = L^i_{\ell m'}$ if and only if $m = m'$. Thus $L^i_{\ell m}$ is a Latin square. Suppose $L^i_{\ell m} = L^i_{rs} = p$ and $L^j_{\ell m} = L^j_{rs} = q$ with $i \neq j$. Then

$$\begin{cases} C_{ip} \cap A_\ell \cap B_m = \{x_{\ell m}\} \\ C_{ip} \cap A_r \cap B_s = \{x_{rs}\} \\ C_{jq} \cap A_\ell \cap B_m = \{x_{\ell m}\} \\ C_{jq} \cap A_r \cap B_m = \{x_{rs}\} . \end{cases}$$

Thus $C_{jq} \cap C_{ip} = \{x_{\ell m}\} = \{x_{rs}\}$, so $\ell = r$ and $m = s$. Hence the $L^i_{\ell m}$ are mutually orthogonal Latin squares.

Conversely, suppose $\{L^i_{\ell m}\}_{i=1}^k$ is a set of mutually orthogonal Latin squares. We consider the n^2 elements in $\mathbb{Z}_n \times \mathbb{Z}_n$. We let $A_i = \{i\} \times \mathbb{Z}_n$ and $B_j = \mathbb{Z}_n \times \{j\}$. We put $(\ell, m) \in C_{ij}$ if and only if

$L^i_{\ell m} = j$. It is easily verified that the partitions $C_i = \{C_{i1},\dots,C_{in}\}$, $1 \le i \le k$, $A = \{A_1,\dots,A_n\}$, and $B = \{B_1,\dots,B_n\}$ generate the desired lattice.

Corollary 11. (See [4; p. 177]). The following statements are equivalent:

(i) The edges of the complete graph K_{n^2} on n^2 points can be decomposed into $n+1$ sets so that each set consists of n components isomorphic to K_n and so that the union of any two sets is a connected graph.

(ii) There exists a projective plane P_n of order n .

(iii) There are $n-1$ mutually orthogonal Latin squares of order n .

(iv) There is a partition lattice L on n^2 elements consisting of $n+1$ mutually complementary elements plus 0 and 1 such that each non-trivial partition has n classes of n elements.

Proof. We shall sketch the proof. The equivalence of (i) and (iv) follows from the method used in the proof of Theorem 7. That is, to each partition $P \neq 0,1$ in L there corresponds a set of edges $L_P = \{\{a,b\}: (a,b) \in P\}$. (Note that each of these partitions turns out to be nothing more than a parallel class of lines in an affine geometry.) The equivalence of (iii) and (iv) follows from the theorem. The proof of the equivalence of (i) and (ii) follows standard lines: Suppose (i) holds. To form P_n add to the points of K_{n^2} the points $c_1,\dots,c_{n+1}$, corresponding to the $n+1$ sets $C_1,\dots,C_{n+1}$. We suppose the components of C_i are $C_{i1},\dots,C_{in}$. The lines of P_n are then the sets $C_{ij} \cup \{c_i\}$, $i = 1,\dots,n+1$, and the set $\{c_1,\dots,c_{n+1}\}$. Conversely, if (ii) holds, let $\{c_1,\dots,c_{n+1}\}$ be a line in P_n . The points of K_{n^2} are then the points of $P_n \setminus \{c_1, \ldots, c_{n+1}\}$. The edge $\{x,y\}$ of K_{n^2} is in the set c_i if x , y and c_i are colinear in P_n .

§ 3 . By Whitman's Theorem (see § 0), every lattice is a sublattice of the lattice of all partitions of some set. If φ is a representation of a lattice L as a lattice of partitions of A , and B is a subset of A , then for every $x \in L$ let $\varphi_B(x)$ be the restriction of the partition $\varphi(x)$ to B . Of course, $\varphi_B(L)$ does not necessarily have to be a sublattice of L . Even if $\varphi_B(L)$ is a sublattice, φ_B does not have to be an isomorphism. If $\varphi_B(L)$ is a sublattice and φ_B is an isomorphism, then the subset B is called faithful.

Remark 5. Every representation of the lattice L_2 has a finite faithful subset. The simplest example of a finite lattice which has a representation without finite faithful subsets is L_3 . The representation is constructed as follows: the points of the set are the vertices of the regular triangular lattice on the plane. Three points form an equivalence class with respect to a given color if they are the vertices of a triangle which has this color (see Figure 4). It is clear that if we take any

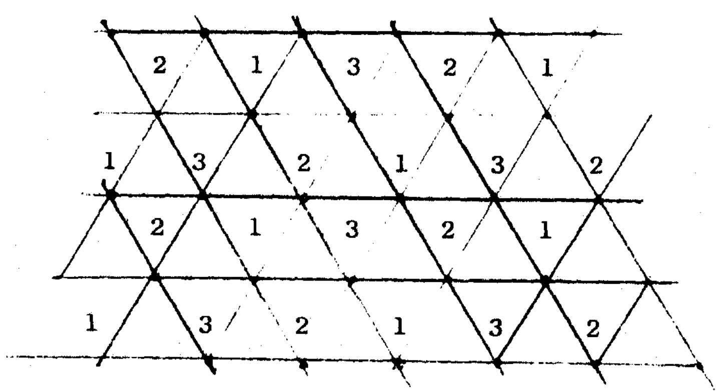


Figure 4.

finite subset S of this triangulation, there will be at least one vertex which appears in only one colored triangle, say color 1 . Thus this vertex is not $2 \vee 3$ equivalent to any other, so S cannot be a faithful subset. We can also show that the lattice of Figure 5 has a representation without finite faithful subsets.

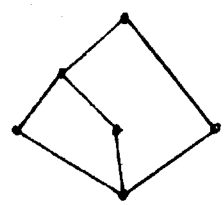


Figure 5.

There exists a finite distributive lattice with a representation without finite faithful subsets. The lattice generated by the partitions induced by the colors 1 , 2 and 3 in Figure 6 is isomorphic to $\{0,1\}^3$.

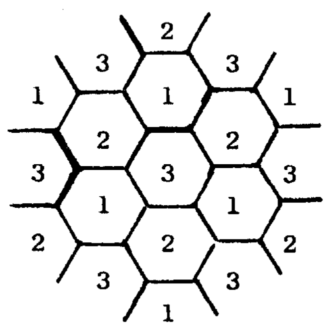


Figure 6.

The lattice L in Figure 7 is a finite lattice with an infinite representation without proper faithful subsets. Partitions A , B , A_1 and B_1 of $\mathbb{Z}$ are formed as follows: A has classes $\{2n , 2n+1\}$ for all $n \in \mathbb{Z}$, B has classes $\{2n-1 , 2n\}$ for all n , A_1 has classes $\{2n-1 , 2n+4\}$ for all n , and B_1 has classes $\{2n+2 , 2n-1\}$

for all n . It is clear that these partitions generate a lattice isomorphic to L . For any proper subset of $\mathbb{Z}$, one of the relations $A \vee B = 1$, $A_1 \vee B_1 = 1$ would fail, so this representation of L has no proper faithful subsets.

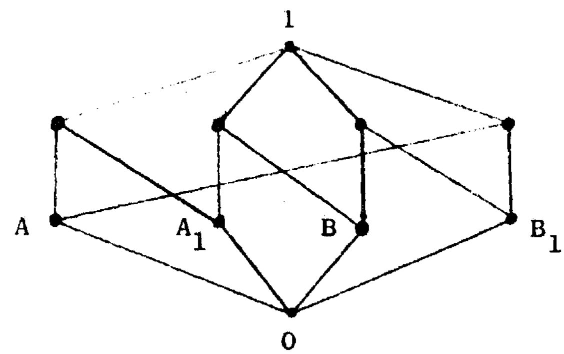


Figure 7.

Problems.

1. Suppose $P \subsetneq Q$ are lattices and P has a representation without finite faithful subsets. Does Q have such a representation? Can a given representation φ of P without finite faithful subsets be extended to a representation $\bar{\varphi}$ of Q such that $\bar{\varphi}$ also does not have finite faithful subsets?

2. Characterize the class of lattices which can be generated by colorings of tesselations of the plane.

3. (See Remark 3.) For what ℓ and m is $\Pi(\ell) \lhd \Pi(m)$?

4. (See Theorems 7, 8 and 9 and [1], [8] and [9].) Find $\mu(L_n)$ for all n .

5. (See Remark 4.) Find $\mu(L(\ell_1, \ell_2, \ldots, \ell_w))$ for all w-tuples of positive integers $(\ell_1, \ell_2, \ldots, \ell_w)$.

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SOME COMBINATORIAL ASPECTS OF LATTICE THEORY*

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This paper will discuss some new lattice-theoretic constructions of combinatorial interest. Throughout, all lattices will be assumed to be finite unless the contrary is stated, and most proofs will be omitted. Proofs and generalizations (e.g. to infinite lattices) are in the author's Doctoral Thesis [13].

After a few technical preliminaries we will discuss a basic representation theorem for lattices and give some applications of it, including a new characterization of distributive lattices and some combinatorial results having to do with the representation of lattices and posets by subsets of the power set of some given set. In Part II, we introduce the poset of join-irreducible and meet-irreducible elements of a lattice, a construction which bears the same relationship to the given lattice, as the poset of join-irreducible elements bears to the corresponding finite distributive lattice. After describing the properties of the poset of join-irreducible

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and meet-irreducible elements, we will give some applications of this construction, including the extension of the work of Crapo and Rota [7] on the factorization of relatively complemented lattices of finite length to all lattices of finite length. We will then discuss the enumeration of the elements of the free distributive lattice on n generators, a problem first proposed by Dedekind [8] in 1897.

Much of the work in Parts I and II has been stimulated by the following question. How much of the structure of a lattice is 'recoverable' from its join-irreducible and meet-irreducible elements? As we shall see, the answer to this question is that by concentrating only on certain relations between join-irreducible and meet-irreducible elements we are able to reconstruct the whole lattice, and can obtain information about the lattice which would be difficult to obtain from the whole lattice directly, such as its factorization.

I. THE BASIC REPRESENTATION THEOREM AND APPLICATIONS

We first introduce some notation. If n is an integer, by $\underline{n}$ we shall mean $\{1,\ldots, n\}$. Of course if $n \leq 0$, $\underline{n} = \emptyset$. If X is a set, we shall denote the cardinality of X by $|X|$, and the power set of X by 2^X. Note that we shall always consider 2^X to be a lattice in the obvious way. We will use $\leq$ and $<$ for set inclusion and proper set inclusion respectively. If L is a lattice, we denote by $J(L)$ the set of all join-irreducible elements of L (recall L is finite) and by $M(L)$ the set of all the meet-irreducible elements of L. $\wedge$ and $\vee$ denote meet and join respectively.

The following representation theorem will be our starting point. It has been used by Zaretskii [18] and is closely related to the dual of the representation by principal dual ideals due to Birkhoff and Frink [2]. It can be generalized quite a bit, and was discovered by the author while he was investigating the structure of the semigroup of binary relations ([14]).

THEOREM 1. Let L be a lattice. The map $f:L\to 2^{M(L)}$ given by $f(a) = \{y\in M(L) \mid y \not\geq a\}$ is injective and join-preserving (and hence order-preserving).

Theorem 1 has a number of consequences. The following corollary is obvious even without Theorem 1.

COROLLARY Let L be a lattice $|J(L)| = j$ and $|M(L)| = k$, then the length of $L \leq \min\{k,j\}$.

The following theorem gives a new combinatorial characterization of finite distributive lattices. It is well known that (c) below implies (a) and (b). But the converse seems to be new.

THEOREM 2. Let L be a finite lattice. The following are equivalent.

(a) L has length n, satisfies the Jordan-Dedekind chain condition, has n join-irreducible elements and n meet-irreducible elements.

(b) L has n join-irreducible elements, n meet-irreducible elements, and every connected (maximal) chain between I and 0 has length n.

(c) L is distributive and has n join-irreducible elements.

Proof: It is easy to see that (a) and (b) are equivalent and it is well known that (c) implies (a) (see Birkhoff [1]). Thus we need only show that (a) implies (c). Let L' be the dual lattice and observe that L' also satisfies (a). From Theorem 1 it follows that we can consider L and L' to be join-sublattices of $\underline{2}^{\underline{n}}$, where by a join-sublattice we mean a subset of $\underline{2}^{\underline{n}}$ closed under arbitrary join (union). Any such subset is of course a lattice with join being union but the meet of two elements is not in general the intersection.

We claim that L and L' are sublattices of $2^{\underline{n}}$ and hence distributive. Let $f:L\to L'$ be an anti-isomorphism.

Note that from (a) it follows that the height of any element of L or L' is equal to its cardinality. We now make a series of observations.

(i) $|f(x)| = n - |x|$ for all $x\in L$, since a connected chain from x to $\underline{n}$ is mapped into a connected chain from ϕ to $f(x)$.

(ii) $|f(y)| - |f(x) \wedge_{L'} f(y)| = |x| - |x \cap y|$ for all $x,y\in L$, since $|f(y)| - |f(x) \wedge_{L'} f(y)|$

$= (n - |y|) - (n - |x \cup y|)$

$= |x \cup y| - |y| = |x| - |x \cap y|$.

(iii) $|f(y)| - |f(x) \cap f(y)| = |x| - |x \wedge_L y|$ for all $x,y\in L$, since $|f(y)| - |f(x) \cap f(y)|$

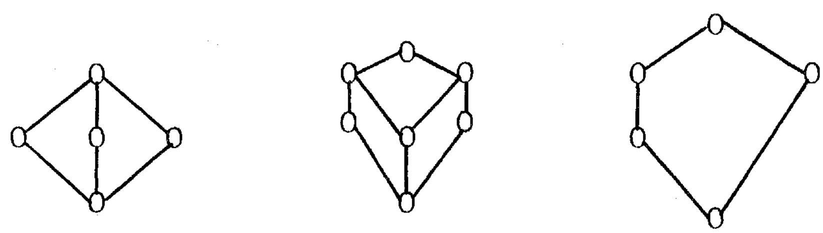
$= |f(x) \cup f(y)| - |f(x)| = (n - |x \wedge_L y|) - (n - |x|)$

$= |x| - |x \wedge_L y|$.

We know that $x \wedge_L y \leq x \cap y$ and $f(x) \wedge_{L'} f(y) \leq f(x) \cap f(y)$ for all $x,y\in L$. Thus $|x| - |x \wedge_L y| \geq |x| - |x \cap y|$ and $|f(y)| - |f(x) \cap f(y)| \leq |f(y)| - |f(x) \wedge_{L'} f(y)|$. But from the last inequality, (ii) and (iii) it follows that $|x| - |x \wedge_L y| \leq |x| - |x \cap y|$. Hence, $|x \wedge_L y| = |x \cap y|$ and $x \wedge_L y = x \cap y$, implying that L is a sublattice of $2^{\underline{n}}$. Similarly, L' is a sublattice.

Remark. Theorem 2 assists in identifying distributive lattices from their Hasse diagrams, since it is usually easy to identify the join-irreducible and meet-irreducible elements, as well as to determine whether a given lattice satisfies the Jordan-Dedekind chain condition. Certainly, a computer can easily be programmed to identify finite distributive lattices.

Theorem 2 can be stated as follows: a finite lattice L with n join-irreducible elements is distributive if and only if (i) it satisfies the Jordan-Dedekind chain condition, (ii) the number of meet-irreducible elements equals the number of join-irreducible elements, and (iii) the length of L is equal to the number of join-irreducible elements. The following three examples show the independence of conditions (i), (ii), and (iii).



(a) (b) (c)

Here $n=3$. (a) satisfies (i) and (ii) only, (b) (i) and (iii) only, and (c) (ii) and (iii) only.

COROLLARY 4. A finite modular lattice L is distributive if and only if its length is equal to $|J(L)|$.

Proof. It is well known that modular lattices satisfy the Jordan-Dedekind chain condition [1]. Also Dilworth has shown [1; 103] that $|J(L)|$ and $|M(L)|$ of any finite modular lattice are equal. Thus the corollary follows directly from Theorem 2 and these additional facts.

Definition. Let L be a lattice. By an embedding of L in $2^{\underline{n}}$ we mean an injective join-preserving map $f: L \to 2^{\underline{n}}$. We shall say that two embeddings f and g are distinct if $f(L) \neq g(L)$.

Theorem 1 shows that L can be embedded in $2^{\underline{n}}$ if $n \geq |M(L)|$. Actually, it is true that L can be embedded in $2^{\underline{n}}$ iff $n \geq |M(L)|$. This was first shown to be true by Zaretskii [18] and later discovered independently by the author [14]. We will not prove this result here.

An obvious question about embeddings is the following. Given a lattice L and an integer n how many distinct embeddings of L in $2^{\underline{n}}$ are there? The results above only tell us when an embedding is possible. The answer to this question follows from work done in exploring the structure of the semigroup of binary relations done by Brandon Butler, D. W. Hardy and the author [3, 4]. It can also be derived from Zaretskii's work [18]. For details about the relationship between lattices and the semigroup of binary relations see [15].

To avoid introducing too much additional theory we simply state the following theorem (which can be generalized to the case of arbitrary join-preserving maps between arbitrary complete lattices [13]).

THEOREM 3. Let L be a finite lattice such that $|L| = p$, $|M(L)| = k$. The number of distinct embeddings of L in $\underline{2}^{\underline{n}}$ is $(1/|\mathrm{Aut}\ L|) \sum_{i=0}^{k} (-1)^i \binom{k}{i}(p-i)^n$, where $\mathrm{Aut}\ L$ is the automorphism group of L. Note that the above quantity is 0 if $n < k$. A purely lattice-theoretic proof can be found in [13].

We will now consider the relationship between some of the material above, and the problem of computing the number of realizations of a given poset by a subset of $2^{\underline{n}}$ for some integer n. By this we mean that if we are given a finite poset P, we wish to know how many subsets of $2^{\underline{n}}$, considered as posets with inclusion being the order, there are which are isomorphic to P. This problem is treated in some detail by Hillman in [11]. We will briefly show how the theorems above apply to this problem. Our method does not always simplify the calculations involved, but it does give "geometrical" insight into the difficulties involved in representing posets by sets.

Definition. Let L be a lattice and P a poset. Then $R(L,n)$ denotes the number of ways of representing L as a join-sublattice of $2^{\underline{n}}$, $R^*(P,n)$ denotes the number of realizations of P by subsets of $2^{\underline{n}}$, and $D(P)$ denotes the distributive lattice of all closed from below subsets of P, while $i:P\to D(P)$ denotes the canonical map $i(a) = \{b\varepsilon P | b \leq a\}$. A subset k of L is called a meet-sublattice of L if it is closed under arbitrary meets (recall that the empty meet is always I).

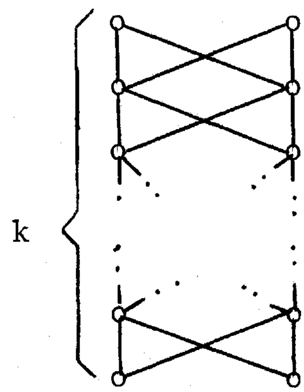
We note here that a meet-sublattice of a lattice is itself a lattice with respect to the induced order.

The key result for applying Theorem 3 to the representation of posets is the following.

THEOREM 4. Let P be a poset, and $\tilde{L}$ the set of all meet-sublattices of $D(P)$ which contain $i(P)$. Pick one representative from each isomorphism class of $\tilde{L}$, say $L_1, \ldots, L_k$. For each $i \varepsilon \underline{k}$ let $m_{L_i} = |\{Q \leq L_i | J(L_i) \leq Q$ and $P \simeq Q$ as posets$\}|$. Then $R^*(P,n) = \sum_{i=1}^{k} m_{L_i} R(L_i,n)$.

All of Hillman's results can be derived starting from Theorem 4, but we will not dwell on this here. Rather we will just give an example in which Theorem 4 supplies the answer more directly than any of Hillman's approaches.

EXAMPLE 1. Let P have the Hasse diagram



Then $R^*(P,n) = 2^{-k} \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (3k+1-i)^n$, since the only meet-sublattice of $D(P)$ containing $i(P)$ is $D(P)$ itself, and $|D(P)| = 3k+1$. Thus in certain cases the lattice method allows one to quickly group the essentials of the situation and arrive at the solution directly. This example illustrates the fact that Theorem 4 often allows one to see quickly how to calculate $R^*(P,n)$ and gives some idea of how complicated the calculation will be, as well as allowing one to calculate $R^*(P,n)$ for a whole class of related posets, as opposed to isolated cases. It is interesting to note that Theorem 4 shows why the poset representation problem is hard in general.

Namely, the poset representation problem involves the representation of a number of lattices which are not "obviously" related to one another. Thus we also see why the coefficients seem to vary so much in the cases that are known. However, Theorem 4 gives us enough information to describe the asymptotic behavior of $R^*(P,n)$, for a fixed P as $n \to \infty$. In particular we have the following corollaries.

COROLLARY 1. $R^*(P,n) \sim \frac{1}{|\mathrm{Aut}(P)|}|D(P)|^n$ asymptotically as $n \to \infty$.

The following corollary is an interesting special case of Corollary 1. It tells us the number of anti-chains of size k in $\underline{2}^{\underline{n}}$ and shows that as $n \to \infty$ almost every subset of $\underline{2}^{\underline{n}}$ of cardinality k is an anti-chain. In the next corollary, A_k is the poset corresponding to the Hasse diagram $\underbrace{0\ 0\ 0 \ldots\ldots 0}_{k}$.

COROLLARY 2. $R^*(A_k,n) \sim \frac{(2^k)^n}{k!} \sim \binom{2^n}{k}$ asymptotically (for fixed k) as $n \to \infty$.

II. THE POSET OF JOIN IRREDUCIBLE AND MEET IRREDUCIBLE ELEMENTS.

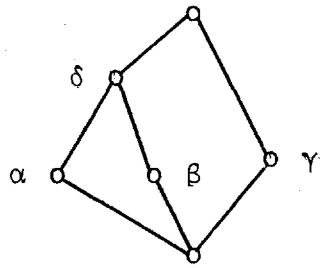
It is standard [1; p.59] that any finite distributive lattice is isomorphic with the ring of all order ideals of the partially ordered set consisting of its join-irreducible elements. Furthermore certain properties of the distributive lattice can be calculated directly from this poset of join-irreducible elements. In particular we have the following results which do not seem to have been generally considered. A proof of Theorem 5(a) can be found in [15].

THEOREM 5. Let L be a finite distributive lattice and P its poset of join-irreducible elements. Let $P = \{v_1, \ldots, v_t\}$. Then:

(a) The map $F:\mathrm{Aut}(P)\to\mathrm{Aut}(L)$ given by $F(f)(\sum_{\Delta} v_i) = \sum_{\Delta} f(v_i)$ for $\Delta \leq t$ is a group isomorphism, i.e., every element of $\mathrm{Aut}(P)$ extends naturally to an element of $\mathrm{Aut}(L)$.

(b) L is decomposable iff P is not connected and the irreducible factors of L may be gotten simply by considering the distributive lattices (i.e., the rings of closed from below subsets) associated with the connected components of P.

REMARK. Theorem 5 does not hold for arbitrary lattices. Consider the lattice L depicted by the following Hasse diagram.



$|\mathrm{Aut}(L)| = 2$ and L is indecomposable while $|\mathrm{Aut}(J(L))| = 6$ and $J(L)$ has 3 components.

We will now describe a poset which can be associated with all finite lattices and which has the same properties with respect to the original lattice that the poset of join-irreducibles has with respect to the corresponding distributive lattice.

<u>Definition</u>. Let L be a lattice. By $P(L)$ we mean the poset $J(L) \dot{\cup} M(L)$ (disjoint union) with the following order. Let $i_1 : J(L) \to P(L)$ and $i_2 : M(L) \to P(L)$ be the canonical injections. For $x, y \in P(L)$, $y > x$ iff (a) $y \in i_2(M(L))$, (b) $x \in i_1(J(L))$, and (c) $i_2^{-1}(y) \not\leq i_1^{-1}(x)$ in L. When talking about $P(L)$, we let $X_1 = i_1(J(L))$ and $X_2 = i_2(M(L))$. We call $P(L)$ the <u>poset of join irreducibles and meet irreducibles</u> of L or simply the <u>poset of irreducibles of L</u>.

$P(L)$ furnishes us with quite a bit of information about L. Since the proofs of the following theorems are somewhat involved we omit them and present the most important properties of $P(L)$.

THEOREM 6. Let L be a lattice and $P(L) = X_1 \cdot X_2$ its poset of irreducibles.

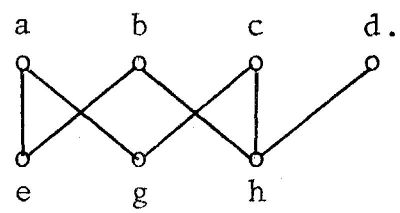
(a) Let $f: X_1 \to 2^{X_2}$ be given by $f(a) = \{b \varepsilon X_2 | b > a\}$. Then $L \simeq \Gamma_L \overset{\text{def}}{=} \{\bigcup_{w\varepsilon\Delta} w | \Delta \leq f(X_1)\}$. (Thus we can reconstruct L from $P(L)$.)

(b) $\mathrm{Aut}(P(L)) \simeq \mathrm{Aut}(L)$.

(c) L is decomposable iff $P(L)$ is not connected. Futhermore, the irreducible factors of L may be gotten by applying the procedure of (a) above to each connected component of $P(L)$.

(d) For each $x \varepsilon X_2$, let $T_x = \mathrm{g.l.b.}_{\Gamma_L} S_x$ where $S_x = \{U \varepsilon f(X_1) | x \varepsilon U\}$, where Γ_L and f are as in (a). Then L is distributive iff for all $V \varepsilon f(X_1)$, $V = {}_{x\varepsilon V}T_x$ iff for all $x \varepsilon X_2$, $x \varepsilon T_x$. To illustrate Theorem 6 we consider the following examples.

EXAMPLE 2. Thus if we construct $P(L)$, where L is the lattice in the remark after Theorem 5, we get



$i_1(\gamma) = h, i_1(\beta) = g, i_1(\alpha) = e,$

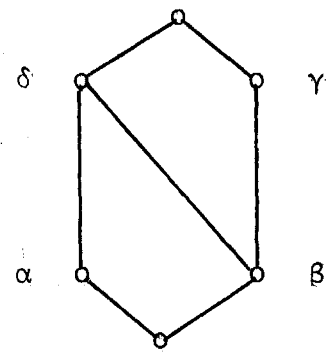
$i_2(\gamma) = a, i_2(\beta) = b, i_2(\alpha) = c,$

and $i_2(\delta) = d$.

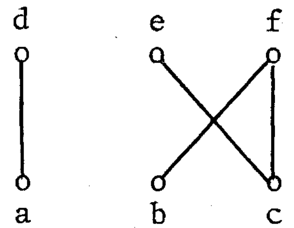
Thus $|\mathrm{Aut}(P(L))| = 2 = |\mathrm{Aut}(L)|$, L is indecomposable.

If we consider $f(L) \leq 2^{\{a,b,c,d\}}$ we see that $f(e) = \{a,b\}$, $f(g) = \{a,c\}$, $f(h) = \{b,c,d\}$. Thus $T_a = \phi$, $T_b = \phi$, $T_c = \phi$, $T_d = \{b,c,d\}$. Consequently, L is not distributive, which of course is no surprise in this case.

EXAMPLE 3. Let L have the following Hasse diagram.

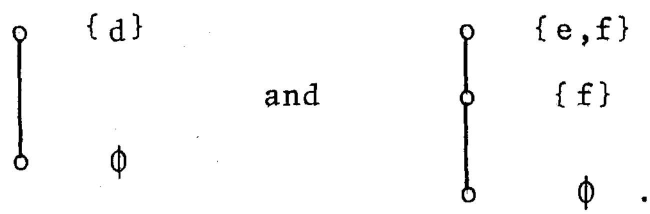


Then P(L) has the following diagram.



Here $i_1(\alpha) = a$, $i_2(\alpha) = f$, $i_1(\beta) = b$, $i_1(\gamma) = c$, $i_2(\gamma) = d$, $i_2(\delta) = e$.

P(L) has two components, so that P(L) has two indecomposable factors corresponding to the diagrams



Note $|Aut(P(L))| = 1 = |Aut(L)|$. Applying Theorem 6 we see that $T_d = \{d\}$, $T_e = \{e,f\}$, and $T_f = \{f\}$, and that consequently L is distributive.

Theorem 6 has some interesting consequences concerning the factorization of lattices. In particular, it leads to a simple characterization of the center of a lattice (see [1; p. 67]). The following fairly immediate corollary of Theorem 6 generalizes and extends the results described by H. Crapo and G.-C. Rota (and which follow from some work of Dilworth) for factorization of relatively complemented lattices with no infinite chains [7; Chapter 12] to the factorization of all lattices with no infinite chains.

COROLLARY. Let L be a lattice and $C(L)$ be the center of L.

(a) $x \in C(L)$ iff x is a separator of L, i.e., if $p \in J(L)$ and $q \in M(L)$ are such that $p \not\leq q$, then either $p \leq x$ or $x \leq q$.

(b) $C(L) \simeq 2^{\underline{k}}$, where k is the number of irreducible (non-trivial) factors of L. (Note L has a unique irreducible factorization.)

(c) $L \simeq [0,c_1] \times [0,c_2] \times \dots \times [0,c_k]$ where $c_1, \dots, c_k$ are the points of $C(L)$.

The author is indebted to Professor Curtis Greene for suggesting that the results of [7; Chapter 12] be considered from the point of view of Theorem 6.

Before we discuss additional aspects of the poset of irreducibles we make the following definition.

Definition. By a bipartite digraph D, we mean a triple (X,Y,A), where X and Y are sets, $X \quad Y = \emptyset$, and $A \leq X \times Y$. A is called the set of arcs. If $S \leq X$, by $Ou(S)$ we mean $\{y \varepsilon Y |$ there exists $x \varepsilon S$ such that $(x,y) \varepsilon A\}$. Similarly, if $T \leq Y$, by $In(T)$ we mean $\{x \varepsilon X |$ there exists $y \varepsilon T$ such that $(x,y) \varepsilon A\}$. If $x \varepsilon X [y \varepsilon Y]$ we write $Ou(x)[In(y)]$ instead of $Ou(\{x\})[In(\{y\})]$. Sometimes we will use the term bidigraph to stand for bipartite digraph.

We will usually think of bidigraphs as being posets with the following ordering. If $w,z \varepsilon D$, then $w > z$ iff $w \varepsilon Y$, $z \varepsilon X$, and $w \varepsilon Ou(z)$.

From Theorem 6 we see that we can associate a "unique" bidigraph $P(L)$ to each lattice L and then recover L from $P(L)$ in a well-defined way. The following theorem shows that to any bidigraph we can associate a lattice. This theorem sets the stage for some interesting questions.

THEOREM 7. Let $D = (X,Y,A)$ be a finite bipartite digraph. Let $f: X \to 2^Y$ be given by $f(x) = Ou(x)$, and let $L_D = \{\bigcup_{w \in \Delta} w \mid \Delta \leq f(X)\}$. Then L_D is a lattice. Let $g: Y \to 2^Y$ be given by $g(y) = \text{l.u.b.}_{L_D} f(X - In(y)) = Ou(X - In(y))$. Then $f(X) \geq J(L_D)$ and $g(Y) \geq M(L_D)$.

We conclude this section by considering the following two questions. First, which finite bidigraphs are isomorphic to $P(L)$ for some lattice L? Second, suppose we are given the Hasse diagram of a finite poset, how can we determine whether or not the poset is a lattice?

The first question is answered by the following theorem.

THEOREM 8. Let $D = (X,Y,A)$ be a finite bidigraph. Then the following are equivalent.

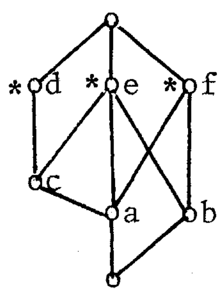
(a) $D \cong P(L)$ for some finite lattice L.

(b) For all $x \in X$, if $\Delta \leq X$ is such that $Ou(x) = Ou(\Delta)$, then $x \in \Delta$. Similarly, for all $y \in Y$, if $\Gamma \leq Y$ is such that $In(Y) = In(\Gamma)$, then $y \in \Gamma$.

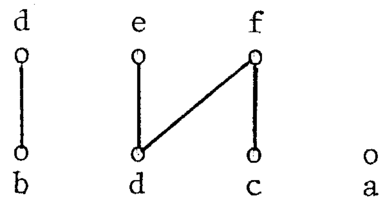
We will not answer the second question formally, but simply show how the techniques described above allow one to systematically attack the second question. The basic idea is that, given a finite P (say in the form of a Hasse diagram) one assumes that it is a lattice and constructs $P(P)$ of Theorem 6 using any element which is only covered by one element as a meet-irreducible element and any element covering only one element as a join-irreducible element. If $P(P)$ does not satisfy (b) of Theorem 8, it follows that P was not a lattice originally. If $P(P)$ does satisfy (b) of Theorem 8 one proceeds to construct $L_{P(P)}$ as in Theorem 7. From the work above, it is clear that P is a lattice iff $L_{P(P)} \cong P$. Often, it is not necessary to construct all of $L_{P(P)}$ to discover that $P \ncong L_{P(P)}$ as will be seen below.

Needless to say, if P has more than one maximal or more than one minimal element, there is no need to test it for being a lattice. Again, it is often easier to test that (b) of Theorem 8 holds for Ou and then construct $L_{P(P)}$, then to see that (b) of Theorem 8 holds for both Ou and In.

EXAMPLE 4. Let P be represented by

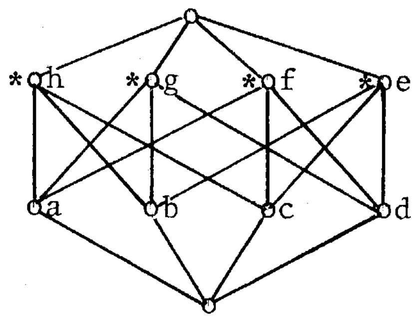


The shaded elements are the "join-irreducible" elements of P determined as above, assuming that P is a lattice. The starred elements are the meet-irreducible elements of P. We will not use i_1 and i_2 when working with P(P), in order to keep notational distractions to a minimum. Here, we have that $Ou(a) = \phi$, $Ou(b) = \{d\}$, $Ou(c) = \{f\}$, and $Ou(d) = \{e,f\}$. Since $Ou(a) = \phi$, (b) of Theorem 8 is not satisfied, since $Ou(\phi) = \phi$ and $a \notin \phi$. Hence P is not a lattice. P(P) can be represented by

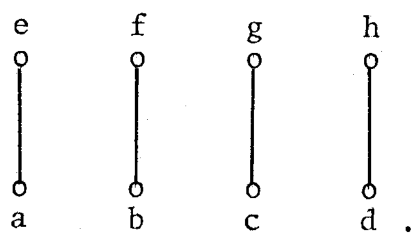


and $L_{P(P)}$ (using Theorem 7) is the lattice of Example 3.

EXAMPLE 5. Let P be represented by

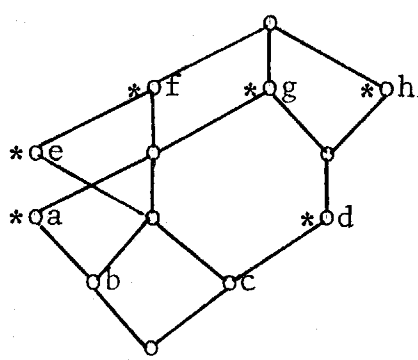


. $P(P)$ can be represented as



. Clearly $L_{P(P)} \cong 2^{\underline{4}} \not\cong P$, so that P is again not a lattice.

EXAMPLE 6. Let P be represented by

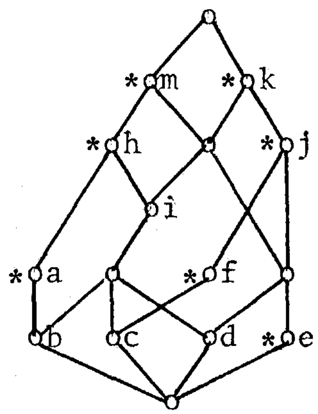


We will not draw $P(P)$, but note that the following are easily obtained from the diagram: $Ou(a) = \{d,e,h\}$, $Ou(b) = \{d\}$, $Ou(c) = \{a\}$, $Ou(d) = \{a,e,f\}$, $Ou(e) = \{a,d,g,h\}$, $Ou(h) = \{a,d,e,f,g\}$. To simplify checking whether (b) of Theorem 13 holds one should arrange the Ou's according to cardinality: $Ou(b)$, $Ou(c)$, $Ou(a)$, $Ou(d)$, $Ou(e)$, $Ou(h)$. In this way, each Ou could only be a union of preceding Ou's. $Ou(b)$ and $Ou(c)$ are singletons and thus satisfy (b). $Ou(a)$ is the first one on the list to contain an "e" or "h", while $Ou(d)$ is the first to contain an "f". "g" first appears in $Ou(e)$. "g" appears only in $Ou(e)$ and $Ou(h)$, but $Ou(e) \nleq Ou(h)$, and hence (b) holds for all the Ou's.

Note that $In(y)$, for $y \in \{a,d,e,f,g,h\}$ is easily constructed since $In(y) = \{x \in \{a,b,c,d,e,h\} \mid y \in Ou(x)\}$. It is also easily verified that (b) holds for $In(y)$. It is easy to construct $L_{P(P)}$, and one quickly sees that $P \simeq L_{P(P)}$.

It is easy to see that $P(P)$ is connected and that therefore P is indecomposable. Furthermore, let $f \in Aut(P(P))$, it is easy to show that f = Identity, since $f(a) = a$ (a considered as belonging to X_2), $f(d) = d$ (d considered as belonging to X_2), etc.

EXAMPLE 7. Let P be represented by



Thus $Ou(c) = \{a,e\}$,

$Ou(d) = \{a,e,f\}$,

$Ou(b) = \{e,f,j\}$,

$Ou(e) = \{a,f,g,h\}$, $Ou(a) = \{e,f,g,j,k\}$, $Ou(f) = \{a,e,g,h,m\}$ $Ou(i) = \{a,e,f,g,j\}$. It is easy to see that Ou satisfies (b) of Theorem 8. However, when constructing $L_{P(P)}$, one notices almost immediately that $Ou(c) \leq Ou(d)$, but that $c \not\leq d$ in P. Thus $L_{P(P)} \not\cong P$, and P is not a lattice.

The above examples actually contain the skeleton of an algorithm for checking posets for being lattices. We will not develop this algorithm further here, but note that it can be refined quite a bit and that some fair-sized examples, e.g., Example 7, can be handled easily using this algorithm.

<u>Remark</u>. Much of the preceding can be generalized to arbitrary lattices. The forms of the theorems vary depending on whether one wants to allow arbitrary joins or just finite joins. In the case of arbitrary joins, the generalization of Theorem 1 allows one to embed every

lattice in a complete lattice, while the generalization of Theorem 6(d) leads directly to some of Raney's results dealing with completely distributive lattices. Both theories are complicated by the fact that arbitrary lattices need not have any join-irreducible or meet-irreducible elements, and by other considerations. Actually all the above theorems hold for lattices of finite length. We have presented everything above in the context of finite lattices so that the underlying ideas would stand out more clearly. We would also like to mention that other classes of lattices (e.g., geometric lattices) can be characterized in terms of properties of their posets of irreducibles as was done in Theorem 6(d) for distributive lattices. For details see [13].

III. THE FREE DISTRIBUTIVE LATTICE ON n GENERATORS

The free distributive lattice on n generators, $FD(n)$, is $D(2^{\underline{n}})$. For basic information about $FD(n)$ the reader should consult [1; pp 34, 59] or [12]. Actually, for an arbitrary set X, $D(2^X)$ is the free completely distributive (complete) lattice on $|X|$ generators.

This contrasts with the result of H. Gaifman and A. W. Hales that there does not exist a free complete Boolean algebra with even countably many generators (see [1; p. 259]). We note that in addition to $FD(|X|)$ for infinite X, it is possible to talk about a free distributive lattice with infinitely many generators (see A. Nerode [16]).

The problem of enumerating $FD(n)$ was first proposed by Dedekind [8] in 1897. Exact answers are known with certainty only for $n \leq 6$. We now show that as is often the case, the problem of enumerating $FD(|X|)$ if X is infinite is much easier than if X is finite.

THEOREM 9. Let X be an infinite set. Then $|FD(|X|)| = |2^{2^X}|$.

<u>Proof</u>: Clearly, $FD(|X|) \leq |2^{2^X}|$. Since X is infinite, there exists a bijection $f:\underline{2} \times X \to X$. If $\gamma \in 2^X$, then we define $\gamma^* = \{f(2,\alpha)|\alpha\varepsilon\gamma\} \quad \{f(1,\alpha)|\alpha\varepsilon X-\gamma\}$. Note that $|\gamma^*| = |X|$, and that if $\gamma_1,\gamma_2 \in 2^X$, $\gamma_1 \neq \gamma_2$,

then $\gamma_1^* \not\leq \gamma_2^*$ and $\gamma_2^* \not\leq \gamma_1^*$.

Define $F: 2^{2^X} \to FD(|X|)$, by $F(S) = \{\Delta \varepsilon 2^X | \Delta \leq \gamma^*$ for some $\gamma \varepsilon S\}$. It is obvious from the definition that for $S \varepsilon 2^{2^X}$, $F(S)$ is closed from below, and hence F is well-defined. We claim that F is injective. Suppose that we have $F(S) = F(T)$, for $S, T \varepsilon 2^{2^X}$. Let $\lambda \varepsilon S$, then $\lambda^* \varepsilon F(S) \Rightarrow$ there exists $\gamma \varepsilon T$ such that $\lambda^* \leq \gamma^*$. But as we saw above this is only possible if $\lambda = \gamma$. Thus $\lambda \varepsilon T$ and consequently $S \leq T$. By symmetry, we get that $S \geq T$, and finally that $S = T$. Thus $|2^{2^X}| \leq |FD(|X|)$ and we are done.

From Theorem 5 we have the following results.

THEOREM 10. $FD(n)$ is irreducible and $Aut(FD(n)) \simeq S_n$ (the symmetric group on n letters).

We note that Theorem 10 is also true for $FD(n)-\{0,I\}$, which is often considered to be the free distributive lattice. This is true since $FD(n) - \{0,I\} \simeq D(2^{\underline{n}} - \{\phi, \underline{n}\})$.

It would be of interest to know the factors of $|FD(n)|$, but the irreducibility of $FD(n)$ suggests that there is no "natural" way to factor $|FD(n)|$. The only result along these lines which is known is Yamamoto's, that if n is even so is $|FD(n)|$ [17]. The converse of this statement is false, e.g., $|FD(3)| = 20$.

We conclude this paper by considering several aspects of the enumeration of FD(n). We wish to briefly sketch the nature of the functions $L_k(n)$, where $L_k(n)$ is the number of elements of FD(n) of cardinality k.

THEOREM 11. $L_k(n) = \sum_{\rho=\lambda_k}^{k-1} C(\rho,k)\binom{n}{\rho}$, where λ_k is an integer such that $2^{\lambda_k} \geq k > 2^{\lambda_k - 1}$ and $C(\rho,k)$ is the number of order ideals of cardinality k of $\underline{2}^{\rho}$ which contain all the singletons of $\underline{2}^{\rho}$.

Remark. Thus we see that, for $k \geq 1$, $L_k(n)$ is a polynomial in n of degree k-1, and since $C(k-1,k) = 1$, the leading coefficient is $1/(k-1)!$. $L_k(n)$ resembles the chromatic polynomial somewhat. Note that $0,1,\ldots,\lambda_k-1$ are among the roots of $L_k(n)$. These are the only possible non-negative integral roots of $L_k(n)$, since if $n \geq \lambda_k$, there exists at least one closed from below subset of $\underline{2}^{\lambda_k}$ having cardinality k. It is possible for $L_k(n)$ to have negative integers as roots, e.g., -1 is a root of $L_4(n)$ and -9 is a root of $L_5(n)$. All the $L_k(n)$ up to $k = 7$ have only real roots each with multiplicity one. Whether this is true in general is not known to the author.

We also observe that $L_k(n) = L_{2^n-k}(n)$ for fixed k and n. Thus if we know $L_k(n)$ for $k = 1,\ldots, m$, for a given n we can calculate the elements on 2m levels of FD(n).

Note also that from Theorem 11, it follows that for fixed k, $L_k(n) \sim \frac{1}{(k-1)!}n^{k-1}$ as $n\to\infty$. Unfortunately, this gives some information about the tail ends of FD(n), but does not help to understand the behavior of the middle terms.

It turns out that the values of the $C(\rho,k)$'s can also be calculated from a polynomial. We will now present the machinery necessary for calculating at least some of the $L_k(n)$ fairly easily.

We should note that a somewhat similar approach to the problem of calculating FD(n) was used by Randolph Church [6], although he fixed n and let k vary. Thus in [6] he obtained the values for $L_k(n)$, $n \leq 5$ and for all k.

<u>Definition</u>. By $P(j,k)$ we shall mean $C(k-j-1,k)$, and by $C_1(a,b)$ we shall mean the number of elements of (a,b) such that no singleton is a maximal element, where (a,b) is the set of all closed from below subsets of cardinality b of $2^{\underline{a}}$ which contain all the singletons of $2^{\underline{a}}$.

<u>Remark</u>. Thus we have that $L_k(n) = \sum_{j=0}^{k-\lambda_k-1} P(j,k)\binom{n}{k-j-1}$. Note also that $P(0,k) = C(k-1,k) = 1$ for all $k \geq 0$.

The following theorem shows that for a fixed j,

$P(j,k)$ is a polynomial in k of degree $2j$.

THEOREM 12. For $j \geq 0$, $P(j,k) = \sum_{i=m_j}^{2j} C_1(i,i+j+1) \binom{k-j-1}{i}$, where m_j is the smallest integer such that $2^{m_j} \geq m_j+j+1 > 2^{m_j-1}$.

The strategies for calculating $C_1(a,b)$, $P(j,k)$, and $L_k(n)$ are involved and rather technical. The author has calculated $C_1(2j-a,3j-a+1)$ explicitly for $0 \leq a \leq 9$ and $P(j,k)$ explicitly for $0 \leq j \leq 10$. Theorem 13 gives the explicit values of $L_k(n)$ for $0 \leq k \leq 16$.

THEOREM 13. For $n \geq 0$,

(1) $L_0(n) = 1;$

(2) $L_1(n) = 1;$

(3) $L_2(n) = \binom{n}{1} = n;$

(4) $L_3(n) = \binom{n}{2} = \frac{n^2-n}{2};$

(5) $L_4(n) = \binom{n}{2} + \binom{n}{3} = \frac{n^3-n}{6};$

(6) $L_5(n) = 3\binom{n}{3} + \binom{n}{4} = \frac{n^4+6n^3-25n^2+18n}{24};$

(7) $L_6(n) = 3\binom{n}{3}+6\binom{n}{4}+\binom{n}{5} = \frac{n^5+20n^4-85n^3+100n^2-36n}{120};$

(8) $L_7(n) = \binom{n}{3} + 15\binom{n}{4} + 10\binom{n}{5} + \binom{n}{6};$

(9) $L_8(n) = \binom{n}{3} + 20\binom{n}{4} + 45\binom{n}{5} + 15\binom{n}{6} + \binom{n}{7};$

(10) $L_9(n) = 19\binom{n}{4} + 120\binom{n}{5} + 105\binom{n}{6} + 21\binom{n}{7} + \binom{n}{8};$

(11) $L_{10}(n) = 18\binom{n}{4}+220\binom{n}{5}+455\binom{n}{6}+210\binom{n}{7}+28\binom{n}{8}+\binom{n}{9};$

(12) $$L_{11}(n) = 13\binom{n}{4}+322\binom{n}{5}+1{,}385\binom{n}{6}+1{,}330\binom{n}{7}+378\binom{n}{8}+36\binom{n}{9}+\binom{n}{10};$$

(13) $$L_{12}(n) = 10\binom{n}{4}+420\binom{n}{5}+3{,}243\binom{n}{6}+6{,}020\binom{n}{7}+3{,}276\binom{n}{8}+630\binom{n}{9}+45\binom{n}{10}+\binom{n}{11};$$

(14) $$L_{13}(n) = 6\binom{n}{4}+500\binom{n}{5}+6{,}325\binom{n}{6}+21{,}014\binom{n}{7}+20{,}531\binom{n}{8}+7{,}140\binom{n}{9}+990\binom{n}{10}+55\binom{n}{11}+\binom{n}{12};$$

(15) $$L_{14}(n) = 4\binom{n}{4}+560\binom{n}{5}+10{,}925\binom{n}{6}+59{,}619\binom{n}{7}+99{,}680\binom{n}{8}+58{,}989\binom{n}{9}+14{,}190\binom{n}{10}+1{,}485\binom{n}{11}+66\binom{n}{12}+\binom{n}{13};$$

(16) $$L_{15}(n) = \binom{n}{4}+600\binom{n}{5}+17{,}345\binom{n}{6}+145{,}050\binom{n}{7}+393{,}540\binom{n}{8}+379{,}848\binom{n}{9}+149{,}115\binom{n}{10}+26{,}235\binom{n}{11}+2{,}145\binom{n}{12}+78\binom{n}{13}+\binom{n}{14};$$

(17) $$L_{16}(n) = \binom{n}{4}+616\binom{n}{5}+25{,}945\binom{n}{6}+314{,}965\binom{n}{7}+1{,}313{,}260\binom{n}{8}+1{,}992{,}144\binom{n}{9}+1{,}226{,}919\binom{n}{10}+341{,}220\binom{n}{11}+45{,}760\binom{n}{12}+3{,}003\binom{n}{13}+91\binom{n}{14}+\binom{n}{15}.$$

Remark. Note that we have enough information to calculate $L_{17}(n)$ entirely, since we know from Theorem 11 that $C(5,17) = L_{17}(5)$ and from the Remark following Theorem 11 that $L_{17}(5) = L_{15}(5)$. All the remaining coefficients can be calculated using the values of $P(j,k)$ for $0 \leq j \leq 10$.

The ideas in this chapter have been applied by Butler and the author [5] to the enumeration of partially ordered sets, to show that when partially ordered sets are broken down into certain classes, each class is enumerated by a polynomial.

We conclude by briefly discussing the problem of finding an accurate upper bound for $|FD(n)|$. The best published result is that of D. J. Kleitman [12] which states that $|FD(n)| \leq 2^{(1+k\, n^{-\frac{1}{2}} \ln n)}\, E_n$ for some constant k, where $E_n = \binom{n}{[\frac{n}{2}]}$. Recently, Kleitman and the author working jointly have been able to improve this upper bound. In particular, we have shown that $|FD(n)| \leq 2^{(1+k\, n^{-1} \ln n)}\, E_n$. The improvement of the upper bound follows from a detailed analysis of Hansel's approach to the problem [10], using a characterization, due to Greene and Kleitman [9], of the partition of $\underline{2}^{\underline{n}}$ into chains used by Hansel. Greene and Kleitman characterize this partition in terms of the way an expression can be parenthesized allowing a certain number of "free" parentheses to remain.

It can be shown that [13] $|FD(n)| \geq 2^{(1 + c2^{-[\frac{n}{2}]})E_n}$, for c a constant on the order of 1 and appropriate n.

The lower bound given in [12] is too large to be supported by the argument given there.

We wish to finish by stating two conjectures. The first is that the order of FD(n) is closer to the lower bound given above than it is to the upper bound given above. The second conjecture concerns the number of anti-chains of $2^{\underline{n}}$ (recall that anti-chains of $2^{\underline{n}}$ correspond in a 1-1 fashion to the sets of maximal elements of elements of $D(2^{\underline{n}})$). This conjecture is due to Garrett Birkhoff and asserts that asymptotically all anti-chains of $2^{\underline{n}}$ consist entirely of subsets of $\underline{n}$ with cardinality between $[\frac{n}{2}]-k$ and $[\frac{n}{2}]+k$, where k is a small fixed integer, perhaps 3, 4 or 5.

The author would like to gratefully acknowledge many stimulating discussions with Garrett Birkhoff, Daniel J. Kleitman, and Curtis Greene, and the many very helpful suggestions which they made.

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ON CLASSES OF LATTICES REPRESENTABLE BY MODULES

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ABSTRACT

For a ring R with 1, let $\mathcal{L}(R)$ denote the class of lattices representable in submodule lattices of R-modules. It is shown that the binary ring predicate $\mathcal{L}(R) \subset \mathcal{L}(S)$ is related to the existence of exact embedding functors R-Mod $\longrightarrow$ S-Mod. The predicate $\mathcal{L}(R) \subset \mathcal{L}(S)$ can be evaluated in general if it can be evaluated for rings with the same characteristic. Furthermore, only rings with zero or prime power characteristic need be considered. Necessary and sufficient conditions on R are given such that $\mathcal{L}(R) = \mathcal{L}(S)$ for S a unitary subring of the field of rationals or for S the ring of integers modulo n, n a prime or a product of distinct primes.

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Given a ring R with unit, a lattice L is "representable by R-modules" if there exists a unitary left R-module M such that L is embeddable in the lattice of submodules of M. Of course, embeddability in the lattice of submodules of M is equivalent to embeddability in the lattice of congruences of M [1: VII, Thm. 1, p. 159]. In the following, we consider the general problem:

For which rings with unit R and S is every lattice representable by R-modules also representable by S-modules?

Our attack on this problem uses abelian category methods in addition to the methods of modular lattice theory. Let us first introduce some notation. Hereafter, R and S will denote rings with unit. The lattice of submodules of a left unitary R-module M will be denoted $\Gamma(M;R)$. The class of all lattices representable by R-modules will be denoted $\mathfrak{L}(R)$. Hence, our general problem is the study of the binary ring predicate $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ for various choices of R and S.

Let R-Mod denote the abelian category of all R-modules and R-linear maps between them. If β is a cardinal number, we will also consider R-Mod(β), the category of all R-modules with cardinality less than β and all R-linear maps between such modules. Note that R-Mod(β) is an exact subcategory of R-Mod if β is infinite. Let card(X) denote the cardinality of a set X.

To digress momentarily, we remark that $\mathcal{L}(R)$ is a "quasivariety" of lattices, that is, $\mathcal{L}(R)$ is the class of all lattices satisfying some set of universal Horn formulas. (A universal lattice Horn formula is of form:

$$(x_1, x_2, \ldots, x_m)\ ((e_1 = e_2 \ \& \ \ldots \ \& \ e_{2n-3} = e_{2n-2}) \Rightarrow e_{2n-1} = e_{2n}),$$

where $e_1, e_2, \ldots, e_{2n}$ are lattice polynomials in the variables $x_1, x_2, \ldots, x_m$.) This was proved by a model theoretic argument in general [4: Thm. 6], and was proved by discovery of a constructive procedure for generating infinite Horn formula axiomatizations of $\mathcal{L}(R)$ in the commutative case [6, 7: Main Thm.]. In [4: Thm. 3], a result is obtained implying that $\mathcal{L}(R)$ is not finitely first-order axiomatizable if R is the ring of integers, or if R is the field of rationals, or if R is any ring between the integers and the rationals (that is, any unitary subring of the rationals). In [10], another model theory approach yields the following results: (1) If R is a ring defined on a recursive set of natural numbers with recursive ring operations of addition and multiplication, then there is a primitive recursive set of universal Horn formulas characterizing $\mathcal{L}(R)$. (2) Suppose that a "term" is 0, 1 or a variable $y_1, y_2, \ldots, y_n$, and an "equation" is $t_1 + t_2 = t_3$ or $t_1 t_2 = t_3$ for terms t_1, t_2 and t_3. For any rings R and S with unit, either $\mathcal{L}(R) \subset \mathcal{L}(S)$ or there exists a system of equations that is true in S for some assignment of elements of S to $y_1, y_2, \dots, y_n$ but which is false in R for every assignment of elements of R to $y_1, y_2, \dots, y_n$.

Let us now return to consideration of the predicate $\mathfrak{L}(R) \subset \mathfrak{L}(S)$. We begin by stating a conjecture:

For any rings R and S with unit, $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if and only if there exists an exact embedding functor R-Mod $\longrightarrow$ S-Mod.

We will not prove this conjecture as stated, but will prove a slightly weaker version for our first theorem. Specifically, we will prove $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ equivalent to the following:

For every infinite cardinal β, there exists an exact embedding functor R-Mod(β) $\longrightarrow$ S-Mod.

The following propositions lead up to the proof of this result.

Prop. 1. If there exists an exact embedding functor R-Mod $\longrightarrow$ S-Mod, then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$.

Prop. 2. If there exists a ring homomorphism $S \longrightarrow R$ preserving 1, then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$.

Prop. 3. If there exists a bimodule M (left S-module, right R-module) which is faithfully flat as an R-module, then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$. (M is "faithful" if $M \otimes_R M_0 = 0$ implies $M_0 = 0$ for all M_0 in R-Mod.)

To prove $\mathfrak{L}(R) \subset \mathfrak{L}(S)$, it suffices to show that $\Gamma(M;R)$ is in $\mathfrak{L}(S)$ for each M in R-Mod. Suppose $F: R\text{-Mod} \longrightarrow S\text{-Mod}$ is an exact embedding functor. Then F induces a lattice embedding $F_0: \Gamma(M;R) \longrightarrow \Gamma(F(M);S)$ defined by $F_0[f] = [Ff]$ for $[f]$ a subobject of M. (See [5: p. 183] for relevant information. Note also that we have identified the lattice of submodules of M with the lattice of subobjects of M in R-Mod, and similarly for F(M) in S-Mod.) This proves Prop. 1.

If there exists a ring homomorphism $h: S \longrightarrow R$ preserving 1, it is well-known (and easily verified) that the "change of rings" operation [2: p. 28ff] $M \longrightarrow M_{(h)}$ induces an exact embedding $R\text{-Mod} \longrightarrow S\text{-Mod}$. So, Prop. 2 follows from Prop. 1.

The hypotheses of Prop. 3, interpreted, assert that $M \otimes_R -$ is an exact functor that reflects zero objects (that is, every inverse image of a zero object is zero). But then $M \otimes_R -$ is an exact embedding functor [11: II, 7.2, p. 57]. Therefore, Prop. 3 also follows from Prop. 1.

To prove that $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ implies existence of an exact embedding functor $R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$, we make use of the "abelian" lattice concept of [5]. By [5: Main Thm.], a functor $\mathbf{A}$ can be constructed, taking an abelian lattice L into a small abelian category $\mathbf{A}_L$, and taking a lattice homomorphism $b: L \longrightarrow M$ of abelian lattices into an exact functor $\mathbf{A}_b : \mathbf{A}_L \longrightarrow \mathbf{A}_M$. If b is a lattice embedding, then $\mathbf{A}_b$ reflects zero objects by [5: 3.16, p. 172], and so $\mathbf{A}_b$ is an embedding functor by [11: II, 7.2, p. 57].

Definition. Let β be an infinite cardinal number, regarded as the set of smaller ordinals. Let R_β be the free R-module with β generators, with free generating set $\{ x_\delta : \delta \in \beta \}$. A submodule M of R_β has "bounded support" if there exists a subset A of β such that $\mathrm{card}(A) < \beta$ and M is contained in the submodule of R_β generated by $\{x_\delta : \delta \in A\}$. Let $\Gamma_b(R_\beta;R)$ denote the set of submodules of R_β with bounded support.

Prop. 4. If β is an infinite cardinal, then $\Gamma_b(R_\beta;R)$ is an ideal of $\Gamma(R_\beta;R)$, and is an abelian lattice.

Proof: Modify the proof of [5: 4.2]. We will only outline the proof that any M in $\Gamma_b(R_\beta;R)$ can be "tripled". Choose $A \subset \beta$ such that $\mathrm{card}(A) < \beta$ and M is contained in the submodule generated by $\{ x_\delta : \delta \in A \}$. Since $\mathrm{card}(\beta - A) = \beta$, we can choose $B \subset \beta - A$ and a bijection $\theta : A \longrightarrow B$. Now each m in M can be expressed uniquely as a sum $\sum_{\delta \in A} r_\delta x_\delta$, where all but finitely many of the coefficients r_δ equal zero. Then define:

$$M_1 = \{ \sum_{\delta \in A} r_\delta x_{\theta(\delta)} : \sum_{\delta \in A} r_\delta x_\delta \in M \},$$

$$M_2 = \{ \sum_{\delta \in A} r_\delta (x_\delta - x_{\theta(\delta)}) : \sum_{\delta \in A} r_\delta x_\delta \in M \}.$$

It is easily shown that M_1 and M_2 are in $\Gamma_b(R_\beta;R)$ and that M, M_1 and M_2 generate the five-element modular lattice of length two.

Prop. 5. If M is in R-Mod and $\beta > \aleph_0 + \mathrm{card}(M)$, then there exists a lattice embedding $\Gamma(M;R) \longrightarrow \Gamma_b(R_\beta;R)$.

Proof: Assume the hypotheses. Let $\gamma = \mathrm{card}(M)$, and extend a bijection $f_0:\gamma \longrightarrow M$ to an R-linear epimorphism $f:R_\gamma \longrightarrow M$ by the free module property. Then $\Gamma(M;R)$ is isomorphic to the interval sublattice $[\ker f, R_\gamma]$ of R_γ. Since R_γ can be regarded as a bounded submodule of R_β, there exists an embedding $\Gamma(M;R) \longrightarrow \Gamma(R_\gamma;R) \longrightarrow \Gamma_b(R_\beta;R)$.

Prop. 6. If M is an R-module with generating set G, then

$$\mathrm{card}(M) \leqq \aleph_0 + \mathrm{card}(R) + \mathrm{card}(G).$$

Proof: Let X be the set $\bigcup_{n=1}^{\infty} (R^n \times G^n)$, and define the function m from X onto M given by:

$$m((r_1, r_2, \ldots, r_n), (g_1, g_2, \ldots, g_n)) = \sum_{i=1}^{n} r_i g_i .$$

If $\gamma = \aleph_0 + \mathrm{card}(R) + \mathrm{card}(G)$, then:

$$\mathrm{card}(M) \leqq \mathrm{card}(X) \leqq \sum_{n=1}^{\infty} (\gamma^n)^2 = \gamma .$$

In the next two propositions, we will use the definitions and notations of [5] without reference.

Prop. 7. Let L be an ideal of some $\Gamma(M;R)$, and Hom_R denote Hom in R-Mod. If $A, B: \mathbf{2} \longrightarrow L$ are r-disjoint, then there is a one-one correspondence $\nu: S(A, B) \longrightarrow \mathrm{Hom}_R(A^1/A^0, B^1/B^0)$ given by:

$$\nu(f)\ (x + A^0) = (x + f^-) \cap B^1,$$

for $f: T \longrightarrow L$ in $S(A, B)$ and $x \in A^1$, and

$$\nu^{-1}(h)^- = \{x - y : x \in A^1,\ y \in B^1,\ h(x + A^0) = y + B^0\},$$

for $h: A^1/A^0 \longrightarrow B^1/B^0$ in R-Mod. Furthermore,

$$\ker \nu(f) = K(f)/A^0 \quad \text{and} \quad \mathrm{im}\ \nu(f) = I(f)/B^0.$$

If $A, B, C: \mathbf{2} \longrightarrow L$ is a mixed sequence, $f \in S(A, B)$ and $g \in S(B, C)$, then $\nu(gof) = \nu(g)\nu(f)$. Also, f is isorepresentative if and only if $\nu(f)$ is an isomorphism, and $\nu(f^{-1}) = \nu(f)^{-1}$ in that case.

Proof: Assume the hypotheses. Using the known relations between A^1, A^0, B^1, B^0 and f^-, we can show that $(x + f^-) \cap B^1$ is a coset in B^1/B^0 for $x \in A^1$, and $\nu(f)$ so defined is R-linear. Straightforward computations prove that $\nu^{-1}(h)^-$ is in L, $A^0 \vee B^0 \subset \nu^{-1}(h)^- \subset A^1 \vee B^1$, $B^1 \vee \nu^{-1}(h)^- = A^1 \vee B^1$ and $B^1 \wedge \nu^{-1}(h)^- = B^0$. So, $\nu^{-1}(h): T \longrightarrow L$ in $S(A, B)$ can be defined as above. We also omit the computations proving that $\nu^{-1}\nu(f) = f$, $\nu\nu^{-1}(h) = h$, $\ker \nu(f) = K(f)/A^0$ and $\mathrm{im}\ \nu(f) = I(f)/B^0$.

Let $k = \nu^{-1}(\nu(g)\nu(f))$ in $S(A, C)$. By definition $k^{-} \subset A^1 \vee C^1$. Suppose $x - z \in k^{-}$, with $x \in A^1$, $z \in C^1$ and $\nu(g)\nu(f)(x + A^0) = z + C^0$. Choose $y \in B^1$ with $\nu(f)(x + A^0) = y + B^0$, and observe that $x - y \in f^{-}$ and $y - z \in g^{-}$, so $x - z \in f^{-} \vee g^{-}$ and

$$k^{-} \subset (g \circ f)^{-} = (f^{-} \vee g^{-}) \wedge (A^1 \vee C^1).$$

So, $k = g \circ f$ by [5: 3.4], proving $\nu(g \circ f) = \nu(g)\nu(f)$. Using the formulas for $\ker \nu(f)$ and $\mathrm{im}\, \nu(f)$ and [5: **3.21**], f is iso-representative if and only if $\nu(f)$ is an isomorphism. If $h: A^1/A^0 \longrightarrow B^1/B^0$ is an isomorphism, then $\nu^{-1}(h)^{-} = \nu^{-1}(h^{-1})^{-}$ by direct computation. Then, $\nu(f^{-1}) = \nu(f)^{-1}$ follows.

Prop. 8. Let β be an infinite nondenumerable cardinal, $\beta > \mathrm{card}(R)$, and let $L = \Gamma_b(R_\beta; R)$. Then there exists a full exact embedding equivalence functor $F: \mathbf{A}_L \longrightarrow R\text{-Mod}(\beta)$, given by $F(A) = A^1/A^0$ and $F([f_2, f_1]) = \nu(f_2)\nu(f_1)$.

Proof: Assume the hypotheses. If $A^0 \subset A^1$ in L, then $\mathrm{card}(A^1/A^0) < \beta$ by Prop. 6, and A^1/A^0 is in $R\text{-Mod}(\beta)$. There is no problem in verifying that F is well-defined and is a full exact embedding functor, by Prop. 7 and [5: 3.13, 3.15, 3.17, 3.19, 3.25]. To prove that F is an equivalence functor, it suffices to show that every M in $R\text{-Mod}(\beta)$ is isomorphic to $F(A)$ for some A in $\mathbf{A}_L$ [9: IV.4, Thm. 1, p. 91]. Let $\mathrm{card}(M) = \gamma < \beta$, and choose an R-linear epimorphism $f: R_\gamma \longrightarrow M$

as in the proof of Prop. 5. Now R_γ can be regarded as a bounded submodule of R_β, so $F(A)$ is isomorphic to M for $A^0 = \ker f \subset R_\gamma = A^1$ in L.

We can now prove:

Theorem 1. Let R and S be rings with unit. Then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if and only if there exists an exact embedding functor $R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ for every infinite cardinal β.

Corollary. $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if and only if there exists an exact embedding functor $C \longrightarrow S\text{-Mod}$ for every small exact subcategory C of $R\text{-Mod}$.

Proof: By a slight modification of the proof of Prop. 1, we can show that $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if there exists an exact embedding functor $R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ for every infinite β.

Assume $\mathfrak{L}(R) \subset \mathfrak{L}(S)$. To prove the theorem, it suffices to show that there exists an exact embedding functor $R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ whenever $\beta > \aleph_0 + \operatorname{card}(R)$. (If $\delta < \gamma$, then $R\text{-Mod}(\delta)$ is an exact subcategory of $R\text{-Mod}(\gamma)$.) So, assume $\beta > \aleph_0 + \operatorname{card}(R)$, and choose $\gamma > \aleph_0 + \operatorname{card}(S)$ such that there exist lattice embeddings:

$$\Gamma_b(R_\beta;R) \xrightarrow{f} \Gamma(M;S) \xrightarrow{g} \Gamma_b(S_\gamma;S),$$

using $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ to obtain f and Prop. 5 to obtain g. Let $L(R)$ denote $\Gamma_b(R_\beta;R)$ and $L(S)$ denote $\Gamma_b(S_\gamma;S)$, and construct an exact embedding $R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ by composing:

$$\text{R-Mod}(\beta) \xrightarrow{F_1} \mathbf{A}_{L(R)} \xrightarrow{F_2} \mathbf{A}_{L(S)} \xrightarrow{F_3} \text{S-Mod}(\gamma) \xrightarrow{F_4} \text{S-Mod}.$$

Here, F_4 is an exact inclusion functor, and F_1 and F_3 are exact embeddings obtained from the equivalences of Prop. 8. The functor F_2 equals $\mathbf{A}_{gf}: \mathbf{A}_{L(R)} \longrightarrow \mathbf{A}_{L(S)}$, which is an exact embedding by the discussion following Prop. 3. This proves Thm. 1.

Half of the corollary is proved by adapting the proof of Prop. 1. Since every small exact subcategory of R-Mod is an exact subcategory of R-Mod(β) for sufficiently large β, the other half of the corollary follows from the theorem.

There is a foundational point worth mentioning. The construction of the reciprocal functor to the equivalence functor $F: \mathbf{A}_L \longrightarrow \text{R-Mod}(\beta)$ in Prop. 8 using [9: IV.4, Thm. 1, p. 91] seems to require the strong axiom of choice (there exists a choice function for the class of all nonempty sets). However, the corollary of Thm. 1 can be proved using a slightly modified version of Prop. 8 requiring only the ordinary axiom of choice. Furthermore, most of the consequences of Thm. 1 hereafter can also be proved using the corollary.

In the remainder of the text, we will sometimes treat integers as members of an arbitrary ring R with unit. In each case, the integer n is identified with the additive multiple $n \cdot 1$ of the ring unit ($2 = 1 + 1$ in R, etc.). Note that an integer n is a central element of R, so nR is a two-sided principal ideal of R. If p is a prime number and $j \geqq 0$, we will say that p is "j-invertible" in R if $p^j(pr - 1) = 0$ for some r in R. If p is 0-invertible in R, then p is invertible as a ring element of R. If p is j-invertible in R, then it is k-invertible for $k > j$. Also, p is k-invertible in R if $\mathrm{char}(R) = p^k m$ for relatively prime p and m.

The next theorem gives some simple tests for proving that $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ is false in various cases.

Theorem 2. Let $\mathfrak{L}(R) \subset \mathfrak{L}(S)$, and let a and b be integers such that b divides a. If $ax + b = 0$ for some x in S, then $ax + b = 0$ for some x in R. Therefore, char(R) divides char(S). Also, for any prime p and $j \geqq 0$, p is j-invertible in R if p is j-invertible in S.

Proof: Assume the hypotheses. Using Thm. 1, choose an exact embedding functor $F: R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ for $\beta > \aleph_0 + \mathrm{card}(R)$. For any S-module M, $\mathrm{im}(b \cdot 1_M) \supset \mathrm{im}(a \cdot 1_M)$ because b divides a. Conversely, $\mathrm{im}(b \cdot 1_M) \subset \mathrm{im}(a \cdot 1_M)$ because $ax + b = 0$ for some x in S. In particular, $\mathrm{im}(b \cdot 1_{F(R)}) = \mathrm{im}(a \cdot 1_{F(R)})$. Since F is an exact embedding, $\mathrm{im}(b \cdot 1_R) = \mathrm{im}(a \cdot 1_R)$ [3: pp. 65-66]. So, $b \in \mathrm{im}(a \cdot 1_R)$, and therefore $ax + b = 0$ for some x in R.

Letting $a = 0$ and $b = \mathrm{char}(S)$, we see that $\mathrm{char}(S) = 0$ in R, and so char(R) divides char(S). (By convention, 0 divides 0.)

If p is j-invertible in S, then $ax + b = 0$ has a solution in S, hence in R, for $a = p^{j+1}$ and $b = -p^j$. Therefore, p is j-invertible in R. This proves Thm. 2.

More information on ring characteristics is given by:

Theorem 3. Let R and S have characteristics m and n, respectively. Then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if and only if $\mathfrak{L}(R) \subset \mathfrak{L}(S/mS)$, and m divides n and $\mathrm{char}(S/mS) = m$ in this case.

Proof: Assume the hypotheses, and that $m \neq 0$ (the case $m = 0$ is trivial). Since $\mathfrak{L}(S/mS) \subset \mathfrak{L}(S)$ by Prop. 2, $\mathfrak{L}(R) \subset \mathfrak{L}(S/mS)$ implies $\mathfrak{L}(R) \subset \mathfrak{L}(S)$. Assume $\mathfrak{L}(R) \subset \mathfrak{L}(S)$, and suppose M is an R-module. By Thm. 1, let $F: R\text{-Mod}(\beta) \longrightarrow S\text{-Mod}$ be an exact embedding for some $\beta > \aleph_0 + \mathrm{card}(R) + \mathrm{card}(M)$. Then F induces an embedding homomorphism $\Gamma(M;R) \longrightarrow \Gamma(F(M);S)$, as usual. Since $\mathrm{char}(R) = m$ and F is additive, $m \cdot 1_{F(M)} = F(m \cdot 1_M) = F(0) = 0$. Therefore, $s_0 x = 0$ if $s_0 \in mS$ and $x \in F(M)$. But then we can make $F(M)$ into a S/mS-module M_0, retaining the additive structure of $F(M)$ and defining $(s + mS)x = sx$ for $s \in S$ and $x \in M_0 = F(M)$. Clearly $\Gamma(M_0;S/mS)$ is isomorphic to $\Gamma(F(M);S)$, and so $\Gamma(M;R)$ is in $\mathfrak{L}(S/mS)$. This proves $\mathfrak{L}(R) \subset \mathfrak{L}(S/mS)$.

If $d = \mathrm{char}(S/mS)$, clearly d divides m. If $\mathfrak{L}(R) \subset \mathfrak{L}(S/mS)$, then m divides d by Thm. 2, and so $m = d$. By Thm. 2, m divides n if $\mathfrak{L}(R) \subset \mathfrak{L}(S)$.

Using Thm. 3, we can evaluate the ring predicate $\mathcal{L}(R) \subset \mathcal{L}(S)$ in general if we can evaluate it for rings with the same characteristic. After some preparation, we will prove that only rings with zero or prime power characteristic need be considered.

Prop. 9. Let $\mathrm{char}(R) = \mathrm{char}(S) = ab$, where a and b are relatively prime positive integers. Then $\mathcal{L}(R) \subset \mathcal{L}(S)$ if and only if $\mathcal{L}(R/aR) \subset \mathcal{L}(S)$ and $\mathcal{L}(R/bR) \subset \mathcal{L}(S)$.

Proof: Assume the hypotheses. Suppose $\mathcal{L}(R) \subset \mathcal{L}(S)$. Then $\mathcal{L}(R/aR) \subset \mathcal{L}(S)$ and $\mathcal{L}(R/bR) \subset \mathcal{L}(S)$ by Prop. 2.

Now assume that $\mathcal{L}(R/aR) \subset \mathcal{L}(S)$ and $\mathcal{L}(R/bR) \subset \mathcal{L}(S)$. Let M be an R-module. Make M/aM into an R/aR-module by defining:

$$(r + aR)(m + aM) = rm + aM \quad \text{for} \quad r \in R \quad \text{and} \quad m \in M,$$

and make M/bM into an R/bR-module similarly. Let $L = \Gamma(M;R)$, and let L_a and L_b denote the interval sublattices $[aM, M]$ and $[bM, M]$ of L, respectively. We can verify that L_a and L_b are isomorphic to $\Gamma(M/aM;R/aR)$ and $\Gamma(M/bM;R/bR)$, respectively. Therefore, there exist lattice embeddings $f:L_a \longrightarrow \Gamma(M_1;S)$ and $g:L_b \longrightarrow \Gamma(M_2;S)$ for some S-modules M_1 and M_2. Then $f \times g:L_a \times L_b \longrightarrow \Gamma(M_1;S) \times \Gamma(M_2;S)$ is a lattice embedding. Also, $i:\Gamma(M_1;S) \times \Gamma(M_2;S) \longrightarrow \Gamma(M_1 \times M_2;S)$ given by $i(N_1, N_2) = N_1 \times N_2$ is a lattice embedding.

Let $au + bv = 1$ for some integers u and v, since a and b are relatively prime. For any $m \in M$, $aum \in aM$ and $bvm \in bM$, so $m = (au + bv)m \in aM \vee bM$, proving $M = aM \vee bM$. Furthermore, if $m \in aM \wedge bM$, then $m = am_1 = bm_2$ for some m_1 and m_2 in M. Therefore, $am = abm_2 = 0 = bam_1 = bm$, since $\mathrm{char}(R) = ab$. But then $m = uam + vbm = 0$, proving $aM \wedge bM = 0$. Finally, suppose $M' \in L$. Then $M' = M' \wedge (aM \vee bM) = (M' \wedge aM) \vee (M' \wedge bM)$, since $m = aum + bvm \in (M' \wedge aM) \vee (M' \wedge bM)$ if $m \in M'$. Therefore, aM, bM and M' generate a distributive sublattice of L [1: Thm. 12, p. 37]. Now define functions as follows:

$$h: L \longrightarrow L_a \times L_b \quad \text{given by} \quad h(M') = (M' \vee aM, M' \vee bM).$$

$$h^*: L_a \times L_b \longrightarrow L \quad \text{given by} \quad h^*(M', M'') = M' \wedge M''.$$

Then $h^*h(M') = (M' \vee aM) \wedge (M' \vee bM) = M' \vee (aM \wedge bM) = M'$ for all $M' \in L$. Also, if $M' \supset aM$ and $M'' \supset bM$, then:

$$hh^*(M', M'') = ((M' \wedge M'') \vee aM, (M' \wedge M'') \vee bM) = (M', M''),$$

since $aM \vee (M'' \wedge M') = (aM \vee M'') \wedge M' = M'$ by modularity and $aM \vee M'' \supset aM \vee bM = M$, and similarly $(M' \wedge M'') \vee bM = M''$. Since h and h^* preserve order, they are reciprocal lattice isomorphisms between L and $L_a \times L_b$. We have proved that L is in $\mathcal{L}(S)$ by the embedding:

$$L \xrightarrow{h} L_a \times L_b \xrightarrow{f \times g} \Gamma(M_1;S) \times \Gamma(M_2;S) \xrightarrow{i} \Gamma(M_1 \times M_2;S).$$

So, $\mathcal{L}(R) \subset \mathcal{L}(S)$, completing the proof.

Prop. 10. Let p be a prime, $t > 0$ and $j = \min A$ for:

$$A = \{t\} \cup \{k\colon p \text{ is } k\text{-invertible in } R\}.$$

Then $\operatorname{char}(R/p^tR) = p^j$. If $\operatorname{char}(R) \neq 0$ and n divides $\operatorname{char}(R)$, then $\operatorname{char}(R/nR) = n$.

Proof: Assume the hypotheses. Since $p^t \in p^tR$, $\operatorname{char}(R/p^tR) = p^d$ for some d, $0 \leqq d \leqq t$. If $0 \leqq k < d$, then p^k isn't in p^tR. But then $p^k(pr - 1) = 0$ is false for all $r \in R$, since otherwise $p^t r^{t-k} = p^k$. So, $d \leqq \min A$. If $d = t$, then $\min A \leqq d$, so assume $d < t$. Then $p^t r = p^d$ for some r in R, so $p^d(pr_0 - 1) = 0$ for $r_0 = p^{t-d-1}r$. So, p is d-invertible in R, and $\min A \leqq d$. This proves that $d = \min A$ in all cases.

Now suppose $m = \operatorname{char}(R) \neq 0$ and n divides m. Let $d = \operatorname{char}(R/nR)$, so d divides n. To prove $\operatorname{char}(R/nR) = n$, it suffices to show that p^k divides n implies p^k divides d, for any prime p and $k > 0$. Assuming that p^k divides n and using n divides m, let $m = xp^k$. Now $\operatorname{char}(R/p^kR) = p^j$ for some j, $0 \leqq j \leqq k$. If $j < k$, then there exists r in R such that $p^{j+1}r = p^j$, by the above. But then $xp^j = xp^k r^{k-j} = mr^{k-j} = 0$ in R, contradicting $m = \operatorname{char}(R)$. Therefore,

$\mathrm{char}(R/p^kR) = p^k$. Since $nR \subset p^kR$ because p^k divides n, there is a ring homomorphism $R/nR \longrightarrow R/p^kR$ preserving 1, and so $\mathfrak{L}(R/p^kR) \subset \mathfrak{L}(R/nR)$ by Prop. 2. Therefore, p^k divides d by Thm. 2, proving that $\mathrm{char}(R/nR) = n$.

We now prove that the predicate $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ can be evaluated for rings with the same nonzero characteristic if it can be evaluated for rings with the same prime power characteristic.

Theorem 4. Let $\mathrm{char}(R) = \mathrm{char}(S) = n \neq 0$, and $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ for distinct primes $p_1, p_2, \ldots, p_t$ and any positive integers $k_1, k_2, \ldots, k_t$. Then $\mathfrak{L}(R) \subset \mathfrak{L}(S)$ if and only if $\mathfrak{L}(R/p_i^{k_i}R) \subset \mathfrak{L}(S/p_i^{k_i}S)$ and $\mathrm{char}(R/p_i^{k_i}R) = \mathrm{char}(S/p_i^{k_i}S) = p_i^{k_i}$ for all $i \leqq t$.

Proof: Assume the hypotheses. Suppose $\mathfrak{L}(R) \subset \mathfrak{L}(S)$, and let $p = p_i$ and $k = k_i$ for some i, $i \leqq t$. Then $\mathrm{char}(R/p^kR) = p^k$ by Prop. 10, and $\mathfrak{L}(R/p^kR) \subset \mathfrak{L}(R) \subset \mathfrak{L}(S)$ by Prop. 2. Therefore, $\mathfrak{L}(R/p^kR) \subset \mathfrak{L}(S/p^kS)$ and $\mathrm{char}(S/p^kS) = p^k$ by Thm. 3.

Now assume that $\mathfrak{L}(R/p^kR) \subset \mathfrak{L}(S/p^kS)$ for $p^k = p_i^{k_i}$, all $i \leqq t$. To prove $\mathfrak{L}(R) \subset \mathfrak{L}(S)$, use induction on t. If $t = 1$, $n = p_1^{k_1}$ and the result is trivial. Assume $t > 1$, and let $a = p_1^{k_1}$ and $b = n/a = p_2^{k_2} \cdots p_t^{k_t}$. If $R' = R/bR$ and $S' = S/bS$, then $\mathfrak{L}(R/p^kR) \subset \mathfrak{L}(S/p^kS)$ implies $\mathfrak{L}(R'/p^kR') \subset \mathfrak{L}(S'/p^kS')$ for $p^k = p_i^{k_i}$, $i = 2, 3, \ldots, t$. Since $\mathrm{char}(R') = \mathrm{char}(S') = b$ by Prop. 10, and b has $t - 1$ prime factors, $\mathfrak{L}(R/bR) = \mathfrak{L}(R') \subset \mathfrak{L}(S') = \mathfrak{L}(S/bS) \subset \mathfrak{L}(S)$, by the induction hypothesis and Prop. 2. Since $\mathcal{L}(R/aR) \subset \mathcal{L}(S)$ by Prop. 2, $\mathcal{L}(R) \subset \mathcal{L}(S)$ by Prop. 9. This completes the proof.

We turn now to consideration of some particular rings. Two types of ring are especially important: the homomorphic images $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ of the ring $\mathbf{Z}$ of integers, and the unitary subrings of the field $\mathbf{Q}$ of rationals.

Let P denote the set of prime numbers, and let P_R denote the subset of P of primes invertible in R:

$$P_R = \{p \in P : p^{-1} \text{ exists in } R\}.$$

Given any subset P_0 of P, let $\mathbf{Q}(P_0)$ denote the unitary subring of $\mathbf{Q}$ generated by $\{p^{-1} : p \in P_0\}$. It is easily proved that the unitary subrings of $\mathbf{Q}$ are in one-one correspondence with the subsets of P via the reciprocal bijections:

$$R \longrightarrow P_R, \qquad P_0 \longrightarrow \mathbf{Q}(P_0).$$

(That is, $P_0 = P_{\mathbf{Q}(P_0)}$ if $P_0 \subset P$, and $R = \mathbf{Q}(P_R)$ if $\mathbf{Z} \subset R \subset \mathbf{Q}$.)

In the next theorem, we give the basic lattice representability relationships between rings of these types, and some other relationships between lattice representability by these special rings and lattice representability for arbitrary rings satisfying certain tests. A proposition preparing for the use of Prop. 3 is inserted first.

Prop. 11. Let M be a flat right R-module, and let kM denote $\mathrm{im}(k\cdot 1_M)$ for $k \geqq 0$. If $R = \mathbf{Z}_n$ for some $n \geqq 2$, then M is faithfully flat if $dM \neq M$ for every proper divisor d of n. If $R = \mathbf{Q}(P_0)$ for some $P_0 \subset P$, then M is faithfully flat if $pM \neq M$ for every prime p not in P_0.

Proof: Assume the hypotheses. To prove M is faithful, it suffices to show that $M \otimes_R R/u \neq 0$ for every proper left ideal u of R. (If M_0 is nonzero, there is an R-linear monomorphism $R/u \longrightarrow M_0$ for some proper or trivial u. Since M is flat, $M \otimes_R R/u \longrightarrow M \otimes_R M_0$ is a monomorphism. Since $M \otimes_R R \approx M \neq 0$, $M \otimes_R R/u \neq 0$ for all proper u implies $M \otimes_R M_0 \neq 0$ whenever $M_0 \neq 0$.) If $R = \mathbf{Z}_n$ or $R = \mathbf{Q}(P_0)$, then every proper left ideal of R equals kR for some $k > 1$. But:

$$M \otimes_R R/kR \approx M \otimes_R (R \otimes_{\mathbf{Z}} \mathbf{Z}_k) \approx (M \otimes_R R) \otimes_{\mathbf{Z}} \mathbf{Z}_k \approx M/kM,$$

using well-known properties of tensor products. So, it suffices to show that $kM \neq M$ if kR is a proper ideal of R. If $R = \mathbf{Z}_n$, every proper ideal of R equals dR for some proper divisor d of n. So, M is faithful if $dM \neq M$ for such d. If $R = \mathbf{Q}(P_0)$, then kR is a proper ideal if there exists a prime p not in P_0 such that p divides k. But then $kM \subset pM$, so M is faithful if $pM \neq M$ for every prime p not in P_0. This proves Prop. 11.

Theorem 5. Suppose n, $m \geqq 2$ and P_1 and P_2 are subsets of P. Then:

(1) $\mathfrak{L}(Z_n) \subset \mathfrak{L}(Z_m)$ iff n divides m.

(2) $\mathfrak{L}(Z_n) \subset \mathfrak{L}(Q(P_1))$ iff no prime in P_1 divides n.

(3) $\mathfrak{L}(Q(P_1)) \subset \mathfrak{L}(Z_n)$ is always false.

(4) $\mathfrak{L}(Q(P_1)) \subset \mathfrak{L}(Q(P_2))$ iff $P_1 \supset P_2$.

(5) If $\mathrm{char}(R) = n$, then $\mathfrak{L}(R) \subset \mathfrak{L}(Z_n)$.

(6) If $\mathrm{char}(R) = n$ and n is a prime or a product of distinct primes, then $\mathfrak{L}(R) = \mathfrak{L}(Z_n)$.

(7) If $\mathrm{char}(R) = 0$, then $\mathfrak{L}(R) \subset \mathfrak{L}(Q(P_R))$.

(8) If R is torsion-free, then $\mathfrak{L}(R) = \mathfrak{L}(Q(P_R))$.

(9) If $\mathrm{char}(R) = 0$, then $\mathfrak{L}(R) = \mathfrak{L}(Q(P_R))$ iff every prime p which is j-invertible in R for some $j \geqq 1$ is invertible in R.

(10) If some unitary subring of R is a field, then $\mathfrak{L}(R) = \mathfrak{L}(Q)$ if $\mathrm{char}(R) = 0$, and $\mathfrak{L}(R) = \mathfrak{L}(Z_p)$ if $\mathrm{char}(R) = p$, p prime.

Proof: Assume the hypotheses. If $\mathrm{char}(R) = n$, then R has a unitary subring isomorphic to $\mathbf{Z}_n$. If $\mathrm{char}(R) = 0$, then R has a unitary subring isomorphic to $\mathbf{Q}(P_R)$, since integers are central elements of R. Using Prop. 2 and Thm. 2, we can then verify parts (1), (3), (4), (5) and (7). If $\mathfrak{L}(\mathbf{Z}_n) \subset \mathfrak{L}(\mathbf{Q}(P_1))$, then each p in P_1 is invertible in $\mathbf{Z}_n$ by Thm. 2, and so p doesn't divide n. This proves half of (2); the converse follows from Prop. 2 and the observation that $\mathbf{Q}(P_1)/n\mathbf{Q}(P_1)$ is isomorphic to $\mathbf{Z}_n$ if no prime in P_1 divides n.

Suppose $\mathrm{char}(R) = n$, where n is prime or a product of distinct primes, and let M denote R considered as a bimodule (left R-module, right $\mathbf{Z}_n$-module). Now $\mathbf{Z}_n$ is a semisimple ring [12: p. 71], hence it is a regular ring [12: Thm. 4.11, p. 78]. Therefore, M is flat as a right $\mathbf{Z}_n$-module [12: Thm. 4.24, p. 86]. Given a proper divisor d of n, d is not invertible in R and so $dM \neq M$. So, M is faithfully flat by Prop. 11. Therefore, $\mathfrak{L}(\mathbf{Z}_n) \subset \mathfrak{L}(R)$ by Prop. 3, and then part (6) follows from part (5).

Suppose R is torsion-free, and let M denote R considered as a bimodule (left R-module, right $\mathbf{Q}(P_R)$-module). Now $\mathbf{Q}(P_R)$ is a principal ideal domain, and so is a Prüfer ring [12: p. 73]. Therefore, M is flat as a right $\mathbf{Q}(P_R)$-module [12: Thm. 4.23, p. 85]. Given p prime not in P_R, p is not invertible in R and so $pM \neq M$. Therefore, M is faithfully flat by Prop. 11, and $\mathcal{L}(\mathbf{Q}(P_R)) \subset \mathcal{L}(R)$ by Prop. 3. Then part (8) follows from part (7).

Suppose $\mathcal{L}(R) = \mathcal{L}(\mathbf{Q}(P_R))$ and p is a j-invertible prime in R for some $j \geqq 1$. Then $p^j(pr - 1) = 0$ for some r in $\mathbf{Q}(P_R)$ by Thm. 2, and so p is invertible in $\mathbf{Q}(P_R)$ since $pr - 1$ must equal 0. Therefore, p is invertible in R. Now suppose $\mathrm{char}(R) = 0$ and every j-invertible prime of R is invertible. Let t denote the two-sided ideal of all torsion elements of R ($r \in t$ if $nr = 0$ for some positive integer n), and let $S = R/t$. Then S is a nontrivial torsion-free ring, and clearly $P_R \subset P_S$. If $p \in P_S$, then $px = 1 + z$ for some x in R and z in t. So, $k(px - 1) = 0$ for some $k > 0$. Let $k = p^j m$, where p and m are relatively prime. So, $pu + mv = 1$ for certain integers u and v. Let $r = mvx + u$ in R. Then $p^j(pr - 1) = p^j(pmvx + pu - pu - mv) = vk(px - 1) = 0$. So, p is j-invertible in R, and therefore $p \in P_R$ by hypothesis. That is, $P_R = P_S$. But then

$$\mathcal{L}(\mathbf{Q}(P_R)) = \mathcal{L}(\mathbf{Q}(P_S)) = \mathcal{L}(S) \subset \mathcal{L}(R) \subset \mathcal{L}(\mathbf{Q}(P_R)),$$

by parts (7) and (8) and Prop. 2. This proves part (9).

Part (10) follows immediately from parts (6) and (8). (If R contains a unitary subring which is a field of characteristic zero, then R is torsion-free and $\mathbf{Q}(P_R) = \mathbf{Q}$.) This proves Thm. 5.

For arbitrary $n \geqq 2$, the author has been unable to establish a necessary and sufficient condition on R so that $\mathcal{L}(R) = \mathcal{L}(Z_n)$. However, the final result sheds some light on this problem.

Prop. 12. Let $\mathrm{char}(R) = p^u$ for prime p and $u > 1$. If there exist r_1 and r_2 in R and integers i, j and k such that $1 \leqq i,\ j,\ k \leqq u-1$, $i + j + k < 2u$, $r_1 r_2 = p^i$, $p^j r_1 = 0$ and $p^k r_2 = 0$, then $\mathcal{L}(R) \neq \mathcal{L}(Z_{p^u})$.

Proof: Assume the hypotheses, and suppose $\mathcal{L}(R) = \mathcal{L}(Z_{p^u})$. By Thm. 1, there exists an exact embedding $F: Z_{p^u}\text{-Mod}(\aleph_0) \longrightarrow R\text{-Mod}$. Let M denote Z_{p^u} as an object of $Z_{p^u}\text{-Mod}(\aleph_0)$. Since $(p^{u-k}\cdot 1_M, p^k\cdot 1_M)$ is exact, so is $(p^{u-k}\cdot 1_{F(M)}, p^k\cdot 1_{F(M)})$. Let v be in $F(M)$. Then $p^k r_2 v = 0$, since $p^k r_2 = 0$ in R. So, $p^{u-k} v_0 = r_2 v$ for some v_0 in $F(M)$. But then $p^{u-1} v = p^{u-1-i} p^i v = p^{u-1-i} r_1 r_2 v = p^{u-1-i} r_1 p^{u-k} v_0 = p^{2u-i-j-k-1} p^j r_1 v_0 = 0$, using the hypotheses. Therefore, $F(p^{u-1}\cdot 1_M) = p^{u-1}\cdot 1_{F(M)} = 0$. But $p^{u-1}\cdot 1_M \neq 0$, contradicting the embedding property for F. This proves Prop. 12.

Given $P_0 \subset P$ and $P_0 \neq P$, one can easily construct a ring R with characteristic zero such that $P_R = P_0$ but $\mathcal{L}(R) \neq \mathcal{L}(Q(P_0))$. For example, choose a prime p not in P_0 and $j \geqq 1$, and let R

denote the quotient of the polynomial ring $\mathbf{Q}(P_0)[y]$ divided by the principal ideal generated by $p^j(py - 1)$. Then $\mathrm{char}(R) = 0$, $P_R = P_0$ and p is j-invertible but not invertible in R. So, $\mathfrak{L}(R) \neq \mathfrak{L}(\mathbf{Q}(P_0))$ by Thm. 5(9).

Another family of counterexamples is related to Prop. 12. Suppose $n \geqq 2$ and n is not square-free, that is, $n = p^2m$ for some prime p and integer m. Let R be the quotient ring of the polynomial ring $\mathbf{Z}_n[y]$ divided by the ideal generated by the polynomials py and $y^2 - pm$. We omit the proof that R is a commutative ring with characteristic n and pn elements; each element of R is representable by a polynomial $uy + v$ with $0 \leqq u < p$ and $0 \leqq v < n$. Assume $\mathfrak{L}(R) = \mathfrak{L}(\mathbf{Z}_n)$, and construct an exact embedding $F: \mathbf{Z}_n\text{-Mod}(\aleph_0) \longrightarrow R\text{-Mod}$. Let M equal $\mathbf{Z}_n$ as an object of $\mathbf{Z}_n\text{-Mod}(\aleph_0)$, and note that $(pm \cdot 1_{F(M)}, p \cdot 1_{F(M)})$ is exact because $(pm \cdot 1_M, p \cdot 1_M)$ is exact. Suppose $v \in F(M)$: since $pyv = 0$ there exists v_0 in $F(M)$ such that $pmv_0 = yv$. But then $pmv = y^2v = ypmv_0 = 0$, since $pm = y^2$ and $py = 0$ in R. Then $F(pm \cdot 1_M) = pm \cdot 1_{F(M)} = 0$ and $pm \cdot 1_M \neq 0$ leads to contradiction. So, R is a ring with characteristic n but $\mathfrak{L}(R) \neq \mathfrak{L}(\mathbf{Z}_n)$. We remark that this R is also a counterexample for the converse of Thm. 2. That is, the equation $ax + b = 0$ for integers a and b has a solution in R if and only if it has a solution in $\mathbf{Z}_n$, but $\mathfrak{L}(R) \neq \mathfrak{L}(\mathbf{Z}_n)$.

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IDEAL COMPLETIONS

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The purpose of this note is to illustrate how some lattice theoretical ideas, which have not been exploited in the context of abstract (ring) ideal theory, can be put to work. Namely, we will exploit the fact that lattices of (ring) ideals are algebraic lattices.

0. Ideal completions of join-semilattices

Let P and Q be posets, $P \subseteq Q$. Q is an extension of P if the ordering of P is the restriction to P of the ordering of Q (i.e., for $x,y \in P$, $x \leq y$ in P if and only if $x \leq y$ in Q). P is join-dense in Q if every $q \in Q$ is representable as the join (in Q) of some subset $M \subseteq P$, $q = \sup_Q M$; one can then take as M the set of all elements $p \in P$ such that $p \leq q$, $M = P \cap (q]$. An element $x \in P$ is called compact if the following condition holds true for each subset $M \subseteq P$:

(0.1) if $x \leq \sup_P M$, then $x \leq \sup_P M'$ for some finite $M' \subseteq M$.

A complete lattice L is said to be algebraic if the set of compact elements, C(L), of L is join-dense. Note

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that in any complete lattice L, C(L) is a join-subsemilattice containing the least element of L.

<u>Theorem 0.1.</u> Let P be a join-semilattice with least element o. Then there exists a complete extension I(P) of P satisfying the following conditions:

(i) P is join-dense in I(P);

(ii) the compact elements of I(P) are exactly the elements of P, P=C(I(P)).

Such I(P) is uniquely determined up to a unique P-isomorphism and is called "the" ideal completion of P. Note that I(P) is an algebraic lattice. As a consequence of condition (i) P is completely meet-faithful in I(P), i.e., if $p=\inf_P M$ where $p\in P$ and $M\subseteq P$, then $p=\inf_{I(P)}M$. So, in particular, if P has e as largest element, then e is also the largest element of I(P). Also, as a consequence of condition (ii) P, being finitely join-closed in I(P), is finitely join-faithful, i.e., if $p=\sup_P M$ where $p\in P$ and M is a finite subset of P, then $p=\sup_{I(P)}M$. Caution: this does not necessarily hold for infinite M. But it does allow us to write $x\vee y$ and $x\wedge y$ for $x,y\in P$ without any risk of ambiguity.

The usual proof of this theorem is by construction. $I\subseteq P$ is called an <u>ideal</u> if I is a lower end (i.e., if $y\in I$ and $x\leq y$ then $x\in I$) and I is closed under finite joins. In particular, $o \in I$. Let $I(P)=\{I \mid I$ is an ideal of $P\}$. For $p \in P$, let $(p]=\{q \mid q \in P,\ q \leq p\}$. Then $(p] \in I(P)$ and the mapping $p \longmapsto (p]$ is an embedding of P into $I(P)$. One shows $I(P)$ satisfies conditions (i) and (ii) of the theorem. For a more detailed exposition, cf. [2].

As an immediate consequence of Theorem 0.1 we obtain the following corollary:

Corollary 1. Each algebraic lattice L is the ideal completion of the semilattice $C(L)$, $L=I(C(L))$.

Henceforth we will use the term semilattice to mean join-semilattice with least element o.

The aforementioned uniqueness of the ideal completion is a special case of the following universal property:

Theorem 0.2. Let L be a complete lattice, P a subsemilattice of L containing the least element. Then the following statements are equivalent:

(i) $L=I(P)$;

(ii) for each complete lattice F, each finitely join-preserving mapping $\varphi : P \longrightarrow F$, there is exactly one completely join-preserving mapping $\psi : L \longrightarrow F$ extending φ.

Note that φ finitely join-preserving means that $\varphi(x \vee y)=\varphi(x) \vee \varphi(y)$ and $\varphi(o)=o$. The statement of this theorem verbatim can be found in Schmidt [4].

The proof of (i)$\Longrightarrow$(ii) is again by construction;

for x in $I(P)$ one defines $\Psi(x)=\sup_F \Phi(P\cap(x])$. Then one checks that Ψ is the unique completely join-preserving extension of Φ. For the proof of (ii)$\Longrightarrow$(i) one uses the standard universal algebra device for universal solutions.

1. Ideal completions of sl-semigroups

A <u>semilattice-semigroup</u> S or, in short, an <u>sl-semigroup</u> is a (join-) semilattice and at the same time a semigroup (in multiplicative notation) subject to the following compatibility conditions:

(i) for any $x,y,z\in S$, $x(y\vee z)=xy\vee xz$, $(y\vee z)x=yx\vee zy$;

(ii) for any $x\in S$, $xo=ox=o$.

(i) and (ii) may be combined in the statement that the product xy as a function of one of its factors is finitely join-preserving. As a consequence, multiplication with an element, be it on the right or the left, is order preserving.

Let $I(S)$ be the ideal completion of S. We would like to extend the multiplication to $I(S)$ so that it also becomes an sl-semigroup.

Note that for $x,y\in S$, $xy=\max\{x'y'\mid x'\in(x]\cap S, y'\in(y]\cap S\}=\sup_{I(S)}\{x'y'\mid x'\in(x]\cap S,\ y'\in(y]\cap S\}$.

Thus, if we define

(1.1) $xy=\sup_{I(S)}\{x'y' \mid x' \in (x]\cap S,\ y' \in (y]\cap S\}$

for $x,y \in I(S)$ we indeed obtain an extension of the multiplication on S. Let $x,y,z \in I(S)$. First, we prove that multiplication by an element is order preserving. Assume $x\leq y$, let $x' \in (x]\cap S$, $z' \in (z]\cap S$. Then $x' \in (y]\cap S$, so, $x'z' \leq yz$, thus, $xz \leq yz$. Similarly, $zx \leq zy$. Next we show that for any $M \subseteq S$,

(1.2) if $y=\sup_{I(S)}M$, then $xy=\sup_{I(S)}xM$, $yx=\sup_{I(S)}Mx$.

Clearly, $xy \geq \sup_{I(S)}xM$. Conversely, let $x' \in (x]\cap S$, $y' \in (y]\cap S$. Then $y' \leq \sup_{I(S)}M$, but by compactness there exists $M' \subseteq M$, finite, such that $y' \leq \sup_{I(S)}M'=\sup_S M'$, so $x'y' \leq x'\sup_S M'=\sup_S x'M' \leq \sup_{I(S)}xM$. The proof of the other half is alike. Now we are ready to prove associativity. By (1.1) $yz=\sup_{I(S)}\{y'z' \mid y' \in (y]\cap S,\ z' \in (z]\cap S\}$, so, by (1.2), $x(yz)=\sup_{I(S)}\{x(y'z') \mid y',z' \text{ as above}\}$. But for $x' \in (x]\cap S$, $x'(y'z')=(x'y')z' \leq (xy)z$. So, $x(y'z') \leq (xy)z$, thus $x(yz) \leq (xy)z$. Similarly, $(xy)z \leq x(yz)$. Finally, since (1.2) implies that $xo=ox=o$, it is enough to show that $x(y\vee z)=xy\vee xz$ and $(y\vee z)x=yx\vee zx$. Clearly, $x(y\vee z) \geq xy\vee xz$. On the other hand, $y\vee z=\sup_{I(S)}\{y'\vee z' \mid y',z' \text{ as above}\}$. Thus, by (1.2), $x(y\vee z)=\sup_{I(S)}\{x(y'\vee z')\}$. But for $x' \in (x]\cap S$, $x'(y'\vee z')=x'y'\vee x'z' \leq xy\vee xz$. Therefore, we have that I(S) with the multiplication defined by (1.1) is an sl-semigroup. Yet we are ready to prove a stronger result than (i). By a <u>strong</u> sl-semigroup we mean a complete sl-semigroup (completeness refers here to the semilattice structure), where multiplication by an element is completely join-preserving. We are now going to show that I(S) is a strong sl-semigroup:

(1.3) $(\bigvee_\alpha y_\alpha)x = \bigvee_\alpha y_\alpha x$, and $x(\bigvee_\alpha y_\alpha) = \bigvee_\alpha xy_\alpha$,

for $x, y_\alpha \in I(S)$. Let $y' \leq \bigvee_\alpha y_\alpha$ and $y' \in S$. Then, by compactness, there exist $y_{\alpha_1}, \dots, y_{\alpha_n}$ such that $y' \leq y_{\alpha_1} \vee \dots \vee y_{\alpha_n}$ so, $xy' \leq x(y_{\alpha_1} \vee \dots \vee y_{\alpha_n}) = xy_{\alpha_1} \vee \dots \vee xy_{\alpha_n} \leq \bigvee_\alpha xy_\alpha$.

Suppose we have defined a multiplication, say $*$, on I(S) such that it extends the multiplication on S and makes I(S) into a strong sl-semigroup; since $x = \sup_{I(S)}(x] \cap S$ and $y = \sup_{I(S)}(y] \cap S$, $x*y = \sup_{I(S)}\{x'*y \mid x' \in (x] \cap S\} = \sup_{I(S)}\{x'y' \mid x' \in (x] \cap S,\ y' \in (y] \cap S\} = xy$. Thus, the following theorem is now clear:

<u>Theorem 1.1.</u> Let S be an sl-semigroup. Then there is exactly one way of extending the multiplication to I(S) so that I(S) becomes a strong sl-semigroup.

The reader may note that the proof of Theorem 1.1 is similar to proving Theorem 0.2 for mappings of two variables. Actually, an alternate proof may be based on Theorem 0.2. However, this would be no shorter than the given one.

Note that I(S) is a commutative semigroup if and only if S is. Also, if S is a monoid, then its identity 1 is also the identity of I(S). Note that 1 need not be the largest element.

Putting Theorem 1.1 together with Corollary 1 of Theorem 0.1 we get:

Corollary 1. Let L be a strong sl-semigroup which is an algebraic lattice. Assume that C(L) is a subsemigroup. Then $L=I(C(L))$.

The equality above is meant not only as lattices, but as sl-semigroups.

We also obtain the following result corresponding to Theorem 0.2:

Theorem 1.2. Let L be a strong sl-semigroup, S an sl-subsemigroup of L. Then the following statements are equivalent:

(i) $L=I(S)$;

(ii) for each strong sl-semigroup F, and each sl-homomorphism $\varphi:S\longrightarrow F$, there is exactly one strong sl-homomorphism $\psi:L\longrightarrow F$ extending φ.

By an sl-homomorphism we mean, of course, a semigroup homomorphism that is finitely join-preserving. If it is completely join-preserving we call it strong. For the proof of (i)$\Longrightarrow$(ii) it is enough to show that the ψ

given by Theorem 0.2 is a semigroup homomorphism. Let $x,y \in I(S)$. Then $\psi(xy)=\psi(\sup_{I(S)}(((x]\cap S)((y]\cap S))=$ $\sup_F \psi(((x]\cap S)((y]\cap S))=\sup_F(\psi((x]\cap S)\,\psi((y]\cap S))=$ $(\sup_F \psi((x]\cap S))(\sup_F \psi((y]\cap S))=\psi(x)\psi(y)$. The proof of (ii)$\Longrightarrow$(i) is, again, by the device for universal solutions.

Let us close with some examples.

First, let us consider an arbitrary complete lattice L, and a pre-fixed non-compact element c L. We make L a strong sl-semigroup by the following multiplication: $xy=c$ when neither x nor y is o and $xy=o$ otherwise. This shows that in a given strong sl-semigroup, the compact elements need not always be a subsemigroup, even if it is algebraic.

Next, let us consider a complete lattice L. Let L* be the set of completely join-preserving mappings of L into itself. L*, then, is, as a subset of the complete lattice L^L, at least a poset. Being closed under arbitrary joins in L^L, L* is actually a complete lattice itself. Composition makes it a strong sl-monoid. The Cayley representation can be used to show that any strong sl-monoid L is embeddable in L*.

Let us now consider a commutative ring R with identity 1. Let K be a unitary (associative) algebra over R. $L(K)$ will denote the lattice of R-submodules of K. $L(K)$ is an algebraic lattice where $C(L(K))$ (which we will write $C(K)$ for short) is the set of finitely generated submodules. For $M,N \in L(K)$ let MN be the submodule generated by the set of all mn where $m \in M$ and $n \in N$. This multiplication makes $L(K)$ into a strong sl-monoid (with identity $R1_K$), where, moreover, $C(K)$ is an sl-submonoid. Thus, by Corollary 1 of Theorem 1.1, $L(K)=I(C(K))$. This was actually the kind of example that led to the present formal considerations.

Finally, let D be an integral domain, and K its field of quotients. So K is an algebra over D. D is a Prüfer domain (cf. [1]) if and only if $C^*(K)$ $(=C(K)\setminus\{o\})$ is a group. But then $C^*(K)$ is a lattice-ordered group (l-group). Thus,

<u>Theorem 1.3.</u> Let D be an integral domain with field of quotients K. Then D is Prüfer if and only if $L(K)=I(G)$ for some Abelian l-group G with o.

By an l-group with o we mean, of course, an l-group with an element o added to it acting both as a zero for the semigroup and the semilattice structures.

By a theorem of Jaffard (cf. [1]), for every Abelian l-group G with o, there exists a Bezout domain D with field of quotients K such that $L(K)=I(G)$ (or, equivalently C(K)=G, or L(D)=I(G-), where G- denotes the negative cone of G). Thus, from the sl-monoid point of view, there is absolutely no difference between Prüfer and Bezout domains. Similarly, there is no difference between Dedekind domains and principal ideal domains. In particular, one cannot detect principal submodules in L(K) (cf. [5]).

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On the equational theory of submodule lattices.

By Christian Herrmann

Equational problems for modular lattices have been studied for a long time, although available results have been established only under significant syntactical difficulties (see e.g. the papers of RALPH FREESE and ALEIT MITSCHKE in this volume). Furthermore, they have been more or less partial in nature. For lattices of submodules things become surprisingly easy, by simply making use of well known algebraic facts. As a by-product algebraic results can be extended by lattice theoretic methods. §1-3 are based on joint work of the author and A.HUHN; the results in §4 have been partially reported in [6].

§1 The two basic lemmas.

For p prime, $n < \infty$, $k \leq \infty$, let L(p,k,n) be the lattice of subgroups of the n-th power of the cyclic group Z_{p^k} of order p^k or of the quasicyclic p-group Z_{p^∞}. If M_R is a unitary R-module, then $L(M_R)$ denotes the lattice of R-submodules of M_R. The lattice varieties generated by all normal subgroup lattices of groups or subgroup lattices of abelian groups or complemented modular lattices will be written as $\mathcal{N}$ or $\mathcal{A}$ or $\mathcal{C}$. $\mathcal{K}$ shall denote the variety generated by $\mathcal{N} \cup \mathcal{A}$.

Lemma 1. $L(M_R)$ is in the variety generated by all lattices $L(p,k,n)$ where $k < \infty$, p^k divides the characteristic of R, and n is less than or equal to the cardinality of a generating set of the P-module M_p, where P is the subring of R generated by the unit element.

Corollary 2. $\mathcal{A}$ is generated by the finite primary lattices $L(p,k,n)$ (p prime, $k,n < \infty$).

Sketch of proof. $L(M_R)$ is a sublattice of $L(M_p)$ and $L(M_p)$ is in the variety generated by the submodule lattices of its finitely generated submodules. By the Homomorphism Theorem these are sublattices of the $L(P^n{}_p)$. Now, if P is finite and $|P| = p_1{}^{k_1} \cdot \ldots \cdot p_m{}^{k_m}$, then $P \cong \times_{i=1}^{m} (Z_{p_i{}^{k_i}})^n$ and $L(P^n{}_p) \cong \times_{i=1}^{m} L(p_i, k_i, n)$. If, finally, P is isomorphic to the ring Z of integers, then we use the fact that a system of linear diophantine equations is solvable in iff it is solvable in all Z_{p^k} and the following construction: To each lattice polynomial w attach a system $\tilde{w}(x_i, y_i^j, \lambda_k)$ of linear equations in variables x_i, y_i^j, λ_k such that for any elements $a_1, \ldots a_n, b_1^1, \ldots b_n^m$ of a R-module M_R $\langle a_1, \ldots a_n \rangle_R \in w(\langle b_1^1, \ldots b_n^1 \rangle_R, \ldots \langle b_1^m, \ldots b_n^m \rangle_R)$ holds iff the system $\tilde{w}(a_i, b_i^j, \lambda_k)$ is solvable with values of the λ_k in R. This can be easily done by induction over the length of w. Hence, if all R-submodules of M_R are generated by at most n elements, the inequality $w \leq v$ is valid in $L(M_R)$ if and only if, for any choice of constants a_i, b_i^j in M_R, the solvability of $\tilde{w}(a_i, b_i^j, \lambda_k)$ implies the solvability of $\tilde{v}(a_i, b_i^j, \mu_k)$ over R .

Lemma 3. If M_R is the $\mathfrak{F}$-ultraproduct of the modules M_{iR_i} , then the $\mathfrak{F}$-ultraproduct of the lattices $L(M_{iR_i})$ is a sublattice of $L(M_R)$, containing all finitely generated R-submodules.

The proof is by the classical model theoretic method of correspondences between classes: consider the structures $(M_R, L(M_R), \phi)$, ϕ being the relation $a \in U$ on $M \times L(M_R)$ (c.f. MAKKAI, McNULTY [13]).

Corollary 4. $\mathcal{C}$ is generated by subspace lattices of finite projective geometries over prime fields and arbitrary nondesarguesian planes.

§2 Lattices generated by a frame.

In [9] A.HUHN introduced the concept of an n-diamond in a lattice: a sequence $a_0, \ldots a_n$ of elements such that any n-element subset is independent in the interval $\left[\prod_{i=0}^{n} a_i, \sum_{i=0}^{n} a_i\right]$. It is called a frame in L, if $\prod a_i = 0_L$

and $\sum a_i = 1_L$. If L is modular, a frame in the usual sense can be derived and vice versa.

<u>Theorem 5.</u> For $n \geq 3$ there is a complete list of all subdirectly irreducible lattices in $\mathcal{N}$ which are generated by an n-diamond:

the rational projective geometry $L(Q^n{}_Q)$;

the lattices $L(p,k,n)$, where p is prime and $k \leq \infty$;

the duals of the $L(p,\infty,n)$, where p is prime.

The generating n-diamond is given, up to automorphism, by the submodules $e_0=(x,\ldots,x)$, $e_i=(0,\ldots,0,x,0,\ldots,0)$, with x in the i-th entry, for $i=1,\ldots n$.

The following notation has been used: For $k_1,\ldots k_n$ in R and variables $x_1,\ldots x_n$ not necessarily distinct we have $(k_1x_1,\ldots,k_nx_n) = \{(k_1a_1,\ldots,k_na_n) \mid a_i \in M \text{ and } x_i=x_j \Rightarrow a_i=a_j\}$, a submodule of ${M_R}^n$.

The proof consists of the following main steps:

1) Reduction to $\mathcal{A}$: If the lattice of normal subgroups of G contains an n-frame $(n \geq 3)$, then G is abelian.

2) If L is a sublattice of any $L(p,k,n)$ $(k < \infty)$ generated by an n-diamond, then L is a $\{0,\ldots n\}$-subdirect product (in the sense of WILLE [18]) of lattices $L(p_i,k_i,n)$, each generated by the diamond $e_0,\ldots e_n$.

3) By Corollary 2 the lattice $F\mathcal{A}(P_n)$ freely generated in $\mathcal{A}$ by an n-diamond is a $\{0,\ldots n\}$-subdirect product of the lattices $L(p,k,n)$ (p prime, $k<\infty$) with generators $e_0,\ldots,e_n$.

4) Any subdirectly irreducible lattice in $\mathcal{A}$ which is generated by an n-diamond is, using the Lemma in JÓNSSON [12], a homomorphic image of the sublattice M generated by $[e_i]\Theta_{\mathcal{F}}$ $(i=0,\ldots n)$ in $F\mathcal{A}(P_n)/\Theta_{\mathcal{F}}$ for a suitable ultrafilter $\mathcal{F}$ on the set $\{p^k \mid p \text{ prime}, k<\infty\}$. Now, if $p\to\infty$ in $\mathcal{F}$, then $M\cong L(Q^n{}_Q)$ is proved with the method of Lemma 3. If, on the other hand, $k\to\infty$ for fixed p, then M is a subdirect product of $L(p,\infty,n)$ and its dual. But the only nontrivial homomorphic image of these lattices is $L(p,1,n)$.

<u>Corollary 6.</u> The subdirectly irreducible lattices in $\mathcal{C}$ generated by an n-diamond are, for $n>3$, exactly those in the above list and , for $n=3$, those in the above list as well as nondesarguesian planes generated by four points.

§3 Applications to equational classes

In order to apply Theorem 5 we need polynomials $d_i(x_o,..,x_3)$ $(i=0,..,3)$ such that in any modular lattice the following holds: For any choice of $x_o,..,x_3$ the $d_i(x_o,..,x_3)$ $(i=0,..,3)$ are equal to each other or form a $\mathbf{3}$-diamond; if $x_o,..,x_3$ is a $\mathbf{3}$-diamond, then $x_i=d_i(x_o,..,x_3)$ $(i=0,..,3)$. Such polynomials are defined in A.HUHN [9] :

$$d_o(x_o,..,x_3):=\prod_{i=1}^{3} b_i(x_o,..,x_3) + a_o(x_o,..,x_3)$$

$$d_i(x_o,..,x_3):=\prod_{j=1,j\neq i}^{3} b_j(x_o,..,x_3) \quad \text{for } i=1,2,3$$

where $a_o(x_o,..,x_3)=x_o\cdot\sum_{i=1}^{3} x_i$, $a_i(x_o,..,x_3)=\sum_{j=1,j\neq i}^{3} x_j$,

$$v(x_o,..,x_3)=\prod_{i=1}^{3} a_o(x_o,..,x_3) + a_i(x_o,..,x_3)$$

$$b_i(x_o,..,x_3)=a_i(x_o,..,x_3)\cdot v(x_o,..,x_3)+\sum_{j=1}^{3} x_o\cdot\sum_{k=1,k=j}^{3} x_k$$

for $i=1,2,3$.

Now we can define inductively $w_o(x_o,..,x_3)=d_3(x_o,..,x_3)$

$$w_{n+1}(x_o,..,x_3)=\{[w_n(x_o,..,x_3) + d_o(x_o,..,x_3)\cdot d_2(x_o,..,x_3)]\cdot [d_1(x_o,..,x_3)+d_2(x_o,..,x_3)] + d_o(x_o,..,x_3)\}\cdot [d_2(x_o,..,x_3) + d_3(x_o,..,x_3)] .$$

<u>Lemma 7.</u> $w_n(e_o,..,e_3)=(0,nx,x)$ in any module M_R^3 .

Theorem 8. The lattice identity $w_n(x_o,..,x_3)=d_3(x_o,..,x_3)$ is valid in $L(M_R)$ if and only if the greatest common divisor of the additive orders of any three weakly independent elements of M divides n.

Theorem 9. Each lattice $L(p,k,n)$ $(n \geqslant 3,\ k<\infty)$ is splitting in $\mathcal{H}$. For $k>1$ splitting universal disjunctions are "length$\leqslant 3$" or $d_3(x_o,..,x_3)\cdot w_p k(x_o,..,x_3) \leqslant d_3(x_o,..,x_3)\cdot w_p k-1(x_o,..,x_3)$, and for $k=1$, they are "length $\leqslant 3$ and $L(p,k,n)$ not order embeddable" or $d_3(x_o,..,x_3)\cdot w_p(x_o,..,x_3) \leqslant x$. We remark that $L(p,1,n)$ is neither projective nor finitely projected nor bounded epimorphic image of a free lattice.

Theorem 10. If $\mathcal{L}$ is any class of lattices contained in $\mathcal{H}$ and containing all sublattices of lattices $L(V_K)$ where V_K is any five dimensional vector space over a field of characteristic zero, then $\mathcal{L}$ cannot be defined by a finite set of first order axioms.

The proof is immediate by the following Lemma 11 and the fact that a nontrivial ultraproduct of $L_{p,q}$'s is embeddable in a $L(V_K)$, (see [5]).

Lemma 11. There is an identity valid in $\mathcal{H}$ which does not hold, for $p \neq q$, in the Arguesian lattice $L_{p,q}=[0,a]\cup[b,1]$ with $b \prec a$, $[0,a]\cong L(p,1,3)$, and $[b,1]\cong L(q,1,3)$ - cf. JONNSON [11].

§4 Lattices with four generators.

In [1] the authors asked for a complete list of subdirectly irreducible modular lattices with four generators; the solution is still distant.

Lemma 12. Any lattice listed in Theorem 5 is generated by four elements.
Furthermore, with the methods of [7] it is possible to construct from a sufficently large partial sublattice of a lattice $L(p,2,3)$ a nondesarguesian uniform Hjelmslevplane with four generators.

The systems of generators and a partial converse of the lemma stem from the work of GELFAND and PONOMAREV [3] on linear spaces with four subspaces.
If V is a linear space with subspaces $V_1, \ldots V_4$, then $(V, V_1, \ldots V_4)$ is called a quadruple. It is called indecomposable, if there is no nontrivial complementary pair A, B of subspaces such that $V_i = A \cap V_i + B \cap V_i$ for $i=1, \ldots 4$.

Lemma 13. If $L = \langle v_1, \ldots v_4 \rangle$ is subdirectly irreducible and can be embedded in the subspace lattice of a linear space of finite dimension, then there is an indecomposable quadruple $(V, V_1, \ldots V_4)$ (with V of finite dimension over an algebraically closed field F) and an isomorphism of L onto the sublattice $\langle V_1, \ldots V_4 \rangle$ of $L(V_F)$ mapping v_i onto V_i for $i=1, \ldots 4$.

Theorem 14. (GELFAND, PONOMAREV [3]) The indecomposable quadruples of finite dimensions over an algebraically closed field F are given (up to isomorphism, permutation, and duality) by the following list:

1) $(F^{2n},(x^n,0^n),(0^n,x^n),(x^n,x^n),(x^1,y^{n-1},\lambda x^1,\lambda y^{n-1}+x^1))$ with λ $F-\{0,1\}$,

2) a) $(F^{2n+1},(x^{n+1},0^n),(0^{n+1},x^n),(x^1,y^n,y^n),(x^n,0^1,x^n))$,

b) $(F^{2n},(x^n,0^n),(0^n,x^n),(x^1,y^{n-1},y^{n-1},0^1),(x^n,x^n)$,

3) a) $(F^{2n+1},(x^n,0^{n+1}),(0^n,x^{n+1}),(x^n,x^n,0^1),(x^n,0^1,x^n)$,

b) $(F^{2n},(x^n,0^n),(0^n,x^n),(0^1,x^{n-1},x^{n-1},0^1),(x^n,x^n)$,

4) $(F^{2n+1},(x^n,0^{n+1}),(0^n,x^n,0^1),(0,x_1,\ ..\ x_{n-1},x_1,\ ..\ x_n,x_n),$
$(x_1,\ ..\ x_n,0,x_1,\ ..\ x_n))$.

Let $S(n,4)$ be the lattice of fig.1 and $FM(J_1^4)$ the modular lattcice freely generated by J_1^4 (cf. fig.2 and [1]) .

Lemma 15. $V_1,\ \dots\ V_4 \subseteq L(V_F)$ is in cases 1)-4) isomorphic to M_4, $S(m,4)$, $L(P^m{}_P)$ where $m=2n+1$ or $m=2n$ and P the prime field of F, respectively.

Theorem 16. If L is subdirectly irreducible in $\mathcal{C}$ and generated by four elements, then L is isomorphic either to a nondesarguesian plane or one of the following lattices: M_4, $S(m,4)$, $L(P^m{}_P)$ P a prime field, $FM(J_1^4)$ or its dual .

Sketch of proof. Similarily as in the proof of Theorem 3 we have to study ultraproducts of lattices listed in Lemma 15. As there is only a finite number of types, we may assume that all components are of the same type. In case 1) there is nothing to do; in 2) the untraproduct is again of breadth two, hence we may use the result of FREESE [2] that any subdirectly irreducible breadth two modular lattice with four generetors is $S(m,4)$, $FM(J_1^4)$ or its dual. But the sublattice M generated by $v_1, \dots v_4$ in the ultraptoduct can be visualized and decomposed in a straightforward manner.

3),4): If m is fixed, then $M \cong L({P^m}_P)$ follows trivially.

If $m \to \infty$, then we consider structures

$({P^m}_P, L({P^m}_P), v_1, \dots, v_4, \Phi, I, J, K, \kappa, \varphi, 1_I, 1_J, \infty_I, \infty_J)$ such that: $({P^m}_P, v_1, \dots, v_4)$ is the given quadruple; Φ is defined as in Lemma 3; $I=\{1, \dots, n\}$, $J=\{n+1, \dots, 2n+1\}$ in case 3a), $J=\{n+1, \dots, 2n\}$ in cases 3b) and 4), $K=\emptyset$ in cases 3 a) and b), and $K=\{2n+1\}$ in cases 4) ; I and J are equipped with the partial poeration of taking the successor; φ is the mapping from I onto J with $\varphi(i)=i+n$; κ is the mapping from $P^m \times (I \cup J \cup K)$ into P such that $\kappa(a,i)$ is the ith coordinate of a ; $1_I=1$, $1_J=n+1$, $\infty_I=n$, $\infty_J=2n+1$ in case 3a), and $\infty_J=2n$ in cases 3b) and 4) .

In any of the cases 3a),3b), and 4) we have formulas $\alpha_1, \dots \alpha_4$ in the first order language of these structurs such that for any m and P $v_i=\{x \mid x \in P^m \text{ and } \alpha_i(x)\}$ holds for $i=1, \dots 4$. Now, in the ultraproduct we have a vector space $V \cong F^{I \cup J \cup K}$ and $v_i=\{x \mid x \in V \text{ and } \alpha_i(x)\}$ is valid as well.

Let I_1 and I_∞ be the subalgebras of $I \cup J$ generated by $\{1_I, 1_J\}$ and $\{\infty_I, \infty_J\}$ resp., and $I_* = (I \cup J) - (I_1 \cup I_\infty)$. Define $A_\gamma = \{f \mid f \in V$ and $(f,i) = 0$ for all $i \in I_\gamma\}$, a subspace of V, for $\gamma \in \{1,*\}$ or $\gamma = \infty$ and case 3a,b) ; $A_\infty = \{f \mid f \in V$ and $(f,i) = 0$ for all $i \in I_\infty \cup K\}$ in case 4) .

Then A_1, A_*, A_∞ yield a direct decomposition of the quadruple $(V, v_1, \ldots v_4)$ into three quadruples $(A_\gamma, v_1^\gamma, \ldots v_4^\gamma)$ $(\gamma = 1, *, \infty)$, thus a subdirect decomposition of M into three factors.

But for $\gamma = 1, *$ or $\gamma = \infty$ and case 3a,b) $v_1^\gamma, \ldots v_4^\gamma$ together with 0 and V form a partial lattice J_1^4 ; hence they generate a lattice $FM(J_1^4)$ (cf. [1]). In case 4) from $v_1^\gamma, \ldots v_4^\gamma$ we get a partial lattice J_3^4 (fig.3) which generates a third subdirect power of $FM(J_1^4)$ (see [14]). In any case, M is a finite subdirect power of $FM(J_1^4)$ and the only subdirectly irreducible epimorphic images of M are M_4 and $FM(J_1^4)$.

§5 Word problems.

HUTCHINSON [10] proved that in a quasivariety $\mathfrak{Q}$ of modular lattices such that $L(R^\omega{}_R) \in \mathfrak{Q}$ for a nontrivial ring R there is a finitely presented lattice with seven generators which has an unsolvable word problem (cf. FREESE [2], too). The attempts on the word problem for free modular lattices FM(n) by SCHÜTZENBERGER [16] and GLUHOV [4] may be regarded as unsuccessful (cf. WHITMAN [17] and HERRMANN [8]). The following solvability results do, however, hold. Here

${}_m\mathcal{M}^n$ denotes the class of all modular lattices of primitive length $\leq n$ and primitive breadth $\leq m$.

Theorem 17. (FREESE 2) In ${}_2\mathcal{M}$ the word problem in four generators is solvable.

Theorem 18. [7] For $n \leq 6$ and $m \leq 3$ or $n < \infty$ and $m \leq 2$ the word problem in ${}_m\mathcal{M}^n$ is solvable.

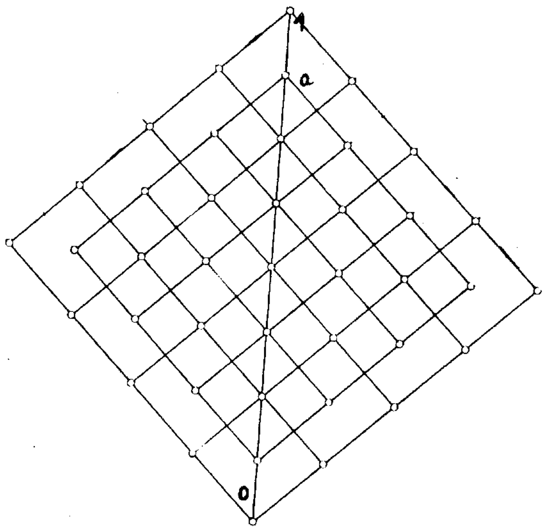
Theorem 19. In $\mathcal{C}$ the word problem in four generators is solvable.

Theorem 20. The word problems for the free lattices $F\mathcal{C}(n)$ and $F\mathcal{A}(n)$ are solvable.

Proof. By Theorem 16 any four generated lattice in $\mathcal{C}$ is embeddable in a complemented modular lattice. Hence a Horn formula in four variables is valid in $\mathcal{C}$ if and anly if it can be derived from the finite set of axioms of complemented modular lattices by a calculus of first order logic. On the other hand the four variable Horn formulas not valid in $\mathcal{C}$ are enumerable by Theorem 16, too.
Theorem 20 is an immediate consequence of Corollaries 2 and 4 and the fact that $\mathcal{C}$ and $\mathcal{A}$ can be defined by enumerable sets of identities. For $\mathcal{C}$ these are just the identities derivable from the axioms of a complemented modular lattice; for $\mathcal{A}$ this

follows by the result of SCHEIN[15] that the class of lattices embeddable in subgroup lattices of abelian groups can be recursively defined.

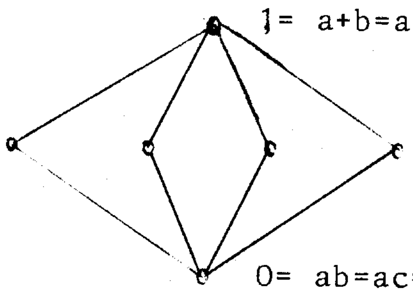
Fig.1



$[0,1] = S(8,4)$

$[0,a] = S(7,4)$

Fig.2

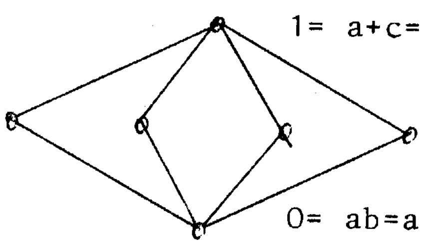


1= a+b=a+c=a+d=b+d=c+d

J_1^4

0= ab=ac=ad=bc=bd=cd

Fig.3



1= a+c=a+d=b+d

J_3^4

0= ab=ac=ad=bc=bd=cd

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Modulare Verbände von Länge $n \leq 6$

Christian Herrmann

1. Einführung

Für einen Verband L wird die Länge $l(L)$ erklärt als das Supremum aller $|K|-1$ (K Kette in L) und die Breite $b(L)$ als das Supremum aller n, für die es eine Abbildung ψ des Booleschen Verbandes $\mathbf{2}^n$ in L gibt, bei der für beliebige $a,b \in \mathbf{2}^n$ $\psi a \leq \psi b$ genau dann gilt, wenn $a \leq b$. Ist L atomistisch und modular, so gilt $l(L)=b(L)$. Ist L modular von endlicher Länge und $b(L) \leq 2$, so heiße **L** quasi-planar.

In [4] wurde für einen modularen Verband $M=(M,+,\cdot)$ endlicher Länge das Skelett $S(M)$ als die Menge der kleinsten Elemente der maximalen atomistischen Intervalle von M eingeführt. Es wurde gezeigt, daß $S(M) = \{x \mid x \in M \text{ und } x^{*+}=x\}$, wobei

$$a^* = \begin{cases} \sup\{b \mid b \succ a\} & \text{für } a<1 \\ 1 & \text{für } a=1 \end{cases} \qquad a^+ = \begin{cases} \inf\{b \mid b \prec a\} & \text{für } a>0 \\ 0 & \text{für } a=0 \end{cases}$$

($a \prec b$ bedeute stets, daß a unterer Nachbar von b ist).

$S(M)$ ist $+$Unterhalbverband von M und ein Verband $(S(M),\vee,\wedge)$ mit $x \vee y = x+y$ und $x \wedge y=(x\cdot y)^{*+}$, mit kleinstem Element 0 und größtem Element 1^+. Es gilt offenbar $l(S(M))<l(M)$ und $b(S(M)) \leq b(M)$. Das duale Skelett $S^\delta(M) = \{x^* \mid x \in S(M)\}$ ist zu $S(M)$ isomorph

mit dem Isomorphismus $x \longmapsto x^*$.
Gemäß dem Hauptsatz von [4] erhält man M aus den Intervallen $M_x = [x,x^*]$ zurück durch die Konstruktion der S-verklebten Summe mit $S=S(M)$.

B. JÓNSSON hat in [8] die modularen Verbände von Länge $n \leq 4$ klassifiziert. Mit dem Begriff des Skeletts läßt sich eine Klassifikation auch für größere n gewinnen:

<u>Satz 1:</u> Ist M ein modularer Verband von Länge ≤ 6 so gilt entweder

a) $S(M)$ ist quasiplanar oder

b) $S(M)$ hat einen modularen Unterverband $S_m(M)$ von Länge 3 so daß die Elemente von $S(M)-S_m(M)$ gleichzeitig Atome und Koatome von $S(M)$ sind.

Hat nun M zusätzlich Breite ≤ 3, so sind alle M_x modulare Verbände von Länge ≤ 3. Daher erhält man durch Satz 1 eine Reduktion von Einbettungsproblemen für solche Verbände - zu einer Klasse $\mathfrak{K}$ von Verbänden ein Verfahren anzugeben, das für jeden endlichen partiellen Verband entscheidet, ob er in einen Verband aus $\mathfrak{K}$ eingebettet werden kann - auf Einbettungsprobleme für modulare Verbände von Breite ≤ 2 (die nach [5] lösbar sind) bzw. von Länge ≤ 3 - hier führt die Konstruktion freier projektiver Ebenen zu einer Lösung.

Satz 2: Zu jedem $n \le 6$ und $m \le 3$ ist das Einbettungsproblem für die Klasse ${}_mM_s^n$ aller modularen Verbände von Länge $\le n$ und Breite $\le m$ lösbar.

Die Klasse ${}_mM_s^n$ ist stabil im Sinne von BAKER [2], also ist jeder Verband in der von ${}_mM_s^n$ erzeugten gleichungsdefinierten Klasse ${}_mM^n$ subdirektes Produkt von Verbänden aus ${}_mM_s^n$. Daher ist auch für ${}_mM^n$ das Einbettungsproblem lösbar. Da für gleichungsdefinierte Klassen nach EVANS [3] das Einbettungsproblem zum Wortproblem äquivalent ist, erhalten wir:

Korollar 3: Zu jedem $n \le 6$ und $m \le 3$ ist das Wortproblem für ${}_mM^n$ lösbar.

HUTCHINSON [7] hat jedoch bewiesen, daß das Wortproblem für eine gleichungsdefinierte Klasse modularer Verbände schon dann unlösbar ist, wenn sie den Verband aller Untermoduln eines Moduls R^I (R nichttrivialer Ring, I unendlich) enthält.

Der Vollständigkeit halber konstatieren wir noch, daß nach [6] die Klasse ${}_mM^n$ durch endlich viele Gleichungen definiert werden kann (n,m beliebig) und daß sie für $m \ge 3$, $n \ge 6$ unendlich viele obere Nachbarn im Verband der gleichungsdefinierten Klassen modularer Verbände hat - nämlich die von den Verbänden L_p^n aus Fig. 1 erzeugten.

2. Das Skelett eines modularen Verbandes von Länge ≤ 6

Wir geben zunächst einige Formeln an, die wir später zur Bestimmung von Skeletten benötigen (M sei modular von endlicher Länge):

(1) Sei $x \in S(M)$ und $a \in M$. a ist genau dann oberer Nachbar von x in $S(M)$, wenn a minimal ist bezüglich der Eigenschaft $x < a \leq x^* < a^*$.

(2) Für jedes $x \in S(M)$ gilt $\sup\{y \mid y \succ x \text{ in } S(M)\} \leq x^*$.

(3) Für jede Kette K aus $S(M)$ und jedes $x \in K$ gilt $l(K) + l([x, \sup\{y \mid y \succ x \text{ in } S(M)\}]) \leq l(M)$.

(4) Ist $S(M)$ modular, so gilt
$l(S(M)) + b(S(M)) \leq l(S(M)) + b(M) \leq l(M)$.

(5) Wird M von n Elementen erzeugt und ist $S(M)$ modular, so wird $S(M)$ von $n+l(M)$ Elementen erzeugt.

Beweis zu (1): Sei a minimal bzgl. der Eigenschaft $x < a \leq x^* < a^*$. Für jedes $y \in S(M)$ mit $x < y \leq a$ gilt $x < y \leq a \leq x^* < y^*$, also wegen der Minimalität von a $y = a$; da es ein solches y gibt $(y := a^{**})$, folgt $a \in S(M)$ und $x \prec a$ in $S(M)$. Gilt umgekehrt $x \prec a$ in $S(M)$, so $a \leq x^*$ nach [4; Lemma 6.3] und $x^* \prec a^*$ in $S^\delta(M)$. Für jedes $b \in M$ mit $x < b \leq a \leq x^* < b^*$ gilt $b^* \leq a^*$ und $b^* \in S^\delta(M)$, also $b^*=a^*$ und $b \geq b^{*+}=a^{*+}=a$. Daher ist a in der gewünschten Weise minimal.

(2) folgt unmittelbar aus (1). Sei K die Kette $x_0<x_1<\ldots<x_i=x<\ldots<x_n$. Dann ergibt sich (3) aus (2) und aus $x_0<x_1<\ldots<x_i=x<x^*<x_{i+1}^*<\ldots<x_n^*$. Ist schließlich $S(M)$ modular, so wähle man $x \in S(M)$ so, daß $b(M_x)$ maximal ist, und eine maximale Kette K von $S(M)$ mit $x \in K$. Wie in (3) folgt $l(K)+b(M_x) \leq l(M)$. Nach [4; Folg. 6.7] gilt jedoch $b(M)=b(M_x)$ und wegen der Modularität $l(S(M))=l(K)$, woraus sich (4) sofort ergibt.

(5) Seien $e_1, \ldots, e_n$ die Erzeugenden von M und $e_i \in M_{x_i}$ für $i=1,\ldots,n$. Man wähle nun eine maximale Kette K von $S(M)$ und betrachte den von $K \cup \{x_1,\ldots,x_n\}$ erzeugten Unterverband S' von $S(M)$. Wegen $l(S(M))=l(S')$ gilt $x \prec y$ in S' genau dann, wenn es in $S(M)$ gilt. Daher bilden die Verbände $M_x (x \in S')$ ein monotones S'-verklebtes System von atomistischen Unterverbänden von M und es folgt aus Korollar 5.4 sowie Lemma 7.1 in [4], daß $M' = \{M_x | x \in S'\}$ ein Unterverband von M ist und $S(M')=S'$. Aber $e_1,\ldots,e_n \in M'$ und deshalb $M'=M$.

<u>Beweis von Satz 1.</u> Sei M ein modularer Verband von Länge ≤ 6. $S(M)$ muß genau eine der drei folgenden Eigenschaften haben:

A: $S(M)$ modular und $b(S(M))\leq 2$

B: $S(M)$ modular und $b(S(M))\geq 3$

C: $S(M)$ nicht modular.

Der Beweis des Satzes (und weitere zur Lösung der Einbettungsprobleme wesentliche Information) ergibt sich dann aus den aus A,B,C abgeleiteten Folgerungen:

Aus A folgt:

(A1) $S(M)$ ist quasiplanar von Länge ≤ 5 und Kette, falls $l(S(M))=5$.

(A2) Hat M Breite ≥ 3, so ist $S(M)$ planar von Länge ≤ 3.

(A3) Ist M endlich erzeugt, so auch $S(M)$.

(A4) Es gibt eine berechenbare Funktion f so, daß für alle M mit n Erzeugenden $|S(M)|\leq f(n)$ gilt.

Aus B folgt:

(B1) $S(M)$ ist modularer atomistischer Verband von Länge 3.

(B2) $S(M)$ ist Unterverband von M_o mit größtem Element $1^{+}=0^{*}$.

(B3) $l(M)=6$ und $b(M)=l(M_x)=3$ für alle $x\varepsilon S(M)$.

Die Verbände M, für die alle M_x irreduzible projektive Ebenen sind und $S(M)=M_o$, $S^{\delta}(M)=M_1$ gilt, sind gerade die Verbände von uniformen projektiven Hjelmslev-Ebenen im Sinne von ARTMANN [1].

Aus C folgt:

(C1) $l(S(M))=3$, $S(M)$ hat einen größten modularen Unterverband $S_m(M)$, $l(S_m(M))=3$, alle $x\varepsilon I(M):=S(M)-S_m(M)\neq\emptyset$ sind zugleich Atome und Koatome von $S(M)$.

(C2) $S(M)\subseteq M_o$, $S_m(M)$ ist Unterverband von M_o , $0^*=1^+$ und $x\prec 0^*\prec x^*$ in M für alle $x\varepsilon I(M)$.

(C3) $l(M)=6$, $b(M)=l(M_x)=3$ für alle $x\varepsilon S_m$, $b(M_x)=2$ für alle $x\varepsilon I(M)$.

(C4) $M'=\bigcup\{M_x | x\varepsilon S_m(M)\}$ ist Unterverband von M und $S(M')=S_m(M)$

(C5) Jedes $a\varepsilon M-M'$ ist irreduzibel und von Rang 3; zu a gibt es genau ein $x\varepsilon I$ mit $a\varepsilon M_x$.

(C6) Für $x\varepsilon I(M)$ hat M_x die Form von Fig. 2 mit $a_i\varepsilon M-M'$.

(C7) Wird M von n Elementen erzeugt, so hat I(M) höchstens n Elemente und wird $S_m(M)$ von $2n+6$ Elementen erzeugt.

Ein Beispiel für einen Verband von Typ B bzw. C findet man in Fig. 3 bzw. 4.

Nun zum Beweis von (A1) - (C7):

(A1) und (A2) folgen sofort aus (3), (A3) und (A4) aus (5) und der Tatsache, daß die Elementanzahl eines quasiplanaren Verbandes aus der Länge und der

Erzeugendenanzahl mit einer berechenbaren Funktion abgeschätzt werden kann ([5; Satz 3.1]).

Zu B. Nach (4) gilt $l(S(M))\leq 3$, also $b(S(M)) = l(S(M)) = 3$ und $S(M)$ ist atomistisch. Hieraus folgt wegen (2) $S(M)\subseteq[0,0^*]=M_o$; also gilt für jede maximale Kette $0\prec x\prec y\prec 1^+$ von $S(M)$ $1^+\leq 0^*<x^*<y^*<1^{*+} = 1$ und somit $0\prec x\prec y\prec 1^+ = 0^*\prec x^*\prec y^*\prec 1$ in M.
Man liest ab, daß $l(M)=6$, $l(M_x)=3$ für alle $x\in S(M)$ und $b(M)=3$ - nach [4; 6.7]. Daß $S(M)$ in M_o auch gegen Schnitte abgeschlossen ist, folgt aus "Platzmangel": sind $x,y\in S(M)$ unvergleichbar, so ist $x\wedge y$ unterer Nachbar von x oder y in $S(M)$, also auch in M_o und somit $x\cdot y=x\wedge y\in S(M)$.

Zu C. Da $S(M)$ nicht modular ist, gibt es einen zu N_5 isomorphen Unterverband U von $S(M)$ - es sei $U=\{u,x,y,z,v\}$, $u<x<y<v$ und $y\wedge z=u$, $x\vee z=v$.
U läßt sich so wählen, daß jedes echte Teilintervall von $[u,v]$ (in $S(M)$) modular ist, daß für alle y',z' mit $y<y'<v$ und $z<z'<v$ gilt $y'\wedge z>u$ und $y\wedge z'>u$ und daß $x\prec y$ in $S(M)$. Es soll nun zunächst gezeigt werden, daß y und z untere Nachbarn von v in $S(M)$ sind.

Wenn y nicht unterer Nachbar von v wäre, gäbe es y' mit $y \prec y' < v$. Dann ist $y' \wedge z > u$, also $y \nleq y \wedge z$ und somit $y \vee (y' \wedge z) = y'$. Da $[u,y']$ modular ist, folgt $y' > x \vee (y' \wedge z) > y' \wedge z$, also hat man einen zu N_5 isomorphen Unterverband $\{y' \wedge z, x \vee (y' \wedge z), z, y', v\}$ in $[y' \wedge z, v]$ im Widerspruch zur Annahme über U.

Sei nun angenommen, daß z nicht unterer Nachbar von v ist. Dann gibt es z' mit $z \prec z' < v$. Es gilt $y \wedge z' > u$. Wäre $x \wedge z' > u$, so folgte aus der Modularität von $[x \wedge z', v]$, daß $x \wedge z' < y \wedge z'$, und man hätte einen zu N_5 isomorphen Unterverband $\{u, x \wedge z', y \wedge z', z, z'\}$ von $[u,z']$ im Widerspruch zur Annahme über U. Also gilt $x \wedge z' = u$. Wegen $x \prec y$ und $z \prec z'$ ergibt sich $x \vee (y \wedge z') = y$ und $z \vee (y \wedge z') = z'$, d.h. $\{u, x, y \wedge z', z, y, z', v\}$ bilden einen Unterverband von $S(M)$ wie in Fig. 5.
Seien schließlich $u_1, u_2, u_3 \varepsilon S(M)$ mit $u \prec u_1 \leq x$, $u \prec u_2 \leq y \wedge z'$ und $u \prec u_3 \leq z$ sowie $v' = u_1 \vee u_2 \vee u_3$, und sei $S' = [u,v']_{S(M)}$. Dann ist $M_x (x \varepsilon S')$ ein S'-verklebtes System, also $S' = S(M')$ mit $M' = \bigcup_{x \varepsilon S'} M_x$ nach [4; 7.1]. Wäre $l(S') \leq 2$, so würde $v = u_1 \vee u_2 \leq y$ und $v = u_2 \vee u_3 \leq z'$, also $u_1 \leq v \leq y \wedge z'$ und somit $u_1 = x \wedge y \wedge z'$ folgen. Daher gilt $l(S') \geq 3$ und man kann mit Lemma (3) schließen, daß $u = 0$ und $v' = v = 1^+$, also $S' = S(M)$ ist. Mit (1) folgt

$0<z<z'<v\leq 0^*<z^*<z'^*<v^*$ und $0<x<y<v\leq 0^*<x^*<y^*<v^*$, also wegen $1(M)\leq 6$ insbesondere $u\prec x$, $u\prec z$ und $v=0^*$. Hieraus ergibt sich durch die Modularität von M_o der Widerspruch $x\vee z=x+z<v$.

Wir haben somit in $S(M)$ einen zu N_5 isomorphen Unterverband $\{u,x,y,z,v\}$ mit $u<x<y<v$, $z\prec v$, $y\wedge z=u$ und $x\vee z=v$ erhalten. Man folgert $u^*<x^*<y^*\prec v^*$, $z^*\prec v^*$, $y^*\wedge z^*=u^*$, $x^*\vee z^*=v^*$ in $S^\delta(M)$ und mit der zu (1) dualen Aussage $v^*>y^*>x^*>u^*\geq v^{*+}=v>y>x>u$. Damit muß $1(M)=6$, $u=0\prec x\prec y\prec v=1^+=0^*$ und $1(M_o)=1(M_{1^+})=3$ gelten, also $S(M)\subseteq M_o$ von Länge 3 sein.

Sei nun $I(M)$ definiert als die Menge aller $z\varepsilon S(M)$ zu denen es $x,y\varepsilon S(M)$ gibt mit $0<x<y<1^+$ und $y\wedge z=0$, $x\vee z=1^+$. Dann gilt für alle $z\varepsilon I(M)$

(*) $0\prec z$ in $S(M)$ und $z\prec {}^*\prec z^*$ in M .

Gäbe es nämlich $z'\varepsilon S(M)$ mit $0<z'<z$, so würde $0^*<z'^*<z^*<1$ und $0^*<x^*<y^*<1$, also y^*,z^* 1 und somit $0^*=(y\wedge z)^*=(y\cdot z)^{*+*}=(y\cdot z)^*=y^*\cdot z^*\succ 0^*$ folgen (nach [4; 6.1 und 6.2]). Wegen $0\prec x$, hat man $x\cdot z=0$ und $x+z=x\vee z=0^*$, also auch $z\prec 0^*$ aus der Modularität von M . Aus $y^*\prec 1^*$, $y^*+z^*=1^*$, $y^*\cdot z^*=y^*\wedge z^*=0^*$ folgt ebenso $z^*\succ 0^*$.

Da nach (*) alle $z\varepsilon I$ in $S(M)$ irreduzibel sind, ist $S_m(M):= S(M)-I(M)$ ein Unterverband von $S(M)$. $S_m(M)$ ist nach der Definition von $I(M)$ der größte modulare Unterverband von $S(M)$. Aus $(*)$ und $l(S(M))=3$ folgt ferner $l(S_m(M))=3$ und wie unter 3, daß $S_m(M)$ Unterverband von M_o ist. Damit sind zunächst (C1) und (C2) voll bewiesen.

Daß M' Unterverband von M und $S_m(M)=S(M')$ ist, ergibt sich aus [4; 5.4 und 7.1] wie im Beweis von (5). Für jede maximale Kette $0 \prec x \prec y \prec 1^+$ in $S_m(M)$ gilt $0 \prec x \prec y \prec 1^+ = 0^* \prec x^* \prec y^* \prec 1$ in M und somit $l(M_x)=3$ für jedes $x \in S_m(M)$.

Ist $x \in I(M)$, so gilt $l(M_x) \leq 2$ nach $(*)$, insgesamt also $b(M)=3$. Sei nun $a \in M$ mit $x \prec a \prec x^*$. Ist $a=z+b$ mit $a \succ b \neq z$, so folgt $z > a^+ \in S(M)$, also $a^+=0$, $a \leq a^{+*} \leq 0^*$ und $a=0^*$. Mit dem dualen Schluß ergibt sich, daß a nur dann reduzibel in M ist, wenn $a=0^*$. Ist $a \neq 0^*$, so gehört a zu keinem M_y mit $y \neq x$: $a \notin M_o$ und $a \notin M_1$ sind schon gezeigt und aus $a \in M_y$ für ein y, das zu x unvergleichbar ist, würde $0^*=1^+=x \vee y \leq a$, also $a=0^*$ folgen. Umgekehrt muß aber auch jedes $a \in M-M'$ zu einem M_x mit $x \in I(M)$ gehören, und $a \nparallel x, x^* \in M_o \cup M_1$ sein. Damit sind auch (C3) - (C6) nachgewiesen und es bleibt (C7) zu zeigen.

Dazu sei M von der n-elementigen Menge E erzeugt. Für $z \in I(M)$ sei $E_z=\{a \mid a \in M, z\ a \neq 0^*\}$. Wegen $E_z \subseteq E$, $|E_z| \geq 1$ und $E_z \cap E_{z'} = \emptyset$ für $z \neq z'$ folgt, daß $I(M)$

höchstens n Elemente hat. Ist $a\varepsilon E_z$, so zieht für jedes $b\varepsilon M$ $b\leq a$ schon $b+a=b+z^*$ und $b\geq a$ schon $b\cdot a=b\cdot z$ nach sich. Daher ist $E'=E\cap M'\cup I(M)\cup\{z^*|z\varepsilon I(M)\}$ eine Erzeugendenmenge von M'. Da $|E'|\leq 2n$ gilt, wird $S_m(M)$ nach (5) von $2n+6$ Elementen erzeugt.

Satz 4: Jeder nach A,B und C mögliche Verband tritt als Skelett eines modularen Verbandes von Länge ≤ 6 auf.

Beweis: Für den Fall A entnimmt man dies unschwer aus der Charakterisierung der quasiplanaren Verbände von Länge ≤ 4 in JONSSON [8]. Sei nun S ein beliebiger atomistischer modularer Verband von Länge 3. Wir setzen $M_x=\{x\}\times S$ für $x=0,1\varepsilon S$ und $M_x=\{x\}\times S^\delta$ $x\varepsilon S$ sonst, wobei S^δ den zu S dualen Verband bezeichne. Für jedes Paar x,y aus S mit $x\prec y$ sei ein Isomorphismus ψ_{yx} gegeben so, daß

$\psi_{yo}:[(0,y),(0,1)]_{M_o}\longrightarrow[(y,1),(y,y)]_{M_y}$ mit

$\psi_{oy}(0,z)=(y,z)$ für alle $z\succ y$

$\psi_{yx}:[(x,y),(x,0)]_{M_x}\longrightarrow[(y,1),(y,x)]_{M_y}$ mit

$\psi_{yx}(x,x)=(y,y)$

$\psi_{1x}:[(x,x),(x,0)]_{M_x}\longrightarrow[(1,0),(1,x)]_{M_1}$ mit

$\psi_{1x}(z,x)=(1,z)$ für alle z x

Für $0=x_1\wedge x_2\prec x_1,x_2\prec y=x_1\vee x_2$ und $a\varepsilon M_o$ ist $\psi_{yx_i}\circ\psi_{x_io}(a)$ genau dann definiert, wenn $a=(0,y)$ bzw. $(0,1)$, und hat dann den Wert $(y,1)$ bzw. (y,y) .

Ebenso ist für $x=y_1 \wedge y_2 \prec y_1, y_2 \prec y=y_1 \vee y_2$ und $a \varepsilon M_x$ $\psi_{yy_i} \circ \psi_{y_i x}(a)$ genau dann erklärt, wenn $a=(x,x)$ bzw. $(x,0)$, und zwar mit dem Wert $(1,0)$ bzw. $(1,x)$. Daher liegt eine lokal S-verbundene Summe im Sinne von [4; Satz 4.3] vor und man erhält einen modularen Verband M mit $S(M)=S$ (nach [4; 7.1]).

Ist schließlich S ein unter C als Skelett zugelassener Verband, so kann man stets einen modularen atomistischen Verband T finden so, daß $S \subseteq T$, der größte modulare Unterverband S_m von S Unterverband von T ist und für alle $z \varepsilon S-S_m$ $z \prec 1_T$ gilt und $x \leq z$ für $x \varepsilon S_m$ schon $x=0$ impliziert.

Nach Fall B können wir annehmen, daß $T=S(L)$ ist für einen Verband vom Typ B. Für jedes $x \varepsilon S-S_m$ sei eine Menge $J_x \neq \emptyset$ mit $J_x \cap L'=\emptyset$ und $J_x \cap J_y=\emptyset$ für $x \neq y$ gegeben. Wählt man $M_x=L_x$ für $x \varepsilon S_m$ und M_x als den Verband mit kleinstem Element x, größtem Element x^* und Atommenge $\{0^*\} \cup J_x$ für $x \varepsilon S-S_m$, so erhält man ein monoton S-verklebtes System im Sinne von [4]. Die S-verklebte Summe M ist dann ein Verband vom Typ C mit $S=S(M)$ und $S_m(M)=S_m$.

3. Zum Einbettungsproblem

Unter einem schwachen partiellen Verband verstehen wir eine Menge L mit partiellen Operationen + und $\cdot$, auf der es eine Halbordnung $\leq$ gibt so, daß gilt: Ist $a+b=c$, so $c=\sup(a,b)$; ist $a\cdot b=c$, so $c=\inf(a,b)$. Dann gibt es auch eine kleinste solche Halbordnung (und diese werden wir stets benutzen). Jeder partielle Verband ist in einen Verband (schwach) einbettbar, nämlich in seinen Idealverband. Umgekehrt ist jede partielle algebraische Struktur $(L,+,\cdot)$, die in einen partiellen Verband (schwach) einbettbar ist, selbst ein solcher. Es liegt auf der Hand, daß es ein Verfahren gibt, um für jedes endliche $(L,+,\cdot)$ zu entscheiden, ob es ein partieller Verband ist, aber auch ob es in einen (modularen) Verband von Länge 1 bzw. 2 eingebettet werden kann.

Lemma 5. Es gibt ein Verfahren, um für jeden endlichen partiellen Verband L und gegebene Elemente $a \in L$ und Zahlen r_a zu entscheiden, ob L in einen modularen Verband M der Länge 3 eingebettet werden kann, so daß a in M den Rang r_a hat.

Beweis. Ist L in ein M einbettbar, so besteht L aus zwei disjunkten Teilmengen P und G und gegebenenfalls noch den Elementen 0 und 1 derart, daß $(P,G,<)$ eine Inzidenzstruktur ist (d.h. daß die Elemente aus P bzw. G jeweils untereinander unvergleichbar sind und es zu $a \neq b \in P$ höchstens ein $c \in G$ gibt mit $a,b<c$ und zu $a \neq b \in G$ höchstens ein $c \in P$ gibt mit $c<a,b$) und daß für $a \neq b \in P$ bzw. $c \neq d \in G$ gilt: ist $a+b$ bzw. $c \cdot d$ in L erklärt, so $a+b \in G$ bzw. $c \cdot d \in P$. Man wähle einfach P als die Menge aller Atome von M, die zu L gehören, und G als die Menge aller in L enthaltenen Koatome von M.

Sei umgekehrt zu einem partiellen Verband L eine Inzidenzstruktur der beschriebenen Art gegeben. Dann ist diese Unterstruktur einer projektiven Ebene $(P',G',<')$ - z.B. der von ihr erzeugten freien projektiven Ebene (s.G.PICKERT [9]) - d.h. insbesondere, daß $P \subseteq P'$, $G \subseteq G'$ und für $a \in P$ und $b \in G$ $a<'b$ genau dann gilt, wenn $a<b$. Durch Hinzunahme von größtem und kleinstem Element entsteht aus $(P' \cup G', \leq')$ ein modularer Verband M, in den $(L,\leq)$ eingebettet ist. Sind a und b aus L unvergleichbar und $a+b=c \in L$ definiert, so gilt $c=\sup_M(a,b)$: sind $a,b \in P$, so $c \in G \subseteq G'$, also $c=\sup_M(a,b)$; ist $a \in P$ $b \in G$ und $a \nless b$, so $c=1=\sup_M(a,b)$; sind $a,b \in G$, so ebenfalls $c=1=\sup_M(a,b)$. Mit der dualen Aussage hinsichtlich "$\cdot$" folgt, daß der partielle Verband $(L,+,\cdot)$ in den Verband M eingebettet ist.

Die folgenden Lemmata geben nun Voraussetzungen an, unter denen "lokale" Einbettungen zu "globalen" zusammengesetzt werden können. Dazu eine Definition: Eine Teilmenge X eines partiellen Verbandes $L=(L,+,\cdot)$ heiße $+$<u>Teilbund</u> von L, falls $a+b$ für alle $a,b\in X$ definiert ist und $a+b\in X$.
X mit der eingeschränkten Operation $\vee=+|X$ wird dann ein Supremum-Halbverband und im Falle endlicher Länge auf kanonische Weise ein Verband $(X,\vee,\wedge)$.

<u>Lemma 6</u>. Sei $L=(L,+,\cdot)$ ein abzählbarer partieller Verband, S ein $+$Teilbund von L, $\bar{S}$ ein $\cdot$Teilbund von L, S und $\bar{S}$ modulare Verbände endlicher Länge, S isomorph zu $\bar{S}$ mit einem Isomorphismus $x\longmapsto\bar{x}$ so, daß $x<\bar{x}$ und aus $x\prec y$ in S $y\leq\bar{x}$ folgt.
Sei L die Vereinigung aller Intervalle $[x,\bar{x}]_L$ $(x\in S)$.
Sei für jedes $x\in S$ $[x,\bar{x}]_L$ eingebettet in einen modularen Verband M_x von Länge ≤ 3 so, daß x das kleinste und $\bar{x}$ das größte Element von M_x ist und für alle $x\prec y$ in S gilt $l([y,\bar{x}]_{M_x}) = l([y,\bar{x}]_{M_y})$.
Sei $A\supseteq S$ ein $+$Teilbund und $B\supseteq\bar{S}$ ein $\cdot$Teilbund von L.
Dann ist der partielle Verband $(L,+|A,\cdot|B)$ einbettbar in einen modularen Verband M mit $b(M)\leq 3$, $S(M)\cong S$ und (wenn K eine beliebige maximale Kette von S ist) $l(M)=$

$$\sum_{x\in K} l(M_x) - \sum_{x\prec y \text{ in } K} l([y,\bar{x}]_{M_x}) .$$

Beweis. Wir können annehmen, daß alle die $M_x (x\varepsilon S)$, die Länge 3 haben, abzählbare projektive Ebenen sind. Daher sind für alle $x \prec y$ aus S die Intervalle $[y,\bar{x}]_{M_x}$ und $[y,\bar{x}]_{M_y}$ entweder beide einelementig, oder beide zweielementig oder aber beide abzählbare Verbände von Länge 2. Also gibt es einen Isomorphismus ψ_{yx} von $[y,\bar{x}]_{M_x}$ auf $[y,\bar{x}]_{M_y}$, der die Elemente von L fest läßt. Die Verbände $M_x (x\varepsilon S)$ mit den Abbildungen $\psi_{yx} (x,y\varepsilon S, x \prec y)$ bilden ein lokal-S-verbundenes System im Sinne von [4; Abschnitt 4]:

Sei $x\wedge y \prec x, y \prec x\vee y$ in S, $a\varepsilon M_{x\wedge y}, \psi_{x(x\wedge y)}\, a$ und $\psi_{y(x\wedge y)} a$ definiert. Dann gilt $x,y \leq a \leq \overline{x\wedge y}$, also $x\wedge y < x < x\vee y \leq a \leq \overline{x\wedge y}$, also $a = x\vee y$ oder $a = \overline{x\wedge y}$; in jedem Falle ist aber $a\varepsilon L$, also $\psi_{x(x\wedge y)}\; a = \psi_{y(x\wedge y)}\; a = a$; wegen $x\vee y \leq a \leq \bar{x}, \bar{y}$ sind auch $\psi_{(x\vee y)x}\; a = \psi_{(x\vee y)y}\; a = a$ definiert.

Sei nun M die lokal-S-verbundene Summe des Systems mit den kanonischen Einbettungen $\pi_x : M_x \longrightarrow M$. Da alle M_x modular von Breite ≤ 3 sind, ist auch M modular von Breite ≤ 3 [4; 3.2 und 3.3] und es liegt die Aussage über die Länge auf der Hand. $S(M) \cong S$ folgt nach [4; 7.1], da alle M_x atomistisch sind und die Verklebung monoton ist.

Da für alle $a\varepsilon L$ $\pi_x\; a = \pi_y\; \psi_{yx}\; a$ gilt, ist $\psi = \bigcup \{\pi_x \mid x\varepsilon S\} | L$ eine injektive Abbildung von L in M. Zum Nachweis, daß ψ mit den partiellen

Operationen $+|A$ und $\cdot|B$ verträglich ist, zeigen wir:

a) $\psi(x) \leq \psi(y)$ für alle $x \leq y$ in S

b) $\psi(x \vee y) = \psi x + \psi y$ für alle $x, y \varepsilon S$

c) $\psi(a+y) = \psi a + \psi y$ für $a \varepsilon A$, x, $y \varepsilon S$ und $x \leq a \leq \bar{x}, x \leq y$

d) $\psi(a+b) = \psi a + b$ für alle $a, b \varepsilon A$

Dabei benutzen wir, daß $\psi|[x,\bar{x}]$ nach Konstruktion ein Homomorphismus ist. Der Beweis ist im wesentlichen derselbe wie bei Satz 5.1-5.2 in [4], es müssen jedoch alle Voraussetzungen des Lemmas ausgenützt werden.

Zu a) Ist $x \prec y$ in S, so gilt $y \leq \bar{x}$, also $\psi x \leq \psi y$. Die Aussage für $x < y$ folgt nun durch Induktion.

Zu b) Gilt $x \wedge y \prec x, y \prec x \vee y$ in S, so folgt $x, y \leq \overline{x \wedge y}$, also $x \vee y \leq \overline{x \wedge y}$ und somit b) in diesem Fall. Die Behauptung ergibt sich nun mit Hilfe der Modularität durch Induktion über die Länge von $[x \wedge y, x \vee y]$ in S .

Zu c) Für $x = y$ folgt wegen $x \leq a \leq \bar{x}$ sofort $\psi(a+y) = \psi a + \psi y$. Für $x < y$ schließt man induktiv: es gibt $z \varepsilon S$ mit $x \leq z \prec y$ und es gilt $\psi(a+y) = \psi(a+z+y) = \psi(a+z) + \psi y = \psi a + \psi z + \psi y = \psi a + \psi y$, da $a+z$ definiert ist, $a \leq \bar{x} \leq \bar{z}$ und somit $a+z, y \varepsilon [z, \bar{z}]$.

Zu d) Ist $a \varepsilon [x,\bar{x}]$, $b \varepsilon [y,\bar{y}]$, so gilt $x \vee y = x+y \leq a+b \leq \bar{x}+\bar{y} \leq \overline{x \vee y}$ und somit

$\psi(a+b) = \psi(a+b+x+y) = \psi(a+x+y) + \psi(b+x+y)$

$= \psi(a) + \psi(x+y) + \psi(b) = \psi(a) + \psi(x) + \psi(y) + \psi(b) = \psi(a) + \psi(b)$.

Lemma 7. Sei $L=(L,+,\cdot)$ ein endlicher partieller Verband, seien $S,\bar{S}\subseteq L$ mit $l(S)=l(\bar{S})=3$, $x\longmapsto\bar{x}$ ein Isomorphismus von $(S,+|S,\cdot|S)$ auf $(\bar{S},+|\bar{S},\cdot|\bar{S})$ mit $x<\bar{x}$. Sei u kleinstes und v größtes Element von S und $v=\bar{u}$. Sei $L=\bigcup\{[x,\bar{x}]_L|x\varepsilon S\}$. Für jedes $x\varepsilon S$ sei $[x,\bar{x}]_L$ eingebettet in einen modularen Verband von Länge 3 so, daß x das kleinste und $\bar{x}$ das größte Element von M_x ist. Maximale Ketten in $S(\bar{S})$ seien maximal in $M_u(M_v)$ und es gelte für alle $x\prec y$ aus S, daß $l([y,\bar{x}]_{M_x}) = l([y,\bar{x}]_{M_y}) = 2$ ist. Sei A ein +Teilbund von L so, daß gilt: zu $a\varepsilon A$ gibt es $x\varepsilon S\cap A$ mit $x\leq a\leq\bar{x}$; sind $x<y$ in $A\cap S$, so gibt es eine maximale Kette $x\prec x_1\prec\ldots\prec x_k=y$ in S mit $x_i\varepsilon A$ für $i=1,\ldots k$. Sei B ein $\cdot$Teilbund mit den entsprechenden Eigenschaften hinsichtlich $\bar{S}$.
$S(\bar{S})$ seien abgeschlossene Teilmengen des mit der Relativstruktur in $M_u(M_v)$ versehenen partiellen Verbandes $(A\cup B)\cap M_u$ $((A\cup B)\cap M_v)$. Dann ist der partielle Verband $(A\cup B,+|A,\cdot|B)$ einbettbar in einen modularen Verband der Länge 6 und Breite 3 mit atomistischem Skelett von Länge 3.

Beweis. M_u kann man insbesondere so wählen, daß das Erzeugnis von $[u,\bar{u}]_{A\cup B}$ die freie projektive Ebene über der Inzidenzstruktur von $[u,\bar{u}]_{A\cup B}$ ist. Dann ist auch das Erzeugnis $\langle S\rangle$ von S in M_u freie projektive Ebene über der Inzidenzstruktur von S.

Wählt man M_v analog, so ist das Erzeugnis $<\bar{S}>$ von $\bar{S}$ in M_v isomorph zu $<S>$ mit einem Isomorphismus der "$-$" fortsetzt.
Sei L' der partielle Verband $A \cup B \cup M_u \cup M_v$.
Dann gilt für alle $x \succ u$ aus $<S>$, daß $[x,\bar{x}]_{L'} = [x,\bar{x}]_{A\cup B} \cup [x,v]_{M_u}$ ist - auch als partielle Verbände. Somit ist $[x,\bar{x}]_{L'}$ in einen modularen Verband der Länge 3 einbettbar so, daß x das kleinste und $\bar{x}$ das größte Element von M_x ist. Entsprechendes gilt für $y \prec v$ und es können alle $M_x (x \in <S>)$ als abzählbare projektive Ebenen gewählt werden. Nach Lemma 6 erhält man eine lokal-$<S>$-verklebte Summe M und eine injektive Abbildung ψ von L' in M, die auf $(A\cup B, +|A, \cdot|B)$ ein Homomorphismus ist. Es gilt $l(M)=6$, $b(M)=3$ und $S(M) \cong <S>$.

Lemma 8. Sei L ein endlicher partieller Verband, seien $S, \bar{S} \subseteq L$ und $x \longmapsto \bar{x}$ ein Isomorphismus von S auf $\bar{S}$ mit $x<\bar{x}$. S habe ein kleinstes Element u und ein größtes Element v und es sei $\bar{u}=v$. Sei ferner $I \subseteq S$ und $\emptyset \neq J \subseteq L-(S\cup\bar{S})$ so, daß gilt:
$L-J = \bigcup\{[x,\bar{x}]_L \mid x \in S-I\}$; der partielle Verband $L-J$ ist in einen modularen Verband M' von Länge 6 und Breite 3 mit modularem Skelett $S(M') \supseteq S-I$ einbettbar, wobei $z \prec v \prec \bar{z}$ in M' für alle $z \in I$; jedes $a \in J$ ist irreduzibel in L und es gibt genau ein

$z\varepsilon I$; mit $z \prec a \prec \bar{z}$ in L . Dann kann L in einen modularen Verband M von Länge 6 und Breite 3 mit nichtmodularem Skelett eingebettet werden.

<u>Beweis</u>. M' kann insbesondere wie im Beweis von Lemma 7 konstruiert werden. Sei S_m der von $S-I$ in M_o' erzeugte Unterverband. Nach Voraussetzung gilt $L-J \subseteq \bigcup\{[x,x^*] \mid x\varepsilon S_m\}$ und, da die projektive Ebene M_o' frei erzeugt ist, für alle $x\varepsilon S_m$ und $z\varepsilon I$, daß aus $x \leq z$ schon $x=0$ folgt. Bestimmt man nun M wie im Beweis von Satz 4 mit $J_z=\{a \mid a\varepsilon J$ und $a \succ z\}$ für $z\varepsilon I$, so ist der partielle Verband L in M eingebettet.

<u>Beweis von Satz 2</u>. Vermöge Lemma 5 ist für jeden endlichen partiellen Verband entscheidbar, ob er die Voraussetzungen von Lemma 6 bzw. 7 erfüllt. Daher erhalten wir eine Lösung des Einbettungsproblems für Verbände vom Typ B oder A von Breite ≤ 3, indem wir eine berechenbare Funktion g angeben, derart, daß jeder partielle Verband P, der in einen solchen Verband M einbettbar ist, so in einen partiellen Verband L mit $|L| \leq g(|P|)$ und ausgezeichneten Teilmengen $S,\bar{S},A,B$ und ggf. K eingebettet werden kann, daß die Voraussetzungen von Lemma 6 bzw. 7 erfüllt sind und, wenn $a+b$ in P definiert ist a und b zu A, wenn $a \cdot b$ in P definiert ist, a und b zu B gehören.

Diese Funktion ist:

$$g(k)=\max\{2^{(k+2f(k))},\ 3\cdot 2^{(2k+2(k+2)^2)+1}\},$$

wobei f die nach (A4) existierende berechenbare Funktion ist, für die $|S(M)|\leq f(|P|)$ gilt, falls M vom Typ A ist und von P erzeugt wird - was wir selbstverständlich hier voraussetzen können.

Ist nun M vom Typ A, so setzen wir $P'=P\cup S(M)\cup S^{\delta}(M)$ und es sei A der von P' in M erzeugte +Teilbund, B der von P' in M erzeugte ·Teilbund, $L=A\cup B$ mit der Relativstruktur in M, $S=S(M)$, $\bar{S}=S^{\delta}(M)$ und K eine beliebige maximale Kette in $S(M)$.

Ist M jedoch vom Typ B, so wähle man $S_o\subseteq S(M)$ so, daß $P=\bigcup\{[x,x^*]_P\mid x\in S_o\}$ und $|S_o|\leq|P|$. Aus S_o erhält man $S_1\subseteq S(M)$ so, daß aus $x\prec y$ in S_o $x\prec y$ in M_o folgt, durch Hinzunahme von höchstens $2\cdot(|S_o|+2)^2$ Elementen von $S(M)$. Sei A der von $P\cup S_1$ erzeugte +Teilbund von M, B der von $P\cup(S_1)^*$ erzeugte ·Teilbund von M. Sei $L'=A\cup B$ und S_2 so gewählt, daß $S_1\subseteq S_2\subseteq S(M)$, $|S_2|\leq|L'|$ und $L'\subseteq\bigcup\{[x,x^*]_{L'}\mid x\in S_2\}$. Sei schließlich $L=L'\cup S_2\cup S_2{}^*$, $S=\langle S_2\rangle_{L'\cap M_o}\cup(\langle S_2^*\rangle_{L^+\cap M_{o^*}})^+$ und $\bar{S}=S^*$.

Insbesondere ist also für einen endlichen partiellen Verband L entscheidbar, ob er die Voraussetzungen von Lemma 8 erfüllt und wir können die Lösung

des Einbettungsproblems für Verbände vom Typ C hierauf zurückführen, da jeder in einen solchen Verband einbettbare partielle Verband P in ein L mit höchstens $3|P|+4$ Elementen eingebettet werden kann.
Es sei bemerkt, daß alle hier verwendeten Algorithmen in den Bereich der primitiv rekursiven Funktionen gehören.

4. Abschließende Bemerkungen.

Bei dem Versuch, die hier entwickelten Methoden für die Lösung von Einbettungsproblemen für weitere Klassen modularer Verbände endlicher Länge nutzbar zu machen, stellen sich folgende Probleme:

Problem 1. Ist das Einbettungsproblem für projektive Geometrien von fester Dimension n lösbar?

Problem 2. Ist das Skelett eines endlich erzeugten modularen Verbandes stets endlich erzeugt?

Problem 3. Ist das Einbettungsproblem für die Klasse aller Skelette von modularen Verbänden fester Länge n lösbar?

Unschwer lassen sich jedoch die folgenden kleinen Ergebnisse herleiten:

(I) Das Einbettungsproblem für pappus'sche projektive Geometrien fester Dimension n ist lösbar: Ist P in den Untervektorraumverband von K^{n+1} - K kommutativer Körper - eingebettet und P endlich, so kann K immer schon als endliche Erweiterung seines Primkörpers gewählt werden. Daher kann man die einbettbaren P aufzählen, andererseits aber, da die Klasse der pappus'schen projektiven Geometrien von Dimension n endlich axiomatisierbar ist, ist auch die Menge der nicht einbettbaren P aufzählbar.

(II) Im Beweis von Satz 2 ist insbesondere enthalten, daß das Einbettungsproblem für modulare Verbände von Breite ≤ 3 mit endlichem Skelett lösbar ist.

(III) Kombiniert man die Methoden von (I) und (II) so erhält man eine Lösung des Einbettungsproblems für pappus'sche Verbände mit endlichem Skelett; dabei ist ein pappus'scher Verband M ein modularer Verband endlicher Länge so, daß M_x pappus'sche (möglicherweise reduzible) projektive Geometrie ist für jedes $x \varepsilon S(M)$.

(IV) Wie im Falle von Verbänden vom Typ B erhält man eine Lösung des Einbettungsproblems für "projektive Hjelmslevebenen n-ter Stufe", d,h, für folgende induktiv definierte Klassen $\mathfrak{P}_n$ von Verbänden: $\mathfrak{P}_o$ bestehe nur aus dem einelementigen Verband; $M \varepsilon \mathfrak{P}_{n+1}$ genau dann, wenn $S(M) \varepsilon \mathfrak{P}_n$, M_x projektive Ebene ist für jedes $x \varepsilon S(M)$ und $S(M)$ und $S^{\delta}(M)$ Intervalle in M sind.

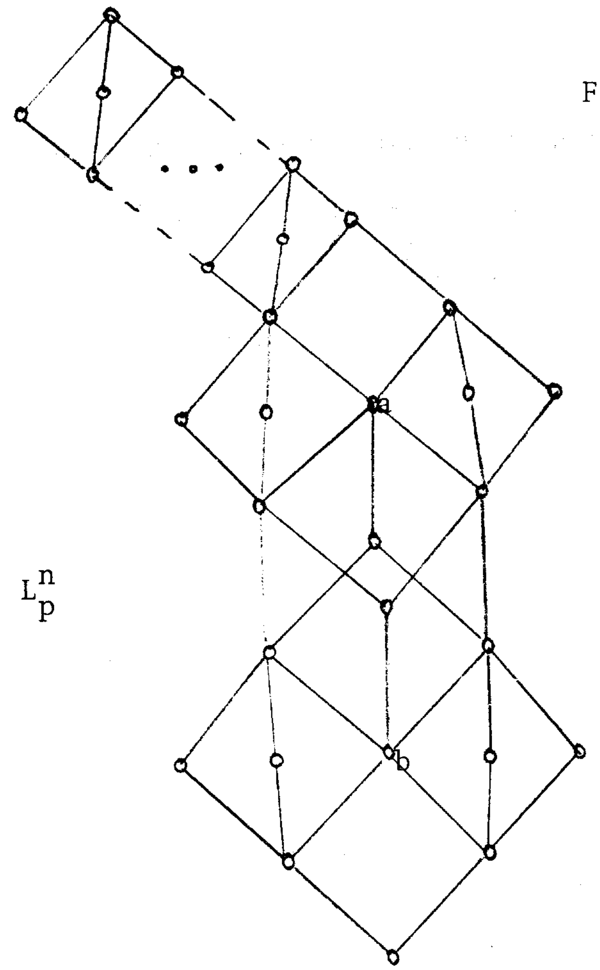
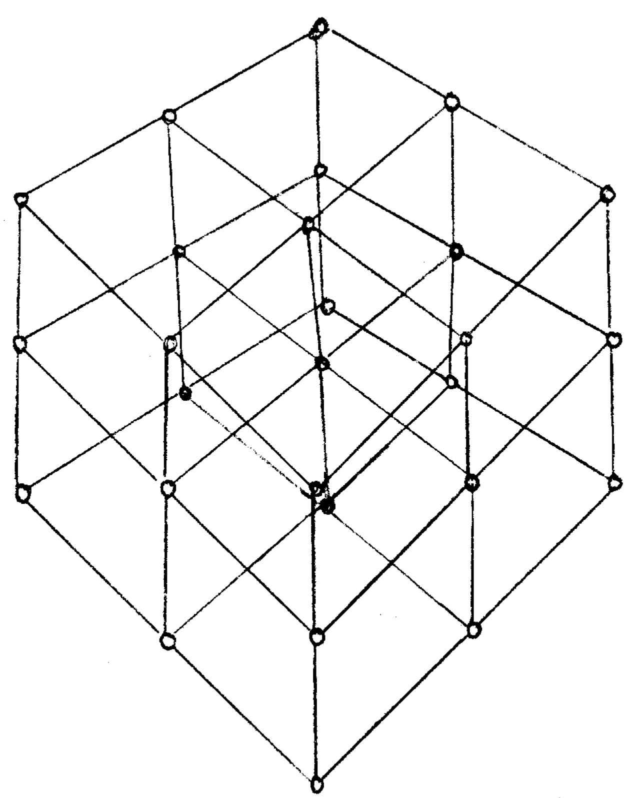


Fig.1

Länge n+1
[a,b] projektive
Ebene über GF(p)

Fig. 3



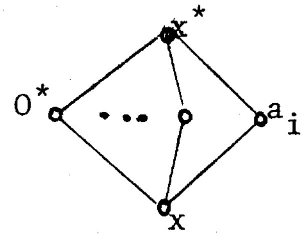


Fig.2

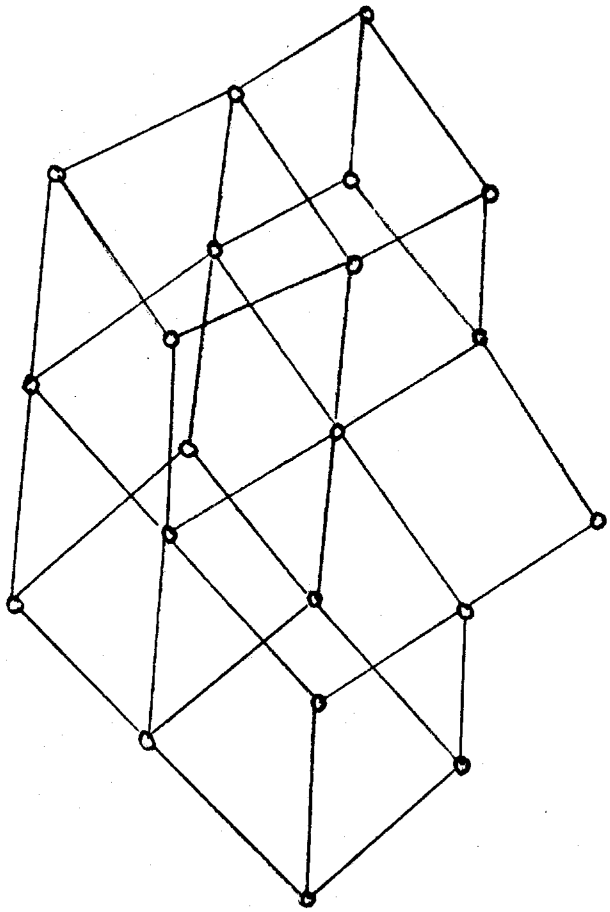


Fig.4

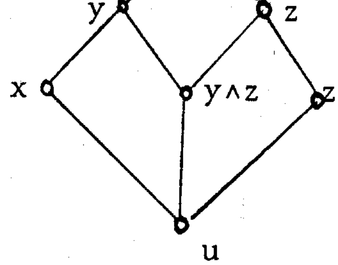


Fig.5

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The Filter Space of a Lattice: Its Rôle in General Topology

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Introduction

A *filter* in a lattice L is a non-void subset F of L for which $x \wedge y \in L$ whenever $x, y \in L$, and $y \in L$ whenever $y \geqslant x$ for some $x \in L$. On the set of all filters F in L one has a naturally arising topology whose basis consists of the sets $\{F \mid a \in F\}$ where $a \in L$; the resulting topological space is the *filter space* ΦL of the lattice L.

If X is a topological space, with topology $\mathfrak{D}X$, we let $\Phi X = \Phi\mathfrak{D}X$, the filter space of $\mathfrak{D}X$ viewed as a lattice (with set inclusion as its partial order). Each $x \in X$ determines the filter $\mathfrak{D}(x) = \{U \mid x \in U \in \mathfrak{D}X\}$ of its *open* neighbourhoods, and thus one has the map $X \to \Phi X$ given by $x \rightsquigarrow \mathfrak{D}(x)$. This map is *continuous* for any X, an *embedding* for exactly the T_0-spaces X, and in general its image is the *reflection* of X in the subcategory, of the category of all topological spaces and continuous maps, given by the T_0-spaces. In the following all spaces are taken to be T_0.

The fundamental significance of the embedding $X \to \Phi X$ lies in the fact that a large class of extensions E of a given space X can be realized within ΦX, i.e. are such that the embedding $X \to \Phi X$ can be lifted to an embedding $E \to \Phi X$. The E in question are exactly the *strict* extensions $E \supseteq X$, i.e. those in which the open sets

$$U^* = \bigcup W \; (X \cap W = U, \; W \in \mathfrak{O}E)$$

form a basis for $\mathfrak{O}E$. This notion goes back to Stone [9]; a detailed account of the rôle of ΦX in this context is given in Banaschewski [1]. The main point about strict extensions of spaces is that many interesting types of extensions (e.g. compactifications, and various of their analogues) are of that kind and hence can be described as, or have actually been explicitly introduced as, suitable subspaces of ΦX.

The use of ΦX in the study of extensions of X has a long history (not to be recalled here); of more recent origin is the result that certain <u>onto</u> maps $E \to X$ can also be realized within ΦX, in such a way that E is embedded into ΦX and the given $E \to X$ corresponds to the operation of taking <u>limits</u> of filter bases in X (Iliadis [7], Banaschewski [2]). This is of importance in the context of projective covers, first introduced for compact and locally compact Hausdorff spaces in Gleason [6].

The purpose of this note is to give an account of the <u>most recent</u> use of the filter spaces ΦX. The notions we are concerned with in this case are the following:

(i) <u>Essential extensions</u>: An extension $E \supseteq X$ of a space X is called <u>essential</u> iff any continuous map $f: E \to Y$ for which $f \mid X$ is an embedding is itself an embedding.

(ii) <u>Injectivity</u>: A space X is called <u>injective</u> (in the category $\mathfrak{T}_0$ of all T_0-spaces and their continuous maps, with respect to embeddings) iff any continuous map f: Y → X lifts to any extension Z ⊇ Y.

(iii) <u>Injective hulls</u>: An essential injective extension of a space X is called an <u>injective hull</u> of X.

These concepts have been investigated, and indeed play an important rôle, in various areas of mathematics. As far as T_0-spaces are concerned, a systematic discussion of injectivity was first given in Scott [8] where the relationship between a particular class of lattices is analyzed in preparation for certain constructions of model theoretic import. Here, we are specifically concerned with the question of the existence of injective hulls and the properties of essential extensions, for which the filter spaces ΦX turn out to provide a natural setting.

The proofs of the results discussed below are given in Banaschewski [3].

<u>1. The Adjointness between Lattices and Spaces</u>.

The correspondence L ⇝ ΦL from lattices to spaces is readily seen to be the object part of a cofunctor (= contravariant functor) from the category $\mathcal{L}$ of all lattices and lattice homomorphisms to the category $\mathcal{T}_0$: For a lattice homomorphism h: L → M, the map Φh: ΦM → ΦL which takes each filter F ⊆ M to the filter $h^{-1}(F)$ is continuous, and the correspondance h ⇝ Φh is functorial. Similarly, one has the "lattice of open sets" cofunctor $\mathfrak{O}$: $\mathcal{T}_0 \to \mathcal{L}$ where $\mathfrak{O}$X, as before, is the topology of X and $\mathcal{O}f: \mathcal{O}Y \to \mathcal{O}X$ is again given, for any continuous $f: X \to Y$, by taking inverse images. Φ and $\mathcal{O}$ are *adjoint on the right*, and the embedding $X \to \Phi X$ introduced above is actually one of the adjunctions. Incidentally, this pair of cofunctors, or some variants of it, provide the starting point for certain studies of duality in Hofmann-Keimel [5]. For the present purpose, the following properties of $\mathcal{O}$ and Φ are worth noting:

Lemma 1. *A continuous map f is an embedding iff $\mathcal{O}f$ is onto, and a lattice homomorphism h is onto iff Φh is an embedding*.

By basic categorical principles, an immediate consequence of this is:

Corollary 1. *If a lattice L is projective then its filter space ΦL is injective*.

Now, for lattices one has the following facts: The two-element chain 2 is projective, and every lattice is a homorphic image of a coproduct of two-element chains. It follows from this that the functor Φ produces the corresponding "dual facts". Moreover, the filter space $\Phi 2$ is actually a familiar object, namely the *Sierpinski space* S, i.e. the two-point space with three open sets:

points: 0,1 ; open sets: $\emptyset$, $\{1\}$, $\{0,1\}$.

Thus one has:

Corollary 2. *S is injective, and every space X can be embedded into a power of S.*

This is well-known (Cech [4], p.485), and can easily enough be proved directly. In the present context it seemed of interest to see how this can be viewed as the counterpart of the rôle of the two-chain among lattices, via the adjointness between $\mathcal{L}$ and $\mathfrak{T}_0$.

As far as the spaces ΦL are concerned, one can actually show much more than the above Corollary 1, but this requires reasoning about specifics rather than general principles. It turns out that the map from the *power set* of a lattice L to ΦL given by generation of filters is continuous if the former is viewed as a power of S; since products and retracts of injectives are injective this proves

Lemma 2. *The filter space of any lattice is injective*.

2. Essential Extensions

Topologies are, of course, *complete* lattices, and for any continuous map f: X $\rightarrow$ Y the associated lattice homomorphism $\mathfrak{O}f: \mathfrak{O}Y \rightarrow \mathfrak{O}X$ does indeed respect some completeness properties - it preserves arbitrary joins. Thus, the lattice-of-open-sets functor can also be considered as going from $\mathfrak{T}_0$ into the category $\mathcal{JCL}$ of complete lattices and their *join-complete* homomorphisms. This viewpoint provides a duality for essential embeddings:

Lemma 3. *A continuous map f: $X \to Y$ in $\mathcal{T}_0$ is an essential embedding iff f: $Y \to X$ is a coessential onto homomorphism in $\mathcal{JCL}$.*

Here, coessential onto for a homomorphism h: $L \to M$ is to mean that $h(K) = M$ iff $K = L$, for any sublattice $K \subseteq L$ in the sense of $\mathcal{JCL}$, i.e. closed with respect to arbitrary joins in L.

A subspace of the filter space ΦL of a lattice L will be called *separating* iff its members distinguish the elements of L, i.e. for any two distinct elements of L there is a filter in the subspace containing one of them but not the other.

Lemma 4. *For separating subspaces Σ and $P \supseteq \Sigma$ of a space ΦL, P is an essential extension of Σ iff each $F \in P$ is the join of all $G \subseteq F$ in Σ*.

Putting these lemmas together, one then obtains, with a few additional arguments:

Proposition 1. *For any extension $E \supseteq X$ of a space X*, *the following conditions are equivalent*:

(1) *E is essential*.

(2) *E is strict, and every trace filter of E on X is a join of filters $\mathcal{D}(x)$*.

(3) *E is superstrict*.

Here, the *trace filters* of E on X are the filters $\{U \cap X \mid U \in \mathcal{D}(y)\}$ for the points $y \in E - X$, and *superstrict* means that any ring of sets $\mathcal{B} \subseteq \mathfrak{O}E$ which yields a basis for $\mathfrak{O}X$ by restriction to X is itself a basis for $\mathfrak{O}E$.

As a fairly direct consequence one obtains:

Proposition 2. *Every space X has a largest essential extension which is unique up to a unique homeomorphism over X, namely the strict extension λX given by the subspace of ΦX consisting of all joins of filters $\mathfrak{O}(x)$*.

3. Injective Hulls.

It is clear that the extension λX of a space X is the only possible candidate for being an injective hull of X, and thus X has an injective hull iff λX is injective. More generally, we first consider subspaces Σ of filter spaces ΦL, for an arbitrary lattice L, which are separating and closed with respect to taking joins of filters. Any such determines a kernel operator $k: \Phi L \to \Sigma$ for which kF is the largest $G \in \Sigma$ contained in F. For such Σ and k one then has:

Lemma 5. *The following conditions are equivalent*:

(1) *Σ is injective*.

(2) *The kernel operator k is continuous*.

(3) *The kernel operator k preserves updirected joins*.

(4) *For each $F \in \Sigma$, $F = \bigvee k(F_a)$ $(a \in F)$ where*

$F_a = \{x \mid x \geqslant a\}$.

A topological criterion for the injectivity of λX which can be derived from this reads as follows:

Proposition 3. *A space X has an injective hull iff, for any $U \in \mathfrak{O}(x)$ $(x \in X)$ there exists a $V \in \mathfrak{O}(x)$ such that $U \cap \Gamma_0 V \neq \emptyset$, where $\Gamma_0 V = \bigcap \Gamma\{z\} (z \in V)$.*

This immediately leads to an "internal" characterization of injectivity itself, and can be used to obtain various further results. For instance: A T_1-space has an injective hull iff it is discrete, and any open subspace of a space which has an injective hull also has an injective hull.

4. Continuous Lattices.

We conclude with some of the results in Scott [8] for which the present setting provides new proofs.

With any partially ordered set S one can associate the space TS whose points are the elements of S and whose topology, the *d-open end topology*, consists of the *ends* $U \subseteq S$ (i.e. $x \geq y$ and $y \in U$ implies $x \in U$) for which $\bigvee A \in U$ implies $A \cap U \neq \emptyset$ for any (up)directed subset $A \subseteq S$.

On the other hand, any space X determines a partially ordered set PX whose elements are the points of X and whose partial order is such that $x \leq y$ iff $\mathfrak{O}(x) \subseteq \mathfrak{O}(y)$.

Finally, a partially ordered set S is called a *continuous lattice* iff S is complete and for any $x \in X$, $x = \bigvee\{\bigwedge U \mid U \in \mathfrak{O}(x)\}$ where $\mathfrak{O} = \mathfrak{O}TS$.

Proposition 4. *(Scott) For any continuous lattice S, TS is an injective space and S = PTS; similarly, for any* injective space X, PX is a continuous lattice and X = TPX.

It should be added to this that the correspondances $S \rightsquigarrow TS$ and $X \rightsquigarrow PX$ between continuous lattices and injective spaces can be extended to a category isomorphism, where the maps between the spaces are the continuous maps and the maps between the lattices are those which preserve updirected joins.

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Disjointness conditions in free products of distributive lattices: An application of Ramsay's theorem.

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1. Introduction. Let L be a lattice. We say that L satisfies the finite disjointness condition if, given any $a \in L$ and any subset $S \subseteq L$ such that $a \notin S$ and such that $x \wedge y = a$ for any distinct $x, y \in S$, it then follows that S is finite. Similarly we say that L satisfies the countable disjointness condition if the above hypotheses imply that S is countable (rather than actually finite). It has long been known that any free Boolean algebra satisfies the countable disjointness condition -- see e.g. R. Sikorski [6], §20, Example L), on page 72, where the countable disjointness condition is called the σ-chain condition. R. Balbes [1] proved that any free distributive lattice satisfies the finite disjointness condition.

In this paper we extend these results to free products in the category $\mathcal{D}$ of distributive lattices and in the category $\mathcal{D}_b$ whose objects are bounded distributive lattices and whose morphisms preserve the bounds. Clearly any free distributive lattice is the free product in $\mathcal{D}$ of a family of one-element lattices, and it is well-known (see [3]) that the

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free Boolean algebra, regarded as a bounded lattice, is the free product in $\mathcal{D}_b$ of a family of four-element lattices. We then generalize the above disjointness conditions by proving the following theorem.

Let $(L_i \mid i \in I)$ *be a family of lattices in* $\mathcal{D}$ (*resp. in* $\mathcal{D}_b$) *and, for each* $i \in I$, *let* L_i *satisfy the finite disjointness condition*. *Then the free product of the family* $(L_i \mid i \in I)$ *in* $\mathcal{D}$ (*resp. in* $\mathcal{D}_b$) *satisfies the finite disjointness condition (resp. the countable disjointness condition)*.

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2. *The word problem.* To accomplish our aim we shall need a characterization of comparability of elements in the free product in $\mathcal{D}$ and in $\mathcal{D}_b$. Let $(L_i \mid i \in I)$ be a family of lattices in $\mathcal{D}$ or $\mathcal{D}_b$ and let L be the free product of $(L_i \mid i \in I)$ in the appropriate category. We take the point of view that each L_i is a sublattice of L ; it follows that in $\mathcal{D}$ $L_i \cap L_j = \emptyset$ whenever $i \neq j$, and that in $\mathcal{D}_b$ $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$. As usual, 0 denotes the lower bound in $\mathcal{D}_b$ and 1 denotes the upper bound. We denote by P the subset $\bigcup(L_i \mid i \in I)$ of L . Note that, in $\mathcal{D}$, if $x, y \in P$ and $x \leq y$ then there is a unique $i \in I$ such that $x, y \in L_i$ and, clearly, $x \leq y$ in that L_i . Similarly, in $\mathcal{D}_b$, if $x, y \in P$ and $x \leq y$ then either $x = 0$ or $y = 1$ or there is a unique $i \in I$ such that $x, y \in L_i$ (and $x \leq y$ in that L_i) .

Since L is distributive, each $a \in L$ can be expressed in the form $\bigvee(\bigwedge X \mid X \in J)$ where J is finite and nonempty, and each $X \in J$ is a finite nonempty subset of P (2). We can always choose each such X to be reduced, that is, to satisfy $|X \cap L_i| \leq 1$ for all $i \in I$, where $|A|$ denotes the cardinality of the set A. In addition, the term "reduced" will be used only for nonempty sets. Note that in $\mathcal{D}_b$ if X is reduced and $0 \in X$ then $X = \{0\}$, and similarly for 1.

Any element of L can also be expressed in the dual form $\bigwedge(\bigvee X \mid X \in J)$, J finite and each X reduced.

LEMMA 1. *Let X, Y be reduced subsets of P. In either category $\mathcal{D}$ or $\mathcal{D}_b$, $\bigwedge X \leq \bigvee Y$ if and only if there are elements $x \in X$ and $y \in Y$ such that $x \leq y$.*

Proof. Assume that for each $\langle x, y\rangle \in X \times Y$, $x \nleq y$. Observe first that $0 \notin X$, $1 \notin Y$ if we are in $\mathcal{D}_b$. In the remainder of the proof it is irrelevant whether we are in $\mathcal{D}$ or in $\mathcal{D}_b$. Let

$$I_1 = \{i \in I \mid |X \cap L_i| = 1,\ |Y \cap L_i| = 0\}$$

$$I_2 = \{i \in I \mid |X \cap L_i| = 0,\ |Y \cap L_i| = 1\}$$

$$I_3 = \{i \in I \mid |X \cap L_i| = |Y \cap L_i| = 1\}$$

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(2) This notation is preferable for our purpose to the equivalent double index notation
$a = (x_1^1 \wedge \cdots \wedge x_1^{n_1}) \vee (x_2^1 \wedge \cdots \wedge x_2^{n_2}) \vee \cdots \vee (x_k^1 \wedge \cdots \wedge x_k^{n_k})$, $x_i^j \in P$.

Let $\underset{\sim}{2}$ be the two-element lattice $\{0, 1\}$ with $0 < 1$. For each $i \in I$ we define a homomorphism $\varphi_i : L_i \to \underset{\sim}{2}$ using the Prime Ideal Theorem:

If $i \in I - (I_1 \cup I_2 \cup I_3)$ φ_i is arbitrary.

If $i \in I_1$, let $x\varphi_i = 1$ where $X \cap L_i = \{x\}$. (This is clearly possible in $\mathfrak{D}$ by taking the constant $L_i \to \underset{\sim}{2}$. In $\mathfrak{D}_b$ we note that $x \neq 0$ and so by the Prime Ideal Theorem we can take $0\varphi_i = 0$, $x\varphi_i = 1$, and, perforce, $1\varphi_i = 1$.)

Similarly, if $i \in I_2$, let $y\varphi_i = 0$ where $Y \cap L_i = \{y\}$.

If $i \in I_3$, let $X \cap L_i = \{x\}$, $Y \cap L_i = \{y\}$. Since $x \nleq y$, we can define φ_i so that $x\varphi_i = 1$, $y\varphi_i = 0$.

The family of homomorphisms $(\varphi_i \mid i \in I)$ then extends to a homomorphism $\varphi: L \to \underset{\sim}{2}$ such that $x\varphi = 1$ for all $x \in X$ and $y\varphi = 0$ for all $y \in Y$. Thus $(\bigvee Y)\varphi = 0 < 1 = (\bigwedge X)\varphi$, showing that $\bigwedge X \nleq \bigvee Y$, and proving the lemma.

A more complete treatment of the word problem can be found in Grätzer and Lakser [3].

3. <u>The finite disjointness condition in</u> $\mathfrak{D}$. If Γ is any set we denote the diagonal $\{\langle\gamma, \gamma\rangle \in \Gamma \times \Gamma\}$ by ω_Γ. We first recall the classic result of Ramsay in the following form:

<u>LEMMA 2</u> (Ramsay's Theorem). <u>Let</u> Γ <u>be an infinite set and let</u> $R_1, \cdots, R_n$ <u>be binary symmetric relations on</u> Γ <u>such that</u> $\omega_\Gamma \cup R_1 \cup \cdots \cup R_n = \Gamma \times \Gamma$.

Then there is a subset $\Gamma' \subseteq \Gamma$ and an $i \leq n$ such that

(i) for any distinct $\alpha, \beta \in \Gamma'$, $\langle \alpha, \beta \rangle \in R_i$;

and

(ii) Γ' is infinite.

For our purposes the following alternative characterization of the finite and countable disjointness conditions is preferable.

LEMMA 3. A distributive lattice L satisfies the finite (resp. countable) disjointness condition if and only if the following condition holds.

Given any $a \in L$ and any subset $S \subseteq L$ such that $x \nleq a$ for all $x \in S$ and such that $x \wedge y \leq a$ for distinct $x, y \in S$, it then follows that S is finite (resp. countable).

Proof. The proof follows immediately by observing that if S satisfies the condition of the lemma then

(i) $x \vee a > a$ for all $x \in S$;

(ii) If $x, y \in S$ are distinct then

$(x \vee a) \wedge (y \vee a) = (x \wedge y) \vee a = a$ (and so the correspondence $x \to x \vee a$ from S to $\{x \vee a \mid x \in S\}$ is one-to-one).

THEOREM 1. Let $(L_i \mid i \in I)$ be a family of lattices in $\mathcal{D}$ satisfying the finite disjointness condition. Then L, the free product in $\mathcal{D}$, also satisfies the finite disjointness condition.

Proof. Let $a \in L$ and let $(s_\gamma \mid \gamma \in \Gamma)$ be any family of elements of L such that

(A) for each $\gamma \in \Gamma$, $s_\gamma \nleq a$;

and

(B) if $\alpha, \beta \in \Gamma$ are distinct then $s_\alpha \wedge s_\beta \leq a$.

We show that Γ must be finite by proving a sequence of statements involving successively weaker hypotheses about the form of the s_γ and of a .

Statement 1. If $a \in P$ and $s_\gamma \in P$ for all $\gamma \in \Gamma$ then Γ is finite.

Let $a \in L_i$ for some $i \in I$ and let α, β be distinct elements of γ . Then, since $s_\alpha \wedge s_\beta \leq a$, it follows that $s_\alpha, s_\beta \in L_i$ by Lemma 1 and condition (A). Thus $\{s_\gamma \mid \gamma \in \Gamma\} \subseteq L_i$ also and perforce Γ is finite since L_i satisfies the finite disjointness condition.

Statement 2. If $a \in P$ and $s_\gamma = \bigwedge X_\gamma$ for each $\gamma \in \Gamma$ where X_γ is a reduced subset of P then Γ is finite.

For each $\gamma \in \Gamma$ and each $x \in X_\gamma$, $x \nleq a$ by Lemma 1 and (A). Let $a \in L_i$. By (B) if $\alpha, \beta \in \Gamma$ are distinct $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq a$. There are thus $x \in X_\alpha \cap L_i$, $y \in X_\beta \cap L_i$ such that $x \wedge y \leq a$. But $|X_\gamma \cap L_i| \leq 1$ for all $\gamma \in \Gamma$. Thus we have a family $(x_\gamma \mid \gamma \in \Gamma)$ such that $x_\gamma \in L_i$ for all $\gamma \in \Gamma$, such that $x_\gamma \nleq a$ for all $\gamma \in \Gamma$ and such that $x_\alpha \wedge x_\beta \leq a$ for distinct α, β . Thus, by Statement 1, Γ is finite.

<u>Statement 3.</u> If $s_\gamma = \bigwedge X_\gamma$, X_γ reduced, for each γ , and if $a = \bigvee Y$, Y reduced, then Γ is finite.

Let $Y = \{y_1, \cdots, y_p\}$. Then for each $j \leq p$ and each $\gamma \in \Gamma$ $\bigwedge X_\gamma \nleq y_j$, by (A). Define binary relations $R_1, \cdots, R_p$ on Γ by setting $\langle \alpha, \beta \rangle \in R_j$ if and only if $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq y_j$. Since, for any distinct $\alpha, \beta \in \Gamma$, $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq \bigvee Y$ it follows, by Lemma 1, that $\omega_\Gamma \cup R_1 \cup \cdots \cup R_p = \Gamma \times \Gamma$. Now let $j \leq p$ and let Γ' be a subset of Γ such that $\langle \alpha, \beta \rangle \in R_j$ for any two distinct $\alpha, \beta \in \Gamma'$. Then, by Statement 2, Γ' is finite. Thus, by Ramsay's Theorem, Γ is finite.

<u>Statement 4.</u> If $a = \bigvee Y_1 \wedge \cdots \wedge \bigvee Y_r$ where each Y_j is a reduced subset of P and if, for each $\gamma \in \Gamma$, $s_\gamma = \bigvee(\bigwedge X \mid X \in J_\gamma)$ for some finite nonempty set J_γ of reduced subsets of P , then Γ is finite.

Since for each $\gamma \in \Gamma$ $s_\gamma \nleq a$ then for each $\gamma \in \Gamma$ there is an $X_\gamma \in J_\gamma$ and a $j(\gamma) \leq r$ such that $\bigwedge X_\gamma \nleq \bigvee Y_{j(\gamma)}$. For each $j \leq r$ let let $\Gamma_j = \{\gamma \in \Gamma \mid j(\gamma) = j\}$. Then if α, β are distinct elements of Γ_j , $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq s_\alpha \wedge s_\beta \leq a \leq \bigvee Y_j$. But, by definition of Γ_j , $\bigwedge X_\gamma \nleq \bigvee Y_j$ if $\gamma \in \Gamma_j$. Thus, by Statement 3, Γ_j is finite. It thus follows that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_r$ is finite, proving Statement 4.

Since each element of L can be expressed in both forms $\bigvee(\bigwedge X \mid X \in J)$ and $\bigwedge(\bigvee Y \mid Y \in K)$, Statement 4 is the statement of the theorem.

4. *The countable disjointness condition in* $\mathcal{D}_b$. The situations in $\mathcal{D}$ and in $\mathcal{D}_b$ differ essentially because of the following fact. In $\mathcal{D}$, if $x, y \in L_i$, if $z \in L_j$, and if $x \wedge y \leq z$ then $i = j$. In $\mathcal{D}_b$, however, it is possible that $i \neq j$; if $z \neq 1$ then $x \wedge y \leq z$ if and only if $x \wedge y = 0$. It is precisely this difference which yields the countable disjointness condition only, rather than finite disjointness. We will also need a more delicate analysis since the argument establishing Statement 2 of Theorem 1 does not apply in $\mathcal{D}_b$ precisely because of this difference.

THEOREM 2. Let $(L_i \mid i \in I)$ *be a family of lattices in* $\mathcal{D}_b$ *satisfying the finite disjointness condition. Then* L, *the free product in* $\mathcal{D}_b$, *satisfies the countable disjointness condition.*

Proof. Let $a \in L$ and let $(s_\gamma \mid \gamma \in \Gamma)$ be any family of elements of L such that

(A) for each $\gamma \in \Gamma$, $s_\gamma \not\leq a$;

and

(B) if $\alpha, \beta \in \Gamma$ are distinct then $s_\alpha \wedge s_\beta \leq a$.

We show that Γ is countable by proving a sequence of statements involving successively weaker hypotheses about the form of the s_γ and of a.

Statement 1. If $a \in P$ and $s_\gamma \in P$ for all $\gamma \in \Gamma$ then Γ is finite.

Let $a \in L_i$. Since, for each $\gamma \in \Gamma$, $s_\gamma \nleq a$ and if $\alpha \neq \beta$ then $s_\alpha \wedge s_\beta \leq a$, it follows that there is a $j \in I$ such that $s_\gamma \in L_j$ for all $\gamma \in \Gamma$. If $i = j$ the finiteness of Γ follows as in Statement 1 of Theorem 1. If $i \neq j$ then $s_\alpha \wedge s_\beta = 0$ for distinct α, β. Since $s_\gamma \nleq a$ implies $s_\gamma \nleq 0$, the finiteness of Γ follows in this case from the fact that L_j satisfies the finite disjointness property.

<u>Statement 2.</u> Let $n \geq 1$ be an integer, let $a \in P$, and let $s_\gamma = \bigwedge X_\gamma$ for each $\gamma \in \Gamma$, where X_γ is a reduced subset of P with $|X_\gamma| = n$. Then Γ is finite.

The case $n = 1$ is Statement 1. We prove Statement 2 by induction on n. Let $n > 1$. First fix $\gamma_0 \in \Gamma$ and let $X_{\gamma_0} = \{x_1, \cdots, x_n\}$. Then there are distinct $i(1), \cdots, i(n)$ in I such that $x_k \in L_{i(k)}$ for each $k \leq n$. For each $k \leq n$ let $\Gamma_k = \{\gamma \in \Gamma \mid X_\gamma \cap L_{i(k)} \neq \emptyset\}$. Now $\Gamma_1 \cup \cdots \cup \Gamma_n = \Gamma$; since $\bigwedge X_{\gamma_0} \nleq a$, $\bigwedge X_\gamma \nleq a$ if $\gamma \neq \gamma_0$, and $\bigwedge X_{\gamma_0} \wedge \bigwedge X_\gamma \leq a$ it follows that, for each γ, $X_\gamma \cap L_{i(k)} \neq \emptyset$ for some k. It suffices thus to prove that each Γ_k is finite. For each $\gamma \in \Gamma_k$ let x_γ be defined by setting $X_\gamma \cap L_{i(k)} = \{x_\gamma\}$ and let $X'_\gamma = X_\gamma - L_{i(k)}$. Then $|X'_\gamma| = n - 1$ and $X_\gamma = X'_\gamma \cup \{x_\gamma\}$. We define two symmetric binary relations R and S on Γ_k. We set $\langle \alpha, \beta \rangle \in R$ if and only if $x_\alpha \wedge x_\beta \leq a$ and we set $\langle \alpha, \beta \rangle \in S$ if and only if $\alpha \neq \beta$ and $\langle \alpha, \beta \rangle \notin R$. Then $\langle \alpha, \beta \rangle \in S$ only if $\bigwedge X'_\alpha \wedge \bigwedge X'_\beta \leq a$. Since $n > 1$ and $|X'_\gamma| = n - 1$ if $\gamma \in \Gamma_k$ we conclude by Ramsay's Theorem and the induction hypothesis that Γ_k is finite for each k. Thus Γ is finite.

Statement 3. Let $n \geq 1$. For each $\gamma \in \Gamma$ let $s_\gamma = \bigwedge X_\gamma$ where X_γ is reduced and $|X_\gamma| = n$. Let $a = \bigvee Y$, Y reduced. Then Γ is finite.

The proof of this statement is a word-for-word duplicate of the proof of Statement 3 of Theorem 1.

Statement 4. Let $a = \bigvee Y_1 \wedge \cdots \wedge \bigvee Y_r$ where each Y_j is a reduced subset of P . For each $\gamma \in \Gamma$ let J_γ be a finite nonempty set of reduced subsets of P such that $s_\gamma = \bigvee(\bigwedge X \mid X \in J_\gamma)$. Then Γ is countable.

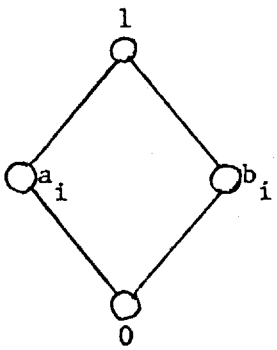
For each $\gamma \in \Gamma$ there is an $X_\gamma \in J_\gamma$ and a $j(\gamma) \leq r$ such that $\bigwedge X_\gamma \nleq \bigvee Y_{j(\gamma)}$. For each $j \leq r$ and $n \geq 1$ let

$$\Gamma_{jn} = \{\gamma \in \Gamma \mid j(\gamma) = j \text{ and } |X_\gamma| = n\} .$$

If α, β are distinct elements of Γ_{jn} then $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq s_\alpha \wedge s_\beta \leq a \leq \bigvee Y_j$. By definition of Γ_{jn} , $|X_\gamma| = n$ if $\gamma \in \Gamma_{jn}$ and $\bigwedge X_\gamma \nleq \bigvee Y_j$. Thus Γ_{jn} is finite by Statement 3. But $\Gamma = \bigcup(\Gamma_{jn} \mid n \geq 1 , 1 \leq j \leq r)$; thus Γ is countable, proving Statement 4.

Statement 4 is the statement of the Theorem.

To complete this section we present an example of a countable family of finite lattices whose free product in $\mathcal{D}_b$ does not satisfy the finite disjointness condition. Let the index set I be the set of positive integers and, for each $i \in I$, let the lattice L_i be the four-element lattice in the diagram.



L_i

Let L be the free product in $\mathcal{D}_b$ of the L_i, $i \in I$. Let $s_1 = b_1$ and for each $n > 1$ let $s_n = a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1} \wedge b_n$. Let $S = \{s_n\}$. Then S is infinite, $0 < s_n$ for each n, and if $m \neq n$, say $m < n$, then $s_m \wedge s_n = 0$, since $s_m \leq b_m$ and $s_n \leq a_m$.

Thus L does not satisfy the finite disjointness condition. Of course, L is just the underlying lattice of the free Boolean algebra generated by a countable set, and this example shows that it need not satisfy the finite disjointness condition.

5. Epilogue. For any infinite cardinal $\mathfrak{m}$ one can of course define the $\mathfrak{m}$-disjointness condition: a lattice L is said to satisfy the $\mathfrak{m}$-disjointness condition if, given any $a \in L$ and any $S \subseteq L$ such that $a \notin S$ and $x \wedge y = a$ for distinct $x, y \in S$, it then follows that $|S| < \mathfrak{m}$. An obvious question is the following:

In either category $\mathcal{D}$ or $\mathcal{D}_b$ is the $\mathfrak{m}$-disjointness condition preserved under free products for $\mathfrak{m} > \aleph_0$?

The methods presented in sections 3 and 4 cannot be applied to answer this question in the affirmative because, as first observed by

Sierpiński [5], the obvious extension of Ramsay's Theorem to infinite cardinals does not hold.

There are Ramsay-type theorems for infinite cardinals; see Erdös, Hajnal, Rado [2] for a rather complete survey. Of particular interest to our problem is the following result of Kurepa [4], under the assumption of the generalized continuum hypothesis:

Let α be any ordinal. Let Γ be a set such that $|\Gamma| \geq \aleph_{\alpha+2}$, and let $R_1, \cdots, R_n$ be binary symmetric relations on Γ such that $\omega_\Gamma \cup R_1 \cup \cdots \cup R_n = \Gamma \times \Gamma$. Then there is a subset $\Gamma' \subseteq \Gamma$ and an $i \leq n$ such that $|\Gamma'| \geq \aleph_{\alpha+1}$ and for any distinct $\alpha, \beta \in \Gamma'$ $\langle \alpha, \beta \rangle \in R_i$.

Using this result in place of Ramsay's Theorem the methods of sections 3 and 4 carry over to prove:

<u>Let</u> $(L_i \mid i \in I)$ <u>be a family of lattices in</u> $\mathfrak{D}$ <u>or</u> $\mathfrak{D}_b$ <u>satisfying the</u> $\aleph_{\alpha+1}$<u>-disjointness condition</u>, $\alpha \geq 0$. <u>Then the free product in</u> $\mathfrak{D}$ <u>or</u> $\mathfrak{D}_b$ <u>satisfies the</u> $\aleph_{\alpha+2}$ <u>-disjointness condition</u>.

Unfortunately I have been unable to construct an example to show that $\aleph_{\alpha+2}$ cannot be replaced by $\aleph_{\alpha+1}$. This is thus to date an open problem.

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THE ORDER-SUM IN CLASSES OF PARTIALLY ORDERED ALGEBRAS

by

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The order-sum of partially ordered algebras will be defined as the solution of a universal problem and as a generalization of the coproduct. We show that the order-sum exists without restriction in quasi-primitive classes, and we investigate some of the properties of the order-sum.

1. Order-sum and lexicographic sum

We will consider classes $\mathfrak{K}$ of partially ordered algebras. The algebras under consideration will be *partial algebras* $(A,(f_i)_{i\varepsilon I})$ of arbitrary finitary or infinitary *type* $\Delta = (K_i)_{i\varepsilon I}$. I.e., the *index-sets* K_i may be finite or infinite, and f_i is a mapping of a subset of A^{K_i} into A. If the domain of f_i is all of A^{K_i}, for each $i\varepsilon I$, we may call $(A,(f_i)_{i\varepsilon I})$ a *complete algebra* of type Δ. A *partially ordered algebra* is a triple $(A,(f_i)_{i\varepsilon I}, \leq)$, where $(A,(f_i)_{i\varepsilon I})$ is a partial algebra and $(A, \leq)$ a partially ordered set. The algebraic structure may be empty, $I=\emptyset$. In that case, the partially ordered algebra is nothing but a partially ordered set, and any class of partially ordered sets is an example of a class of partially ordered algebras. Since, on the other hand, the partial order may be total disorder, we can also interpret any class of partial algebras as a class of partially ordered algebras.

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We do not require any kind of compatibility postulates to hold between the algebraic structure and the partial order. But there is no ban on compatibility conditions either.

A *homomorphism* of the partially ordered algebra $(A,(f_i)_{i\varepsilon I}, \leqslant)$ into the partially ordered algebra $(B,(g_i)_{i\varepsilon I}, \leqslant)$ - of the same type Δ - is a mapping $\phi:A \longrightarrow B$ that is *order-preserving:*

(1.1) $$\text{if } x \leqslant y, \text{ then } \phi(x) \leqslant \phi(y) ,$$

for all elements $x,y\varepsilon A$, and at the same time an *algebraic homomorphism:*

(1.2) $$(f_i(a_\kappa | \kappa\varepsilon K_i)) = g_i(\phi(a_\kappa) | \kappa\varepsilon K_i),$$

for each index $i\varepsilon I$, for each sequence $(a_\kappa)_{\kappa\varepsilon K_i}$ in the domain of f_i (making the left side exist - it is understood that the right side will then exist too).

In the sequel, $\mathfrak{K}$ will always be a class of partially ordered algebras of the same type Δ.

Suppose T is a partially ordered set, and assume that a *partially ordered family* of partially ordered algebras $P_t \varepsilon \mathfrak{K}$ is given. A family of homomorphisms $\phi_t:P_t \longrightarrow P$, where P is also supposed to be in $\mathfrak{K}$, is called a *T-family* provided that the following condition holds true for all indices $s,t\varepsilon T$:

(1.3) $$\text{if } s < t \text{ (in T), then } \phi_s(x) \leqslant \phi_t(y) \text{ (in P)},$$

for all elements $x\varepsilon P_s$, $y\varepsilon P_t$. The *order-sum* is now simply a universal T-family. I.e., the T-family $\phi_t:P_t \longrightarrow P$ is an order-sum if, for each algebra $Q\varepsilon \mathfrak{K}$ and each T-family $\psi_t:P_t \longrightarrow Q$, there is a unique homomorphism $\psi:P \longrightarrow Q$ such that $\psi \circ \phi_t = \psi_t$, for all indices $t\varepsilon T$.

In the special case where the index-set T is totally unordered, the order-sum coincides with the coproduct. Clearly, the order-sum (if it exists) will be unique up to unique isomorphism, and in that sense it is justified to talk about "the" order-sum.

Assume now that $\mathfrak{K}$ is the class of all partially ordered algebras of a given type $\Delta = (K_i)_{i\varepsilon I}$, and assume further that the type Δ is without constants, i.e. $K_i \neq \emptyset$ for each $i\varepsilon I$. The algebraic lexicographic sum of partially ordered algebras that we are going to define, is a combination of the partial direct sum of partial algebras (cf. Schmidt [9]) and of the well-known lexicographic sum of partially ordered sets (cf. Birkhoff [2], Schmidt [7],[8]).

For a partially ordered family of partially ordered algebras P_t, we define $\underset{t\varepsilon T}{L} P_t$ to be the set of all ordered pairs (t,x), where $t\varepsilon T$ and $x\varepsilon P_t$, endowed with the *lexicographic order* :

(1.4) $\qquad (s,x) \leq (t,y)$ iff $s < t$ or $s = t$ and $x \leq y$

The *natural mappings* $i_t : P_t \longrightarrow LP_t$, defined by $i_t(x) = (t,x)$, are obviously order-preserving, even order- embeddings. On LP_t, there exists now the "weakest" algebraic structure $(f_i)_{i\varepsilon I}$ such that the natural mappings i_t become algebraic homomorphisms, i.e. the final structure for the mappings i_t (cf. Bourbaki[3], Schmidt [9]).

LP_t with the lexicographic order $\leq$ and this algebraic structure - and with the natural mappings i_t - will be called the *algebraic lexicographic sum* of the partially ordered algebras P_t $(t\varepsilon T)$.

If T and all algebras P_t are totally disordered, then the algebraic lexicographic sum coincides with with the partial direct sum of the algebras P_t (cf. Schmidt [9]). On the other hand, if $I = \emptyset$, our algebraic lexicographic sum is nothing but the ordinary lexicographic sum of partially ordered sets P_t.

<u>Theorem 1.1</u> In the class $\mathfrak{K}$ of all partially ordered algebras of type Δ, the algebraic lexicographic sum $i_t : P_t \longrightarrow L P_t$ is the order-sum.

In the class of partially ordered topological spaces, a topological lexicographic sum can be defined in a similar manner as for partially ordered algebras: It will be a combination of the topological sum of the spaces and the lexicographic sum of the partially ordered sets. An exact analogue of Theorem 1.1 holds true.

Unfortunately, we had to restrict ourselves so far to the case where Δ is a type without constants. This is to a good extend due to

<u>Theorem 1.2</u> Suppose $\psi_t : P_t \longrightarrow P$ is a T-family of order-preserving mappings. Assume that for each $t \in T$, there is an $a_t \in P_t$ such that $\psi_t(a_t) = a$, where a is independent of t. Suppose $s < t$ in T. Then $\max \psi_s(P_s) = \min \psi_t(P_t) = a$.

<u>Corollary 1.</u> $\max \psi_s(P_s) = a$ if s is not maximal in T, $\min \psi_s(P_s) = a$ if s is not minimal in T. $\psi_s(P_s)$ collapses into $\{a\}$ if s is not extremal in T (neither maximal nor minimal).

<u>Corollary 2.</u> If s is not maximal in T, and $\min P_s = a_s$, then again $\psi_s(P_s)$ collapses into $\{a\}$.

Let us show which damage Theorem 1.2 does to the order-sum in the presence of constants: Let $\mathfrak{K}$ be the class of partially ordered algebras with least and greatest elements, the latter explicitly listed among the constants. I.e., the homomorphisms in $\mathfrak{K}$ are supposed to preserve both least and greatest elements. We now assume that T contains a pair of comparable elements $s < t$. Consider a T-family $\psi_t : P_t \longrightarrow P$. Since s is not maximal, $\psi_s(P_s)$ consists of the least element of P only, according to Corollary 2. On the other hand, it contains the greatest element of P. So the latter has to coincide with the least element, thus squeezing P down to one element. In such a class, the old coproduct will be the only meaningful order-sum. If we give up insisting on the preservation of extrema, however, other order-sums become highly meaningful.

Theorem 1.3 Let the class $\mathfrak{K}$ be closed under taking subalgebras. Let $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathfrak{K}$. Then the union $\bigcup \mathrm{im}\, \phi_t$ generates P.

2. The algebraic lexicographic sum with constants

We want to extend the notion of the algebraic lexicographic sum to the general case where the type Δ may now contain some constants, $K_i = \emptyset$ for some $i \in I$. This should be done in such a way that Theorem 1.1 remains true. The construction is similar to the construction of the partial direct sum of algebras (cf. Schmidt [9]), but somewhat more involved in the presence of partial orders.

Throwing out the indices $i \in I$ standing for constants, we arrive at the reduced index-domain $I^* = \{ i \mid K_i \neq \emptyset \}$ and the corresponding *reduced type* Δ^*, without constants. The partially oredered algebras P_t are turned into partially ordered algebras P_t^* of type Δ^*. We can consider the algebraic lexicographic sum of the latter, $\bigsqcup P_t^*$. In order to arrive at an appropriate factorization, we consider quasi-orders ρ of $\mathsf{L}P_t^*$ which are admissible in the sense that the following three conditions hold:

(i) $\rho \cap \rho^{-1}$ is a congruence relation of the algebra $\mathsf{L}P_t^*$;

(ii) ρ contains the lexicographic order of $\mathsf{L}P_t^*$;

(iii) ρ takes care of the constants insofar as $(s,f_{si})\ \rho\ (t,f_{ti})$, for each $s,t\in T$ and for each $i\in I\setminus I^*$.

It is easy to see that there is a least admissible quasi-order, say σ. The contraction $\mathsf{L}P_t^*/\sigma \cap \sigma^{-1}$ is then a partially ordered algebra of type Δ^*, and the natural projection $p:\mathsf{L}P_t^* \longrightarrow \mathsf{L}P_t^*/\sigma \cap \sigma^{-1}$ is a homomorphism between them. One makes $\mathsf{L}P_t^*/\sigma \wedge \sigma^{-1}$ an algebra of type Δ by introducing the constants $g_i = p(t,f_{ti})$, for each $i\in I$, this definition is independent of t. The partially ordered algebra $\mathsf{L}P_t^*/\sigma \wedge \sigma^{-1}$ so enriched may be called the *algebraic lexicographic sum* of the partially ordered algebras P_t $(t\in T)$ and again be denoted by $\mathsf{L}P_t$. Clearly, in the case without constants, $I^* = I$, $\Delta^* = \Delta$, $P_t^* = P_t$, nothing has happened at all. We introduce the mappings $j_t = p\circ i_t : P_t \longrightarrow \mathsf{L}P_t$, which are homomorphisms by construction (in particular, they preserve the constants).

Theorem 2.1 In the class of all partially ordered algebras of type Δ, the algebraic lexicographic sum $j_t : P_t \longrightarrow \mathsf{L}P_t$ is the order-sum.

Note that the homomorphisms j_t need no longer be one-one since p can not be expected to be one-one. Indeed, σ and $\sigma \wedge \sigma^{-1}$ may become the universal relation in P_t^*, forcing $\mathsf{L}P_t$ to collapse into one element.

3. A general existence theorem

The order-sum in a class $\mathfrak{K}$ of partially ordered algebras can be build up in two steps. The first one of these has been described in sections 1 and 2. In general, of course, the algebraic lexicographic sum of algebras P_t will not be in $\mathfrak{K}$. So the second step will consist in associating with the latter a universal object in $\mathfrak{K}$.

<u>Theorem 3.1</u> Consider a T-family of homomorphisms $\phi_t : P_t \longrightarrow P$ in $\mathfrak{K}$ and the associated homomorphism $\phi : \coprod P_t \longrightarrow P$ (which exists according to Theorem 2.1). Then the following two conditions are equivalent:

(i) $\phi_t : P_t \longrightarrow P$ is the order-sum in $\mathfrak{K}$;

(ii) $\phi : \coprod P_t \longrightarrow P$ is the universal homomorphism of $\coprod P_t$ into a $\mathfrak{K}$-algebra.

As in universal algebra without partial order, a *quasi-primitive class* of partially ordered algebras will be a class closed under taking cartesian products, subalgebras, and isomorphic images. After reinterpretation of the partial orders as partial operations, such a class will become a quasi-primitive class in the ordinary sense of universal algebra.

<u>Theorem 3.2 (Existence of Order-sums)</u>
In a quasi-primitive class all order-sums exist.

4. When is the order-sum an extension of the lexicographic sum?

Suppose that $\mathfrak{K}$ is a class of partially ordered algebras. Suppose $\phi_t : P_t \longrightarrow P$ to be an order-sum in $\mathfrak{K}$ and $i_t : P_t \longrightarrow \coprod P_t$ the lexicographic sum. Let $\phi : \coprod P_t \longrightarrow P$ be the universal homomorphism of Theorem 3.1.

In order to avoid the difficulties connected with the constants, we shall assume from now on that the type Δ be *without constants* ($K_i \neq \emptyset$ for each $i \in I$).

Theorem 4.1 Equivalent are:

(i) $\phi: \mathsf{L} P_t \longrightarrow P$ is one-one;

(ii) the homomorphisms $\phi_t: P_t \longrightarrow P$ are one-one, and their images are pairwise disjoint;

(iii) there is a $\mathfrak{K}$-algebra Q and a one-one homomorphism $\psi: \mathsf{L} P_t \longrightarrow Q$.

Theorem 4.2 Equivalent are:

(i) $\phi: \mathsf{L} P_t \longrightarrow P$ is an order-embedding;

(ii) the homomorphisms $\phi_t: P_t \longrightarrow P$ are order-embeddings, and the indexed family of their images is not only pairwise disjoint, but a "lexicographic decomposition" of the partially ordered set $\bigcup_{t \in T} \operatorname{im} \phi_t$ $(= \operatorname{im}\phi)$;

(iii) there is a $\mathfrak{K}$-algebra Q and an order-embedding (and algebraic homomorphism) $\psi: \mathsf{L} P_t \longrightarrow Q$.

Theorem 4.3 Equivalent are:

(i) the homomorphisms $\phi_t: P_t \longrightarrow P$ are one-one;

(ii) for all indices $s \in T$ and all elements $x, y \in P_s$ such that $x \neq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t: P_t \longrightarrow Q$ separating x and y, $\psi_s(x) \neq \psi_s(y)$.

Theorem 4.4 Equivalent are:

(i) the homomorphisms $\phi_t: P_t \longrightarrow P$ are order-embeddings;

(ii) for all indices $s \in T$ and all elements $x, y \in P_s$ such that $x \nleq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t: P_t \longrightarrow Q$ such that $\psi_s(x) \nleq \psi_s(y)$.

Whenever the universal homomorphism ϕ is an order-embedding, we can replace it by the inclusion mapping of the lexicographic sum into an isomorphic copy of P, due to the well-known Zermelo - van der Waerden replacement procedure.I.e., the order-sum can be considered as an extension of the lexicographic sum. $\bigsqcup P_t$ becomes a subset of P, the lexicographic order of $\bigsqcup P_t$ is the restriction of the partial order of P. However, the inclusion of the partial algebra $\bigsqcup P_t$ into the algebra P will only be a homomorphism, not necessarily an embedding. I.e., $\bigsqcup P_t$ may only be a *weak* relative algebra of P. We will refer to this situation by saying that P is an *order-extension* of $\bigsqcup P_t$ (algebraically, it may only be a *weak extension*). If all $\mathfrak{K}$-algebras are complete, at least the inclusions of the pieces $i_t(P_t)(= \phi_t(P_t))$ into P are strong, the pieces are then genuine subalgebras of the complete algebra P.

We now find convenient sufficient conditions on the class $\mathfrak{K}$ to garantee that our mapping $\phi: \bigsqcup P_t \longrightarrow P$ will be an order-embedding.

(I) $\mathfrak{K}$ is *non-trivial*, i.e. it contains a *non-trivial algebra* Q in the sense that Q contains a pair of distinct comparable elements.

(II) All constant mappings between $\mathfrak{K}$-algebras are homomorphisms.

(III) For every $\mathfrak{K}$-algebra P and all elements $x,y \in P$ such that $x \nleq y$, there is a $\mathfrak{K}$-algebra Q and a homomorphism $\alpha: P \longrightarrow Q$ such that $\alpha(y) = \min \alpha(P) < \alpha(x) = \max \alpha(P)$ ("separability").

Condition (III) may be replaced, for our purposes, by the following:

(III') Every $\mathfrak{K}$-algebra is embeddable into a non-trivial $\mathfrak{K}$-algebra with least and greatest element.

Note that in the class $\mathfrak{K}$ of all distributive lattices, all four conditions hold. In the class of modular lattices, at least (I), (II), (III') hold.

Theorem 4.5 Let $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathcal{K}$. Suppose that $\mathcal{K}$ fulfills the conditions (I), (II), and (III) or, alternatively (III'). Then the universal homomorphism $\phi : \sqcup P_t \longrightarrow P$ is an order-embedding.

Corollary Suppose is a class of complete algebras fulfilling conditions (I), (II), and one of (III) or (III'). Then the order-sum P (provided it exists) is an order-extension (and weak algebraic extension) of the lexicographic sum $\sqcup P_t$, and the pieces $i_t(P_t)$ are subalgebras of P.

Recall that the order-sum exists, if $\mathcal{K}$ is quasi-primitive.

5. The order-sum extends the lexicographic sum in some nice classes

Theorem 5.1 In the class $\mathcal{K}$ of semilattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.2 In the class $\mathcal{K}$ of lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.3 In the class $\mathcal{K}$ of distributive (modular) lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.4 In the class $\mathcal{K}$ of semilattices (lattices, distributive, modular lattices), the order-sum over a chain T coincides with the lexicographic sum of the partially ordered sets.

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ARITHMETIC PROPERTIES OF RELATIVELY FREE PRODUCTS

By Stephen D. Comer

Arithmetic properties of direct products have been studied for many years. W. Hanf showed in [3] that the cancellation law, Cantor-Bernstein and square-root properties fail for direct products of Boolean algebras. The present note contains some observations concerning analogous problems for free products. Free product is understood to mean coproduct where the canonical injections are monic. Unlike direct products, the free product of algebras depends on the variety where it is formed and it may not even exist. Free products are assumed to exist in any variety considered. Consider the following three properties for algebras A, B, C in a variety V.

(1) $A * B \cong A * C$ implies $B \cong C$.

(2) $A \cong B * D$ and $B \cong A * C$ implies $A \cong B$.

(3) $B * B \cong C * C$ implies $B \cong C$.

Properties (1),(2),(3) are known as the cancellation law, Cantor-Berstein property and square-root property, respectively. These properties are established in section 1 under suitable finiteness assumptions. Counterexamples to (1),(2),(3) are given for Boolean algebras in section 2. In section 3 the results from section 2 are applied to derive counterexamples for other classes of algebraic structures.

1. We first consider the cancellation property (1) for a variety V. Normally this property will obviously fail if we do not require some finiteness condition on A. For example, it usually fails if we let A be a V-free algebra generated by an infinite set and let B,C be V-free algebras generated by finite sets with different cardinalities. For a subclass K of V we say that A _cancels for_ K if (1) holds for all B,C in K. The results below give conditions under which A cancels for the class of all finite members of V.

Theorem 1.1. Suppose $A * B \cong A * C$ for A,B,C in V and, in addition, B,C are finite and $0 < |\mathrm{Hom}(A,X)| < \omega$ for every subalgebra X of B and every subalgebra X of C. Then $B \cong C$.

Proof. From the conditions on A,B,C,X and the fact that

$$|\mathrm{Hom}(A,X)| \cdot |\mathrm{Hom}(B,X)| = |\mathrm{Hom}(A*B,X)| = |\mathrm{Hom}(A,X)| \cdot |\mathrm{Hom}(C,X)|$$

it follows that

(1.2) $|\mathrm{Hom}(B,X)| = |\mathrm{Hom}(C,X)|$ for every $X \in S\{B,C\}$.

Let $\{X_i : i < n\}$, for some n, be a listing without repetition of the maximal proper subalgebras of X. For an algebra D the principle of inclusion-exclusion gives

$$|\mathrm{Epi}(D,X)| = |\mathrm{Hom}(D,X)| - \sum_i |\mathrm{Hom}(D,X_i)| + \sum_{i,j} |\mathrm{Hom}(D,X_i \cap X_j)| - \cdots .$$

The right sides of the two equations obtained by letting $D = B$ and $D = C$ are the same by (1.2). Thus,

(1.3) $|\mathrm{Epi}(B,X)| = |\mathrm{Epi}(C,X)|$ for every $X \in S\{B,C\}$.

Setting $X = B$ in (1.3) gives $0 \neq \mathrm{Epi}(B,B) = \mathrm{Epi}(C,B)$ so $|C| \geq |B|$. Similarly, setting $X = C$ in (1.3) gives an epimorphism from B onto C. $|C| \geq |B|$ implies this map is an isomorphism so $B \cong C$.

The above proof was obtained by "dualizing" the proof of the analogous result for direct products due to L. Lovász [5]. The following statements are immediate corollaries of (1.1).

(1.4) If $A * B \cong A * C$ for A,B,C in V where B,C are finite, A finitely generated and $\mathrm{Hom}(A,X) \neq 0$ for every $X \in S\{B,C\}$ then $B \cong C$.

(1.5) If every member of V has a one element subalgebra, then every finitely generated member cancels for the finite members of V.

(1.6) Finitely generated V-free algebras cancel for the finite members of V.

In particular, (1.5) applies for any variety of lattices or groups. By (1.4) finite Boolean algebras cancel for the class of finite BA's.

We now consider the Cantor-Bernstein property (2) and the square-root property (3) for the finite members of a variety. The message of (1.7) and (1.8) is that the finite versions hold. The proof of (1.7) depends on the following lemma due to Bjarni Jónsson.

Lemma. Suppose A,B in V and A is not isomorphic to a proper subalgebra of itself. Then $A \cong A * B$ if and only if, for every extension E of A and every homomorphism $h:B \to E$, h maps B into A.

Proof. Given E and h let $k = 1_A * h:A * B \to E$. Let i,j denote the canonical embeddings of A,B into $A * B$. By the assumption on A, the monic $A \xrightarrow{i} A * B \cong A$ is onto so $i(A) = A * B$. Hence $h(B) = k(j(B)) \leq k(i(A)) = A$ as desired. Conversely, consider $i:A \to A * B \leftarrow B:j$ and let $E = A * B$ (identifying $i(A)$ with A) and $h = j$. Then $j(B) \leq i(A)$ and so $A \cong A * B$.

Theorem 1.7. The Cantor-Bernstein property (2) holds in V whenever A is not isomorphic to a proper subalgebra of itself. In particular, it holds whenever A is finite.

Proof. (2) implies $A \cong A * C * D$; so $C * D$ satisfies the condition of the Lemma and there exist a map of D into A. Then C satisfies the condition of the lemma; so $A \cong A * C$ as desired.

Two proofs of the square-root property for finite algebras are given below. The first one uses the argument in 1.1. The second was communicated to me by Jan Mycielski who reported that it was discovered a few years ago by A. Ehrenfeucht (unpublished). Ehrenfeucht's proof is outlined below since it illustrates an alternative way of giving the counting argument basic to both 1.8 and 1.1.

<u>Theorem 1.8.</u> The square-root property (3) holds whenever B,C are finite members of V.

<u>Proof.</u> For every $X \in S\{B,C\}$,

$$|\mathrm{Hom}(B,X)|^2 = |\mathrm{Hom}(B * B,X)| = |\mathrm{Hom}(C * C,X)| = |\mathrm{Hom}(C,X)|^2;$$

hence, (1.2) holds. Thus, $B \cong C$ by the same argument used in the proof of 1.1.

The key to 1.8 and also 1.1 is to show that, for finite algebras B and C, (1.2) implies (1.3).

Ehrenfeucht proved (1.3) by induction on $|X|$. For $|X| = 1$ the result is trivial. Now, for any algebra A,

$$|\mathrm{Hom}(A,X)| = |\mathrm{Epi}(A,X)| + \sum_{D \subsetneqq X} |\mathrm{Epi}(A,D)|.$$

In the two equations obtained by letting $A = B$ and $A = C$, the left sides are equal by (1.2) and the right terms on the right side are equal by the induction hypothesis. Hence $|\mathrm{Epi}(B,X)| = |\mathrm{Epi}(C,X)|$ follows.

2. We now turn our attention to some counterexamples. To minimize our work we introduce property (4) below. Property (2) is clearly equivalent to the statement that $A \cong A * C * D$ implies $A \cong A * C$. This statement, in turn, implies

(4) $A \cong A * C * C$ implies $A \cong A * C$.

Observe that (3) also implies (4); for if there exist A and C where $A \not\cong A * C$ but $A \cong A * C * C$, then $(A * C) * (A * C) \cong A * A$ while $A \not\cong A * C$.

For BA's we will show that (1) fails with A finite and (4) fails with C finite (and thus, (2), (3) also fail). The examples in (2.4), (2.5) are based on those due to Hanf and Tarski in [3] for direct products. We need the following from [3].

(2.1) There exist denumerable BA's B, C such that $B^2 \cong C^2$ and $B \not\cong C$.

(2.2) For each integer $n > 1$ there exist a BA H_n such that $H_n \cong H_n \times 2^n$ but $H_n \not\cong H_n \times 2^k$ for $k = 1,\ldots,n-1$. The H_n's are uncountable and $H_n^2 \cong H_n$.

The following simple observation is crucial.

(2.3) If A, B are BA's and B is finite with n atoms then $A * B \cong A^n$ (the direct product of A with itself n times).

<u>Proof</u>. The dual space of $A * B$ is the cartesian product of the dual X of A times an n element discrete space. Thus, it is also a disjoint union of n copies of X.

<u>Theorem 2.4</u> (1) The four element BA 2^2 does not cancel for the class of all denumerable BA's.

(2) The two element BA is the only finite BA to cancel for the

class of BA's.

Proof. (1) Let $A = 2^2$ and choose B,C from (2.1). By (2.3), $A * B \cong B^2 \cong C^2 \cong A * C$ but $B \not\cong C$.

(2). Suppose $A = 2^n$ for $n > 1$. Let $B = H_n$ and $C = H_n \times 2$. From (2.2), $B \not\cong C$. However, using (2.3) and (2.2), $A * C \cong (H_n \times 2)^n \cong (H_n \times 2)^n \cong H_n^n \times 2^n \cong H_n^n \cong A * B$.

Theorem 2.5. Property (4) fails for BA's with C finite.

Proof. Let $C = 2^2$ and $A = H_3 \times 2$. By (2.3) and (2.2), $A * C * C \cong (H_3 \times 2)^4 \cong H_3^4 \times 2^4 \cong (H_3^3 \times 2^3) \times 2 = A$ but $A * C \cong H_3 \times 2^2 \not\cong A$.

It is worth noting the counterexample to (3) obtained from (2.5) is $A = H_3 \times 2$ and $B = H_3 \times 2^2$. By passing to the dual spaces we get Boolean spaces X,Y which are not homeomorphic however, $X \times X$ is homeomorphic to $Y \times Y$. This answers an old question posed by Halmos in [2].

3. We can obtain counterexamples to the arithmetic properties (1), (2), and (3) for other classes of algebras from the results in the previous section by constructing appropriate functors.

Proposition 3.1. Suppose $\mathcal{C}$ is a category (with free products) and Γ is a full embedding of the category of BA's into $\mathcal{C}$ that preserves free products. Then (1), (2), (3), and (4) fail in $\mathcal{C}$. Moreover, $\Gamma(2^n)$ does not cancel for $\mathcal{C}$ for finite $n > 2$.

The proof is straightforward using the fact that a full embedding has the property: $A \cong B$ iff $\Gamma A = \Gamma B$.

We apply (3.1) below to obtain examples for the variety generated by a primal algebra, bounded distributive lattices, and rings. In each case the functor used is one that arises in the study of sectional representations over Boolean spaces. Let X_B denote the Stone space of a BA B. For a universal algebra A and a Boolean space X let $\Gamma(X,A)$ denote the algebra of all continuous functions from X into A (given the discrete topology). For each of the varieties (categories) $\mathcal{C}$ to be considered a natural algebra A is selected in $\mathcal{C}$. The functor Γ from BA's into $\mathcal{C}$ is defined for a BA B by $\Gamma(B) = \Gamma(X_B,A)$. Γ does the natural thing to homomorphisms. This functor Γ is always an embedding. To apply (3.1) we have to check in the categories below that Γ is full and preserves free products.

(3.2) $\mathcal{C}$ is the variety generated by a primal algebra A.

In this case the functor Γ establishes an equivalence between the category of BA's and $\mathcal{C}$ (see Hu[4]). Thus, (3.1) applies and its conclusion holds for $\mathcal{C}$. In particular, A^2 does not cancel for $\mathcal{C}$.

(3.3) $\mathcal{C}$ is the variety of (0,1)-distributive lattice.

Let A be the two element distributive lattice with 0 and 1 distinguished. The functor Γ in this situation essentially just forgets the complementations operation. The embedding is full since any 0,1 preserving lattice homomorphism between BA's also preserves complements. The following lemma (3.4) implies that Γ preserves free products.

(3.4) If B_i is a Boolean subalgebra of a BA B $(i \in I)$ and $B = \Pi^*_I B_i$ (as BA's), then $B = \Pi^*_I B_i$ as (0,1)-distributive lattices.

The proof of (3.4) easily follows from the internal description of free products of BA's and a similar description for bounded distributive lattices due to Grätzer and Lakser [1].

Thus, (3.1) applies and its conclusion holds for the class of bounded distributive lattices. In particular, 2^n does not cancel for this class for $1 < n < \omega$.

(3.5) Rings.

Let A denote a fixed field and $\mathcal{C}$ the category of all commutative rings that contain A as a subring. Mappings in $\mathcal{C}$ are A-homomorphisms. The free product operation in $\mathcal{C}$ is the tensor product $\otimes_A$ over A (see Zariski, Samuel [6]). The ring $\Gamma(B) = \Gamma(X_B, A)$ is in $\mathcal{C}$ when A is identified with the subring of constant functions on X_B. Observe that $\Gamma(B)$ is commutative and B is isomorphic to the BA $B(\Gamma(B))$ of all idempotent elements of $\Gamma(B)$. In fact, $B(\Gamma(B))$ is just the double dual space of B. It easily follows that the embedding Γ is full. We need the following lemma.

(3.6) For Boolean spaces $X, Y, \Gamma(X,A) \otimes_A \Gamma(Y,A) \cong (X \times Y, A)$.

The obvious projections of $X \times Y$ onto X and Y induce embeddings $f: \Gamma(X,A) \to \Gamma(X \times Y, A)$ and $g: \Gamma(Y,A) \to \Gamma(X \times Y, A)$. Let $R = f(\Gamma(X,A))$ and $S = g(\Gamma(Y,A))$. Note that $\sigma \in \Gamma(X \times Y, A)$ is in R if and only $\{\sigma^{-1}(a) : a \in A\}$ partitions $X \times Y$ into disjoint sets of the form $N \times Y$ where N is a clopen subset of X. A similar description of S also holds.

For N, N' clopen subsets of X, Y respectively, let $c_{N \times N'}$ denote the characteristic function of $N \times N'$. For $a \in A$ $ac_{N \times N'} = (ac_{N \times Y}) \cdot (c_{X \times N'})$ is in the subring generated by R and S; thus, R and S generate $\Gamma(X \times Y, A)$.

Below we need the observation that $(f(\sigma') \cdot g(\tau'))(x,y) = \sigma'(x) \cdot \tau'(y)$ for all $\sigma' \in \Gamma(X,A)$, $\tau' \in \Gamma(Y,A)$, $x \in X$, $y \in Y$.

To verify (3.6) it remains to show R and S are linearly disjoint over A(cf., Zariski-Samuel [6]). Suppose $\sigma_1,\ldots,\sigma_n \in R$ and $\tau_1,\ldots,\tau_m \in S$ are each linearly independent sets over A. For each i,j choose $\sigma_i' \in \Gamma(X,A)$ and $\tau_j' \in \Gamma(Y,A)$ such that $f(\sigma_i') = \sigma_i$ and $g(\tau_j') = \tau_j$. In addition consider $c_{ij} \in A$ such that $\sum_{i,j} c_{ij}\sigma_i\tau_j$.

For $(x,y) \in X \times Y$, the observation above shows that

$$0 = \sum_{i,j} c_{ij}(\sigma_i\tau_j)(x,y) = \sum_i \left(\sum_j c_{ij}\tau_j'(y)\right)\sigma_i'(x).$$

Since $\sigma_1',\ldots,\sigma_n'$ are linearly independent in $\Gamma(X,A)$ over A, it follows that $0 = \sum_j c_{ij}\tau_j'(y)$ for each j and $y \in Y$. The linear independence of $\tau_1',\ldots,\tau_m'$ in $\Gamma(Y,A)$ over A now implies that $c_{ij} = 0$ for each i,j. Thus, $\{\sigma_i\tau_j : i = 1,\ldots,n;\ j = 1,\ldots,m\}$ is linearly independent over A; so R and S are linearly disjoint and consequently (3.6) holds.

From (3.6) it follows that Γ preserves free products and hence (3.1) applies. So (1), (2), (3), and (4) fail in $\mathcal{C}$. In particular, A^n $(1 < n < \omega)$ does not cancel for the category of all commutative rings that contain the field A.

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On the dimensional stability of compact zero-dimensional semilattices

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Introduction: Let $\underline{Z}$ be the category of compact zero-dimensional semilattice monoids and identity preserving homomorphisms. We consider the question when an object $S \in \underline{Z}$ has the property that each homomorphic image is also in $\underline{Z}$. Equivalently, for which $S \in \underline{Z}$ is $S/R \in \underline{Z}$ for every closed congruence R on S? Lawson [2] has recently considered this question for more general S, and he shows that each finite dimensional locally connected compact semilattice has no dimension raising homomorphisms. However, when applied to objects in $\underline{Z}$, this result only shows that finite objects in $\underline{Z}$ are dimensionally stable.

Our results are not comprehensive, indeed, they are somewhat scattered. However, they do serve our purpose, which is to provide an interesting and informative application of the duality theory developed in [1].

Specifically, we assume that for $S \in \underline{Z}$, $\hat{S} = \underline{Z}(S,2) \subset \underline{S}$ (the category of discrete semilattice monoids and identity preserving homomorphisms); dually, that for $S \in \underline{S}$, $\hat{S} = \underline{S}(S,2) \in \underline{Z}$; and that for $S \in \underline{Z}$ or $\underline{S}$, $S \simeq \hat{\hat{S}}$. Moreover, that for $S \in \underline{Z}$, $\hat{S} \simeq (K(S),\vee)$, where $K(S)$ is the set of local minima of S, and, that for $k_1,k_2 \in K(S)$, $k_1 \vee k_2 = \wedge(\uparrow k_1 \cap \uparrow k_2)$, $\uparrow k_i$ being the set of points $s \in S$ with $k_i s = k_i$.

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Definition: An object $S \in \underline{Z}$ is *stable* if for each closed congruence R on S, $S/R \in \underline{Z}$. Otherwise, S is called *instable*.

Probably the most natural example of an instable object in $\underline{Z}$ is C, the Canter set in the unit interval I, under min multiplication. Indeed, if $\rho : C \to I$ is the Carathéodory map, then ρ is a continuous surmorphism of semilattices.

Moreover, the property of having I as a semilattice quotient is decisive for instable objects of $\underline{Z}$. Clearly the condition is sufficient. Conversely, if $S \in \underline{Z}$ is instable, then there is a compact semilattice T with a non-degenerate component K and a surmorphism $f : S \to T$. T is a Lawson semilattice since S is ($\underline{Z}(S,2)$ separates the points of S), and so, if $a,b \in K$, $a \neq b$, there is a homomorphism $g : T \to I$ with $g(a) \neq g(b)$. Assuming $g(a) < g(b)$, $g(T) \supseteq [g(a),g(b)]$, and, if $r : I \to [g(a),g(b)]$ is the canonical semilattice retraction, we then have $r \circ g \circ f : S \to [g(a),g(b)]$ is the desired surmorphism. We have proved:

Proposition 1: $S \in \underline{Z}$ is instable if and only if there is a continuous surmorphism $f : S \to I$.

This is a rather simple characterization of the instable objects in $\underline{Z}$; in fact too simple. It sheds little light on the structure of instable objects, and it utilizes an object outside the category $\underline{Z}$ to characterize this notion. We now explore the possibility for a more inherent characterization and we begin by establishing some properties of instable objects.

Proposition 2: If $S \in Z$ is instable, then there is a perfect nondegenerate chain $C \subseteq S$.

Proof. Let S in $\underline{Z}$ be instable. Then, by Proposition 1, there is a surmorphism $f : S \to I$. Define $\tilde{f} : I \to S$ by $\tilde{f}(s) = \wedge f^{-1}(s)$. Clearly f is monotone, (i.e. $t \leq t' \in I$ implies $\tilde{f}(t) \leq \tilde{f}(t')$), $f \circ \tilde{f} = 1_I$, and $\tilde{f}(f(s)) \leq s$ for each $s \in S$. Moreover, if $\{t_\alpha\}_{\alpha \in D} \subseteq I$ with $\{t_\alpha\} \underset{\alpha \in D}{\to} t \in I$, and $t_\alpha \leq t$ for each $\alpha \in D$, then

then $\{\tilde{f}(t_\alpha)\}_{\alpha \in D} \to \tilde{f}(t)$ and $\tilde{f}(t_\alpha) \leq \tilde{f}(t)$ for each $\alpha \in D$.

Let $C_o = \tilde{f}(I)^*$. C_o is a compact chain, and if $c \in C_o \setminus \tilde{f}(I)$, then c is not isolated in C_o. If $c = \tilde{f}(t)$ for $t > 0$, then $c = \lim_{n\to\infty} \tilde{f}(t-\frac{1}{n})$, and so c is again not isolated in C_o. Thus $0 = \wedge C_o$ is the only possible isolated point of C_o. We let $C = C_o$ if 0 is not isolated in C_o, and $C = C_o \setminus \{0\}$ otherwise. C is clear by the desired chain.

Corollary: If $S \in \underline{Z}$ is instable, then there is a surmorphism $f \in \underline{S}(\hat{S},D)$ where D is an order dense chain.

Proof: We let C be the chain guaranteed in Proposition 2. Then $i : C \subseteq S$ implies $\hat{i} = f : \hat{S} \twoheadrightarrow \hat{C}$. Since C is a compact perfect chain, $\hat{C} = D$ in an order dense chain.

The question of whether $S \in \underline{Z}$ is instable if and only if S contains a compact perfect nondegenerate chain is settled in the negative by the following.

Lemma: Let X be a set. Then $2^X \in \underline{Z}$ is stable.

Proof: Let $f : 2^X \to I$ be a homomorphism. Since $X = \varinjlim\{F \subseteq X : F \text{ is finite}\}$, $1 = \lim\{\chi_F : F \text{ is finite}\}$ Thus, if $t < 1$, there is some finite $F \subseteq X$ with $t < f(\chi_F)$. Now, if $y \in 2^X$ with $f(y) \leq f(\chi_F)$, then $f(\chi_F \cdot y) = f(\chi_F)f(y) = f(y)$. Therefore $f(\chi_F \cdot 2^X) = f(2^X) \cap [0,f(\chi_F)]$ and since F is finite, $\chi_F \cdot 2^X$ is finite. Thus f is not surjective.

Now, let $\mathbb{Q} = \{r \in (0,1] : r \text{ is rational}\}$. Then $\mathbb{Q}_2 \in \underline{S}$ and $\mathbb{Q}_2 = \widehat{2^{\mathbb{Q}}}$. As we have just seen, $2^{\mathbb{Q}}$ is stable. However, there is a surmorphism $\mathbb{Q}_2 \to \mathbb{Q}$ which extends the identity map on $\mathbb{Q}$, and so, by duality, $2^{\mathbb{Q}}$ contains a compact perfect non-degenerate chain.

Note that for $S \in \underline{Z}$, if there is a surmorphism $f : S \to C$, C the Canter semilattice, then S is instable. Moreover, by duality, this is equivalent to there being a monomorphism $\hat{f} : \mathbb{Q} \hookrightarrow \hat{S}$, i.e. that there is a countable order-dense chain $C_o \subseteq (K(S),\vee)$ with $0 \in C_o$. It is not unreasonable to conjecture that this property characterizes the instable objects in $\underline{Z}$.

As we shall see, this is not the case, but we do have the following.

Theorem 1: Let $S \in \underline{Z}$ and suppose $\hat{S}$ is complete. Then the following are equivalent.

a) S is instable.

b) There is a surmorphism $f \in \underline{Z}(S,C)$. Moreover, if $f : S \to I$ is any surmorphism, then there is a surmorphism $\bar{f} : S \to C$ with $\rho \cdot \bar{f} = f$, $\rho : C \to I$ being the Carathéodory map.

Proof: Clearly b) implies a). Let $f : S \to I$ be a surmorphism. Define $\tilde{f} : I \to S$ as in Proposition 2. Note that, as I is connected, the points of discontinuity of $\tilde{f}$ must be dense in I. Let $0 < t < 1$ be one such point, and set $s = \tilde{f}(t)$. Also, let $u = \wedge\tilde{f}(t,1]$, and $k = \vee\{k' \in K(S) : k' \leq s\}$, where the supremum is taken in $K(S)$. As $s = \vee\{k' \in K(S) : k' \leq s\}$, where this supremum is in S, we have $s \leq k$. But, $s = \tilde{f}(t) = \lim_{t' \uparrow < t} \tilde{f}(t')$ implies $s \notin K(S)$, whence $s < k$.

Note that, for $x \in S$ with $f(x) \geq t$, $f(xs) = f(x)f(s) = t$, and so $s = \tilde{f}(t) \leq xs \leq x$. Hence, if $k' \in K(S)$ with $f(k') > t$, $s \leq k'$, so $k' \geq k''$ for each $k'' \in K(S)$ with $k'' \leq s$. Therefore $k \leq k'$. Now, $f^{-1}(t,1]$ is open in S as f is continuous, and so if $x \in f^{-1}(t,1]$, then $x = \vee(K(S) \cap Sx)$ implies there is $k' \in K(S)$ with $k' \leq x$ and $f(k') > t$. Thus $k \leq k'$, so $k \leq x$. Therefore $k \leq u$.

Let $I_1, I_2 \subseteq I \times 2$ by $I_1 = \{(x,y) : x \leq t$ and $y = 0\}$ and $I_2 = \{(x,y) : t \leq x$ and $y = 1\}$. Define $f_t : S \to I_1 \cup I_2$ by

$$f_t(x) = \begin{cases} (f(x),\ 1) & \text{if } k \leq x \\ (f(x),\ 0) & \text{if } k \nleq x \end{cases}.$$ Note that

f_t is a continuous surmorphism of S onto $I_1 \cup I_2$. If $\pi : I_1 \cup I_2$ is a projection on the first coordinate, we clearly have $\pi \circ f_t = f$.

To finish our proof, we now find points $0 < t_1 < t < t_2 < 1$ where f, and hence f_t are discontinuous, and split I_1 and I_2 each into subintervals in analogous fashion to what we just did for I. We continue this process by induction to obtain a system of intervals whose limit is C, the Cantor set. This induces the desired factorization.

To see that the above Theorem is false in general, we construct the following example. Let $\rho : C \to I$ be the Carathéodory map, and let

$$S_o = \{(\rho(x),x) : x \in C\} \subseteq I \times I.$$

For each local minimum $0 \neq k \in C$, choose a sequence $\{p_{kn}\}_{n\in\omega} \subseteq Ck \setminus \{k\}$ with $p_{kn} < p_{kn+1}$ and $\vee p_{kn} = \vee(Ck \setminus \{k\})$. Finally, let $S = S_o \cup \{(\rho(p_{kn}),k) : k \in K(C), n\in\omega\}$. The local minima are precisely the points $(\rho(p_{kn}),k)$. Moreover, if $\pi : S \to I$ is the projection on the first coordinate, there is no factorization of π through C. Indeed, suppose $f : S \to C$ with $\rho \circ f = \pi$. Let $c \in C$ be a local minimum with $c \neq 0$. Then $f^{-1}[c,1]$ is an open-closed subsemilattice of S, and so $p = \wedge f^{-1}[c,1] \in K(S)$. Hence $p = (\rho(p_{kn}),k)$ for some $k \in K(C)$ and $n \in \omega$. If $k' = \vee(Ck \setminus \{k\})$, then $p_{kn} < k'$ and, therefore, $\rho(p_{kn}) < \rho(k') = \rho(k)$. Thus, $\pi(\rho(p_{kn}),k)\pi(\rho(k'),k') = \rho(p_{kn})$. But $\pi(\rho(k'),k') = \rho(k') = \rho(k) = \pi(\rho(k),k)$, whence $f(\rho(k'),k') = f(\rho(k),k)$ or $f(\rho(k),k) \in K(C)$ and $f(\rho(k'),k') = \vee(Cf(\rho(k),k) \setminus \{f(\rho(k),k)\})$. Since $(\rho(p_{kn}),k) < (\rho(k),k)$, we have $t = f(\rho(p_{kn}),k) \leq f(\rho(k),k)$. But $\rho(p_{kn},k) = \wedge f^{-1}\{t\}$ implies $f(\rho(k'),k') \neq f(\rho(k),k)$ since $(\rho(p_{kn}),k) \cdot (\rho(k'),k') \neq (\rho(p_{kn}),k)$. Hence, the only other possibility is $f(\rho(k),k) = t$ and $f(\rho(k'),k') = \vee(Ct \setminus \{t\})$. But, then

$$\begin{aligned}
&\pi((\rho(p_{kn}),k)(\rho(k'),k')) = \rho(f((\rho(p_{kn}),k)(\rho(k'),k'))) \\
&= \rho(f(\rho(p_{kn}),k)f(\rho(k'),k')) = \\
&= \rho(f(\rho(k'),k')) = \pi(\rho(k'),k') = \rho(k') \neq \\
&\neq \pi(\rho(p_{kn}),k)\pi(\rho(k'),k'),
\end{aligned}$$

contradicting that π is a morphism.

We note that this example does not, on the surface, show that an instable object in $\underline{Z}$ must have some map onto the Cantor semilattice. We feel that this property does hold for instable objects, but we are not able to settle the question.

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LAWSON SEMILATTICES DO HAVE A PONTRYAGIN DUALITY

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The category *L* of *Lawson semilattices* is the category of all compact topological semilattices S with identity, whose continuous interval morphisms $S \to I$ (where $I = [0,1]$ has the min multiplication) separate the points and whose morphisms are continuous identity preserving semigroup morphisms.

We say that *a category A has a Pontryagin duality* iff there is a category $\underline{B}$ which is dual to $\underline{A}$ in a way analogous to the fashion in which discrete abelian groups and compact abelian groups are dual. Specifically, there are functors

$$\begin{tikzcd} \underline{A} \arrow[r, shift left, "F"] \arrow[d, "U"'] & \underline{B}^{op} \arrow[l, shift left, "G"] \arrow[d, "V"] \\ \mathrm{Set} \arrow[r, shift left, "\mathrm{Set}(-,T)"] & \mathrm{Set}^{op} \arrow[l, shift left, "\mathrm{Set}(-,T)"] \end{tikzcd}$$

Such that the following conditions are satisfied:

(1) There are natural isomorphisms $\eta : I_A \to GF$ and $\varepsilon : I_B \to FG$.

(2) U and V are grounding functors (i.e. faithful functors into the category of sets) and $VF \cong \underline{A}(-,A)$. $UG \cong \underline{B}(-,B)$ with distinguished

objects A and B such that $UA \cong VB \cong T$. (i.e. the functors F and G are given by hom-ing into cogenerating objects which are based on one and the same set).

(3) There are natural monics $\mu_X : FX \to B^{UX}$ and $\nu_Y : GY \to A^{VY}$ such that the diagram

$$\begin{tikzcd}
VFX \arrow[r, "V\mu_X"] \arrow[dd, "\cong"'] & V(B^{UX}) \arrow[r, "\mathrm{nat}"] & (VB)^{UX} \arrow[d, "\cong"] \\
 & & T^{UX} \arrow[d, equal] \\
\underline{A}(X,A) \arrow[r, "U"'] & \underline{Set}(UX,UA) \arrow[r, "\cong"] & \underline{Set}(UX,T)
\end{tikzcd}$$

(with nat denoting the natural morphism $V(\Pi Y_j) \to \Pi VY_i)$ commutes as well as a similar one for ν.

(This says, intuitively, that FX is defined on the set $\mathrm{Hom}(X,A)$ by inheriting the structure from the product B^{UX} (which itself is based on the function set $\underline{Set}(UX,T)$).

(4) Let $e : S \to \underline{Set}(\underline{Set}(S,T),T)$ be the function defined by evaluation $e(s)(f) = f(s)$. Then the diagram

$$\begin{tikzcd}
UX \arrow[r, "U\eta_X"] \arrow[dd, "e"'] & UGFX \cong \underline{B}(FX,B) \arrow[d, "V"] \\
 & \underline{Set}(VFX,VB) \arrow[d, "\cong"] \\
\underline{Set}(\underline{Set}(X,T),T) & \underline{Set}(\underline{A}(X,A),T) \arrow[l, "\underline{Set}(U,T)"]
\end{tikzcd}$$

commutes. A similar condition holds for ν.

(This condition expresses, in categorical terms, the fact that the isomorphism X identifying an object X in

$\underline{A}$ with its double-dual GFX is obtained by evaluation when everything is viewed in terms of functions and sets.)

Let us remark here that these conditions, which in functorial terms describe the distinguishing features of Pontryagin dualities, are by no means artificial. In fact one has the following

PROPOSITION 1. *Let* $\underline{A} \underset{G}{\overset{F}{\rightleftarrows}} \underline{B}^{op}$ *be a pair of functors between categories with products, such that* F *is left adjoint to* G *and that there are objects* A' *in* $\underline{A}$ *and* B' *in* $\underline{B}$ *such that* $X \longmapsto \underline{A}(A',X)$, $Y \longmapsto \underline{B}(B',Y)$ *are faithful set functors* U, *resp.* V. *Then with front adjunction* η *and back adjunction* ε, *conditions* (2), (3), (4) *are satisfied, with* $A = GB'$ *and* $B = FA'$.
(See Hofmann and Keimel [2], Section 0.)

Note that functors U and V which are of this form are called *representable* and that objects with the properties of A' and B' always exist if the categories $\underline{A}$ and $\underline{B}$ have free functors (i.e. left adjoints to the grounding functors into $\underline{Set}$).

We will also say that a duality such as it is described in (1) through (4) is *a Pontryagin duality*.

The original Pontryagin duality between the category $\underline{A}$ of discrete abelian groups and the category $\underline{B}$ of compact abelian groups satisfies these conditions with FX = character group $\underline{A}(X, \mathbb{R}/\mathbb{Z})$ with the compact topological group structure inherited from the inclusion $\underline{A}(X, \mathbb{R}/\mathbb{Z}) \to (\mathbb{R}/\mathbb{Z})^{UX}$, with GT = character group $\underline{B}(Y, \mathbb{R}/\mathbb{Z})$ with the group structure inherited from the inclusion $\underline{B}(Y, \mathbb{R}/\mathbb{Z}) \to (\mathbb{R}/\mathbb{Z})^{VY}$ (where U and V are the "underlying set"-functors and where we (ambiguously) denote the discrete circle group A, the compact circle group B and the underlying set T with the same symbol $\mathbb{R}/\mathbb{Z}$.

The following further dualities are Pontryagin dualities:

(a) Stone duality between Boolean algebras and compact zero dimensional spaces.

(b) Gelfand duality between commutative C*-algebras with identity and compact spaces (although this is not entirely obvious).

(c) The duality between discrete and compact semilattices.

(d) The duality between complete lattices in which every element is a meet of primes and arbitrary sup-preserving morphisms on one hand and the category of spectral spaces on the other (Hofmann and Keimel).

(e) Tannaka duality for compact groups(this again is not obvious).

The duality between the category of all compact topological semigroups and the category of commutative C*-bigebras co-semigroups in the symmetric monoidal category of commutative C*-algebras with identity introduced by Hofmann is not a Pontryagin duality, since neither category has a cogenerator.

The purpose of this note is to point out that the category of Lawson semilattices, contrary to the suspicions of people who have worked in the area, does have a Pontryagin duality. This duality may not be as useful and as applicable as some of the dualities listed above because the dual category of $\underline{L}$ is rather involved; our observation may still serve a useful purpose since it will avoid futile efforts to prove the contrary. In view of the Hofmann duality mentioned above it is known that Lawson semilattices indeed do have a dual category. Let us record this fact in the following

PROPOSITION 2. *The category* E* *of commutative and co-commutative* C*-*bialgebras* Y *with identity, idempotent comultiplication* $c : Y \to Y \otimes^* Y$ *and co-identity* $u : A \to C$ (where idempotency of the comultiplication is expressed in the commutativity of the diagram

$$\begin{tikzcd} Y \arrow[rr, "1_Y"] \arrow[dr, "c"'] & & Y \\ & Y \otimes Y \arrow[ur, "\mathrm{mult}"'] & \end{tikzcd},$$

together with all C^*-bialgebra morphisms is dual to the category of all compact topological semilattices with identity. If $C(I)$ denotes the dual of the interval semilattice in this category, then the full subcategory $\underline{E}^*_I$ of E^* generated by $C(I)$ is dual to the category $\underline{L}$ of Lawson semilattices. The duality is given by the functors $C : \underline{L} \to E^*$ *and* $\mathrm{Spec} : E^* \to \underline{L}$.
(For more details of the Hofmann duality see Hofmann [1], notably pp. 136-139.)

The rest of the note is now concerned with a discussion of

THEOREM 3. *The duality between* $\underline{E}^*_I$ *and* $\underline{L}$ *is a Pontryagin duality.*

Since $\underline{E}^*_I$ does not appear to have a free functor the routine proof which works in many of the examples does not apply. However, due to the special property of having a generator, the category $\underline{E}^*_I$ has a representable grounding functor. The category $\underline{L}$ even has a free functor hence a representable grounding functor. In view of Proposition 3, the hypotheses of Proposition 1 are satisfied; thus Theorem 3 is proved.

Let us see some of the details. Let $2 \in \mathrm{ob}\,\underline{L}$ be the two element semilattice. Then $S \longmapsto \underline{L}(2,S)$ clearly is (equivalent to) the "underlying set" functor U. The grounding functor V of $\underline{E}^*_I$ is given by $VY = \underline{E}^*(C(I),Y)$. Since $C(I)$ generates $\underline{E}^*_I$, then this functor is faithful on $\underline{E}^*_I$, We note $VC(2) = E^*(C(I),C(2)) \cong \underline{L}(2,I) \cong [0,1] \cong \cong UI \cong U\,\mathrm{Spec}\ C(I)$.
Thus, as is to be expected for a Pontryagin duality the objects which serve as the fixed domains for the hom set representation of the duality functors are modelled on one

and the same set, namely the unit interval $[0,1]$. The one in $\underline{L}$ is I, the unit interval semilattice, the one in E_I^* is C(2). Note that the underlying C*-algebra of C(2) is $C \times C$. The monic $\mu_S : C(S) \to C(2)^{L(2,S)}$ of condition (3) is given by $\mu_S(f)(\phi) = f \circ \phi$; similarly $\nu_Y : \text{Spec } Y \to I^{E^*(C(I),Y)}$ is given by $f(\nu_Y(\psi)(\alpha)) = (\psi \circ \alpha)(f)$, $f \in C(I)$, $\psi : Y \to C$ an element of Spec Y, $\alpha : C(I) \to Y$. The verification of the commuting of the diagrams in (3) and (4) (which we know on the basis of the general Proposition 2) is left as an exercise.

The deviation from the more customary Pontryagin duality theories is due to the somewhat unfamiliar grounding functor for the category E_I^*, which is not the underlying set functor, nor the unit ball functor (which is the natural underlying set functor for C*-algebras in the context of Gelfand duality). This accounts for the somewhat curious appearance of the unit interval in the guise of C(2) in E_I^*.

Let us conclude with the remark, that a similar theory holds for semilattices without identity. In that case the generator for the category of Lawson semilattices without identity is 1, the one element semilattice, and the cogenerator for the dual category is $C(1) \cong C$.

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INTRINSIC LATTICE AND SEMILATTICE TOPOLOGIES

Jimmie D. Lawson

I. Intrinsic Topology Functors.

Lattices and semilattices differ from many other algebraic structures in that there are several rather natural ways to define topologies from the algebraic structure. This chapter is devoted to describing several of these constructions and deriving some of their elementary properties. Some of the proofs that are quite straightforward are omitted.

Definition 1.1. A topology $\mathcal{U}$ on a lattice L is intrinsic if $\mathcal{U}$ is preserved by all automorphisms of L, i.e., if $\alpha \in \mathrm{Aut}(L)$ and $U \in \mathcal{U}$, then $\alpha(U) \in \mathcal{U}$.

Proposition 1.2. The following are equivalent for a topology $\mathcal{U}$ on a lattice L:

(1) $\mathcal{U}$ is intrinsic on L;

(2) Each automorphism of L is continuous w.r.t. $\mathcal{U}$;

(3) Each automorphism of L is a homeomorphism w.r.t. $\mathcal{U}$.

Proposition 1.3. The intrinsic topologies are a complete sublattice containing 0 and 1 of the lattice of topologies on L.

Proof. It is immediate that the discrete and indiscrete topologies are intrinsic, that the intersection of any collection of intrinsic topologies is again an intrinsic topology, and that the join of two intrinsic topologies is again an intrinsic topology. Hence the proposition follows.

Definition 1.4. Let $\mathcal{L}$ denote the category whose objects are lattices and whose morphisms are lattice homomorphisms. Let $\mathcal{L}_E$ denote the subcategory of $\mathcal{L}$ consisting of the same objects and those morphisms which are isomorphisms. Let $\mathcal{LT}$ denote the category whose objects are pairs $(L,\mathcal{U})$ where L is a lattice and $\mathcal{U}$ is a topology on L and whose morphisms from $(L_1,\mathcal{U})$ to $(L_2,\mathcal{V})$ are lattice homomorphisms which are continuous. An intrinsic topology for lattices is a functor from $\mathcal{L}_E$ to $\mathcal{LT}$ which assigns to an L in $\mathcal{L}_E$ an object $(L,\mathcal{U})$ in $\mathcal{LT}$ and to a morphism $\alpha:L_1 \to L_2$ the morphism defined by α from $(L_1,\mathcal{U})$ to $(L_2,\mathcal{V})$ in $\mathcal{LT}$. Hence isomorphisms in $\mathcal{L}$ must be continuous $\mathcal{LT}$.

A non-empty subset A of a lattice or lower semilattice is lower complete if every non-empty subset B of A has a greatest lower bound which is again in A. Upper complete subsets are defined dually. A non-empty subset which is both upper and lower complete is complete.

There are several natural definitions of convergence in a lattice.

Definition 1.5. Convergence in lattices.

(1) A net $\{x_\alpha\}_{\alpha \in P}$ in a lattice L is said to be ascending if $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$. An ascending net $\{x_\alpha\}$ is said to ascend (or converge) to x if $x = \sup \{x_\alpha : \alpha \in D\}$. The notions of ''descending net'' and ''descend to x'' are defined dually. The notation is $x_\alpha \uparrow x (x_\alpha \downarrow x)$ if $\{x_\alpha\}$ ascends (descends) to x.

(2) A net $\{x_\alpha\}_{\alpha \in D}$ in a lattice L is said to order converge to x (denoted $x_\alpha \to x$) if there exists nets $\{y_\alpha\}_{\alpha \in D}$, $\{z_\alpha\}_{\alpha \in D}$ such that $z_\alpha \leq x_\alpha \leq y_\alpha$ for all $\alpha \in D$ and $z_\alpha \uparrow x$ and $y_\alpha \downarrow x$.

(3) A net $\{x_\alpha\}_{\alpha \in D}$ is said to order-star converge to x if every subnet of $\{x_\alpha\}$ has a subnet of it which order converges to x. Order-star convergence is denoted $x_\alpha \overset{*}{\to} x$.

(4) For a net $\{x_\alpha\}_{\alpha \in D}$ in a complete lattice L, by definition,

$$\liminf x_\alpha = \bigvee_\alpha \bigwedge_{\beta \geq \alpha} x_\beta ,$$

$$\limsup x_\alpha = \bigwedge_\alpha \bigvee_{\beta \geq \alpha} x_\beta .$$

A net $\{x_\alpha\}$ is said to lower star converge to x if every subnet of $\{x_\alpha\}$ has a subnet of it which has x as its lim inf. Upper star convergence is defined dually.

A non-empty subset A of a lattice or semilattice is lower Dedekind complete if every descending net in A descends to some element of A. Upper Dedekind completeness and Dedekind completeness are defined in the predictable way.

Proposition 1.6. Let L be a lattice.

(1) If $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$, then $x_\alpha \to x$.

(2) If $x_\alpha \to x$, then so does any subnet.

Hence $x_\alpha \to x$ implies $x_\alpha \overset{*}{\to} x$.

(3) If L is complete, then $x_\alpha \to x$ if and only if

$$x = \limsup x_\alpha = \liminf x_\alpha .$$

Proof. Parts (1) and (2) are straightforward. See [6, p. 244] for part (3).

There are two basic methods of defining intrinsic topology functors. The first of these is declaring a set closed if it contains all of its limit points with respect to some convergence criterion. If the convergence criterion satisfies the condition that any convergent net still converges to the same limit point if the domain of the net is restricted to a cofinal subset, then the closed sets defined in this way actually form a topology of closed sets. The four convergence criteria given in Definition 1.5 all satisfy this condition.

Definition 1.7.

(a) The Dedekind topology (D) .

(b) The order topology (O) .

(c) The lower star topology (L_*) .

A subset A of a lattice L is closed in the Dedekind resp. order resp. lower star topology if whenever $\{x_\alpha\}$ is a net in A which ascends or descends resp. order converges resp. lower star converges to x , then $x \in A$.

(d) The chain topology (χ) . A subset A of a lattice L is closed in the chain topology if for all chains C in A , A also contains $\sup C$ and $\inf C$ if they exist.

A second method of defining intrinsic topologies in lattices is by declaring a certain collection of sets defined in order theoretic terms to be a subbasis for the closed (or open) sets.

First however, we need to introduce certain terminology. If A is a subset of a lattice L ,

$$L(A) = \{x \in L: x \leq a \quad \text{for some} \quad a \in A\}$$
$$M(A) = \{y \in L: a \leq y \quad \text{for some} \quad a \in A\} .$$

The non-empty subset A is a semi-ideal if $L(A) = A$ and an ideal if it is both a semi-ideal and a sublattice. A proper ideal I is completely irreducible if whenever I

is the intersection of a collection of ideals, then I is in the collection. Equivalently, I is an ideal which is maximal with respect to not containing some element $x \in L$, $x \neq 0$.

Definition 1.8 (continued).

(e) The interval topology (I). A subbase of closed sets are all sets of the form $L(x)$ and $M(x)$, $x \in L$.

(f) The complete topology (K). A subbase of closed sets is defined by taking as a subbase for the closed sets all sets which contain all inf's and sup's which exist of its non-empty subsets. In complete lattices, these are precisely the complete subsets.

(g) The lower complete topology (LK). A subbase of closed sets is defined by taking all Dedekind closed sets which are lower subsemilattices.

(h) The semi-ideal topology (Σ). A subbase for the closed sets is given by all Dedekind closed semi-ideals together with sets satisfying the dual condition, i.e., $M(A) = A$ and A is Dedekind closed.

(i) The ideal topology (Δ). A subbase for the open sets consists of all completely irreducible ideals and sets which satisfy the dual conditions.

Before defining the last intrinsic topologies, we need to define another notion of convergence.

Definition 1.9. If A is a subset of a lattice L, let $A^{\wedge}$ denote the smallest Dedekind closed lower semilattice containing A. $A^{\vee}$ is defined dually. The net $\{x_\alpha\}$ weakly order converges to x (denoted $x_\alpha \rightarrowtail x$) if

(i) $x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^{\wedge} \subset L(x)$ and dually

(ii) $x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^{\vee} \subset M(x)$.

The net $\{x_\alpha\}$ weakly order star converges to x (denoted $x_\alpha \overset{*}{\rightarrowtail} x$) if every subnet has a subset which weakly order converges to x.

The net $\{x_\alpha\}$ weakly lower star converges to x (denoted $x_\alpha \overset{*}{\hookrightarrow} x$) if condition (i) is satisfied for some subnet of every subnet of $\{x_\alpha\}$. Weak upper star convergence is defined dually.

Definition 1.10.

(j) The weak order topology (WO).

(k) The weak lower star topology (WL_*).

A set A is closed in the weak order resp. weak lower star topology if whenever $\{x_\alpha\}$ is a net in A which weak order star resp. weak lower star converges to x, then $x \in A$.

Proposition 1.11. All the topologies (a)-(k) define intrinsic topology functors.

Definition 1.12. If $(L,\leq)$ is a lattice, the dual lattice is $(L,\geq)$. There is a factor δ on $\mathcal{L}$ which assigns to a lattice the dual lattice and to a morphism the morphism between the dual lattices which as a function is the same as the original function. The functor δ restricted to $\mathcal{L}_E$ is still a functor. A functor δ_T corresponding to δ can be defined on $\mathcal{LT}$ by assigning to $(L,\leq,\mathcal{U})$ the triple $(L,\geq,\mathcal{U})$. Given an intrinsic topology functor Γ, one can define a dual functor Γ' by $\Gamma' = \delta_T\Gamma\delta$. The functor Γ is self-dual if $\Gamma = \Gamma'$.

Proposition 1.13. Let Γ be an intrinsic topology functor which is self-dual. Then any anti-isomorphism between two lattices is continuous.

Proof. Let $\alpha:(L,\leq) \to (L',\leq)$ be an anti-isomorphism. Then $\alpha:(L,\geq) \to (L',\leq)$ is an isomorphism and hence is continuous. Since the topology Γ assigns to $(L,\leq)$ and $(L,\geq)$ are the same, $\alpha:(L,\leq) \to (L',\leq)$ is continuous.

Proposition 1.14. Of the intrinsic topologies (a)-(k), only L_*, LK and WL_* fail to be self-dual. We denote their duals by U_*, UK and WU_* resp.

Definition 1.15. The intrinsic topology functor Γ is finer than the intrinsic topology functor Λ if for each lattice L the topology Γ assigns to L is finer (i.e., has more open sets) than the topology Λ assigns to L. We write

$\Lambda \leq \Gamma$. The relation of ''finer than'' is a partial order on any subset of intrinsic topology functors.

Definition 1.16. An intrinsic topology functor Γ is linear if for any lattice L and any maximal chain M in L , the topology which Γ assigns to L restricted to M is the same as the topology generated by taking the open intervals of M as a basis.

Proposition 1.17. All topologies (a) - (k) except Δ are linear. If Γ is a linear intrinsic topology functor, then $\Gamma \leq \chi$.

Definition 1.18. A topology $\mathcal{U}$ on a lattice L is order compatible if it contains the interval topology and is contained in the Dedekind topology. An intrinsic topology functor Γ is order compatible if $I \leq \Gamma \leq D$.

Proposition 1.19. The intrinsic topology functors (a)-(k) are all order compatible except for Δ and χ .

Proposition 1.20. Let L be a lattice, and let $\{x_\alpha\}$ be an ascending net in L .

(a) If $\mathcal{U}$ is a topology on L courser than the Dedekind topology and if $x_\alpha \uparrow x$, then $\{x_\alpha\}$ converges to x in the topology of $\mathcal{U}$.

(b) If $\mathcal{U}$ is a topology on L finer than the interval topology and $\{x_\alpha\}$ clusters to x in $\mathcal{U}$, then $x_\alpha \uparrow x$.

Proof. (a) First we show that $\{x_\alpha\}$ converges to x in the Dedekind topology. Let U be an open set in the Dedekind topology which contains x. If $\{x_\alpha\}$ is not residually in U, then cofinally it lies in the complement of U. This cofinal collection of $\{x_\alpha\}$ also ascends to x, and since the complement of U is Dedekind closed, $x \in L\backslash U$, a contradiction. Hence $\{x_\alpha\}$ is residually in U. Thus $\{x_\alpha\}$ converges to x in the Dedekind topology, and hence in any coarser topology.

(b) Since $\mathcal{U}$ is finer than the interval topology $M(x_\alpha)$ is closed for each α. If $x \notin M(x_\alpha)$, then the complement of $M(x_\alpha)$ would be an open set containing x such that $\{x_\alpha\}$ is residually not in this open set, an impossibility. Hence $x_\alpha \leq x$ for all α. Now suppose $x_\alpha \leq y$ for all α. Since $L(y)$ is closed and $x_\alpha \in L(y)$ for all α, it follows that $x \in L(y)$, i.e., $x \leq y$. Thus x is the least upper bound, i.e., $x_\alpha \uparrow x$.

Corollary 1.21. Let L be a lattice, $\{x_\alpha\}$ an ascending net in L, $x \in L$, and $\mathcal{U}$ an order compatible topology on L. The following are equivalent.

(1) $x_\alpha \uparrow x$;

(2) $\{x_\alpha\}$ converges to x;

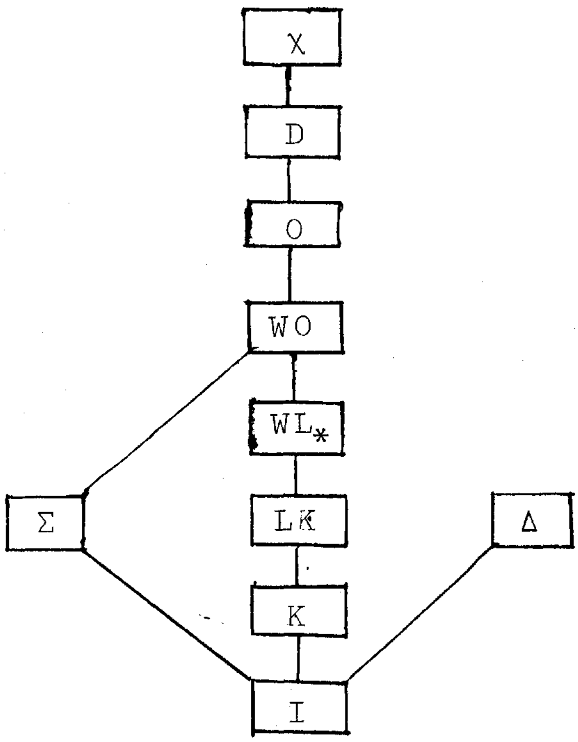
(3) $\{x_\alpha\}$ clusters to x.

Proof. (1) ⇒ (2) . Proposition 19(a).

(2) ⇒ (3) . Immediate.

(3) ⇒ (1) . Proposition 19(b).

Proposition 1.22. The following is a Hasse diagram of the intrinsic topology functors considered for arbitrary lattices (since L_* is defined for complete lattices, it is omitted).



II. Convexity.

Definition 2.1. The convexity functors c, c', σ are functors from $\mathcal{LT}$, the category of lattices with topologies and continuous homomorphisms, back into $\mathcal{LT}$. The functors are defined as follows:

(a) The functor c assigns to an object $(L,\mathcal{U})$ the object $(L,c(\mathcal{U}))$ where $c(\mathcal{U})$ is the topology generated by the open, convex elements of $\mathcal{U}$;

(b) The functor c' assigns to an object $(L,\mathcal{U})$ the object $(L,c(\mathcal{U}))$ where $c'(\mathcal{U})$ is the topology whose closed sets are those generated by the closed, convex sets in $(L,\mathcal{U})$.

(c) The functor σ assigns to an object $(L,\mathcal{U})$ the object $(L,\sigma(\mathcal{U}))$ where $\sigma(\mathcal{U})$ is generated by those open sets of $\mathcal{U}$ which are increasing or decreasing, i.e., those $U \in \mathcal{U}$ such that $M(U) = U$ or $L(U) = U$.

Proposition 2.2. The functor c resp. c' resp. σ is a reflection (categorically) from $\mathcal{LT}$ into the full subcategory of lattices with a basis of open, convex sets, resp. with topology generated by closed, convex sets resp. with topology generated by open, increasing and open, decreasing sets. (For the functor c this means that the following triangle can always be filled in uniquely to be a commutative diagram for any morphism into a locally convex lattice $(M,\mathcal{V})$

$$\begin{tikzcd}
(L, c(\mathcal{U})) \arrow[dr, dashed] & \\
(L, \mathcal{U}) \arrow[u] \arrow[r] & (M, \mathcal{V})
\end{tikzcd} \quad .$$

Similar statements hold for c' and σ .)

<u>Definition 2.3</u>. An intrinsic topology functor Γ is <u>convex</u> resp. <u>closed-convex</u> resp. <u>fully convex</u> if Γ composed with c resp. c' resp. σ is again Γ .

<u>Proposition 2.4</u>. The topology $\sigma(\mathcal{U})$ is coarser than $c(\mathcal{U})$ and $c'(\mathcal{U})$; furthermore $c(\sigma(\mathcal{U})) = c'(\sigma(\mathcal{U})) = \sigma(\mathcal{U})$. Hence if an intrinsic topology functor is fully convex, it is both convex and closed-convex.

<u>Proof</u>. Since increasing and decreasing sets are convex, every member of $\sigma(\mathcal{U})$ will be a member of $c(\mathcal{U})$. The complements of the open increasing or open decreasing sets are closed decreasing or closed increasing sets. Hence the closed sets are generated by closed convex sets, and hence every element of $\sigma(\mathcal{U})$ is one of $c'(\mathcal{U})$. The rest of the proposition follows easily.

<u>Proposition 2.5</u>. The intrinsic topology functors Σ, I, and Δ are all fully convex, hence convex and closed-convex.

<u>Proposition 2.6</u>. Let Γ be an order compatible intrinsic

topology functor. If Γ is convex, then $I \leq \Gamma \leq c(D)$. If Γ is fully convex, then $I \leq \Gamma \leq \Sigma$.

Proof. If $\Gamma \leq D$ and Γ is convex, then $\Gamma = c(\Gamma) \leq c(D)$. Similarly if Γ is fully convex, then $\Gamma \leq \sigma(D)$. By Proposition 1.22 $\Sigma \leq D$; hence $\sigma(D) \geq \sigma(\Sigma) = \Sigma$. On the other hand a subbasic closed set in $\sigma(D)$ is an increasing or decreasing set which is Dedekind closed, and hence in Σ . Thus $\sigma(D) \leq \Sigma$. Hence $\Sigma = \sigma(D)$ and $\Gamma \leq \Sigma$.

III. Complete Lattices.

The main purpose of this paper is to study intrinsic topologies in complete lattices and in compact topological lattices and semilattices. A basic and non-trivial result is the following result of Rennie ([19] or [20]).

Theorem 3.1. For a complete lattice L , $c(\chi) \leq O$. (Note: Rennie denotes the topology $c(\chi)$ by L ; we shall call it the convex order topology. Rennie actually proved this theorem for conditionally complete lattices.)

Corollary 3.2. For a complete lattice L , $c(O) = c(D) = c(\chi)$.

Proof. Since c is a functor on $\mathcal{LT}$, it follows from Proposition 1.22 that $c(O) \leq c(D) \leq c(\chi)$. But since

$c(\chi) \leq O$ and c is a reflection, $c(\chi) \leq c(O)$.

<u>Diagram 3.3</u>. The following is a Hasse diagram for the principal intrinsic topology functors which we have considered for the category of complete lattices. M. Stroble is preparing a master's thesis which contains a much more exhaustive account of relationships between intrinsic topologies. All dominations in the diagram are fairly easy to establish either by straightforward arguments or using earlier results.

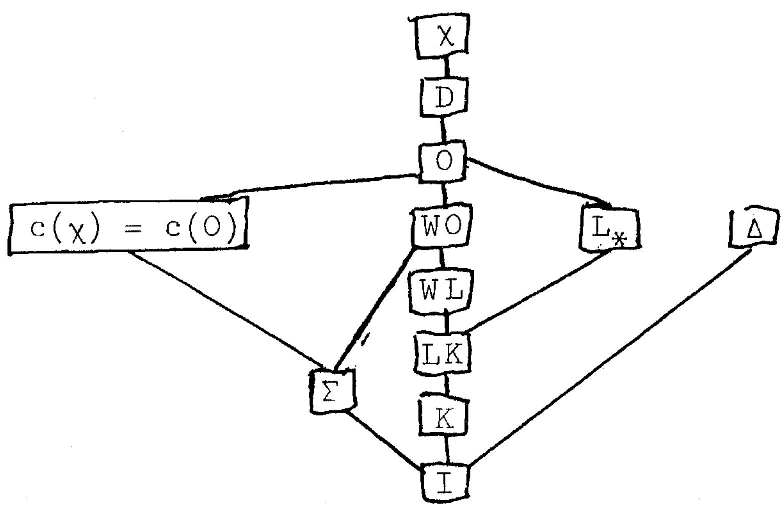


Diagram 3.3

<u>Proposition 3.4</u>. For complete lattices,

$$\Sigma = \sigma(\chi) = \sigma(D) = \sigma(O) = \sigma(WO) = \sigma(WL_*) = \sigma(c(\chi)) .$$

<u>Proof</u>. By Proposition 2.6 we have $\sigma(D) \leq \Sigma$. Since σ is a reflection and Σ is fully convex

$\Sigma = \sigma(\Sigma) \leq \sigma(O), \sigma(WO), \sigma(WL_*), \sigma(c(\chi)) \leq \sigma(D) \leq \Sigma$, which shows all equalities except $\Sigma = \sigma(\chi)$. Now $\Sigma = \sigma(\Sigma) \leq \sigma(\chi) = \sigma\sigma(\chi) \leq \sigma c(\chi) \leq \Sigma$; hence $\Sigma = \sigma(\chi)$.

Because of the extensive collapsing that takes place at $c(O)$ and Σ , these two intrinsic topology functors are of special interest. They are the finest linear topologies that are convex and fully convex resp.

We turn now to consideration of the behavior of these topologies with respect to subspaces, products, and homomorphic images.

A. Subspaces. Most intrinsic topologies of Diagram 3.3 are hereditary for complete sublattices.

Proposition 3.5. Let L be a complete lattice and let M be a complete subset of L . Then for the functors χ, D, O, WO, WL_*, LK, K, L_*, and I the topology assigned to M agrees with the one assigned to L restricted to M . For Σ and $c(O)$ the identity function on M is continuous from the topology assigned to M to the subspace topology and vice-versa for Δ .

Proof. All the verifications are quite straight-forward. For Δ note that an ideal P in M maximal with respect to missing $x \in M$ can be extended to an ideal in L maximal with respect to missing x whose intersection with M is P.

Note that a complete sublattice is closed in the K topology and hence in any finer topology on L.

B. Products. Proposition 3.6. Let $\{L_\alpha : \alpha \in A\}$ be a collection of complete lattices. For all intrinsic topology functors of Diagram 3.3 the identity function from ΠL_α with the topology assigned to it by the functor to the product topology of the topologies assigned to each coordinate is continuous. The interval topology I is productive.

C. Homomorphic images. Continuity in intrinsic topologies we are considering is closely related to the preservation of limits of increasing and decreasing nets.

Definition 3.7. Let L and M be complete lattices. An order-preserving function f from L into M is linearly complete if for any chain C in L $f(\inf C) = \inf f(C)$ and $f(\sup C) = \sup (f(C))$; f is complete if for any $x_\alpha \uparrow x$ and $y_\beta \downarrow y$ we have $f(x_\alpha) \uparrow f(x)$ and $f(y_\beta) \downarrow f(y)$. Note that for the case f is a homomorphism, f is complete if and only if f preserves arbitrary joins and meets.

Proposition 3.8. Let f be an order-preserving function from L to M which is continuous from the Dedekind (chain) topology on L to the interval topology on M. Then f is complete (linearly complete).

Proof. Let $x_\alpha \uparrow x$. By Proposition 1.20(a) $\{x_\alpha\}$ converges to x in the Dedekind topology. Hence $\{f(x_\alpha)\}$ converges to $f(x)$ in the interval topology on M . By 1.20(b) $f(x_\alpha)\uparrow f(x)$. Similarly $x_\alpha \downarrow x$ implies $f(x_\alpha)\downarrow f(x)$. Hence f is complete.

The linearly complete case is analogous.

Proposition 3.9. Let L and M be complete lattices and let f be an order-preserving function from L into M . The following are equivalent:

(1) f is continuous from L into M for the intrinsic topology Γ where $\Gamma = \chi$, D, $c(\chi) = c(D)$, or $\Sigma = \sigma(\chi) = \sigma(D)$;

(2) f is complete;

(3) f is linearly complete.

Proof. Suppose f is continuous for $c(D)$. Then $(L,D) \to (L,c(D)) \xrightarrow{f} (M,c(D)) \to (M,I)$ is continuous. Hence by 3.8 f is complete.

Conversely suppose f is complete. Let U be a basic convex open set in M which has Dedekind closed complement. Then $f^{-1}(U)$ is convex and it follows easily since f is complete $L\setminus f^{-1}(U) = f^{-1}(M\setminus U)$ is Dedekind closed since

$M \setminus U$ is. Hence $f^{-1}(U)$ is open in L. Thus f is continuous from $(L, c(D))$ to $(M, c(D))$. Hence if $\Gamma = c(D)$, (1) is equivalent to (2).

In a strictly analogous manner (1) is equivalent to (3) if $\Gamma = c(\chi)$. But since for complete lattices, $c(\chi) = c(D)$, we have (2) is equivalent to (3).

The proofs that (1) is equivalent to (2) for $\Gamma = D$ and (1) is equivalent to (3) for $\Gamma = \chi$ and $\sigma(\chi) = \Sigma$ follow the pattern of the previous proofs.

Proposition 3.10. Let L and M be complete lattices and f a lower homomorphism (i.e., $f(x \wedge y) = f(x) \wedge f(y)$) from L into M. Then f is complete if and only if f is continuous for the intrinsic topology Γ where $\Gamma = L_*$, WL_*, or LK.

Proof. That f is complete if f is continuous for Γ follows from 3.8 in a fashion analogous to the use of 3.8 in the proof of 3.9.

Conversely, suppose f is complete. For any non-empty subset A of L, let $a = \inf A$. We show $f(a) = \inf f(A)$. Now the set $\{a_1 \wedge \ldots \wedge a_n : n \in \omega, a_i \in A \text{ for } i = 1, \cdots, n\}$ directed by itself descends down to a. Then the image net $\{f(a_1) \wedge \ldots \wedge f(a_n) : n \in \omega, a_i \in A \text{ for } i = 1, \cdots, n\}$ descends to $f(a)$ (by completeness) and to $\inf (f(A))$ (by its definition).

Since f is complete and we have just seen that f preserves arbitrary inf's, it follows easily that f is continuous for L_* since $f(\bigvee_\alpha \bigwedge_{\beta \geq \alpha} x_\beta) = \bigvee_\alpha(f(\bigwedge_{\beta \geq \alpha} x_\beta)) =$

$= \bigvee_\alpha \bigwedge_{\beta \geq \alpha} f(x_\beta)$.

Let B be a lower subsemilattice of Y which is Dedekind closed. Since f is complete $f^{-1}(B)$ is Dedekind closed and since f is a lower homomorphism $f^{-1}(B)$ is a subsemilattice. Hence f is continuous for LK .

To show continuity for WL_* , we first note that the inverse image of a point is a Dedekind closed lower subsemilattice and hence contains its inf . Suppose that the net $\{x_\alpha\}$ weakly lower order converges to x , i.e.,

$$x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^\wedge \subset L(x) .$$

Let B be a Dedekind closed lower semilattice containing residually many of the set $\{f(x_\alpha)\}$. Then $f^{-1}(B)$ contains residually many of the set $\{x_\alpha\}$ and we have just seen that $f^{-1}(B)$ is a Dedekind closed lower subsemilattice. Hence it must be the case $x \in f^{-1}(B)$, and hence $f(x) \in B$. Thus

$$f(x) \in \bigcap_\alpha \{f(x_\beta) : \beta \geq \alpha\}^\wedge .$$

Let A be a Dedekind closed lower semilattice containing residually many of $\{x_\alpha\}$. Since f is a lower homomorphism, $f(A)$ is a lower semilattice. Let $\{y_\gamma\}$ be an ascending resp. descending net in $f(A)$ converging to y. Then $f^{-1}(y_\gamma) \cap A$ is a Dedekind closed lower semilattice, and hence has a least element a_γ. If $y_\gamma \leq y_\delta$, then $f(a_\gamma \wedge a_\delta) = f(a_\gamma) \wedge f(a_\delta) = y_\gamma \wedge y_\delta = y_\gamma$; hence $a_\gamma = a_\gamma \wedge a_\delta$, i.e., $a_\gamma \leq a_\delta$. Thus $\{a_\gamma\}$ is an ascending resp. descending net in A. Since A is Dedekind closed, the limit a of $\{a_\gamma\}$ is in A. Since f is complete $y = f(a) \in f(A)$. Thus $f(A)$ is Dedekind closed.

Now let $b \in \bigcap_\alpha \{f(x_\beta) : \beta \geq \alpha\}^\wedge$.

Then for any α, $f(\{x_\beta : \beta \geq \alpha\}^\wedge)$ is a Dedekind closed lower subsemilattice of M and hence contains $\{f(x_\beta) : \beta \geq \alpha\}^\wedge$, and in particular contains b. Let t_α be the least element of $f^{-1}(b) \cap \{x_\beta : \beta \geq \alpha\}^\wedge$. Then $\{t_\alpha\}$ is an increasing net and increases to some element t. Since $f^{-1}(b)$ is Dedekind closed, $t \in f^{-1}(b)$. Since $\{t_\alpha\}$ is eventually in any set $\{x_\beta : \beta \geq \alpha\}^\wedge$, then $t \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}$. Thus $t \leq x$. Hence $b = f(t) \leq f(x)$. Thus $\{f(x_\alpha)\}$ weakly lower order converges to $f(x)$. From this fact it follows easily that f is continuous for WL_*.

<u>Proposition 3.11</u>. Let L be a complete lattice and f a homomorphism from L into M. Then f is complete if and only if f is continuous for the intrinsic topology Γ where $\Gamma = O$, K, or I.

Proof. That f continuous implies f is complete follows as in 3.9 and 3.10. We saw in 3.10 that a complete lower homomorphism preserves arbitrary meets. Hence f preserves arbitrary joins and meets. It follows easily that the inverse of a closed set is closed with respect to the order topology; hence f is continuous for O.

We also saw in 3.9 that the inverse of a Dedekind closed set is Dedekind closed, and since f is a homomorphism, the inverse of a lattice is a lattice. Hence f is continuous with respect to K.

If $L(y)$ is a subbasic closed set in M with the interval topology, then $f^{-1}(L(y))$ is a Dedekind closed sublattice, and hence has a largest element x. Then $f^{-1}(L(y)) = L(x)$ and hence is closed. Dually $f^{-1}(M(y))$ is closed. Thus f is continuous.

Proposition 3.12. Let f, L and M as in 3.11. If f is a complete homomorphism, then f is continuous for the intrinsic topology Δ.

Proof. Let U in M be a subbasic open set, an ideal maximal with respect to missing y. Then $f^{-1}(U)$ is an ideal maximal with respect to missing x, the least element of $f^{-1}(y)$. Hence f is continuous.

Propositions 3.9 through 3.12 allow one to consider the

IV. Complete Semilattices.

A meet semilattice S is said to be complete if every non-empty subset has a greatest lower bound and if every ascending net ascends to some element of S. For complete semilattices S if $x \in S$ then $L(x)$ is a complete lattice (if $\emptyset \neq A \subset L(x)$, then $\sup A = \inf \{b : a \in A \text{ implies } a \leq b\}$). Hence if S has a 1, S is a complete lattice.

Many of the intrinsic topologies for complete lattices together with their properties transfer to complete semilattices. As a matter of fact the ones which are not self-dual were motivated by the semilattice case. Also the functors c, c', and σ can be defined for the category of complete semilattices.

Proposition 4.1. Let S be a complete semilattice. Then $c(D) = c(\chi)$ on S.

Proof. Since $D \leq \chi$ we have $c(D) \leq c(\chi)$. We show $c(\chi) \leq D$. It will then follow that $c(\chi) = c(c(\chi)) \leq c(D)$, completing the proof.

Let U be a basic convex open set in $c(\chi)$. We show U is open in D by showing that its complement is closed. Suppose $\{x_\alpha\}$ is a net in $S\setminus U$ and $x_\alpha \downarrow x$ where $x \in U$. Let A be a maximal descending i.e., downward directed family containing the set $\{x_\alpha\}$ in $M(x)\setminus U$.

intrinsic topologies within a larger framework. They can be viewed as functors from the category of complete lattices with morphisms complete homomorphisms to the category of complete lattices with a topology and morphisms continuous homomorphisms. We summarize some of the results of this section.

<u>Proposition 3.13</u>. Let f be a homomorphism from a complete lattice L into a complete lattice M.

The following are equivalent:

(1) f is continuous for any intrinsic topology Γ of Diagram 3.3 except Δ;

(2) f is complete;

(3) f is linearly complete;

(4) the inverse of a point has a least and greatest element.

<u>Proof</u>. That (1) and (2) are equivalent follows from 3.9, 3.10, and 3.11. That (2) and (3) are equivalent follows from 3.9. From the proof of 3.10 it follows that (2) implies (4). From the proof that (2) $\Rightarrow$ (1) for the interval topology I, all that was needed was that f satisfy (4). Hence (4) implies (1) for $\Gamma = I$.

<u>Problem 3.14</u>. Given the hypotheses of Proposition 3.13, for which intrinsic topologies is f a closed map?

We note first that $M(A) = A$. For if A is descending, then $M(A)$ is descending. Also if $b \geq a$ for some $a \in A$, then we have $b \geq a \geq x$. Since $a \notin U$ and U is convex, we have $b \notin U$. Hence $M(A)$ is a descending family in $M(x)\backslash U$ which contains A . Since A is maximal, $M(A) = A$.

Secondly we note A is a subsemilattice. For if $a,b \in A$, then since A is descending there exists $c \in A$ such that $a \geq c$ and $b \geq c$. Thus $a \wedge b \geq c$. Since $A = M(A)$, $a \wedge b \in A$.

Thirdly we note that if $p \in M(x)$, then $p \in A$ if $p \wedge a \notin U$ for all $a \in A$. For in this case $(p \wedge A) \cup A$ is a descending set missing U and containing A , and hence must be A by maximality of A .

Now let P be a maximal chain in A , and let $p = \inf P$. Since $A \cap U = \emptyset$ and U is open in $c(\chi)$ and hence χ , we have $p \notin U$. Let $a \in A$. Then by the second note $a \wedge P \subset A$. Since $p = \inf P$, we have $a \wedge p = \inf (a \wedge P)$. But again since U is χ open, $a \wedge p \notin U$. Hence by the third note $p \in A$. Hence by the second note if $b \in A$, then $b \wedge p \in A$. But $b \wedge p \cup M$ is then a chain; thus $b \wedge p \in M$ by maximality of M in A . Thus $b \wedge p = p$ since $p = \inf M$. Hence $p = \inf A$. But $x = \inf \{x_\alpha\} \geq \inf A = p$. Since $p \in M(x)$, $x \leq p$. Hence $x = p$. This is impossible since $x \in U$ and $p \notin U$. Thus if $x_\alpha \downarrow x$, $\{x_\alpha\} \subset S\backslash U$, then $x \in S\backslash U$.

If $x_\alpha \uparrow x$ where $\{x_\alpha\} \subset S \backslash U$, then applying the techniques of the preceding part of the proof to the complete lattice $L(x)$ and the open set $U \cap L(x)$, we obtain that $x \notin U$. Hence U is open in D , which is the needed result to complete the proof.

Diagram 4.2. The following is a diagram of intrinsic topology functors for complete semilattices.

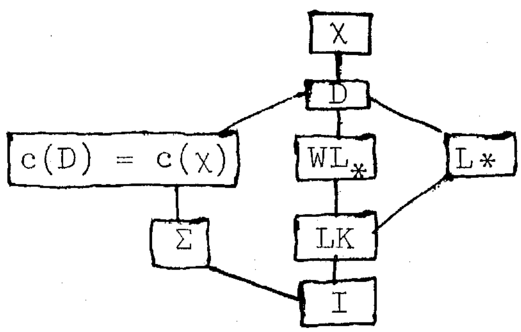


Diagram 4.2

All of these topologies were considered for complete lattices in section 3. Analogous results remain valid for complete semilattices and the proofs require only minor modification. The following two propositions are examples.

Proposition 4.3. Let S and T be complete semilattices and let f be a homomorphism from S into T . The following are equivalent:

(1) f is complete;

(2) f is linearly complete;

(3) f is continuous for the intrinsic topologies Γ assigns to S and T where Γ is any intrinsic topology of

Diagram 4.2 except I .

Proof. The proofs that f being complete is equivalent to f being continuous for D , WL_* , L_* , LK, c(D) or Σ are the same as in section 3; also the same proof holds to show f being linearly complete is equivalent to f being continuous for χ or $c(\chi)$. Since by 4.1 $c(\chi) = c(D)$, we have (2) is equivalent to (1).

Proposition 4.4. Let S be a complete semilattice and T a complete subsemilattice. Then the topology that the intrinsic topology functor Γ assigns to T agrees with the one restricted to T that Γ assigns to S for $\Gamma = WL_*, L_*, LK, D$ and χ .

Proof. Straightforward.

We now define a functor from the category of complete semilattices and complete (semilattice) homomorphisms to the category of complete distributive lattices and complete (lattice) homomorphisms.

For a complete semilattice S let $\mu(S)$ be the set of all non-empty semi-ideals that are Dedekind closed ordered by inclusion. Since the finite union and arbitrary intersection of Dedekind closed semi-ideals is another such, $\mu(S)$ is a complete distributive lattice. If S and T are complete semilattices and f is a complete homomorphism from S into T, define $\mu(f):\mu(S) \to \mu(T)$ by $\mu(f)(A) = L(f(A))$.

Proposition 4.5. The μ defined in the preceding paragraph is indeed a functor from the category of complete semilattices and complete morphisms to the category of complete distributive lattices and complete morphisms.

Before the proof of the theorem, we first establish two lemmas.

Lemma 1. If A and B are semi-ideals in S, then $A \wedge B = A \cap B$.

Proof. Since $A \wedge B \subset A$ and $A \wedge B \subset B$, we have $A \wedge B \subset A \cap B$. Conversely if $x \in A \cap B$, then $x = x \wedge x \in A \wedge B$.

Lemma 2. If A is a Dedekind complete subsemilattice of a semilattice S, then $L(A)$ is Dedekind closed.

Proof. The set $L(A)$ clearly contains limits of descending nets. Let $\{x_\alpha\}$ be an ascending net in $L(A)$, $x_\alpha \uparrow x$. Since A is a subsemilattice and Dedekind complete, $M(x_\alpha) \cap A$ has a least element a_α for each α. Then $\{a_\alpha\}$ is an ascending net which ascends to $a \in A$, since A is Dedekind complete. Then $x \leq a$, and hence $x \in L(A)$.

Proof (of Proposition 4.5). We have seen already that $\mu(S)$ is is a complete distributive lattice if S is a complete semilattice. Let $f:S \to T$ be a complete homomorphism of complete semilattices. Let A be a Dedekind closed semi-ideal in S. Then $f(A)$ is a subsemilattice of T. Since a complete semilattice homomorphism preserves arbitrary meets, $f(A)$ is lower complete. Let $\{y_\alpha\}$ be a net in $f(A)$, $y_\alpha \uparrow y$. For each α, $\exists a_\alpha \in A$ such that $f(a_\alpha) = y_\alpha$. Since A is a semi-ideal the least element b_α of $f^{-1}(y_\alpha)$ is also in A. The net $\{b_\alpha\}$ is increasing, and hence increases to $b \in A$. Since f is complete $y = f(b) \in f(A)$. Thus $f(A)$ is Dedekind closed (and thus Dedekind complete). Hence by Lemma 2 $L(f(A))$ is Dedekind closed. Thus $\mu(f)$ is indeed a function from $\mu(S)$ to $\mu(T)$.

Let A and B be Dedekind closed ideals in S. Then

$$L(f(A \cup B)) = L(f(A) \cup f(B)) = L(f(A)) \cup L(f(B))$$

and using Lemma 1

$$L(f(A \cap B)) = L(f(A \wedge B)) = L(f(A) \wedge f(B)) = L(f(A)) \wedge L(f(B))$$
$$= L(f(A)) \cap L(f(B))\ ; \text{ hence } \mu(f) \text{ is a homomorphism.}$$

Let $\{A_\alpha\}$ be a descending family of Dedekind closed semi-ideals. Then $A_\alpha \downarrow A$ where $A = \bigcap_\alpha A_\alpha$. We have easily that

$$L(f(A)) = L(f(\bigcap_\alpha A_\alpha)) \subset L(\bigcap_\alpha f(A_\alpha)) \subset \bigcap_\alpha L(f(A_\alpha))$$

Conversely let $y \in \bigcap_\alpha L(f(A_\alpha))$. For each α, let a_α

be the least element of A_α such that $y \leq f(a_\alpha)$. For indices α , β , then $f(a_\alpha \wedge a_\beta) = f(a_\alpha) \wedge f(a_\beta) \geq y$. Hence since $a_\alpha \wedge a_\beta \in A_\alpha \wedge A_\beta = A_\alpha \cap A_\beta$, we have $a_\alpha \wedge a_\beta = a_\alpha = a_\beta$. Hence $\{a_\alpha\}$ is a constant net $a \in \bigcap_\alpha A_\alpha = A$. Thus $y \in L(f(A))$.

Now let $\{A_\alpha\}$ be a net increasing to A . Then for all α , $A_\alpha \subset A$ implies $L(f(A_\alpha)) \subset L(f(A))$. Hence $L(f(A))$ is an upper bound. Suppose B is a Dedekind closed semi-ideal in T containing all $L(f(A_\alpha))$. Since f is complete, $f^{-1}(B)$ is Dedekind closed and a semi-ideal which contains A_α for all α . Thus $f^{-1}(B) \supset A$, and hence $B \supset L(f(A))$. Thus $L(f(A))$ is the join of the set $\{L(f(A_\alpha))\}$.

Thus $\mu(f)$ is a complete homomorphism. The other functorial properties for μ follow easily.

V. Algebraically Continuous Operations.

Definition 5.1. Let S be a complete semilattice. Then the meet operation is said to be algebraically continuous if for any $x_\alpha \uparrow x$ and any $y \in S$, then $x_\alpha \wedge y \uparrow x \wedge y$. In this case S is said to be meet-continuous.

One has always that if $x_\alpha \downarrow x$, then $x_\alpha \wedge y \downarrow x \wedge y$, or more generally, if $x_\alpha \downarrow x$ and $y_\beta \downarrow y$, then $x_\alpha \wedge y_\beta \downarrow x \wedge y$.

Proposition 5.2. The meet operation in a complete semilattice S is algebraically continuous if and only if $x_\alpha \uparrow x$ and $y_\beta \uparrow y$ implies $x_\alpha \wedge y_\beta \uparrow x \wedge y$.

Proof. The second condition easily implies the first by taking the constant net consisting of the element y. Conversely, let $x_\alpha \uparrow x$ and $y_\beta \uparrow y$. Then $x \wedge y \geq x_\alpha \wedge y_\beta$ for all choices of α and β. Suppose $z \geq x_\alpha \wedge y_\beta$ for all α, β. If α is fixed, $x_\alpha \wedge y_\beta \uparrow x_\alpha \wedge y$. Hence $z \geq x_\alpha \wedge y$ for all α. But $x_\alpha \wedge y \uparrow x \wedge y$; hence $z \geq x \wedge y$, i.e., $x \wedge y$ is the join of $\{x_\alpha \wedge y_\beta\}$.

Proposition 5.3. The meet operation in a complete lattice is algebraically continuous if and only if $\{x_\alpha\}$ order converges to x and $\{y_\beta\}$ order converges to y implies $\{x_\alpha \wedge y_\beta\}$ order converges to $x \wedge y$.

Proof. See [6, p. 248].

Proposition 5.4. In a complete semilattice (lattice) S the following are equivalent:

(1) S is meet continuous;

(2) For $y \in Y$, the function from S into S which sends x to $x \wedge y$ is continuous for the intrinsic topology Γ;

(3) The meet operation is continuous from $S \times S$ with the Γ topology to S with the Γ topology for the intrinsic topology functor Γ.

For the semilattice case Γ may be chosen to be any topology of Diagram 4.2 except I, and for the lattice case any topology of Diagram 3.3 except K, I or Δ.

Proof. Since the function $x \to x \wedge y$ for a fixed y is a semilattice homomorphism and since one has always $x_\alpha \downarrow x$ implies $x_\alpha \wedge y \downarrow x \wedge y$, the function is a complete homomorphism for every y if and only if S is meet continuous. The equivalence of (1) and (2) now follows from Proposition 4.3 for the semilattice case, and Propositions 3.9 and 3.10 cover all the lattice cases except O. Proposition 5.3 shows that if the lattice S is meet continuous then translation by y is a continuous function in the order topology for each y (show the inverse of a closed set is closed). If translation by y is continuous in the order topology for each y then Proposition 3.8 implies each translation is complete, and hence that S is meet continuous.

The meet operation from $S \times S$ to S is a semilattice homomorphism which satisfies if $(x_\alpha, y_\alpha) \downarrow (x,y)$, then $x_\alpha \wedge y_\alpha \downarrow x \wedge y$. Hence by Proposition 5.2 the meet operation is complete if and only if S is meet continuous. The proof that (1) and (3) are equivalent now parallels the proof that (1) and (2) were equivalent.

Lemma 5.5. Let S be a semilattice endowed with a topology $\mathcal{U}$ for which the functions $x \to x \wedge y$ are continuous for every $y \in S$. If $U \in \mathcal{U}$, then $M(U) \in \mathcal{U}$.

Proof. $M(U) = \bigcup_{y \in U} \{x : x \wedge y \in U\}$; each set in the union is open since translation by y is continuous.

Proposition 5.6. Let L be a complete lattice which is both meet- and join-continuous. Then on L the $c(D)$ and Σ topologies.

Proof. Let U be open and convex in the $c(D)$ topology. By 5.4 the translation functions $x \to x \wedge y$ are continuous for the $c(D)$ topology. Hence by Lemma 5.5 and its dual, $M(U)$ and $L(U)$ are open in $c(D)$. Hence since $\Sigma = \sigma(D) = \sigma(c(D))$, $L(U)$ and $M(U)$ are open in Σ. Since U is convex, $U = L(U) \cap M(U)$ is open Σ. Since continuity always holds in the reverse direction, the proposition is established.

VI. Topological Semilattices and Lattices.

The central and most difficult results of the paper lie in this and the last section. They concern the problem of starting with a compact topology on a semilattice or lattice and trying to identify it as an intrinsic topology.

Definition 6.1. Let S be a semilattice endowed with a topology $\mathcal{U}$. If the function from S into S defined by $x \to x \wedge y$ is continuous for each $y \in S$, then S is a semitopological semilattice. If the meet operation from $S \times S$ with the product topology into S is continuous and S is Hausdorff, then S is a topological semilattice. A lattice

L endowed with a topology $\mathcal{U}$ is a <u>semitopological</u> (<u>topological</u>) <u>lattice</u> if L is a semitopological (topological) semilattice with respect to both the meet and the join operations.

<u>Proposition 6.2</u>. Let $(S,\mathcal{U})$ be a compact Hausdorff semitopological semilattice. Then S is complete and $\mathcal{U}$ is order compatible.

<u>Proof</u>. Let $x \in S$. Then $L(x) = S\wedge\{x\}$ is compact since S is compact and translation is continuous; thus $L(x)$ is closed since S is Hausdorff. Now $M(x) = \{y:y\wedge x = x\}$ is closed since $\{x\}$ is closed and translation by x is continuous. Thus we have the identity function from $(S,\mathcal{U}) \to (S,I)$ is continuous.

Now let $\{x_\alpha\}$ be an increasing net in S. Then $\{x_\alpha\}$ clusters to x for some $x \in S$ since S is compact. By Proposition 1.20(b), $x_\alpha \uparrow x$. A similar result holds for decreasing nets. Hence S is Dedekind complete (and hence complete) and the net $\{x_\alpha\}$ must converge to its least upper bound if increasing and greatest lower bound if decreasing. This implies $(S,D) \to (S,\mathcal{U})$ is continuous, i.e., $(S,\mathcal{U})$ is order compatible.

<u>Proposition 6.3</u>. Let S be a compact Hausdorff first countable semitopological semilattice. If a sequence $\{x_n\}_{n=1}^{\infty}$ clusters to x, then there exists a subsequence for which x is the

lim inf of both the subsequence and any subsequence of the subsequence.

Proof. Let $\{W_i\}$ be a countable base at x. Set $V_0 = W_1$. Pick V_1, an open set, such that $x \in V_1 \subset V_1^* \subset W_1$ and $x \wedge V_1 \subset W_1$. Pick an open set O such that $x \wedge O \subset V_1$. Pick $y_1 = x_{n_1} \in O \cap V_1$. Suppose $\{V_i\}_{i=0}^{k-1}$ and $\{y_i = x_{n_i}\}_{i=1}^{k-1}$ have been chosen satisfying for all $i=1,\cdots,k-1$:

(1) V_i is open;

(2) $x \in V_i \subset V_i^* \subset W_i \cap V_{i-1}$;

(3) $V_i \wedge y_{i-1} \subset V_{i-1}$;

(4) $y_i \in V_i$ and $x \wedge y_i \in V_i$.

Then by regularity there exists an open set U such that $x \in U \subset U^* \subset W_k \cap V_{k-1}$. Since $x \wedge y_{k-1} \in V_{k-1}$, there exists an open set $V_k \subset U$ such that $x \in V_k$ and $V_k \wedge y_{k-1} \subset V_{k-1}$. Pick an open set P such that $P \subset V_k$ and $x \wedge P \subset V_k$. Pick $y_k = x_{n_k} \in P$. Continuing the process inductively one gets a sequence of open sets $\{V_i\}_{i=0}^{\infty}$ and a subsequence $\{y_i\}_{i=1}^{\infty}$ of $\{x_n\}$ satisfying (1)-(4).

Now for positive integers n and k,

$$\begin{aligned} y_n \wedge y_{n+1} \wedge \cdots \wedge y_{n+k} &\in y_n \wedge \cdots \wedge y_{n+k-2} \wedge (y_{n+k-1} \wedge V_{n+k}) \\ &\subset y_n \wedge \cdots \wedge y_{n+k-2} \wedge V_{n+k-1} \\ &\subset y_n \wedge \cdots \wedge V_{n+k-2} \\ &\;\;\vdots \\ &\subset V_n \end{aligned}$$

For a fixed n, $z_{n,k} = y_n \wedge \ldots \wedge y_{n+k}$ is a decreasing sequence. Hence by Proposition 6.2 we have $z_{n,k} \downarrow z_n$ and the sequence $\{z_{n,k}\}$ converges to z_n. Since each $z_{n,k} \in V_n$, we have $z_n \in V_n^* \subset W_n$. If $n \leq m$, then $z_n \leq z_m$ (since $z_n = \bigwedge_{j=n}^{\infty} y_j$ and $z_m = \bigwedge_{j=m}^{\infty} y_j$). Hence $\{z_n\}$ increases to some z, and hence converges to z. Since the sequence is eventually in V_n^* for each n, we have $z \in \bigcap_n V_n^* \subset \bigcap_n W_n$. Hence $z = x$. This shows x is the lim inf of the subsequence $\{y_n\}$. By techniques analogous to those already employed, one shows that any subsequence of $\{y_n\}$ has lim inf in $\cap W_n$; hence the lim inf must be x.

Definition 6.4. If S is a semilattice, the graph of the partial order on S is the set

$$\mathrm{Gr}(\leq) = \{(x,y) \in S \times S\colon x \leq y\}.$$

A basic fact concerning topological semilattices is the following well-known result.

Proposition 6.5. A topological semilattice has closed graph in the product topology.

Proof. Let S be a topological semilattice. Then S is is Hausdorff; so the diagonal Δ of $S \times S$ is a closed set. Define a continuous function $f\colon S \times S \to S \times S$ by

$f(x,y) = (x, x\wedge y)$; then $Gr(\leq) = f^{-1}(\Delta)$ and hence is closed.

Theorem 6.6. Let S be a compact Hausdorff semitopological semilattice. Then S is a topological semilattice (and hence $Gr(\leq)$ closed).

Proof. First we assume S is in addition metrizable. In this case we wish to show that $Gr(\leq)$ is closed. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $Gr(\leq)$ which converges to (x,y) in the product topology of $S \times S$. By Proposition 6.3 there exists a subsequence $\{x_{n_i}\}$ such that

$$x = \bigvee_i \bigwedge_{j\geq i} x_{n_j} .$$

For the subsequence of the $\{y_n\}$ corresponding to the one chosen for $\{x_n\}$, there exists by 6.3 again a subsequence of this subsequence with y as the lim inf . Denote this sub-sub-sequence by $\{y_n'\}$ and the corresponding one for $\{x_n\}$ by $\{x_n'\}$ (by 6.3 the latter still has x as its lim inf). Then

$$x = \bigvee_n \bigwedge_{m\geq n} x_n' \leq \bigvee_n \bigwedge_{m\geq n} y_n' = y .$$

Thus $(x,y) \in Gr(\leq)$. Now by Proposition 7.4 of [16] a compact Hausdorff semitopological semilattice with closed graph is a topological semilattice. This concludes the proof for the case S is metrizable.

The non-metrizable case follows from a reduction to the metric case. The reduction is analogous to that given in Theorem 5.1 of [16]. I am currently preparing for future publication a general reduction technique which will include both cases.

Proposition 6.7. Let S be a complete semilattice (lattice). Then the graph of the partial order is closed in the topology Γ assigns to $S \times S$ for $\Gamma = \chi$, D, L_*, WL_*, and LK($\Gamma = \chi$, D, O, WO, L_*, WL_*, and LK).

Proof. Most of the proofs follow easily from the definition of the topology. Assume $\{(x_\alpha, y_\alpha)\}$ is a net in $Gr(\leq)$ which weakly lower converges to (x,y). Then $x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^\wedge \subset \bigcap_\alpha L(\{y_\beta : \beta \geq \alpha\}^\wedge)$; the latter containment holds because $L(\{y_\beta : \beta \geq \alpha\}^\wedge)$ is lower complete and Dedekind closed (as we saw in Lemma 2 of Proposition 4.5) and contains $\{x_\beta : \beta \geq \alpha\}$. Let z_α be the least element of $\{y_\beta : \beta \geq \alpha\}^\wedge$ which is larger than x. Then $\{z_\alpha\}$ is an increasing net which is eventually in each $\{y_\beta : \beta \geq \alpha\}^\wedge$; thus if $z \in S$ is the point such that $z_\alpha \uparrow z$, then

$$x \leq z \in \bigcap_\alpha \{y_\beta : \beta \geq \alpha\}^\wedge \subset L(y) .$$

Hence $x \leq y$. The case for weak lower star convergence follows from the above by taking subnets.

Note that $Gr(\leq)$ is lower complete, and hence closed in the LK topology.

If $Gr(\leq)$ is closed, then $Gr(\geq)$ is closed for an intrinsic topology since the coordinate reversing function is an automorphism. Hence $\Delta = Gr(\leq) \cap Gr(\geq)$ is closed. If the intrinsic topology is also productive, then of necessity it must be Hausdorff. Since E. E. Floyd [8] has given an example of a non-Hausdorff complete lattice with respect to every linear topology, it follows that any lattice topology in Proposition 6.7 is not productive for this lattice (that the order is productive is incorrectly stated in [9]).

The next proposition is a key tool in identifying a topology as an intrinsic topology. First, however, we introduce some additional notation. If A is a subset of a complete semilattice S, then

$$A^+ = \{x: \text{ there is a net } \{x_\alpha\} \text{ in } A \text{ with } x_\alpha \uparrow x\}$$

and A^- is defined dually.

Proposition 6.8. Let S be a compact topological semilattice and let T be a subsemilattice of S. Then $T^* = T^{-+-+}$. If T is a semi-ideal then $T^* = T^{++}$.

Proof. Since T^* is closed, it is Dedekind closed by Proposition 6.2. Hence $T^* \supset T^{-+-+}$.

Conversely, let $x \in T^*$. Choose by continuity of multiplication a sequence $\{W_n : n \in \omega\}$ satisfying for all n ,

(i) $x \in W_n^\circ$, $W_n = W_n^*$

(ii) $W_n \wedge W_n \subset W_{n-1}^\circ$.

Choose for each n an element $x_n \in W_n \cap T$. By techniques analogous to those in 6.3 we have $z_n = \bigwedge_{i=n}^{\infty} x_i \in W_n$. Hence $z = \bigvee_n z_n = \bigvee_n \bigwedge_{m \geq n} x_n$ is in $\bigcap_{n\in\omega} W_n$. Since T is a subsemilattice, we have $z_n \in T^-$. Hence $z \in T^{-+}$. Thus $(T^{-+}) \cap (\bigcap_{n\in\omega} W_n) \neq \emptyset$.

Now T^{-+} is a subsemilattice since $a_\alpha \downarrow a$, $b_\alpha \downarrow b$ implies $a_\alpha \wedge b_\alpha \downarrow a \wedge b$ (always) and $a_\alpha \uparrow a$ and $b_\alpha \uparrow b$ implies $a_\alpha \wedge b_\alpha \uparrow a \wedge b$ (by joint continuity). Also by condition (ii) $\bigcap_{n\in\omega} W_n$ is a compact semilattice. Hence w , the meet of $(T^{-+}) \cap (\bigcap_{n\in\omega} W_n) \neq \emptyset$ is the limit of a descending net in T^{-+} (and hence is in T^{-+-}) and is in $\bigcap_{n\in\omega} W_n$.

We show that w , the meet of $(T^{-+}) \cap (\bigcap_{n\in\omega} W_n)$, is less than or equal to x . If not, by closed graph, there exists an open set A containing w and an open set B containing x such that if $a \in A$ and $b \in B$, then $a \not\leq b$.

However, when one was choosing all the $\{W_n\}$ in the earlier part of the proof, they could have been chosen so that $W_n \subset B$ for all n. If z was again the lim inf of $\{x_n\}$, each $x_n \in W_n$, then $z \in B$ and $z \in T^{-+}$. Hence $w \leq z$, a contradiction.

Now let D be the set of all sequences $\{W_n : n \in \omega\}$ satisfying (i) and (ii). If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geq \{V_n\}$ if $W_n \subset V_n$ for all n. It is straightforward to verify that $(D, \leq)$ is a directed set. For each $\{W_n\}$, choose w, the meet of $(T^{-+}) \cap (\bigcap_{n \in \omega} W_n)$. This defines an ascending net with all elements in the net below x. Since any closed neighborhood of x can be chosen as a W_1 for some sequence in the net, and the w chosen for this sequence will be in W_1, then eventually the net is in any open set around x. Thus it ascends to x. Hence $x \in T^{-+-+}$. Thus $T^* = T^{-+-+}$.

If T is a semi-ideal, then $T^- = T$. To finish the proof, we show $T^{+-} = T^+$. We actually show T^+ is a semi-ideal. Let $a \in T^+$ and let $x \leq a$. Then there exists an ascending net $\{a_\alpha\}$ in T such that $a_\alpha \uparrow a$. By continuity of the meet operations $a_\alpha \wedge x \uparrow a \wedge x = x$. Since T is a semi-ideal, $a_\alpha \wedge x \in T$ for all α. Hence $x \in T^+$. This concludes the proof.

<u>Theorem 6.9</u>. Let $(S, \mathcal{U})$ be a compact topological semilattice

If $\{x_\alpha\}$ converges to x in $\mathcal{U}$, then $\{x_\alpha\}$ weakly lower converges to x. Conversely if $\{x_\alpha\}$ weakly lower star converges to x, then $\{x_\alpha\}$ converges to x in $\mathcal{U}$. Hence the topology $\mathcal{U}$ is the WL_* topology.

Proof. Suppose $\{x_\alpha\}$ converges to x in $\mathcal{U}$. By Proposition 6.8 for any α, the set $\{x_\beta\colon \beta \geq \alpha\}^\wedge$ is closed in $\mathcal{U}$ (since it is a Dedekind closed semilattice). Hence $x \in \bigcap_\alpha \{x_\beta\colon \beta \geq \alpha\}^\wedge$.

Suppose $y \in \bigcap_\alpha \{x_\beta\colon \beta \geq \alpha\}^\wedge$. If $y \not\leq x$, then there exist by closed graph open sets A and B such that $y \in A$, $x \in B$ and if $a \in A$, $b \in B$, then $a \not\leq b$. There exists an index γ such that if $\alpha \geq \gamma$, then $x_\alpha \in V$, where $x \in V^\circ \subset V^* \subset B$. Then $P = \{x_\alpha\colon \alpha \geq \gamma\}^* \subset B$. Now P is closed, hence compact. Then $L(P)$ is closed [17, p. 44], and hence Dedekind closed. $L(P)$ is also a subsemilattice. If $t \in L(P)$, then there exists $b \in B$ such that $t \leq b$. Hence $t \notin A$. Thus $\bigcap_\alpha \{x_\beta\colon \beta \geq \alpha\}^\wedge \subset \{x_\beta\colon \beta \geq \gamma\}^\wedge \subset L(P)$. But $y \notin L(P)$, a contradiction. Thus $\bigcap_\alpha \{x_\beta\colon \beta \geq \alpha\}^\wedge \subset L(x)$. Hence $\{x_\alpha\}$ weakly lower converges to x.

Conversely, let $\{x_\alpha\}$ weakly lower star converge to x. If $\{x_\alpha\}$ fails to converge to x, then there exists $y \in S$, $y \neq x$ such that $\{x_\alpha\}$ clusters to y. Then a subnet of the $\{x_\alpha\}$ converges to x. Since $\{x_\alpha\}$ weakly lower star converges to x, a subnet of this subnet weakly lower converges to x. But this sub-subnet still converges to y, and hence by the first part of the proof weakly lower converges to y. Since a net can weakly lower converge to a most one point, $x = y$, a contradiction. Thus $\{x_\alpha\}$ converges to x.

<u>Theorem 6.10</u>. Let $(L,\mathcal{U})$ be a compact topological lattice. Then $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ weakly order converges to x. The $c(O)$, WO, WL_*, and Σ topologies agree and are equal to $\mathcal{U}$.

<u>Proof</u>. If $\{x_\alpha\}$ converges to x, then by Theorem 6.9 and its dual it weakly lower converges and weakly upper converges to x. Hence $\{x_\alpha\}$ weakly order converges to x.

Conversely let $\{x_\alpha\}$ weakly order converge to x. Then if $\{x_\alpha\}$ fails to converge to x, some subnet converges to $y \neq x$. Then the subnet weakly order converges to y by the first part of the proof. Hence

$$y \in \bigcap_{\alpha_j} \{x_{\beta_i} : \beta_i \geq \alpha_j\}^\wedge \subset \bigcap_{\alpha_j} \{x_\beta : \beta \geq \alpha_j\}^\wedge \subset L(x) ;$$

similarly $y \in M(x)$. Thus $y = x$, a contradiction.

It now follows immediately that $\mathcal{U} = WO$. By Theorem 6.9, $\mathcal{U} = WL_*$. By Diagram 3.3, $(L,WO) \to (L,\Sigma)$ is continuous. By Proposition 5.6, $\Sigma = c(O)$. By [17, p. 48], since L has closed graph, the closed semi-ideals of L and the closed dual semi-ideals form a subbase for the closed sets. Hence $(L,\Sigma) \to (L,\mathcal{U})$ is continuous, i.e., they agree.

An alternate proof that $\mathcal{U} = c(O)$ may be found in [15].

<u>Problem 6.11</u>. Must $\mathcal{U}$ in 6.10 also be the O-topology?

The characterizations in this section are quite useful in the study of topological semilattices and lattices. They reduce the study to certain algebraic categories with continuous homomorphisms corresponding to complete homomorphisms.

VII. <u>Small Semilattices and Lattices</u>.

An important class of topological semilattices (lattices) are those which possess a basis of neighborhoods at each point which are subsemilattices (sublattices). We say such semilattices (lattices) have small semilattices (lattices). Some of the basic properties of semilattices with small semilattices may be found in [13].

<u>Proposition 7.1</u>. Let $(S,\mathcal{U})$ be a compact topological semilattice with small semilattices. Then the L_*, WL_* and LK topologies agree and are all equal to $\mathcal{U}$. Furthermore a net $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ lower star converges to x.

Proof. We begin with the last assertion first. Let $\{x_\alpha\}$ converge to x in $\mathcal{U}$. Then for any fixed α, let $y_\alpha = \bigwedge_{\beta \geq \alpha} x_\beta$. Since $\{(y_\alpha, x_\beta): \beta \geq \alpha\}$ is a subset of $Gr(\leq)$ for any fixed α and $Gr(\leq)$ is closed, we have $y_\alpha \leq x$ for all α. Given any neighborhood N of x, there exists a neighborhood M of x such that $M^* \subset N$ and M is a subsemilattice. There exists an index γ such that $x_\alpha \in M$ for $\alpha \geq \gamma$. Since M is a subsemilattice all finite meets of the set $\{x_\alpha: \alpha \geq \gamma\}$ are again in M; hence $y_\beta \in M^*$ for $\beta \geq \gamma$. Then $\{y_\alpha\}$ is eventually in any open set around x, and so must ascend to x. Thus x is the lim inf of the net $\{x_\alpha\}$.

Conversely, let $\{x_\alpha\}$ lower star converge to x. If $\{x_\alpha\}$ clusters to y, then there is a subnet which converges. By the first of this proof any subnet of this subnet must have y for its lim inf. Hence $y = x$, and thus $\{x_\alpha\}$ converges to x.

It follows easily from what we have just shown that $\mathcal{U} = L_*$. By 6.9 $\mathcal{U} = WL_*$.

We show $\mathcal{U} = LK$ by showing LK is Hausdorff; this will be sufficient since $(S,\mathcal{U})$ is compact and $(S,\mathcal{U}) = (S,L_*) \to (S,LK)$ is continuous (Diagram 4.2).

Let $x,y \in S$, $x \neq y$. We may assume $x \nleq y$. Since S has small semilattices, there exists $z \leq x$, $z \in S \backslash L(y)$, such that $x \in M(z)^{\circ}$. Then $M(z)$ and $(S\backslash M(z))^{*}$ are lower complete sets. Their complements are open sets in LK separating x and y .

<u>Proposition 7.2</u>. Let $(L,\mathcal{U})$ be a compact topological lattice such that each point has a basis of neighborhoods which are sub-lattices. Then $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ order converges to x . Furthermore $\mathcal{U} = O = I$ and all topologies in between O and I in Diagram 3.3.

<u>Proof</u>. If $\{x_\alpha\}$ converges to x in $\mathcal{U}$, then by the first part of the proof of 7.1 and its dual, $\{x_\alpha\}$ order converges to x .

Conversely suppose $\{x_\alpha\}$ order converges to x . Then $\{x_\alpha\}$ converges to x in the order topology and hence converges to x in the WO topology (Diagram 3.3) which is the $\mathcal{U}$ topology (6.10). Hence it follows that $\mathcal{U}$ is the order topology.

Again we complete the proof by showing I is Hausdorff. Suppose $\{x_\alpha\}$ is a net in L . Then since L is compact Hausdorff, some subnet converges to some x , and hence by the first part of the proof order converges to x . By a result of K . Atsumi [4, Theorem 3] L with the interval topology is Hausdorff.

We now turn our attention to the converse problem. We wish to postulate algebraic conditions which will be sufficient to insure that a semilattice admits a topology with small semilattices. First, however, we give a preliminary result concerning compactness. This generalizes results of Frink, who showed the interval topology was compact in a complete lattice [9], and Insel, who showed the complete topology was compact in a complete lattice [11].

Proposition 7.3. Let S be a complete semilattice. Then S with the LK topology is compact.

Proof. Let $\{A_\alpha\}$ be a collection of Dedekind closed lower subsemilattices of S with the finite intersection property. For each finite subset $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ pick the least element of $\bigcap_1^n A_{\alpha_i}$. With the finite subsets ordered by inclusion, these least elements form an ascending net and hence ascend to some element a. Since each A_α is Dedekind closed, $a \in \bigcap_\alpha A_\alpha$. Since the Dedekind closed lower subsemilattices form a subbase for the closed sets of S, by Alexander's lemma, L is compact.

We are now ready for a converse to Proposition 7.1.

Proposition 7.4. Let S be a complete semilattice in which the meet operation is algebraically continuous and the

LK topology is Hausdorff. Then S with the LK topology is a compact topological semilattice with small semilattices.

Proof. By 7.3 S is compact. By 5.4 the meet operation is separately continuous for LK. Hence by 6.6 S is a topological semilattice.

We show now that S has small semilattices. We first consider the case that S has a largest element 1, and show S has small semilattices at 1. Let U be an open set, $1 \in U$. Since $Gr(\leq)$ is closed, by a result of Nachbin [17], there is a convex open set V with $1 \in V \subset U$. Then $A = S \backslash V$ is compact and decreasing.

Since the LK topology is Hausdorff, for each $a \in A$, there exist basic open sets P_a and Q_a in the LK topology with $a \in P_a$, $1 \in Q_a$, and $P_a \cap Q_a = \emptyset$. P_a is the complement of finitely many complete subsemilattices. Finitely many of the $\{P_a : a \in A\}$ cover A, say $A \subset \bigcup_1^n P_i$. Let $Q = \bigcap_1^n Q_i$. For each P_i, let $S \backslash P_i = S_{i,1} \cup \ldots \cup S_{i,m_i}$ be the representation of the complement of P_i in terms of complete subsemilattices. Consider all possible sets of the form $S_{1,j_1} \cap \ldots \cap S_{n,j_n}$ where $1 \leq j_i \leq m_i$ for each i. Each such intersection is a subset of V and the union of all such intersections contain Q. Since there are only finitely many such intersections and each such is closed, some such intersection, call it T, must have an interior. Since T is a complete subsemilattice T has a least element t.

By continuity of the meet operation, $M(T^\circ)$ is an open set containing 1. Note that $M(t) \supset M(T^\circ)$; hence $M(t)$ is a neighborhood of 1. Since A is decreasing, $M(t) \subset V$. Since $M(t)$ is a subsemilattice, S has small semilattices at 1.

Now let $x \in S$. It follows easily that the LK topology restricted to $L(x)$ agrees with the LK topology on $L(x)$. Since x is the largest element of $L(x)$, it follows from the first part that $L(x)$ has small semilattices at x. By [13] this implies S has small semilattices at x.

Proposition 7.5. Let L be a complete lattice in which the meet and join operations are algebraically continuous and the complete topology K is Hausdorff. Then L equipped with the complete topology is a compact topological lattice with a basis of sublattices.

Proof. Since LK is compact, K is Hausdorff, and $(L,LK) \to (L,K)$ is continuous, the K and LK topologies agree. Hence by 7.4 and its dual L is a compact topological lattice with a basis of subsemilattices with respect to each operation.

Let $x \in L$, and U an open neighborhood of x. Let V be an open, convex set such that $x \in V \subset U$. Then there exists a lower subsemilattice T such that $x \in T^{\circ} \subset T \subset T^{*} \subset V$. Then T^{*} will be a compact lower subsemilattice and hence have a least element t. Let P be an upper subsemilattice such that $x \in P^{\circ} \subset P^{*} \subset T^{\circ}$. Let p be the greatest element of P^{*}. Then $x \in P^{\circ} = P^{\circ} \cap T^{\circ} \subset L(p) \cap M(t) = [t,p] \subset V$, the last inclusion holding since V is convex. Then $[t,p]$ is a sublattice, a neighborhood of x, and a subset of U. Hence L has a basis of sublattices.

Propositions 7.4 and 7.5 would be significantly improved if it were possible to find a ''reasonable'' algebraic condition to replace the hypothesis that LK or K be Hausdorff.

VIII. Comments - Historical and Otherwise

Intrinsic topologies in lattices first appeared with G. Birkhoff's definition of the order topology in the late 1930's [5]. Shortly thereafter O. Frink [9] introduced the interval topology and studied basic properties of the order and interval topologies.

Interest revived in intrinsic topologies in the middle 50's with the work of B. C. Rennie [19, 20], Frink's introduction of the ideal topology [10], the work of A. J. Ward [23] and E. E. Floyd's examples of lattices with pathological intrinsic topologies [8]. Rennie's work contains germs of several of the developments pursued here.

About this time A. D. Wallace initiated interest in topological lattices [22] and early investigations in this area were carried out by L. W. Anderson [1, 2, 3] in the late 50's. During this same period E. S. Wolk introduced the concept of order compatible topologies [24], and T. Naito gave a necessary and sufficient condition for all such topologies to be identical in a complete lattice [18].

In the 60's A. J. Insel introduced and studied the complete topology [11, 12]. D. Strauss [21] appears to be the first to investigate intrinsic topologies in compact topological lattices. Some additions were given by T. H. Choe in [7]. Recently I had shown that any compact topological lattice has the $c(0)$ topology [15]. An implicit algebraic characterization of the topology of a compact topological semilattice is also included.

A problem of recurring interest in intrinsic topologies relates to the Hausdorffness of certain topologies, in particular the interval topology. Ward [23] and K. Atsumi [4] for instance treat this latter problem. Floyd's example [8] shows that the order topology may fail to be Hausdorff. Insel [12] gave necessary and sufficient conditions for the complete topology to be Hausdorff. Strauss [21] characterized those compact topological lattices in which the interval topology is Hausdorff. Propositions 7.2 and 7.5 are essentially her results. Recently I published an example of a compact distributive topological lattice in which the interval topology is not Hausdorff [14].

The preceding is by no means an exhaustive account of the work in intrinsic topologies, but rather should be considered as a background out of which this paper grew.

There are several directions for future investigation. The Hasse diagram of the relation between the various intrinsic topologies needs to be rigorously verified for the following classes; complete lattices and semilattices, complete algebraically continuous lattices and semilattices and compact topological semilattices and lattices. A complete list of counter-examples even for Diagram 3.3 to show it is the best possible is not known. Other interesting classes in which to study intrinsic topologies might be vector lattices and equationally compact semilattices and lattices. A complete semilattice S inherits many topologies as a subspace of the complete lattice $\mu(S)$ (see Section 4). How do these relate to the topologies already given to S directly?

The ideal topology has been frequently ignored in the considerations of this paper. In particular, what can be said about it in compact topological lattices?

The considerations of this paper may be generalized to arbitrary lattices in a variety of ways. Many of the functors considered are already defined for all lattices. Another method of extension of an intrinsic topology functor defined on complete lattices is to take the completion by cuts of an arbitrary lattice and give the lattice the subspace topology. Alternately, one may declare a set open if and only if its intersection with each complete sublattice is open with respect to some intrinsic topology functor Γ . Most of even the basic properties of these extensions remain unexplored.

Finally, the definition of intrinsic topology given here (automorphisms are continuous) is somewhat artificial. A precise definition of intrinsic topologies in terms of generating a topology from the algebra needs to be given in logic and set language and basic properties of such topologies explored.

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ON THE DUALITY OF SEMILATTICES AND ITS APPLICATIONS TO LATTICE THEORY

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This article reports on a monograph in which the authors discuss the duality between the category $\underline{S}$ of semilattices with identity and identity preserving morphisms on one hand and the category $\underline{Z}$ of compact zero dimensional topological semilattices with identity and identity preserving continuous morphisms.

In itself, this duality theory is not new. Various authors discovered the duality on objects some time ago and the full duality theory itself together with various ramifications was described in the context of other duality theories by Hofmann. **The** duality theory for discrete and compact abelian groups was introduced by Pontryagin with the express purpose of immediate applications to algebraic topology. It was soon applied in group theory, topology and analysis. Thus it became fruitful by producing results in either of two directions: from the discrete theory to the topological one and indeed also vice versa. By contrast, the duality of semilattices has not been noticed as a vehicle for applications at all. We hope to demonstrate that it, too can have useful applications to discrete and topological lattice theory and to the theory of compact semilattices as a part of compact abelian semigroup theory.

I. As a first step we set apart a chapter describing basic functorial properties of the categories $\underline{S}$ and $\underline{Z}$ such as their completeness, cocompleteness, their having biproducts, and the existence of free functors (i.e. left adjoints for the obvious grounding functors into the category Set of sets). We then give a proof of the duality theory which is based on a fairly general, yet useful functorial device which e.g. has been applied recently by Roeder to give a new proof of the self duality of locally compact abelian groups. This proof is based on some generalities on functorial density and continuous (i.e. limit or colimit preserving functors) which we describe in a preliminary chapter, preceding Chapter I, which in itself does not refer to semilattices. The proof of the duality theorem then proceeds as follows: We show that the category F of finite semilattices is co-dense in S and dense in Z. It is very elementary to show that F is naturally dual to itself. Then we push the button and the functorial machinery yields the desired duality. The advantage is that this method allows generalizations beyond the application we have in mind. Alternative proofs of the duality are available in the literature.

II. In the second chapter we view the duality theory as an instance of a character theory, thereby exhibiting its closeness to Pontryagin duality theory for abelian groups. This requires that we first give a description of the category Z from the view point of compact topological semigroups. We record a characterization theorem for zero dimensional compact semilattices known to semigroupers for some time, in which the existence of small semilattices, the existence of sufficiently many ultra-pseudometrics, and the separation of points by characters (and some other properties) are used to characterize the objects of Z. We introduce the concept of a local minimum $m \in S$ and give different semigroup theoretical characterizations: Indeed m is a local minimum iff $\{m\}$ is isolated in Sm iff $\uparrow m$ (the set $\{s \in S \mid sm = m\}$) is open. Further $m \in S$ is called a strong local maximum iff there is a local minimum $n \in S$ such that m is maximal in the ideal $S \setminus \uparrow n$. We observe that the set of local minima is dense in S and in fact even in every principal ideal Ss, and that the set of strong local maxima is dense.

In the second part of the chapter we correlate the concepts of characters and filters; *a character* of S is a morphism $S \to 2$ (in $\underline{S}$, respectively $\underline{Z}$), a filter $F \subseteq S$ is a subsemilattice such that $s \in F$ implies $\uparrow s \in F$. Since a function $f : S \to 2$ for a discrete S is a character iff $f^{-1}(1)$ is a filter, we have the following.

PROPOSITION. *The character semilattice* $\hat{S}$ *of a discrete semilattice* S *is naturally isomorphic to the filter semilattice* $\mathcal{F}(S)$ *under intersection as operation.*

The search for a concrete realization of the character semilattice of a $T \in \text{ob}\ \underline{Z}$ is a bit more involved. Firstly we observe that the underlying semilattice of T is in fact a complete lattice. We then prove the following

PROPOSITION. *Let* $k \in T$, *where* T *is a compact zero dimensional semilattice. Then the following statements are equivalent:*

(1) k *is a local minimum of the topological semilattice* T.

(2) k *is a compact element of the underlying complete lattice.*

We denote the sup-semilattice of all compact elements of a semilattice T by $K(T)$; recall that an element k of a semilattice T is *compact* iff $k \le \sup X$ for some $X \subseteq T$ implies the existence of a finite subset $F \subseteq X$ with $k \le \sup F$. For each $k \in K(T)$ there is precisely one $\underline{T}$-character $f : T \to 2$ with $k = \min f^{-1}(1)$, and each $\underline{T}$-character is so defined.

PROPOSITION. *The character semilattice* $\hat{T}$ *of a compact zero dimensional semilattice* T *is naturally isomorphic with the* (sup) *semilattice* $K(T)$ *of compact elements of the underlying lattice of* T.

By our earlier observation we know that for $T \in \text{ob}\ \underline{Z}$ the set of local minima, hence $K(T)$ is dense in every principal ideal Tt of T. This rather directly implies that the underlying lattice of T is *algebraic*, i.e. is a complete lattice in which every element is a sup of the elements in $K(T)$ which it dominates. We prove, conversely, that every algebraic lattice has a unique compact zero dimensional semilattice topology relative to which $K(T)$ is the set of local minima. Since it is not hard to see that a semilattice morphism $T \to T'$ between algebraic lattices is continuous relative to these topologies iff it is an order continuous lattice morphism, i.e. iff arbitrary infs and sups of upward directed sets are preserved, we have the following

THEOREM. *The category $\underline{Z}$ of compact zero dimensional semilattices and continuous identity preserving semilattice morphisms is isomorphic to the category of algebraic lattices and order continuous semilattice morphisms (and this latter category is then dual to the category $\underline{S}$ of discrete semilattices and identity preserving semilattice morphisms).*

If we call a lattice T *arithmetic* if it is algebraic and if in addition $K(T)$ is a sublattice, we have the

COROLLARY. *The category of lattices with identity and identity preserving semilattice morphisms is dual to the category of arithmetic lattices and order continuous semilattice morphisms.*

III. The third chapter contains various applications of the duality theory to lattice theory. We begin with a preliminary section in which we record simple consequences of the duality, such as e.g. the following: If $f \in \underline{S} \cup \underline{Z}$, then $\hat{f}$ is injective iff f is surjective. A family $S \to S_j$ of morphisms is a product diagram in one of the two categories iff the family $\hat{S}_j \to \hat{S}$ is a coproduct diagram in the other. (In fact this holds for arbitrarily limits and colimits). Quotients are dual concepts for subobjects.

We proceed to discuss concepts which are of key importance in lattice theory.

A fundamental role is played by the prime elements in a semilattice. We say that $p \in S$ is *prime* iff $ab \leq p$ implies $a \leq p$ or $b \leq p$, and we call the set of primes Prime S. We say that S is *primally generated* iff Prime S generates S (in either $\underline{S}$ or $\underline{Z}$; note that $T \in \mathrm{ob}\, \underline{Z}$ is generated by $A \subseteq T$ iff T is the smallest closed subsemilattice of T containing A). A morphism $f : S \to T$ between semilattices will not automatically preserve primes; if indeed we have $f(\text{Prime } S) \subseteq \text{Prime } T$, then we call f *a prime-morphism*. A prime-morphism into 2 is a *prime-character*. A *prime filter* is a prime element in the filter semilattice.

In a semilattice finite sups need not exist. Nevertheless, various concepts of distributivity are possible. We say that a semilattice is distributive iff $\uparrow a(\uparrow b \cap \uparrow c) = \uparrow ab \cap \uparrow ac$ for all a, b, c. We say that a morphism $f: S \to T$ is a *sup-morphism* iff $f^{-1}(Q)$ is a prime filter in S for every prime filter Q of T. These morphisms do preserve existing sups if the prime filters of T separate the points. Thus all *sup-characters* of S (i.e. sup-morphisms $S \to 2$) preserve existing sups. The duality theory sheds light on the mutual relation of these concepts:

THEOREM. *A morphism $f \in \underline{S} \cup \underline{Z}$ is a prime morphism iff its dual $\hat{f}$ is a sup-morphism. If $S \in \text{ob } \underline{S}$ and $T \in \text{ob } \underline{Z}$ is its dual then the following statements equivalent:*

(1) S is a distributive semilattice. (2) The sup-characters of S separate the points. (3) S is a subsemilattice of a distributive lattice (such that the inclusion preserves sups). (4) T is primally generated. (5) T is a distributive lattice. (6) T is a Brouwerian lattice.

Further, the following statements are equivalent:

(i) S is primally generated. (ii) T is a topological distributive lattice. (iii) The lattice characters of T separate the points. (iv) $K(T)$ is primally generated.

Finally, the following are equivalent:

(I) S is a distributive lattice. (II) T is an arithmetic distributive lattice.

At this point we can easily tie in results of other duality theories which are exemplified by recent results of Keimel and Hofmann (Memoir of the Amer. Math. Soc. 122 (1972)). We exemplify the amalgamation of these two theories by the following

THEOREM. *The subcategory in Z of distributive lattices and lattice morphisms is dual to the category of continuous maps between topological spaces* X *having the following properties*:

(a) X *is a* T_o*-space in which every irreducible set is a singleton closure*. (A set Z is irreducible in X if it is closed and not contained in the union of two closed subsets unless at least one of the two contains Z.)

(b) X *has a basis of quasicompact open sets* (i.e. every open set is the union of the quasi-compact open subsets which it contains).

Thus the category of these spaces is equivalent to the category of distributive semilattices and prime morphisms.

Remark. The spaces described in (a) and (b) have been called spectral spaces since they occur, e.g., as the spectra of commutative rings.

In a subsequent section we proceed to discuss Boolean lattices in *S* and in *Z* (a Boolean lattice in *Z* is a Boolean topological lattice and as such is equivalent to a compact topological Boolean algebra). Recall that a semilattice in S is free (over *Set*) iff it is the $\cup$-semilattice of all finite subsets of some set X. We denote such a semilattice by $^X2$ (since indeed it is the coproduct of X copies of 2). The category *Z* has a free functor from the category *ZComp* of compact zero dimensional spaces (which is left adjoint to the forgetful functor). It associates with a space $X \in$ *ZComp* the $\cup$-semilattice C(X) of all closed subsets of X with the Hausdorff topology. We say that such a semilattice is *free over ZComp*. We have the

THEOREM. *Let* $S \in$ ob *S* *and* $T \in$ ob *Z* *its dual*. *Then*

(a) S *is a Boolean lattice iff* T *is free over ZComp*

(b) S *is free over* *Set* *iff* T *is a Boolean topological lattice*.

In particular, the compact topological Boolean lattices are precisely the 2^X *for some set* X.

A morphism $f \in$ *S* $\cup$ *Z* *between Boolean objects in* _either category is a Boolean morphism_ (i.e. preserves complements) _iff its dual_ $\hat{f}$ _is co-atomic_ (i.e. maps all co-atoms in its domain into the set of co-atoms of the co-domain.) (A _co-atom_ a is an element which is maximal relative to the property $a < 1$.).

In a further section we complement the work of Kimura and Horn about the injectives and projectives in S.

The results are as follows:

THEOREM. Let $S \in \text{ob } \underline{S}$ _and_ $T \in \text{ob } \underline{Z}$ _be its dual. Then the following are equivalent._
(1) S _is projective in_ $\underline{S}$ (2) S _is a retract of some_ $^{X}2$
(3) S _is a distributive lattice with_ $\uparrow s$ _finite for all_ $s \in S$ (4) S _is primally generated and_ $\uparrow p$ _is finite for all_ $p \in \text{Prime } S$ (5) T _is injective in_ $\underline{Z}$ (6) T _is a retract of some_ 2^X (7) T _is a distributive arithmetic lattice such that_ Tk _is finite for all_ $k \in K(T)$ (8) T _is a distributive arithmetic and topological lattice such that_ Tk _is finite for all_ $k \in K(T)$.
Furthermore, the following conditions are equivalent:
(i) S _is injective in_ $\underline{S}$. (ii) S _is a retract of a complete Boolean lattice_. (iii) S _is a complete Brouwerian lattice_. (iv) T _is projective in_ $\underline{Z}$. (v) T _is a retract of some free object (over_ _Set_). (vi) T _is a retract of some_ $C(E)$ _with an extremally disconnected space_ E.

IV. In a final chapter we discuss application of duality theory to the theory of compact semilattices. A portion of this is presented in another contribution (K. H. Hofmann and M. Mislove, Stability in compact zero dimensional semilattices). As an example of material not presented at this conference which will be discussed in detail in the monograph let us mention the following results. If X is a topological space we may associate with it two cardinals, its weight $w(X) = \min \{a \mid$ there is a basis for the topology of X of cardinality $a\}$ and its density character $d(X) = \min \{a \mid$ there is a dense subset of X of cardinality $a\}$. These cardinals in a sense describe the size of the space X. We then have the following

THEOREM. *Let* T *be a compact zero dimensional semilattice and* S *its dual semilattice. Then* $w(T) = \text{card } S \leq 2^{d(T)}$. *In fact if, for a cardinal* a *we let* log a *denote the smallest cardinal* b *with* $a \leq 2^b$, *then* $d(T) = \log \text{card } S$.

We also use duality to characterize extremally disconnected objects in $\underline{Z}$:

THEOREM. *Let* T *be a zero dimensional compact semilattice. Then the following are equivalent statements.*

1) T *is extremally disconnected.*
2) *Every converging sequence is finally constant.*
3) T *satisfies the ascending chain condition and for each* t *the set of minimal elements in* $\uparrow t \setminus \{t\}$ *is finite.*
4) T *is finite.*

An account of the history of the subject and detailed references are to follow in the complete presentation of the material indicated in this report.

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Distributive Topological Lattices

(Dedicated to L. W. Anderson)

by

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It is our purpose to discuss some recent developments in the theory of distributive topological lattices. As is usual in such discussions the topics most interesting to the author are those in which he has made some contributions. We shall carry forward that hallowed tradition. L. W. Anderson, to whom this paper is dedicated, gave a survey of the theory of topological lattices in 1961 [1]. We shall take that survey as a foundation for our subsequent remarks.

We begin with some results about lattices of (semilattice) ideals of compact semilattices. Aside from the intrinsic value of such lattices we begin our discussion here because such lattices provide us with examples and counter-examples needed in later sections. We then construct representations for compact, distributive lattices of finite breadth. This topic leads naturally to questions involving compactification of lattices which will be discussed in section 3. We conclude with some remarks about the congruence extension property for compact lattices.

Let $\mathcal{P}$ be the category of Hausdorff topological spaces equipped with closed partial orders. The morphisms of $\mathcal{P}$ will be continuous order-preserving maps. For $(S,\leq)$ in $\mathcal{P}$ and $x,y \in S$, $x \vee y = \text{l.u.b.}\ \{x,y\}$ and $x \wedge y = \text{g.l.b.}\ \{x,y\}$ where they exist. $\mathcal{S}$ is the subcategory of $\mathcal{P}$ consisting of those objects S for which $x \wedge y$ exists for all $x,y \in S$ and the map $\wedge: S \times S \longrightarrow S$ is continuous. The morphisms of $\mathcal{S}$ will be those $\mathcal{P}$-morphisms which in addition preserve $\wedge$. $\mathcal{L}$ will be that subcategory of $\mathcal{S}$ consisting of those objects L for which $x \vee y$ exists for all $x,y \in L$ and the map $\vee: L \times L \longrightarrow L$ is continuous. The morphisms of $\mathcal{L}$ will be those $\mathcal{S}$-morphisms which also preserve $\vee$. $\mathcal{DL}$ will be the full subcategory of distributive lattices in $\mathcal{L}$. By $\mathcal{CS}$ we shall mean the full subcategory of compact semilattices, $\mathcal{CL}$ and $\mathcal{CDL}$ are defined accordingly. For a lattice L, $J(L)$ will be the set of join-irreducible elements of L and $M(L)$ will be the set of meet-irreducible elements of L.

1. <u>The lattice of ideals of a compact semilattice.</u>

Suppose that S is an object of $\mathcal{CS}$. We define $\mathcal{M}(S)$ to be the set of all closed (semilattice) ideals of S i.e. closed subsets A of S such that if $a \in A$, $s \in S$ and $s \leq a$ then $s \in A$. When $\mathcal{M}(S)$ is ordered by set-theoretic inclusion and endowed with the Vietoris topology it becomes a compact, distributive topological lattice. There is a natural imbedding $\rho_S: S \longrightarrow \mathcal{M}(S)$ defined by $\rho_S(s) = s \wedge S$. (cf. [7]). For $f: S \longrightarrow T$ a morphism in $\mathcal{CS}$ we define $\mathcal{M}(f)\, \mathcal{M}(S) \longrightarrow \mathcal{M}(T)$ by $\mathcal{M}(f)(A) = f(A) \wedge T$. $\mathcal{M}$ is a covariant functor of $\mathcal{CS}$ to $\mathcal{CDL}$. The following results appear in [13] and [10] or can be derived from results therein.

(1) $\rho_s(S)$ is the set of join-irreducible elements of S.

(2) The set of meet-irreducible elements of $\mathcal{M}(S)$ is the set of closed prime ideals of S.

(3) If I is the closed real interval from 0 to 1 with its natural order then $\mathcal{M}((I,\wedge))$ is isomorphic with $(I,\wedge,\vee)$.

(4) The following are equivalent

(a) $\mathcal{S}(S,I)$ separates points.

(b) $\mathcal{L}(\mathcal{M}(S),I)$ separates points.

(c) Every point of $\mathcal{M}(S)$ is a meet of members of $M(\mathcal{M}(S))$.

(d) [6] S is a Lawson semilattice (i.e. the topology of S has a neighborhood base of subsemilattices of S).

(e) The topology of $\mathcal{M}(S)$ has a neighborhood base of lattices.

We define $\mathcal{K}(\mathcal{K}')$ to be the full subcategory of objects K of $\mathcal{CDL}$ such that the topology of K has a neighborhood base of $\vee$-semilattices (lattices).

(5) If $L \in \mathcal{CDL}$ then L is the lattice of ideals of an object of $\mathcal{CS}$ if and only if (a) $L \in \mathcal{K}$ (b) $J(L) \in \mathcal{CS}$ and (c) every element of L is a join of a subset of $J(L)$. In this case L is isomorphic with $\mathcal{M}(J(L))$.

(6) Let $L \in \mathcal{CDL}$. Define $\Phi : \mathcal{M}((L,\wedge)) \longrightarrow (L,\wedge,\vee)$ by $\Phi(A) = \vee A$ (i.e. the sup of A when A is considered as a subset of $(L,\wedge,\vee)$). Then $\Phi : \mathcal{M}(L) \longrightarrow L$ is an $\mathcal{L}$-morphism if and only if $L \in \mathcal{K}$.

It was pointed out to the author by J.D. Lawson and J.W. Stepp that from (6) and [6] we have

(7) $(\mathcal{K}, \mathcal{LS}, U, \mathcal{M})$ and $(\mathcal{K}', \mathcal{CLS}, U, \mathcal{M})$ are adjunctions where U in each case is the suitable restriction of the functor $U : \mathcal{CDL} \longrightarrow \mathcal{CS}$ which forgets the $\vee$-operation and $\mathcal{CLS}$ is the category of compact Lawson semilattices.

2. Compact distributive lattices of finite breadth.

A lattice L has breadth n (n a positive integer) if (1) given any finite subset A of L there is $B \subseteq A$ such that card $B \leq n$ and $\wedge B = \wedge A$ and (2) there is $A \subseteq L$ with card $A = n$ and for $B \subseteq A$ with $B \neq A, \wedge B \neq \wedge A$. The main result of this section is a representation theorem obtained by the author and Kirby Baker in [2]. The steps of this theorem are of some independent interest.

(2.1) If L is a complete lattice of finite breadth n and the operations of L are continuous with respect to order convergence then every element of L is the meet of a subset A of M(L) with card $A \leq n$. Also each element of L is a join of a subset B of J(L) with card $B \leq n$.

(2.2) If L satisfies the hypotheses of (2.1) and is distributive then by applying the Dilworth coding theorem [4] and Zorn's lemma $J(L) = K_1 \cup \ldots \cup K_n$ where each K_i is a maximal chain in J(L). When endowed with the interval topology each K_i becomes a compact chain.

(2.3) If L satisfies the hypotheses of (2.2) the maps $\sigma_i : L \longrightarrow K_i$ defined by $\sigma_i(x) = \vee\{k \in K_i ; k \leq x\}$ are continuous lattice homomorphisms.

(2.4) If L satisfies the hypotheses of (2.2) then L is a member of $\mathcal{CDL}$ with either the interval on order topology (which coincide). Moreover the map $\sigma_i \times \ldots \times \sigma_n : L \longrightarrow K_1 \times \ldots \times K_n$ is an imbedding of L into a product of n compact chains.

A. C. Dempster by a very different method had obtained the representation theorem for lattices of breadth two at about the same time [3].

From (2.4) the class of those objects L of $\mathcal{CDL}$ such that $B_n(L) \leq n$ and the class of all closed sublattices of products of n compact chains coincide.

A result similar to (2.3) was obtained by the author and E. D. Shirley in [9].

(2.5) Let L and M be locally compact, connected topological lattices of finite breadth and let M be distributive. If $\varphi: L \longrightarrow M$ is a $\wedge$ and $\vee$ preserving map of L onto M then φ is continuous.

We will now give some examples to show what happens to these results when some of the hypotheses are dropped.

(2.6) J. D. Lawson in [7] gave an example of a metric, connected, one-dimensional object of $\mathcal{CS}$, which we denote by Law, with the property that $\mathcal{S}(\text{Law}, I)$ is trivial. Then $\mathcal{L}(\mathcal{M}(\text{Law}), I)$ is also trivial. Hence not every element of $\mathcal{M}(\text{Law})$ is the meet of meet-irreducibles. In fact, if $\mathcal{M}(\text{Law})$ does not already have this property, it is possible to create an object $L \in \mathcal{CDL}$ with $B_r(L) = \infty$ such that $M(L) = \{1\}$. Since $\mathcal{M}(\text{Law})$ is a lattice of ideals every element of $\mathcal{M}(\text{Law})$ is a join of join-irreducibles. $\mathcal{M}(\text{Law})^{op}$ ($\mathcal{M}(\text{Law})$ with order reversed) has the opposite properties.

(2.7) Let $T = \{1 - \frac{1}{n} ; n = 1,2,\ldots\} \cup \{1\} \leq I$. With the inherited order from I T is a compact chain. Form $S = T \times I/_{T \times \{0\}}$. (The Rees quotient of $T \times I$ by $T \times \{0\}$ i.e. $T \times \{0\}$ is shrunk to a point). Then $T \in \mathcal{CS} \backslash \mathcal{CL}$. Let I_o be the unique chain from 0_S to 1_S. Define $\varphi: S \longrightarrow I_o$ by $\varphi(s) = \vee\{u \in I_o; u \leq s\}$. Then φ is a $\wedge$-preserving map of S onto I but is not continuous because $\varphi(S \backslash I_o) = 0$. φ induces a lattice homomorphism $\Phi: \mathcal{M}(S) \longrightarrow (I_o) = I_o$ defined by $\Phi(A) = \vee\{\rho_S(u) \in \rho_S(I_o); \rho_S(u) \leq A\}$. $\rho_S(I_o)$ is a maximal chain in $J\mathcal{M}(S))$ and the map $\Phi: \mathcal{M}(S) \longrightarrow I_o$ satisfies all of the hypotheses of (2.5) except for finite breadth. However Φ cannot be continuous because Φ restricted to $\rho_S(S)$ is the same as $\varphi: S \longrightarrow I_o$.

Thus (2.3) and (2.5) do not hold without finite breadth. This example is found in [9].

3. Locally convex lattice

A subset A of a lattice L is called convex if whenever $x,y \in A$ and $x < y$ then $[x,y] \subseteq A$. A topological lattice is called locally convex if its topology has a neighborhood base of convex sets. To see that many lattice are locally convex we have:

(3.1) The following classes of lattices are locally convex

(a) Compact lattices (Nachbin [8])

(b) Locally compact and connected lattices (L.W. Anderson [1]).

(c) Discrete lattices

(d) Sublattices of locally convex lattices.

To see that some lattices are not locally convex we have:

(3.2) In the plane let $L = \{(1,\frac{1}{2n});\ n = 1,2,\ldots\} \cup \{(0,\ \frac{1}{2n+1});n = 1,2,\ldots\} \cup \{(1,0)\}$. An order $\leq$ is defined on L by setting $(x_1,y_1) \leq (x_2 y_2)$ if and only if $y_1 \leq y_2$. With the topology L inherits from the plane it becomes a topological lattice, in fact a chain. However L is not locally convex.

From [11] we have

(3.3) Let L be an object of $\mathcal{DL}$ which is locally convex and is of finite breadth n. Then L can be imbedded in a member of $\mathcal{CDL}$. Hence from (2.4) L can be imbedded in a product of n compact chains.

(3.3) characterizes all sublattices of finite products of compact chains in the same way that (2.3) characterizes all closed sublattices of finite products of compact chains.

For infinite breadth we have more difficulty. First we note that $\mathcal{M}$(Law) is an object of $\mathcal{CDL}$ which cannot be imbedded in any product of compact chains.

(3.4) Let L be a locally compact and connected distributive topological lattice then L can be imbedded in a compact lattice [14].

(3.5) Let L be the product of countably many copies of the two point lattice. When L is given the discrete topology it cannot be imbedded (as a topological lattice) in any compact lattice [14].

4. <u>Congruence extension property for $\mathcal{CDL}$.</u>

It is well-known that the congruence extension property characterizes distributive lattices (cf. [5]). We make the obvious modification of this property to $\mathcal{L}$ as follows: $L \in \mathcal{L}$ has the congruence extension property (c.e.p.) if given A a closed sublattice of L and $\varphi : A \longrightarrow B$ an $\mathcal{L}$-morphism on A there is an $\mathcal{L}$-morphism $\Phi : L \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "\Phi"] & M \\ A \arrow[u, "i"] \arrow[r, "\varphi"] & B \arrow[u, "j"'] \end{tikzcd}$$

where i the inclusion map of A into L and j is an imbedding of B into M.

For $L \in \mathcal{CL}$ let $\mathcal{C}(L)$ be the lattice of closed congruence on L. From [12] we have the following results:

(4.1) If $L \in \mathcal{CDL}$ and $Br(L) < \infty$ then $\mathcal{C}(L)$ is a distributive lattice.

(4.2) If $L \in \mathcal{CDL}$ and $Br(L) < \infty$ then L has c.l.p. It seems likely that the following conjecture should hold

(C) If $L \in \mathcal{CDL}$ and $\dim L = 0$ then $\mathcal{C}(L)$ is a distributive lattice.

However the general question remains

(Q) If $L \in \mathcal{L}(\mathcal{CL})$ is $\mathcal{C}(L)$ distributive?

(4.3) Let **X** be a countable product of copies of the two point lattice endowed with the Cartesian product topology. (as such $X \in \mathcal{CTL}$). Then X can have no dimension-raising, continuous $\wedge$-preserving maps.

Then because the usual chain lattice C in the Cantor set can be imbedded in X and C has dimension-raising lattice homomorphisms it follows that

(4.4) X does not have c.e.p.

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REPRESENTATIONS OF LATTICE-ORDERED RINGS

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In this paper we present two typical representation theorems for archimedean lattice-ordered rings with identity, a classical one by means of continuous extended real valued functions and a less classical one by means of continuous sections in sheaves.

0. Introduction.

The oldest question in the theory of lattice-ordered rings, groups, and vector spaces probably is the question of representations by real valued functions. In the forties F. MAEDA and T.OGASAWARA [17], H. NAKANO [19], T. OGASAWARA [2o] and K. YOSIDA [23] and probably others established such representation theorems by continuous functions for vector lattices, M.H. STONE [22] and H. NAKANO [18] for lattice-ordered real algebras. (See also R.V. KADISON [13].) In the sixties, this question has been taken up in a more general and modern presentation e.g. by S.J. BERNAU [1], M. HENRIKSEN and D.G. JOHNSON [9], D.G. JOHNSON [11], D.G. JOHNSON and J. KIST [12], J. KIST [15].

Our first theorem has been proved in various ways and various generality in almost all of the papers listed above. Our proof might contain some new aspects: It is a self-contained proof not using any ideal theory, based on a notion of characters like GELFAND's representation theorem for commutative C -algebras. In the case of lattice-ordered groups this idea is implicitely used by D.A. CHAMBLESS [4], in the case of Banach lattices it is explicitely used by H.H. SCHAEFER [24].

Our second representation theorem as well as its proof is inspired by GROTHENDIECK's construction of the affine scheme of a commutative ring with the one exception that to some extent the lattice operations are used instead of the ring operations. The sheaf associated with a lattice-ordered ring also reminds the sheaf of germs of continuous functions, although this second theorem applies to a much bigger class of lattice-ordered rings than that representable by extended real valued functions. As references for theorem 2 we give [7], [14], [15].

1. Representation by continuous extended real valued functions.

In this paper, rings are always supposed to have an identity e ; but commutativity is not required (although archimedean f-rings turn our to be commutative).

DEFINITION 1. A lattice-ordered ring is a ring A endowed with a lattice order $\leq$ in such a way that $a+b \geq 0$ and $ab \geq 0$ for all elements $a \geq 0$ and $b \geq 0$ in A . We denote by $A_+ = \{a \in A \mid a \geq 0\}$ the positive cone of A , and by $\vee$ and $\wedge$ the lattice operations.

If A and A' are lattice-ordered rings, a function $f:A \to A'$ is called an ℓ-homomorphism, if f is a ring and a lattice homomorphism (preserving the identity).

Unfortunately, only few things can be said about lattice-ordered rings in general. Usually one considers a more special class of lattice-ordered rings:

DEFINITION 2. A lattice-ordered ring A is called an abstract function ring (shortly f-ring) if A is a subdirect product of totally ordered rings.

BIRKHOFF and PIERCE [3] have shown that a lattice-ordered ring A is an f-ring if and only if one has:

$a \wedge b = 0$ implies $a \wedge bc = 0 = a \wedge cb$ for all $c \in A_+$.

In a fist approach we call concrete function ring every ℓ-subring (i.e. subring and sublattice) of the f-ring $C(X)$ of all continuous real valued functions on some topological space X. The answer to the question, whether every abstract function ring is isomorphic to a concrete function ring is obviously negative; for a non-archimedean field cannot be represented in this way.

DEFINITION 3. A lattice-ordered ring A is called <u>archimedean</u>, if for every pair of elements a,b in A with $a \neq 0$ there is an integer n such that $na \nleq b$.

BIRKHOFF and PIERCE [3] have shown that an archimedean lattice-ordered ring is an f-ring if and only if the identity e is a weak order unit, i.e. $e \wedge x > 0$ for every $x > 0$.

Every archimedean abstract function ring can be represented as a concrete function ring, if one generalises slightly the notion of concreteness: Let X be a topological space. Denote by $E(X)$ the set of all continuous functions $f:U_f \to \mathbb{R}$, where U_f is any open dense subset of X. We identify two such functions $f:U_f \to \mathbb{R}$, $g:U_g \to \mathbb{R}$, if f and g agree on $U_f \cap U_g$. (Note that the intersection of two open dense subsets is open and dense.) Then $E(X)$ is an f-ring.

A more formal construction of $E(X)$ goes as follows: Let $\mathcal{U}$ be the collection of all open dense subsets of X.

For each $U \in \mathcal{U}$ consider $C(U)$, the f-ring of all continuous real valued functions defined on U. If $U, V \in \mathcal{U}$ and $V \subset U$, define the ℓ-homomorphism $\rho_U^V : C(U) \to C(V)$ to be the restriction map $f \mapsto f|V$. Then

$$E(X) = \varinjlim_{U \in \mathcal{U}} C(U) .$$

With the exception of some rather special classes of spaces X, the f-ring $E(X)$ cannot be represented in any $C(Y)$, as one may conclude from some results of CHAMBLESS [5].

If we call <u>concrete function ring</u> every ℓ-subring of some $E(X)$, we can state:

THEOREM 1. <u>Every archimedean f-ring with identity can be represented as a concrete function ring</u>.

One can prove something more precise by using the extended real line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty , +\infty \},$$

endowed with the usual order and topology; we also use the usual conventions for addition and multiplication with $\pm\infty$, as far as reasonable.

A continuous function $f : X \to \overline{\mathbb{R}}$ is called an almost finite extended real valued function, if the open set $U_f = \{x \in X \mid f(x) \neq \pm\infty \}$ is dense in X. The set $D(X)$ of all these functions can be naturally embedded in $E(X)$ by the assignment $f \mapsto f|U_f$. This allows us to consider $D(X)$ as a subset of $E(X)$. $D(X)$ always is a sublattice of $E(X)$, but it need not be a subring. Every ℓ-subring of $E(X)$ contained in $D(X)$ will be called an f-ring of continuous extended almost finite real valued functions. Now we state:

Theorem 1'. *Every archimedean f-ring (with identity e) is isomorphic to a lattice-ordered ring of continuous extended almost finite real valued functions defined on some compact Hausdorff space.*

The proof is carried out in several steps. In a sense, the whole proof is based on the following result credited to PICKERT [22] by FUCHS [6], but probably known for quite some time:

(a) THEOREM (αρχιμεδησ ([1]) ?). If A is an archimedean totally ordered ring with identity, then there is a unique order preserving isomorphism from A onto some subring of $\mathbb{R}$.

(b) Let A be any f-ring with identity e. A function $\omega : A \to \overline{\mathbb{R}}$ is called a *character* of A, if it satisfies:

(1) $\omega(e) = 1$;

(2) $\omega(a \vee b) = \omega(a) \vee \omega(b)$, $\omega(a \wedge b) = \omega(a) \wedge \omega(b)$

(3) $\omega(a+b) = \omega(a) + \omega(b)$, $\omega(ab) = \omega(a)\omega(b)$, whenever the right hand side is defined in $\overline{\mathbb{R}}$.

Let X denote the set of all characters of A. Note that X is a subset of $\overline{\mathbb{R}}^A$. Endow $\overline{\mathbb{R}}^A$ with the product topolo-

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([1]) Archimedes, Greek mathematician (287? to 212 b.c.)

gy which is compact Hausdorff. It is easily checked that X is a closed subset of $\overline{\mathbb{R}}^A$. Consequently, X is a compact Hausdorff space, called the <u>character space</u> of A.

(c) For every a in A define a function $\hat{a}: X \to \overline{\mathbb{R}}$ by $\hat{a}(\omega) = \omega(a)$ for all $\omega \in X$. As $\hat{a}$ is the a-th projection $\overline{\mathbb{R}}^A \to \overline{\mathbb{R}}$ restricted to X, it is a continuous function.

(d) For all a, b in A we have:

$$(a \vee b)\hat{} = \hat{a} \vee \hat{b} \quad \text{and} \quad (a \wedge b)\hat{} = \hat{a} \wedge \hat{b}.$$

For all $\omega \in X$ on has indeed $(\hat{a} \vee \hat{b})(\omega) = \hat{a}(\omega) \vee \hat{b}(\omega) =$ $= \omega(a) \vee \omega(b) = \omega(a \vee b) = (a \vee b)\hat{}(\omega)$, and likewise for $\hat{a} \wedge \hat{b}$. In the same way one shows that

$$(a + b)\hat{}(\omega) = (\hat{a} + \hat{b})(\omega) \quad \text{and} \quad (ab)\hat{}(\omega) = \hat{a}(\omega)\hat{b}(\omega)$$

whenever $\hat{a}(\omega) + \hat{b}(\omega)$ and $\hat{a}(\omega)\hat{b}(\omega)$, respectively, are well defined in $\overline{\mathbb{R}}$.

(e) PROPOSITION. Let B be the ℓ-subring of all bounded elements of A, i.e. B is the set of all $a \in A$ such that $-ne \leq a \leq ne$ for some $n \in \mathbb{N}$. Then the assignment $a \mapsto \hat{a}$ gives an ℓ-homomorphism from B into $C(X)$ the kernel of which is the set of all a such that $na \leq e$ for all integers n. In particular, if A is archimedean, this ℓ-homomorphism is injective.

Indeed, if $a \in B$, then $\hat{a}(\omega) = \omega(a) \in \mathbb{R}$ for every character ω. By (c) and (d), $a \mapsto \hat{a}$ is then an ℓ-homomorphism from B into $C(X)$. The assertion about the kernel follows from the following lemma:

(f) LEMMA. If a is an element of A such that $na \nleq e$ for some integer n, then there is a character ω of A such that $\omega(a) \neq 0$.

Proof. Let $na \nleq e$. As A is a subdirect product of totally ordered rings, there is an ℓ-homomorphism α from A onto some totally ordered ring $\bar{A}$ such that $\alpha(na) > \alpha(e)$. Denote $\bar{x} = \alpha(x)$ for all x. Now let $\bar{B}$ be the ring of all bounded elements of $\bar{A}$ and $\bar{I}$ the set of all $\bar{x}$ with $n\bar{x} < \bar{e}$ for all integers n. Then $\bar{I}$ is a convex ideal of $\bar{B}$ and $\bar{B}/\bar{I}$ is an archimedean totally ordered ring with identity. Using (a) we can find an order preserving homomorphism $\bar{\omega}:\bar{B} \to R$ such that $\bar{\omega}(\bar{e}) = 1$, whence $\bar{\omega}(a) \neq 0$. By defining

$$\bar{\omega}(\bar{x}) = \begin{cases} +\infty & \text{if } \bar{x} < n\bar{e} \text{ for all } n > 0, \\ -\infty & \text{if } \bar{x} > n\bar{e} \text{ for all } n > 0, \end{cases}$$

we have extended $\bar{\omega}$ to a character of $\bar{A}$. Then $\omega = \bar{\omega} \circ \alpha$ is a character of A such that $\omega(a) \neq 0$.

In order to achieve the proof of theorem 1' we need two more lemmas. As in the preceding lemmas, we are working an an f-ring with identity, not necessarily archimedean.

(g) LEMMA. The sets of the form

$$V(f) = \{\omega \in X \mid \hat{f}(\omega) = \omega(f) > 0\}, \quad 0 \leq f \leq e, \ f \in A,$$

constitute a basis of the topology on X.

Proof. We first note that, by the definition of the product topology on $\mathbb{R}^A$, the sets $\overline{V}(f,q) = \{\omega \in X \mid \omega(f) > q\}$ and $\underline{V}(f,q) = \{\omega \in X \mid \omega(f) < q\}$ with $f \in A$ and $q = \frac{n}{m} \in \mathbb{Q}$ form a subbasis of the topology on X. As $\omega(f) > \frac{n}{m}$ iff $\omega(mf) > n = \omega(ne)$ iff $\omega(mf - n) > 0$, we conclude that $\overline{V}(f,q) = \overline{V}(mf-ne,0) = V(mf-ne)$; likewise $\underline{V}(f,q) = V(ne-mf)$. Thus, the $V(f)$, $f \in A$, already form a subbasis. They even form a basis, as $V(f) \cap V(g) = V(f \wedge g)$. As $V(f) = V((f \vee 0) \wedge e)$, we may restrict our attention to elements f with $0 \leq f \leq e$.

(h) LEMMA. If A is archimedean, one has $a = \bigvee_{n \in \mathbb{N}} (a \wedge ne)$ for all $a \in A_+$.

Proof. By the way of contradiction, we suppose that there is an element b in A such that $a \wedge ne \leq b < a$ for all $n \in \mathbb{N}$. As $0 < a-b$ and as e is a weak order unit, $e \wedge (a-b) > 0$. The element $d = e \wedge (a-b)$ satisfies $0 < d \leq e$ and $d \leq a$. Under the hypothesis that $(n-1)d \leq a$, we can conclude that $(n-1)d \leq (n-1)e \wedge a \leq b$, which together with $d \leq a-b$ implies $nd \leq a$. Thus, we have shown by induction that $nd \leq a$ for all $n \in \mathbb{N}$ which is incompatible with the archimedean hypothesis.

(j) Now we are ready to achieve the proof of theorem 1': We first show that $\hat{a} = \hat{b}$ implies $a = b$. As $a = (a \vee 0) - (-a \vee 0)$, it suffices to consider the case where $a,b \geq 0$. If $\hat{a} = \hat{b}$, then $\hat{a} \wedge n\cdot 1 = \hat{b} \wedge n\cdot 1$ for all $n \in \mathbb{N}$, whence $(a \wedge ne)^\wedge = (b \wedge ne)^\wedge$ for all $n \in \mathbb{N}$ by (d). As $a \wedge ne$ and $b \wedge ne$ are bounded, we conclude that $a \wedge ne = b \wedge ne$ for all $n \in \mathbb{N}$ by (e). Hence, $a = b$ by (h). Now we prove that $\hat{a} \in D(X)$: If U is an open subset of X such that, for exemple, $\hat{a}(\omega) = +\infty$ for all $\omega \in U$, then by (g) we may suppose that $U = V(f)$ for some f in A with $0 \leq f \leq e$, and we conclude that $\hat{a} = (a+f)^\wedge$. Consequently, $f = 0$ by the preceding, i.e. $U = V(f) = \emptyset$. Finally, (d) shows that $a \mapsto \hat{a}$ is an ℓ-homomorphism.

REMARKS. 1. Using property (g), one can show easily that $(\bigvee_{i\in I} a_i)^\wedge = \bigvee_{i\in I} \hat{a}_i$, whenever $\bigvee_{i\in I} a_i$ exists in A. The same holds for arbitrary meets.

2. Every archimedean f-ring without nilpotent elements can be embedded in an f-ring with identity which is archimedean, too. Consequently, all archimedean f-rings with identity have representations as concrete function rings.

3. Let $\psi: Y \to X$ be a continuous map of topological spaces such that $\psi^{-1}(U)$ is dense in Y for every dense open subset U of X. For every $f \in E(X)$ the function $f\circ\psi$ belongs to $E(X)$. Thus, we obtain an ℓ-homomorphism $E(\psi): E(X) \to E(Y)$; moreover, $D(X)$ is mapped into $D(Y)$. If, in addition, the image $\psi(Y)$ is dense in X, then $E(\psi)$ is injective. This gives the idea, how to obtain

representations of A on other spaces Y from the above representation on the character space X. We list two cases:

Let $\pi : P \to X$ be the projective cover of the character space X of the archimedean f-ring A (cf. GLEASON [8]). Then π is surjective and has the property required above. Moreover, P is extremally disconnected, compact and Hausdorff. Thus, we obtain a representation of A in $E(P)$ for some extremally disconnected compact Hausdorff space P. One can show that this representation of A is just the representation of BERNAU [1].

In a similar way one can obtain JOHNSON's [10] and KIST's [15] representation theorems from theorem 1'; for the character space X is homeomorphic with the "space of maximal ℓ-ideals"; further there is a continuous map from the space of all "prime ℓ-ideals" of A onto X which has all the required properties.

2. Representation by continuous sections in sheaves.

This section is not as self-contained as the first. But the proofs are complete. We refer to [14] and [15] for further information.

Let A be an arbitrary f-ring (with identity e). A subset I of A is called an ℓ-ideal, if I is a ring ideal and a convex sublattice. For an ℓ-ideal I, the

the quotient ring A/I becomes an f-ring by defining $a+I \leq b+I$ if there is an $x \in I$ with $a \leq b+x$. For every subset C of A, we define $C^{\perp} = \{x \in A \mid |x| \wedge |c| = 0 \ \forall c \in C\}$. Then $C^{\perp}$ is an ℓ-ideal, called <u>polar</u> ℓ-<u>ideal</u>.

DEFINITION 4. The f-ring A is called <u>quasi-local</u>, if A has a unique maximal ℓ-ideal.

DEFINITION 5. A <u>sheaf of</u> [<u>quasi-local</u>] f-<u>rings</u> is a triple $\mathcal{F} = (E,\eta,X)$, where E and X are topological spaces and $\eta : E \to X$ is a local homeomorphism; moreover, every <u>stalk</u> $E_x = \eta^{-1}(x)$, $x \in X$, has to bear the structure of a [quasi-local] f-ring in such a way that the functions

$$(x,y) \mapsto x+y \ , \ (x,y) \mapsto xy \ , \ (x,y) \mapsto x \wedge y$$

from $\bigcup_{x \in X}(E_x \times E_x)$ into E are continuous, where $\bigcup_{x \in X}(E_x \times E_x) \subset E \times E$ is endowed with the topology induced from the product space $E \times E$.

DEFINITION 6. Let $\mathcal{F} = (E,\eta,X)$ be a sheaf of [quasi-local] f-rings. Call <u>section</u> of $\mathcal{F}$ every continuous function $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for all $x \in X$. Denote by $\Gamma\mathcal{F}$ the set of all sections of $\mathcal{F}$. By defining on $\Gamma\mathcal{F}$ addition, multiplication and order pointwise, $\Gamma\mathcal{F}$ becomes an f-ring, in fact, an ℓ-subring of the direct product of the stalks.

Now we are ready to state:

THEOREM 2. <u>For every f-ring</u> A <u>(with identity</u> e) <u>there is a sheaf</u> $\mathcal{F} = (E,\eta,X)$ <u>of quasi-local f-rings over a compact Hausdorff space</u> X <u>such that</u> A <u>is isomorphic to the f-ring</u> $\Gamma\mathcal{F}$ <u>of all</u> (continuous global) <u>sections of</u> $\mathcal{F}$.

The proof is carried out in several steps. Let B be the f-ring of all bounded elements of A . We use the character space X of A and the representation $a \mapsto \hat{a}: B \to C(X)$ established in Proposition (e) of section 1.

(a) For every $\omega \in X$, let I_ω be the union of all the polars $a^\perp$, where a runs through all elements of A such that $\omega(a) > 0$. Then I_ω is an ℓ-ideal. Let $A_\omega = A/I_\omega$.

(b) CONSTRUCTION. Let E be the disjoint union of the quotient rings A_ω , $\omega \in X$. For every $a \in A$, define

$$\tilde{a}: X \to E \quad \text{by} \quad a(\omega) = a + I_\omega \in A_\omega .$$

It is easily shown that the sets of the form $\tilde{a}(U)$ with $a \in A$ and $U \subset X$ open, form a basis of a topology on E such that the triple $\mathcal{F} = (E,\eta,X)$ is a sheaf of f-rings, where $\eta: E \to X$ is the obvious projection which maps A_ω onto ω . The stalks of $\mathcal{F}$ are the f-rings A_ω . Moreover, every $\tilde{a}$ is a section of $\mathcal{F}$ and the assignment $a \mapsto \tilde{a}: A \to \Gamma\mathcal{F}$ is an ℓ-homomorphism.

(c) LEMMA. Let U be an open neighborhood of $\omega_o \in X$. There is an element p in A_+ such that $\tilde{p}(\omega_o) = \tilde{e}(\omega_o)$ and $\tilde{p}(\omega) = 0$ for all $\omega \notin U$.

Proof. By lemma (g) in section 1, there is an element f in A_+ such that $\omega_o \in V(f) \subset U$. Then $\omega_o(f) > 0$ and $\omega(f) = 0$ for all $\omega \notin U$. After replacing f by $nf \wedge e$ for a suitably large n, we may suppose that $\omega_o(f) = 1$. Now let $g = 3f-e$ and $h = 2f-e$. We use the notation $x_+ = x \vee 0$ and $x_- = -x \vee 0$ and note that $x_+ \wedge x_- = 0$. Let

$$P = g_+^{\perp} \text{ and } Q = h_+^{\perp} .$$

We have $\omega_o(h_+) = (2\omega_o(f) - \omega_o(e)) \vee 0 = 1$, whence $Q = h_+^{\perp} \subset I_{\omega_o}$. For every $\omega \notin U$, one has $\omega(g_-) = \omega(e-3f) \vee 0$ $= (\omega(e) - 3\omega(f)) \vee 0 = 1$, Hence, $P^{\perp} \subset g_-^{\perp} \subset I_{\omega}$. The ℓ-ideal $P^{\perp}+Q$ contains $g_+ + h_- = (3f-e) \vee 0 + (e-2f) \vee 0$, and this element is not contained in any proper ℓ-ideal of A, as its image in every non zero totally ordered ring is easily seen to be strictly positive. Consequently, $P^{\perp}+Q = A$. Thus, there are positive elements $p \in P^{\perp}$ and $q \in Q$ such that $p+q = e$. This means that $p+Q = e+Q$ and consequently $p+I_{\omega_o} = e+I_{\omega_o}$ and $p \in I_{\omega}$ for all $\omega \notin U$; thus, p has the required properties.

(d) LEMMA. A_{ω} is a quasi-local f-ring for every $\omega \in X$.

Proof. We first note that I_{ω} is contained in $\ker \omega$. From (c) it follows that $I_{\omega} \not\subset \ker \omega'$ for every $\omega' \neq \omega$. Let M_{ω} be the greatest ℓ-ideal of A contained in $\ker \omega$, i.e. M_{ω} is the sum of all ℓ-ideals contained in $\ker \omega$. Then M_{ω} is a maximal ℓ-ideal of A. It is the unique maximal ℓ-ideal containing I_{ω}; indeed, every maximal

ℓ-ideal is easily seen to be contained in the kernel of some character.

(e) LEMMA. $\bigcap_{\omega \in X} I_\omega = \{0\}$.

Proof. Suppose that $b \in I_\omega$ for all $\omega \in X$. Then $b \in a_\omega^\perp$ for some element a_ω satisfying $\omega(a_\omega) > 0$. After replacing a_ω by na_ω for a suitable n , we may suppose that $\omega(a_\omega) > 1$. The sets $W(a_\omega) = \{\omega' \mid \omega'(a_\omega) > 1\}$ are open in X and cover X . Hence, there is a finite subset F in X such that $X = \bigcup_{\omega \in F} W(a_\omega)$. Let $a = \bigvee_{\omega \in F} a_\omega$. Then $\omega(a) > 1$ for all $\omega \in X$,whence $a > e$; further $|b| \wedge a = 0$ as $b \in a_\omega^\perp$ for all ω . As e and consequently a is a weak order unit, this implies $b = 0$.

(f) The proof of theorem2 will be achieved, if we show that the assignment $a \mapsto \tilde{a} : A \to \Gamma F$ is bijective. The injectivity is a straightforward consequence of lemma (e). For the surjectivity let σ be an arbitrary section of F . We want to find an element a in A such that $\tilde{a} = \sigma$. As $\sigma = (\sigma \vee 0) + (\sigma \wedge 0)$, we may restrict ourselves to the case $\sigma \geq 0$. By the construction of the sheaf F , for every $\omega \in X$ there is an element $a_\omega \in A_+$ such that $\tilde{a}_\omega(\omega) = \sigma(\omega)$. If two sections of a sheaf coincide in a point, they agree in a whole neighborhood; hence, there is a neighborhood U_ω of ω such that $\sigma | U_\omega = \tilde{a}_\omega | U_\omega$. By lemma. By lemma (c), there is an element $p_\omega \in A_+$ such that $\tilde{p}_\omega(\omega) = \tilde{e}(\omega)$ and $\tilde{p}_\omega(\omega') = 0$

for all $\omega' \notin U_\omega$. One may suppose $p_\omega \le e$. Let $b_\omega = a_\omega p_\omega$; then $\tilde{b}_\omega(\omega) = \sigma(\omega)$ and $\tilde{b}_\omega \le \sigma$. Let V_ω be an open neighborhood of ω such that $\tilde{b}_\omega | V_\omega = \sigma | V_\omega$. The V_ω, $\omega \varepsilon X$, form an open covering of X . As X is compact, we may find a finite subset $F \subset X$ such that the V_ω with $\omega \varepsilon F$ already form a covering of X . Let $a = \bigvee_{\omega \varepsilon F} b_\omega$. Then $\tilde{a} = \sigma$.

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MODULAR CENTERS OF ADDITIVE LATTICES

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The modular center of a lattice $L(\mathfrak{M}(L))$ is defined to be $\{x | x \in L$ and $\mathfrak{M}(a,x)$ for all $a \in L\}$ where the symbol $\mathfrak{M}(a,x)$ means that a and x form a modular pair. A lattice L is said to be <u>additive</u> iff whenever p is an atom of L such that $p \leq x \vee y$, then there exists atoms x_1 and y_1 in L with $x_1 \leq x$ and $y_1 \leq y$ such that $p \leq x_1 \vee y_1$. The lattice L of convex subsets of a vector space V over an ordered division ring is additive, and in this case $\mathfrak{M}(L)$ is the affine subsets of V.

If L is atomistic and additive, then $\mathfrak{M}(L)$ is a complete lattice in its own right, with the meet operation in $\mathfrak{M}(L)$ being the meet operation in L.

We present conditions in L which guarantee that $\mathfrak{M}(L)(p,1)$ is a projective geometry whenever p is an atom of L, and then give conditions on L which imply that $\mathfrak{M}(L)$ is the affine subsets of a vector space V over a decision ring R. In the latter case we show further that the R is ordered and that L is the lattice of the convex subsets of V.

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Structure of Archimedean Lattices

by Jorge Martinez

Abstract An archimedean lattice is a complete algebraic lattice L with the property that for each compact element $c \in L$, the meet of the maximal elements in the interval $[0, c]$ is 0. L is hyper-archimedean if it is archimedean, and for each $x \in L$, $[x, 1]$ is archimedean. The structure of these lattices is analysed from the point of view of their meet irreducible elements. If the lattices are also Brouwer, then the existence of complements for the compact elements characterizes a particular class of hyper-archimedean lattices.

The lattice of ℓ-ideals of an archimedean lattice ordered group is archimedean, and that of a hyper-archimedean lattice ordered group is hyper-archimedean, In the hyper-archimedean case those arising as lattices of ℓ-ideals are fully characterized.

Finally, we examine the role played by these lattices in representations by lattices of open sets of some topological space. We point out a duality between algebraic, Brouwer lattices and certain T_0-spaces with bases of compact open sets.

Notation and terminology Our set theoretic notation is as follows: if A and B are subsets of a set X then $(A \subset B)$ $A \subseteq B$ denotes (proper) containment of A in B; $A \setminus B$ is the complement of B in A.

Our lattice theoretic and topological terminology is standard, except where expressly noted that it is not. The terminology from the theory of lattice ordered groups is for the most part that of Conrad [5].

1. Structure of archimedean and hyper-archimedean lattices   We will be dealing exclusively with algebraic lattices: complete lattices generated by compact elements. We call an algebraic lattice archimedean if for each $c \in c(L)$, the semilattice of compact elements, the interval $[0, c]$ has the property that the meet of its maximal elements is 0. The motivation for this notion comes from the theory of ℓ-groups (abbreviation for lattice ordered groups): among the abelian ℓ-groups the archimedean ℓ-groups are characterized precisely by the condition that the lattice of its ℓ-ideals be archimedean as defined above. (Recall: an ℓ-group G is archimedean if for each pair $0 \leq a, b \in G$ $na \not\leq b$, for some natural number n.) This observation concerning the lattice of ℓ-ideals of an archimedean ℓ-group first appeared in [3], and is due to Roger Bleier.

Let us call an algebraic lattice L hyper-archimedean if it is archimedean and for each $x \in L$ $[x, 1]$ is archimedean. Again, here we are motivated by the theory of ℓ-groups: an ℓ-group G is hyper-archimedean if it is archimedean, and each ℓ-homomorphic image of G is archimedean. It is immediate then that G is hyper-archimedean if and only its lattice of ℓ-ideals is hyper-archimedean.

We shall call an element t of a lattice L meet-irreducible if $t = \bigwedge_{\lambda \in \Lambda} x_\lambda$ implies that $t = x_\mu$, for some $\mu \in \Lambda$. The notion of finite meet irreducibility is defined in the obvious manner.

Below, let L be an algebraic lattice; the first three lemmas are well known. See [2] or [7].

1.1 Lemma: If $x < 1$ in L then x is the meet of meet-irreducible elements.

1.2 Lemma: The meet of all the meet-irreducible elements of L is 0.

1.3 Lemma: L is a Brouwer lattice if and only if L is distributive.

(Note: A complete lattice B is Brouwer if and only if the following distributive law holds in B: $a \wedge (\vee_\lambda b_\lambda) = \vee_\lambda (a \wedge b_\lambda)$.)

Now the first structure theorem on archimedean lattices!

1.4 Proposition: Let L be an archimedean lattice, and $0 < c < d \in c(L)$. Then c and d have a value in common. Conversely, if L is a modular algebraic lattice, and any two comparable compact elements have a value in common, then L is archimedean.

(Remark: $p \in L$ is a value of $c \in c(L)$ if p is maximal with respect to not exceeding c. If p is a value of some compact element then p is meet-irreducible, and conversely.

We shall provide a converse to show that we cannot dispense with modularity in the converse of 1.4 .)

Proof: Suppose L is archimedean and $0 < c < d \in c(L)$. There is a maximal element m of [0, d] such that $c \nleq m$. Using Zorn's lemma pick $y \leq m$ so that it is a value of c; one can easily show then that y is a value of d as well.

Conversely, suppose L is modular, and $c, d \in c(L)$ with $0 < c < d$. If p is a value of both c and d, then by modularity $d \wedge p$ is maximal in [0, d] and $c \nleq d \wedge p$. This suffices to show L is archimedean.

1.5 Theorem: Suppose L is a hyper-archimedean lattice; then the subset of meet-irreducibles is trivially ordered. Conversely, if L is modular and the set of meet-irreducibles is trivially ordered, then L is hyper-archimedean.

Proof: Suppose first that L is hyper-archimedean. A meet-irreducible element t (in any complete lattice) always has a <u>cover</u> $\bar{t}$: namely, the meet of all the eleeents that exceed t properly. Here we show that $\bar{t} = 1$ for each meet-irreducible element t. [t, 1] is an archimedean lattice in which $\bar{t}$ is the unique atom; if $\bar{t} < 1$, one can show that a compact element d of [t, 1] exceeds $\bar{t}$. This contradicts the fact that [t, 1] is archimedean.

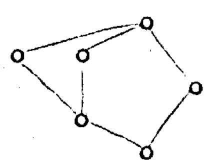
Conversely, suppose L is modular and the set $\{ t_\lambda \mid \lambda \in \Lambda \}$ is the trivially ordered set of meet-irreducibles. Then each one is maximal and their meet is 0 by lemma 1.2, so if $c \in c(L)$ and $c > 0$ then some t_μ fails to exceed c. By modularity $c \wedge t_\mu$ is maximal in [0, c] for each such t_μ, and the intersection of all these $c \wedge t_\mu$ is 0. This proves L is archimedean.

If one observes that for each $x < 1$ $\{ t_\lambda \mid t_\lambda \geq x \}$ is the complete set of meet-irreducibles of [x, 1] the argument of the preceding paragraph shows [x, 1] is archimedean, and hence that L is hyper-archimedean.

1.6 <u>Examples</u>: a) If E is any vector space, the lattice $\underset{\sim}{V}(E)$ of subspaces of E is a hyper-archimedean, modular lattice. In fact, if R is any semisimple, Artinian ring and M is a left R-module then the lattice of submodules of M is hyper-archimedean. The author will explore this matter further elsewhere.

b) Examples can be found of non-modular archimedean and hyper-archimedean lattices; see [9].

c) Below we exhibit a lattice satisfying the condition of proposition 1.4 which is not modular and not archimedean.



Notice also that this lattice satisfies the condition of theorem 1.6, but is not hyper-archimedean.

We now direct our attention to archimedean, Brouwer lattice. Recall that in a complete Brouwer lattice it is true that for each pair of elements x and y the set $\{ z \mid x \wedge z \leq y \}$ has a unique largest element. In particular if $y = 0$, there is a largest element x' such that $x \wedge x' = 0$. It is well known that this "complementation" is an auto-Galois connection on the Brouwer lattice. The set of all elements with the property that $x = x''$ form a Boolean algebra in which the meet operation agrees with that of the underlying lattice. We shall refer to it as the <u>Boolean algebra of polars</u> and to its elements as <u>polars</u>.

1.7 <u>Proposition</u>: Let L be an algebraic, Brouwer lattice. Then L is archimedean if and only if $c' = \wedge \{$ all values of c $\}$, for each $c \in c(L)$.

Proof: Suppose L is archimedean, and $0 < c \in c(L)$ and let $\{ p_\lambda \mid \lambda \in \Lambda \}$ be the set of values of c; since $c \wedge c' = 0$ and p_λ is prime, $p_\lambda \geq c'$, for each $\lambda \in \Lambda$. If $c' < \wedge p_\lambda$ there is a compact element $d \leq \wedge p_\lambda$ so that $d \nleq c'$, ie. $d \wedge c > 0$. Since L is archimedean there is an m maximal below c such that $d \wedge c \nleq m$. Let y be the largest element of L such that $y \wedge c = m$; then y is a value of c, and so $y = p_\mu$, for some $\mu \in \Lambda$. But then $d \leq y$ and hence $d \wedge c \leq y \wedge c = m$, a contradiction. Thus $c' = \wedge p_\lambda$.

Using the same notation of the preceding paragraph, let us assume the indicated condition holds. It is not hard to see that the elements $c \wedge p_\lambda$ are precisely the maximal elements of $[0, c]$. Now $\wedge_\lambda (c \wedge p_\lambda) = c \wedge (\wedge_\lambda p_\lambda) = c \wedge c' = 0$, and so L is archimedean.

If $\underset{\sim}{L}(G)$ is the lattice of ℓ-ideals of a hyper-archimedean ℓ-group G then the set of prime elements of $\underset{\sim}{L}(G)$ is trivially ordered; see [6]. As we shall see this is not true of any hyper-archimedean, Brouwer lattice. Also $\underset{\sim}{L}(G)$ (for any abelian ℓ-group G) has the property that the meet of two compact elements is compact; once again this is not true in general in the abstract lattice setting. The above considerations may serve to motivate the following definitions. If L is an algebraic, Brouwer lattice we say it has the finite intersection property (FIP) if the meet of any two compact elements is compact. L has the compact splitting property (CSP) if each compact element of L is complemented, ie. if $c \vee c' = 1$, for each $c \in c(L)$.

Our next theorem ties things together properly.

1.8 Theorem: Let L be an algebraic, Brouwer lattice; the following are equivalent:

(a) L has the CSP.

(b) L has the FIP, and the set of primes of L is trivially ordered.

In particular, with either of these conditions L is hyper-archimedean.

Proof: (a) $\rightarrow$ (b) Suppose $c, d \in c(L)$ and $c \wedge d = \vee_{i\varepsilon I}\, x_i$, where the x_i are upward directed. $1 = d \vee d'$, so $c = (c \wedge d) \vee (c \wedge d')$, and hence $c = \vee_{i\varepsilon I}(x_i \vee (c \wedge d'))$. But then $c = x_{i_o} \vee (c \wedge d')$ for a suitable index i_o; this implies that $c \wedge d = x_{i_o}$. This suffices to show $c \wedge d$ is compact.

If $p < q$ are both prime, there is a $c \in c(L)$ with $c \leq q$ yet $c \not\leq p$. Since $c \wedge c' = 0$, $c' \leq p$, and so $1 = c \vee c' = q \vee p = q$, a contradiction.

The converse of theorem 1.8 requires a technical lemma which we shall not prove; its proof may be found in [9].

1.9 Lemma: Suppose L has the FIP; there is a one to one correspondence between minimal primes of L and ultrafilters of $c(L)$.† This correspondence is given as follows: if p is a minimal prime, let $N(p) = \{ c \in c(L) \mid c \not\leq p \}$; its inverse assigns to an ultrafilter M of $c(L)$ the element $\vee \{ c' \mid c \in M \}$.

(† Filter here means proper filter; an ultrafilter is a maximal filter.)

1.9.1 Corollary: If L has the FIP, then $p \in L$ is a minimal prime if and only if $p = \vee\{ c' \mid c \not\leq p, c \in c(L) \}$. If p is a minimal prime and $p \geq d \in c(L)$, then $p \not\geq d'$.

Now let us prove that (b) implies (a) in theorem 1.8: suppose $c \in c(L)$ yet $c \vee c' < 1$. Let p be a meet irreducible so that $p \geq c \vee c'$; by assumption p is a minimal prime, and so by 1.9.1 $p \geq c \rightarrow p \not\geq c'$, a contradiction. This completes the proof of theorem 1.8 .

We should check that the pair of conditions contained in (b) of 1.8 are irredundant. So consider an infinite set X with the finite complement topology, and let $L = \underset{\sim}{O}(X)$, the lattice of open sets of X; this is a hyper-archimedean, Brouwer lattice (interpreting infinite meets as interiors of intersections of open sets.) However, L has the FIP (each $x \in L$ is compact) while 0 is prime.

On the other hand let $X = \{ x_1, x_2, \ldots, y, z \}$, and $X' = X \setminus \{y, z\}$. Any subset of X' shall be open , and the open neighbourhoods of y (resp. z) are the sets with a finite complement in X'. Again let $L = \underset{\sim}{O}(X)$; L is a hyper-archimedean, Brouwer lattice in which every prime is maximal, yet if $U = X \setminus \{y\}$ and $V = X \setminus \{z\}$, then U and V are compact whereas $X' = U \cap V$ is not. The author owes this example to Jed Keesling.

We close this section with a rather striking analogue of a well known result about archimedean ℓ-groups. For its proof we refer the reader to [9].

1.10 Theorem: Suppose L is an archimedean, Brouwer lattice and $x \varepsilon L$ is a polar. Then $[x, 1]$ is archimedean.

2. Realizations of hyper-archimedean, Brouwer lattices as lattices of ℓ-ideals We were motivated to study this concept of an archimedean lattice in order to discover which lattices arise as the lattice $\underset{\sim}{L}(G)$ of ℓ-ideals of an archimedean ℓ-group G. Although some necessary conditions become obvious rather early in the game, (such as: the lattice must be an archimedean, Brouwer lattice with the FIP plus a good deal more), the problem is in general quite hard. In the case of hyper-archimedean ℓ-groups the matter as a lot simpler; we can fully characterize those lattices arising as the lattice of ℓ-ideals of a hyper-archimedean ℓ-group.

2.1 Theorem: A hyper-archimedean, Brouwer lattice L arises as the lattice of ℓ-ideals of an ℓ-group if and only if L has the CSP.

Proof: The necessity is well known (see [6]), so we pass to a sketch of the proof of the sufficiency; further details may be found in [9]. Let $\{ p_\lambda \mid \lambda \varepsilon \Lambda \}$ be the family of primes of L, and G* be the ℓ-group of integer-valued functions on Λ with finite range; alternatively, the ℓ-group of integral step functions on Λ. We define a mapping σ: $c(L) \to G^*$ by: $c\sigma_\lambda = 1$, if $c \not\leq p_\lambda$, and 0 if $c \leq p_\lambda$. It is easy to verify that σ is a lattice embedding.

Let G be the ℓ-subgroup of G* generated by $c(L)\sigma$, and $\underset{\sim}{P}(G)$ denote its lattice of principal ℓ-ideals; these are the compact elements of $\underset{\sim}{L}(G)$. Define a mapping

$\tau: c(L) \to \underset{\sim}{P}(G)$ by letting $c\tau = G(c\sigma) \equiv$ the ℓ-ideal generated by $c\sigma$ in G. Once again it is easily verified that τ is a lattice embedding, so one is only left with proving that τ is onto. Once this is done $c(L)$ and $\underset{\sim}{P}(G)$ are isomorphic lattices, and hence so are L and $\underset{\sim}{L}(G)$. It is here that one uses the full force of the CSP, in the following way: if $0 \neq g \in G$ expressible by $g = m_1(c_1\sigma) + \dots + m_k(c_k\sigma)$, then this expression can be rewritten so that the compact elements of L that appear are pairwise disjoint.

2.1.1 Corollary: If G is a hyper-archimedean ℓ-group then one cannot tell from the lattice of ℓ-ideals whether G is embeddable as an ℓ-subgroup of a group of real valued step functions.

3. Topological realizations of algebraic, Brouwer lattices and dualities

For further amplification on the material in this section the reader is urged to consult Bruns [4], Hofmann & Keimel [8], Martinez [10] and Schmidt [12], plus probably many, many others.

If L is an algebraic, Brouwer lattice, let $\underset{\sim}{I}(L)$ denote the set of meet-irreducibles, and $\underset{\sim}{P}(L)$ denote the set of primes of L. Topologize $\underset{\sim}{P}(L)$ by taking for its open sets the sets $P(x) = \{ p \in \underset{\sim}{P}(L) \mid p \not\geq x \}$, for all $x \in L$; topologize $\underset{\sim}{I}(L)$ with the subspace topology. Then $\underset{\sim}{P}(L)$ is a T_0-space with a base of compact, open sets, (it is spectral in the terminology of [8],) and $\underset{\sim}{I}(L)$ also has a base of compact, open sets and is t_1: every point is isolated in its closure; Bruns [4] first dealt with this separation axiom and called it $T_{1/2}$. Moreover, L is isomorphic with the lattice of open sets of both $\underset{\sim}{P}(L)$ and $\underset{\sim}{I}(L)$.

Let us say that a topological space X coordinatizes L if $L \simeq \underset{\sim}{O}(X)$, the lattices of open sets of X. Bruns [4] showed that if X is a T_o-coordinatization of L then X is homeomorphic to a set $\underset{\sim}{B}$, with $\underset{\sim}{I}(L) \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{P}(L)$, having the subspace topology of $\underset{\sim}{P}(L)$. It can easily be shown that $\underset{\sim}{I}(L)$ is (up to homeomorphism) the only t_1-coordinatization, and likewise $\underset{\sim}{P}(L)$ the only spectral one. The author will take up coordinatizations of non-algebraic lattices elsewhere.

Coordinatizations by $\underset{\sim}{P}(L)$ gives rise to a duality between the category of algebraic, Brouwer lattices and lattice homomorphisms preserving all joins, and the category of spectral spaces with bases of compact, open sets, together with all continuous mappings, see [8]. Coordinatization by $\underset{\sim}{I}(L)$ also gives rise to a duality; qua objects, a one to one correspondence between algebraic, Brouwer lattices and t_1-spaces with bases of compact, open sets. The morphism-classes pertinent to this duality are so restricted so as not to merit discussion here. Presumably, any "canonical" association of a set B, with $\underset{\sim}{I}(L) \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{P}(L)$, with L will produce a new duality, and it is a reasonable question whether every duality arises in this manner.

The theorem below interprets in terms of the $\underset{\sim}{I}(L)$-duality what topological conditions go with some of the lattice-theoretic notion discussed in this paper.

3.1 Theorem: Let L be an algebraic, Brouwer lattice.

i) L is archimedean if and only if each basic compact, open set of $\underset{\sim}{I}(L)$ has in the subspace topology a dense set of points whose closures are singletons.

ii) L is hyper-archimedean if and only if $\underset{\sim}{I}(L)$ is T_1.

iii) L satisfies the CSP if and only if $\underset{\sim}{I}(L)$ is Hausdorff.

iv) $\underset{\sim}{I}(L)$ is discrete if and only if L is Boolean.

For proofs of these consult [10].

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INSIDE FREE SEMILATTICES*

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Abstract. Necessary and sufficient conditions are derived for a given semilattice to be embeddable in a free semilattice.

§0. Introduction

I'd like to talk today about a circle of ideas concerning free semilattices. The problems involved are fairly concrete, and yet in them you will see echoes of several higher-level concepts dealt with in other papers at this conference.

As you well know, the very structure of free lattices and free modular lattices presents some very difficult questions. The basic structure of free distributive lattices is somewhat more transparent, and yet still eludes even a simple count of elements in the finite case.

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In contrast, the structure of free semilattices seems utterly trivial -- so much so, in fact, that it is hard at first to imagine how a free semilattice could give rise to any interesting questions at all.

Specifically, let us consider join-semilattices $(S, \vee)$, not necessarily with a 0-element or a 1-element. An example of such a semilattice is $\mathrm{Fin}(X)$, the semilattice of nonempty finite subsets of an arbitrary nonempty set X, with set-union being the operation. Our basic fact is that, for any nonempty set X of generators, the free semilattice $\mathrm{FSL}(X)$ on X is isomorphic to $\mathrm{Fin}(X)$. The isomorphism is the obvious one: For any $x_1, \ldots, x_n \in X$, the element $x_1 \vee \ldots \vee x_n$ of $\mathrm{FSL}(X)$ corresponds to $\{x_1, \ldots, x_n\} \in \mathrm{Fin}(X)$.

§1. Horn's Problem

A. Horn posed the following tempting "lunch-table problem."

Problem 1. Clearly, $\mathrm{FSL}(X)$ and its subsemilattices obey the condition

(*) every principal ideal is finite.

Is (*) also a sufficient condition for a semilattice S to be isomorphically embeddable in a free semilattice?

One indication pointing in the direction of an affirmative answer is that every semilattice can be isomorphically realized as a semilattice of subsets of itself; therefore the answer is always positive for finite semilattices. In a sense, then, the problem asks whether local embeddability is sufficient for global embeddability.

The answer, interestingly, is no. A counterexample is the "ladder" R depicted in Figure 1a.

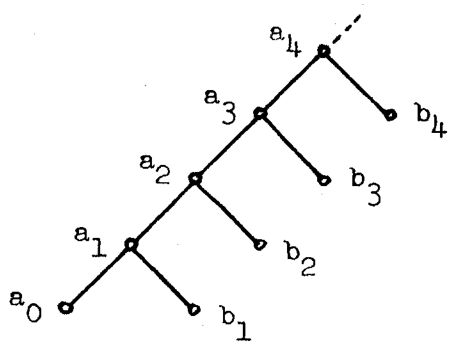

Figure 1a: the "ladder" R

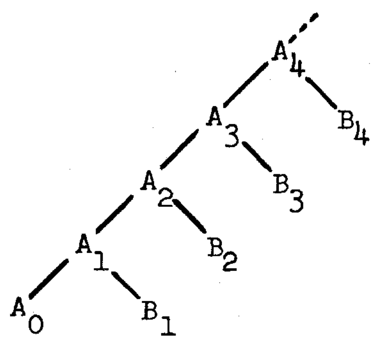

Figure 1b: subsets of X

As a ladder, R has certain deficiencies, but as a semilattice, R will be a useful example throughout this talk.

To show that R is a genuine counterexample, let us suppose, on the contrary, that R could be embedded in $FSL(X)$ for some X. Then there would be a corresponding subsemilattice of $Fin(X)$, consisting of finite subsets of X with the inclusion relations indicated by Figure 1b. For each n, $A_0 \cup B_n = A_n$, so that B_n can differ from A_n by at most a few elements of A_0, a fixed finite set. Thus, if we watch $A_0 \cap B_n$ as n varies, we must arrive at i and j $(i < j)$ such that $A_0 \cap B_i = A_0 \cap B_j$. In other words, to go from A_i to B_i we lose the same elements as in going from A_j to B_j. Since $A_i \subseteq A_j$, we conclude that $B_i \subseteq B_j$, in contradiction to Figure 1b.

This proof settles Problem 1, but it simultaneously raises another question, to be known, out of turn, as

Problem 3. Characterize those semilattices which can be embedded in a free semilattice.

An equivalent problem, of course, is to characterize those semilattices which can be isomorphically represented by finite subsets of some set, under the union operation. A logical setting for an attack on this problem is therefore the general theory of representations of semilattices by sets.

§2. Representations of semilattices

Let us review this theory. Many of the basic ideas are simply semilattice adaptations of the early distributive-lattice set-representations invented by Birkhoff and turned into a pretty, topological duality theory by Stone. Birkhoff and Frink [3] discussed meet-representations of arbitrary lattices, by ideals, which extend naturally to the semilattice case. Bruns [5,6], developed and surveyed these ideas further, placing them in their most natural context. Recently, such ideas have been studied in terms of category theory and duality and there further developed. Several speakers at this conference have followed this approach, although the specific categories used have differed, in varying degrees, from the ones I'll be using implicitly now.

Let S be a join-semilattice and let X be a set. Although our ultimate interest is representations by finite subsets, we must work now with $\mathrm{Pow}(X)$, the set of <u>all</u> subsets [power set] of X. We regard $\mathrm{Pow}(X)$ as a semilattice under $\cup$.

<u>Definition</u> 2.1. A <u>representation</u> of S on X is a semilattice homomorphism $\sigma : S \to \mathrm{Pow}(X)$ such that

(i) the sets $\sigma(s)$ distinguish points of X, i.e. no two distinct elements of X are contained in exactly the same subsets $\sigma(s)$, $s \in S$; and

(ii) the sets $\sigma(s)$ cover X, i.e., $\bigcup_{s\in S} \sigma(s) = X$.

If σ is one-to-one, i.e., an isomorphism, let us call σ "faithful."

Bruns [5,6] does not initially require conditions (i) and (ii), but they will be convenient for our purposes and are not really restrictive. For example, if S can be embedded in a free semilattice on a set Y of generators, then, as we noted, S is isomorphic to a semilattice of finite subsets of Y; if Y is "reduced" to a smaller set X by deleting elements not used and by identifying elements not distinguished by the finite subsets used, then we get a genuine faithful representation of S by finite subsets of X.

For a given semilattice S, there are three "famous representations" of S, all faithful:

1. The "regular" representation, σ_{reg}. Here $X = S$ and $\sigma_{reg}(s) = \{t \in S : s \not\leq t\}$.
2. The "ideal representation," σ_{id}. Here $X = Id(S)$, the set of ideals of S (including $\emptyset$), and $\sigma_{id}(s) = \{I \in Id(S) : s \notin I\}$.

3. The "CMI" representation, σ_{cmi}. Here $X = CMI(Id(S))$, the set of nonempty, completely meet-irreducible (c.m.i.) ideals of S, and again $\sigma_{cmi}(s) = \{I \in CMI(Id(S)) : s \notin I\}$.

(An element $m < 1$ of a complete lattice L is said to be completely (or strictly) meet-irreducible if m is not the meet of any set of strictly larger elements [2, p. 194]. Equivalently, there is a least element $c > m$ in L. Notice that c covers m. $Id(S)$ is an algebraic lattice, so has many c.m.i. elements; in fact, every element of an algebraic lattice is a meet of c.m.i. elements. If S has a 0-element, then $\emptyset$ is a legitimate c.m.i. element of $Id(S)$, but for technical reasons we'll always explicitly exclude $\emptyset$ in discussion of c.m.i. ideals.)

Each of the representations (1), (2), (3), has its own virtues:

(1) is the simplest, most natural representation. (The dual version of (1) is even more natural: Each element of a meet-semilattice is represented by the principal ideal it generates.)

(2) is the ultimate parent representation, in that <u>any</u> representation of S is equivalent to a "subrepresentation" of σ_{id}, obtained by restricting attention to some subset of $Id(S)$.

(I'll clarify this terminology in a moment.) For example, σ_{reg} corresponds to the set of principal ideals, and of course σ_{cmi} corresponds to the set of c.m.i. ideals. The association of each representation with a subset of Id(S) also provides a handy way of comparing the "size" of representations: Informally, we can write "$\sigma \subseteq \tau$" when the associated subsets of Id(S) are so related.

(3) is an especially economical, efficient representation, as Birkhoff and Frink point out in the case of semilattice representation of lattices [2].

Before considering an example, let's clarify the terminology just used: Two representations σ_1, σ_2 of S on sets X_1, X_2 are said to be <u>equivalent</u> if there is a one-to-one correspondence between X_1 and X_2 which makes $\sigma_1(s)$ correspond to $\sigma_2(s)$ for each $s \in S$. For a representation σ of S on X, a <u>subrepresentation</u> of σ is any representation τ of S obtained by taking a subset Y of X and setting $\tau(s) = \sigma(s) \cap Y$. To be more graphic, we can say that "τ is the intersection of σ with Y." Of course, even for faithful σ, it is possible to "lose faith" in passing from σ to τ, if we strip away too many elements of X in forming Y. An obvious necessary and sufficient condition for τ to be faithful is that σ be faithful and that of any two representing sets $\sigma(s_1) \neq \sigma(s_2)$, there be an element of Y in one and not the other. If $\sigma = \sigma_{id}$, this condition is fulfilled if Y is the set of principal ideals or the set of nonempty c.m.i. ideals.

If σ is a representation of S on a set X, the equivalent subrepresentation of σ_{id} is easily constructed: each element $x \in X$ corresponds to the ideal $I_x = \{t \in S : x \notin \sigma(t)\} \in Id(S)$, and Y is the set of such ideals. This same correspondence shows up as the basis of categorical duality theory, where ideals may appear as characters and $Id(S)$ as the dual space of S.

Let's look at all three standard representations in one particular setting.

Example 2.2. Let S be $Fin(X)$ itself, for some set X, and let $\sigma : S \to Pow(X)$ be simply the inclusion map. Thus, the elements of S are finite subsets of X; the ideals of S correspond naturally to arbitrary subsets of X. The subset A of X corresponds to the ideal $I_A = \{F \in Fin(X) : F \subseteq A\}$ of $Fin(X)$. For each element of S, i.e., for each nonempty finite subset F of X, $\sigma_{reg}(F)$ consists of all finite subsets of X which do not contain F; $\sigma_{id}(F)$ consists of ideals corresponding to all subsets of X which do not contain F; and it is not hard to determine that $\sigma_{cmi}(F)$ consists of ideals corresponding to those "cosingleton" subsets $X - \{x\}$ for which $x \in F$. Of the three, only the CMI representation has finite representing subsets even when X is infinite, so its pretense to economy is borne out in this instance.

By the way, one feature of this example, namely, that ideals of S are "represented" by subsets of the same set X, leads to a generalization, in which $\mathrm{Id}(S)$ is regarded as a semilattice:

<u>Observation</u> 2.3. If σ is a representation of a semilattice S on a set X, then σ^* is a representation of $\mathrm{Id}(S)$ on X, where $\sigma^*(I) = \bigcup_{s\in I} \sigma(s)$ for each $I \in \mathrm{Id}(S)$. Even if σ is faithful, though, σ^* may not be, as can be seen by representing $\mathrm{Pow}(X)$ on X by the identity map, for an infinite set X.

§3. Economy of representation.

We have now reviewed the three basic representations of a semilattice S. To judge from the example of the preceding section, the CMI representation, with its economy, will be the most useful for studying representations by finite subsets. In this connection, we have left one question as yet unanswered:

Problem 2. In what sense is the CMI representation the most economical?

Once this problem is settled, we'll be in a stronger position to investigate embeddings in free semilattices.

A natural conjecture in answer to Problem 2 would be that "$\sigma_{cmi} \subseteq \sigma$" for all faithful representations σ of S. A glance at the example of the preceding section shows the falsity of this conjecture, however: For an infinite set X and $S = \mathrm{Fin}(X)$, $\sigma_{cmi} \not\subseteq \sigma_{reg}$, even though $\sigma_{cmi}(s)$ is always finite.

Here's another try. The topological analogue of a finite set is a compact set, and, happily, compact subsets form a semilattice under union, in any topological space. (The intersection of two compact sets may not be compact.) Topological representation theories, on the other hand, most naturally represent structures having a join operation by open subsets. Stone early showed the advantage of performing a marriage of these two properties by considering <u>open</u> <u>compact</u> subsets; among spaces with many such subsets, the prime example - in fact the ideal example - is the Stone representation space of a Boolean algebra [20,21]. The Stone space is Hausdorff; for semilattices, T_0 spaces constitute a natural setting.

An investigation provides the following solution to Problem 2, with a few added frills.

<u>Theorem</u> 3.1. Let $\sigma : S \to \mathrm{Pow}(X)$ be a faithful representation of a semilattice S on a set X. Then the following conditions on σ are equivalent:

(1) "$\sigma_{cmi} \subseteq \sigma$";

(2) under some topology on X, every set $\sigma(s)$ is compact and open;

(3) σ^* is a faithful representation of $\mathrm{Id}(S) \setminus \{\emptyset\}$ on X;

(4) each (nonempty) c.m.i. ideal I of S has the form I_x for some $x \in X$, where $I_x = \{t \in S : x \notin \sigma(t)\}$.

[A proof of Theorem 3.1 is supplied in the Appendix.]

Thus the CMI representation is the smallest faithful representation by open compact subsets.

In particular, every semilattice has a faithful representation by open compact subsets.

The theorem immediately gives a fact, reminiscent of Example 2.2 ($S = \mathrm{Fin}(X)$), which is exactly what we need:

Corollary 3.2. For a semilattice S, the following are equivalent.

(1) S can be embedded in some free semilattice;

(1') S has a faithful representation by finite subsets of some set X;

(2) the CMI representation of S is itself a representation by finite subsets.

The only implication needing proof is $(1') \Rightarrow (2)$. All we have to do for this proof is to give X of $(1')$ the discrete topology and quote $(2) \Rightarrow (1)$ of Theorem 3.1.

The theorem 3.1 gives us useful information even in the case where S is finite. For such an S, all nonempty ideals are principal and so correspond to elements. The nonempty CMI ideals correspond to the "uniquely covered" elements - elements covered by exactly one other element. For convenience, let $\mathrm{NUC}(S)$ denote the Number of Uniquely Covered elements of S. In the CMI representation, then, $\sigma_{cmi}(t)$ consists of ideals corresponding to uniquely covered elements not $\geq t$. It follows that $|\sigma_{cmi}(t)| = NUC(S) - NUC[t,1]$, where 1 is the top element of S, $[t,1]$ denotes the closed interval $\{s \in S : t \leq s \leq 1\}$, and $|A|$ denotes the cardinality of a set A. Thus we obtain the following fact.

Corollary 3.3. Let σ faithfully represent a finite semilattice S on a set X. then for each $t \in S$, $|\sigma(t)| \geq NUC(S) - NUC[t,1]$.

Proof. Again we put the discrete topology on X and quote $(2) \Rightarrow (1)$ of Theorem 3.1. X is necessarily finite.

Here we have implicitly observed that for finite semilattices, the CMI representation really is "contained" in any faithful representation. Of course, the CMI representation for finite semilattices is really nothing more than a dualized version of the familiar expression of lattice elements as joins of join-irreducibles. A direct proof of Corollary 3.3 would not be difficult.

§4. The Characterization.

Recall that our goal has been a solution of

Problem 3. Characterize those semilattices which can be embedded in a free semilattice.

Actually, Corollary 3.2 deserves to be called an answer, in that it gives a criterion which is "intrinsic" to S (namely, that the CMI representation of S is itself a representation by finite subsets). By rephrasing this criterion, we obtain

Solution 1. A semilattice S can be embedded in a free semilattice if and only if each element of S is contained in all except finitely many completely meet-irreducible ideals of S.

In most situations, this criterion would be cumbersome. It does apply nicely, though, to our original "ladder" semilattice R of Figure 1a. There the principal ideal generated by each b_i is plainly c.m.i., and none of these ideals contains a_0. Thus, the condition of Solution 1 fails, and R is not embeddable in a free semilattice. (Actually, Solution 1 was developed first and R was invented to conform to a failure of that criterion.)

One ingredient is missing from Solution 1: The requirement that all principal ideals be finite. This property is especially useful, because Corollary 3.3 gives us potentially relevant information about faithful representations of such a finite ideal, if not the whole semilattice. The following conjecture is natural: For each element $t \in S$, look at representations of the various principal ideals containing t, regarded as semilattices in their own right. (For each principal ideal, choose the most economical faithful representation possible.) If the size of subsets representing t remains bounded as the principal ideals get larger and larger, then S should have a faithful representation which represents t by a finite set. If not, t should not be so representable.

Let us incorporate this conjecture, for all $t \in S$, into a proposed solution, using the estimate of Corollary 3.3. The principal ideal generated by an element s can be denoted by $(s]$.

<u>Solution</u> 2. A semilattice S can be embedded in a free semilattice if and only if the following two conditions are met:

(a) Every principal ideal of S is finite, and

(b) for each $t \in S$, $\mathrm{NUC}(s] - \mathrm{NUC}[t,s]$ is bounded as s runs through $\{s : s \geq t\}$.

This conjectured solution is true. Half of the proof, at least, is immediate: Suppose S <u>can</u> be embedded in a free semilattice. Then S has a representation σ by finite subsets of a set X. For any $t \in S$ and $s \geq t$, σ restricted to $(s]$ is an isomorphism of $(s]$ into $\mathrm{Pow}(X)$. This restriction might not meet our technical requirements for being a representation, but by discarding some elements of X and identifying others, as discussed in Section 2, we get a genuine representation $\sigma^{(s)}$ of $(s]$ on a "smaller" set X_0. By Corollary 3.3, $NUC(s] - NUC[t,s] \leq |\sigma^{(s)}(t)|$, which is at most $|\sigma(t)|$, a bound not depending on s.

For the other half of the proof, I'd like to describe a method which is simple and pretty, if a knowledge of ultraproducts is presupposed: Suppose S satisfies (a) and (b) (and is not itself finite). For each $s \in S$, let $\sigma^{(s)}$ be a representation of the ideal $(s]$ on a set $X^{(s)}$, with $\sigma^{(s)}$ being equivalent to the CMI representation of $(s]$. S can be embedded in an ultraproduct of its principal ideals by taking a suitable ultrafilter $\mathcal{U}$ on S (one among whose members are all principal dual ideals of S [10, Corollary, p. 27]); thus $S \hookrightarrow \Pi_s(s]/\mathcal{U}$. The corresponding ultraproduct of the representation $\sigma^{(s)}$ is a faithful representation σ of $\Pi_s(s]/\mathcal{U}$ on the set $X = \Pi_s X^{(s)}/\mathcal{U}$. "Restricted" to S, σ becomes a faithful representation, with $\sigma(t)$ being essentially $\Pi_{s \geq t}\sigma^{(s)}(t)/\mathcal{U}$. Since $|\sigma^{(s)}(t)| = NUC(s] - NUC[t,s]$, which is bounded as s runs through $\{s : s \geq t\}$, the ultraproduct expression for $\sigma(t)$ yields a finite set.

(Does there exist an alternate proof which constructs the representation of S explicitly, while avoiding any form of the axiom of choice?)

§5. Applications.

Let's apply Solution # 2 in several cases.

Example 5.1. Let S be the "ladder" semilattice R of Figure 1a. For $t = a_0$, s runs through the a_n. $NUC(a_n] - NUC[a_0,a_n] = 2n - n = n$, which is unbounded. Therefore R is not embeddable, as we know.

Example 5.2. Let S be the semilattice depicted in Figure 2. S is really a modular lattice consisting of $N \times N$ $(N = \{0,1,2,\ldots\})$ with additional elements c_i adjoined.

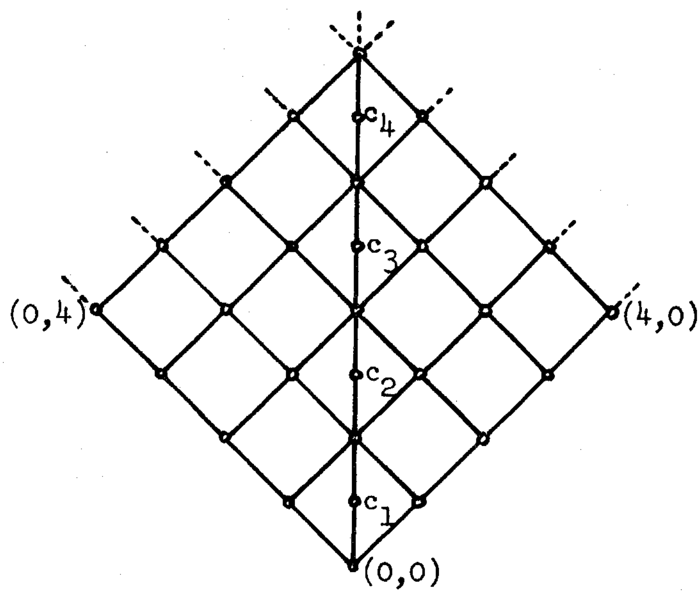

Figure 2

For $t = (m,n)$ and $s = (m',n')$, we get $NUC(s] - NUC[t,s] =$ $(m' + n' + [\text{a certain number of } c_i]) - ([m' - m] + [n' - n] + [\text{a certain number of } c_i]) = m + n + |\{c_i : c_i \leq (m',n'),$ $c_i \not\geq (m,n)\}| \leq m + n + |\{c_i : c_i \not\geq (m,n)\}| = m + n + \max(m,n)$. The computations where s and/or t is among the c_i differ by at most 1 from the same answer, for suitable $m = n$ or $m' = n'$. Thus $NUC(s] - NUC[t,s]$ is bounded, for each t, and the semilattice of Figure 2 is embeddable in a free semilattice.

Example 5.3. Let V be an infinite-dimensional vector space over a finite field $GF(q)$, and let S be its (semi-) lattice of finite-dimensional subspaces. Because $(s]$ (i..e, $[0,s]$) is relatively complemented, the only uniquely covered elements are its "coatoms." Since $[0,s]$ is self-dual, we can count its atoms (one-dimensional subspaces) instead; if s is a space of dimension n, this count is $(q^n-1)/(q-1)$, the number of nonzero vectors divided by the number of vectors in a one-dimensional subspace. If t is k-dimensional, $[t,s]$ is isomorphic to the subspace lattice of an $(n-k)$-dimensional vector space, so that the same kind of calculation applies. Thus $NUC(s] - NUC[t,s] = [(q^n-1)/(q-1)] - [(q^{n-k}-1)/(q-1)]$ $= q^{n-k}(q^k-1)/(q-1)$, which is unbounded for fixed k as $n \to \infty$. Therefore S cannot be embedded in a free semilattice, even though its principal ideals are finite.

Further examples, for which the calculations are interesting but will not be carried out here, are these:

Example 5.4. Let T be an infinite set, and let S be the (semi-) lattice consisting of those partitions of T which have only finitely many nontrivial classes. In other words, S is the semilattice of compact elements of the full partition lattice of T.

Example 5.5. Again let T be an infinite set and let S be the dual of the meet-semilattice of "cocompact" partitions of T; i.e., the partitions of T into finitely many pieces.

Finally, let us consider this case:

Example 5.6. Let S be any distributive lattice in which all principal ideals are finite. In a finite distributive lattice D, the number of meet-irreducible elements equals the length $\ell(D)$; therefore $\mathrm{NUC}(s] - \mathrm{NUC}[t,s] = \ell([0,s]) - \ell([t,s]) = \ell([0,t])$, a fixed, hence bounded, quantity as s varies. Thus such a lattice, regarded as a semilattice, can always be embedded in a free semilattice. (Horn and Kimura [12] have shown that any distributive lattice of this type is projective as a semilattice, from which the embeddability is also immediate.)

Appendix: Proof of Theorem 3.1.

Let's follow the order $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (3)$: By assumption, σ is equivalent to the intersection of σ_{id} with some subset Y of $Id(S)$ such that $CMI(Id(S)) \subseteq Y$. Then σ^* is equivalent to "${\sigma_{id}}^* \cap Y$." (${\sigma_{id}}^*$ is nothing more than the regular representation of $Id(S)$.) Since each nonempty ideal I of S is an intersection of nonempty c.m.i. ideals and so is uniquely identifiable by which c.m.i. ideals do or do not contain I, ${\sigma_{id}}^* \cap Y$, and hence σ^*, is one-to-one on $Id(S) \setminus \{\emptyset\}$.

$(3) \Rightarrow (2)$: Let X be given the topology for which the sets $\sigma(s)$ themselves form a subbase for the open sets. Since σ^* is a complete join -isomorphism, taking joins in $Id(S) \setminus \{\emptyset\}$ to unions in $Pow(X) \setminus \{\emptyset\}$, the fact that the principal ideals $(s]$ are compact elements of S [10; Lemma 2, p. 21] translates into the statement that any covering of one of the chosen subbasic sets by other subbasic sets has a finite subcover. Alexander's Subbase Theorem [13, p. 139] then asserts that each subbasic set $\sigma(s)$ is compact in the generated topology adopted for X.

(2) $\Rightarrow$ (4): Without loss of generality, we may assume that S consists of open compact subsets of the topological space X; the members of S cover X. Let I be a nonempty c.m.i. ideal of S. We must find an $x \in X$ such that $I = I_x$, where $I_x = \{s : x \notin s\}$. Let I^+ be the unique smallest ideal properly containing I, and let s_0 be an element of I^+ not in I. The members of I do not cover s_0; if they did, the union of the members of some (nonempty) finite subcover would contain s_0 and would also be in I, forcing $s_0 \in I$, contrary to assumption. Let x, then, be a point of s_0 not covered by any member of I. By definition, $I_x \supseteq I$. To prove $I_x = I$, let us consider $s \notin I$ and show $s \notin I_x$, i.e., $x \in s$: The join of I and the principal ideal $(s]$, $I \vee (s]$, properly contains I, so $I^+ \subseteq I \vee (s]$. $s_0 \in I^+$ implies that $s_0 \in I \vee (s]$, in other words, that $s_0 \subseteq t \cup s$ for some $t \in S$. Since $x \in s_0$ and $x \notin t$, we must have $x \in s$, as desired.

(4) $\Rightarrow$ (1): It suffices to consider the case where $X \subseteq \mathrm{Id}(S)$ and $\sigma = \sigma_{id} \cap X$. But in this case, for each x, the ideal I_x coincides with x itself. Thus the condition of (1), that X include all c.m.i. ideals, reduces to (4).

Remark. Our choice of conventions regarding $\emptyset$ as an ideal, etc., become relevant in the proof just concluded. To have σ^* be an isomorphism on all of Id(S) in (3) of Theorem 3.1, for instance, we could either (a) include $\emptyset$ as a c.m.i. ideal, or (b) exclude $\emptyset$ as an ideal. If (a), then representing sets $\sigma(s)$ cannot be allowed to be empty, or else (1) fails; furthermore, c.m.i. ideals no longer correspond only to uniquely covered elements in the case of a finite lattice, so that the "NUC" calculations must be altered. If (b), then σ_{id} no longer contains all representations, unless the representing sets $\sigma(s)$, $s \in S$, are required to have empty intersection - a condition with other side effects. Of course, the conventions adopted do, unhappily, give Id(S) one more element than S when S is a finite lattice.

Acknowledgement: The author wishes to thank A. Horn, G. Bruns, J. Martinez, and J. Schmidt for their helpful comments.

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General references for lattice theory are [2] and [11]; for universal algebra, [10]. Relevant category - theoretical ideas are to be found in [5] and [6] (implicitly), in [9], [14], [16], [18], [19], and in the Conference talks of B. Banaschewski and K. H. Hofmann. Ideas related to spaces of ideals are treated in [14], and in the Conference talks of J. Martinez, R. Mena, and H. Werner. For further references, see the bibliographies of [5], [16], and [18].

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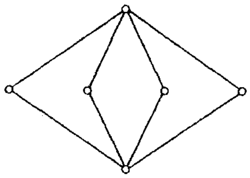
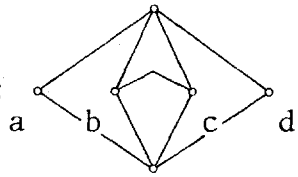
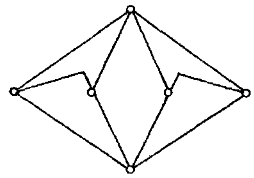
ON FREE MODULAR LATTICES OVER PARTIAL LATTICES WITH FOUR GENERATORS

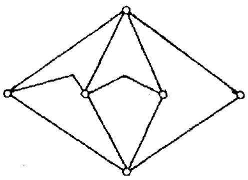
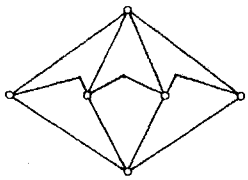
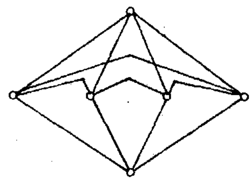
by Günter Sauer, Wolfgang Seibert, and Rudolf Wille

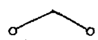
1. Main Results: This paper is a continuation of DAY, HERRMANN, WILLE [2]. It also wants to give some contribution to the word problem for the free modular lattice with four generators by examining free modular lattices over partial lattices with four generators. By a partial lattice we understand a relative sublattice of any lattice, that is a subset together with the restrictions of the operations $\wedge$ and $\vee$ to this subset (e.g. GRÄTZER [3 ; Definition 5.12]). The principal results proved in this paper are the following theorems.

Theorem 1: Let J^4 be a partial lattice $(\{0,g_1,g_2,g_3,g_4,1\};\wedge,\vee)$ with $g_i \vee 1=1$ and $g_i \wedge g_j=0$ for $i \neq j\,(1 \leq i,j \leq 4)$. Then every modular lattice which has J^4 as generating relative sublattice is freely generated by J^4 if and only if J^4 is described by one of the following diagrams:

M_4:  J^4_1: 

$J^4_{1,1}$: 

J^4_2:  J^4_3:  J^4_4: 

(an angle of the form  means that the join of the connected elements is deleted from the lattice M_4).

By $FM(J^4)$, we denote the free modular lattice over the partial lattice J^4, that is a certain modular lattice freely generated by J^4. In DAY, HERRMANN, WILLE [2] the free modular lattice $FM(J^4_1)$ is extensively examined; especially, it is shown that $FM(J^4_1)$ is an infinite, subdirectly irreducible lattice with one non-trivial congruence relation $\Theta(FM(J^4_1)/\Theta \tilde{=} M_4)$. For the formulation of Theorem 2 we still need some notations:

Z is the set of all integers;

N is the set of all positive integers ($N_o := N \cup \{0\}$);

G is the free abelian group with countably many generators;

$(e_i \mid i \in Z)$ and $(f_i \mid i \in N)$ are some basis of G;

S_G is the lattice of all subgroups of G.

Theorem 2: The free modular lattices $FM(J_1^4)$, $FM(J_{1,1}^4)$, $FM(J_2^4)$, $FM(J_3^4)$ and $FM(J_4^4)$ are isomorphic to subdirect powers of $FM(J_1^4)$ and sublattices of S_G; generators of the subdirect powers and sublattices, respectively, are described by the following list:

	n	generators in $FM(J_1^4)^n$	generators in S_G
$FM(J_1^4)$	1	a b c d	$\langle e_{2i} \mid i \in Z\rangle$ $\langle e_{2i}+e_{2i+1} \mid i \in Z\rangle$ $\langle e_{2i-1}+e_{2i} \mid i \in Z\rangle$ $\langle e_{2i-1} \mid i \in Z\rangle$
$FM(J_{1,1}^4)$	2	(a,c) (d,b) (b,d) (c,a)	$\langle e_{4i}, e_{4i-1}+e_{4i+1} \mid i \in Z\rangle$ $\langle e_{4i+2}, e_{4i+1}+e_{4i+3} \mid i \in Z\rangle$ $\langle e_{4i+3}, e_{4i-2}+e_{4i} \mid i \in Z\rangle$ $\langle e_{4i+1}, e_{4i}+e_{4i+2} \mid i \in Z\rangle$
$FM(J_2^4)$	2	(a,b) (b,c) (c,d) (d,a)	$\langle f_{2i} \mid i \in N\rangle$ $\langle f_{2i}+f_{2i+1} \mid i \in N\rangle$ $\langle f_{2i-1}+f_{2i} \mid i \in N\rangle$ $\langle f_{2i-1} \mid i \in N\rangle$
$FM(J_3^4)$	3	(d,a,b) (a,b,c) (b,c,d) (c,d,a)	$\langle e_{2i} \mid i \in Z\rangle$ $\langle e_{2i}+e_{2i+1}, e_{2j-1}+e_{2j} \mid i>0\geq j\rangle$ $\langle e_{2i-1}+e_{2i}, e_{2j}+e_{2j+1} \mid i>0>j\rangle$ $\langle e_{2i-1} \mid i \in Z\rangle$
$FM(J_4^4)$	4	(a,b,c,d) (b,c,d,a) (c,d,a,b) (d,a,b,c)	$\langle e_{2i}, e_{2j-1} \mid i>0\geq j\rangle$ $\langle e_{2i}+e_{2i+1} \mid i \in Z\rangle$ $\langle e_{2i-1}+e_{2i} \mid i \in Z\rangle$ $\langle e_{2i-1}, e_{2j} \mid i>0>j\rangle$

Theorem 3: The congruence lattice of $FM(J_1^4)$, $FM(J_{1,1}^4)$, $FM(J_2^4)$, $FM(J_3^4)$ and $FM(J_4^4)$, respectively, is a 2^n-element Boolean lattice with a new greatest element where n is the number of undefined joins in the generating partial lattice.

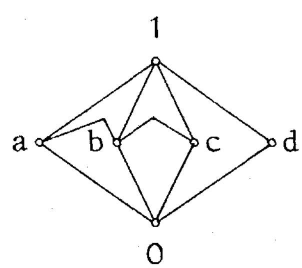
Proof of Theorem 1: Let M be a modular lattice freely generated by J^4, where J^4 as relative sublattice of M is described by one of the listed diagramms. There is a homomorphism ψ from $FM(J^4)$ onto M whose restriction to J^4 is the identity. By Theorem 3, ψ has to be injective (otherwise, there are more joins in ψJ^4 than in J^4). Thus, M is freely generated by J^4 . For the converse, we recall that every projective plane Π_p over a prime field is generated by four points a_1, a_2, a_3, a_4 no three on a line. Obviously, the elements 0, a_1, a_2, a_3, a_4 and 1 form a relative sublattice J_6^4 of Π_p in which $a_i \vee a_j$ is not defined for $i \neq j (1 \leq i,j \leq 4)$. Since projective planes over different prime fields are not isomorphic, no such plane is freely generated by J_6^4 . This argument can simiarly be applied to the remaining partial lattices $J_3^{3,1}$ (3 undefined joins), $J_{3,1}^{3,1}$ (4 undefined joins) and J_5^4 (5 undefined joins). First, we take a line g in Π_p which does not contain a_1, a_2 and a_3. Then, the relative sublattice $J_3^{3,1}$ consisting of the elements 0, a_1, a_2, a_3, g and 1 generates Π_p, because a_1, a_2, $(a_1 \vee a_3) \wedge g$ and $(a_2 \vee a_3) \wedge g$ are four points no three on a line. $J^{3,1}_{3,1}$ is represented in $\Pi_p \times FM(J^4_1)$ by $\{(0,0),(a_1,d),(a_2,a),(a_3,b),(g,c),(1,1)\}$, and J^4_5 is represented in $\Pi_p \times \Pi_p$ by $\{(0,0),(a_1,g),(a_2,a_3),(a_3,a_2),(g,a_1),(1,1)\}$. By the above argument, no of the described subsets freely generates its generated sublattice. Thus, there are no more partial lattices J^4 with the desired properties.

Proof of Theorem 2: It can be easily seen that the generators together with the smallest and the greatest element of $FM(J^4_1)^n$ and S_G, resp., form a relative sublattice isomorphic to the corresponding partial lattice J^4. Thus, by Theorem 1, the sublattice generated by the described elements is isomorphic to $FM(J^4)$.

Proof of Theorem 3: This proof will cover the rest of the paper. In section 2 the assertion is proved for $FM(J^4_2)$ by solving the word problem for $FM(J^4_2)$. Using these results, the congruence lattice of $FM(J^4_4)$ is determined in section 3. This result immediately gives us the congruence lattices of the remaining lattices.

2. $FM(J_2^4)$: The goal of this section is to solve the word problem for $FM(J_2^4)$ in a similar manner as the word problem is solved for $FM(J_1^4)$ in DAY, HERRMANN, WILLE [2]. The elements of $FM(J_2^4)$ will be represented by quadruples of natural numbers and ∞. By Proposition 19, meets and joins are described in terms of these quadruples. As consequence of Proposition 19 and [2 ; Theorem 4 and Theorem 5], we get Theorem 3 for J_2^4. It should be mentioned that the lattice $FM(J_2^4)$ appears first in BIRKHOFF [1 ;p.70] where the generators in S_G are described as in Theorem 2.

As in DAY, HERRMANN, WILLE [2], the method which makes computations practicable is to introduce suitable endomorphisms of $FM(J_2^4)$. An detailed study of these endomorphisms by several lemmata prepares the proof of Proposition 19. Since it does not make any confusion, we choose the same notation for the elements of J_2^4 as for the elements of J_1^4 :



Lemma 4: There are endomorphisms μ and ν of $FM(J_2^4)$ such that

$$(1) \quad \begin{aligned} \mu 0 &= 0 \\ \mu a &= b \\ \mu b &= a \wedge (b \vee c) \\ \mu c &= d \wedge (b \vee c) \\ \mu d &= c \\ \mu 1 &= b \vee c \end{aligned} \qquad\qquad (2) \quad \begin{aligned} \nu 0 &= 0 \\ \nu a &= d \wedge (a \vee b) \\ \nu b &= c \wedge (a \vee b) \\ \nu c &= b \\ \nu d &= a \\ \nu 1 &= a \vee b \end{aligned}$$

Proof: By modularity, it can be easily seen that (1) and (2) define homomorphisms from J_2^4 into $FM(J_2^4)$. Since $FM(J_2^4)$ is freely generated by J_2^4, those homomorphisms can be (uniquely) extended to endomorphisms μ and ν of $FM(J_2^4)$.

Lemma 5: $\mu\nu = \nu\mu$

Proof:
$$\mu\nu a = \mu(d \wedge (a \vee b)) = c \wedge (b \vee (a \wedge (b \vee c))) = c \wedge (a \vee b) = \nu b = \nu\mu a ,$$
$$\begin{aligned} \mu\nu b &= \mu(c \wedge (a \vee b)) = (d \wedge (b \vee c)) \wedge (b \vee (a \wedge (b \vee c))) = d \wedge (b \vee c) \wedge (a \vee b) \\ &= d \wedge (a \vee b) \wedge ((c \wedge (a \vee b)) \vee b) = \nu(a \wedge (b \vee c)) = \nu\mu b , \end{aligned}$$
$$\mu\nu c = \nu\mu c \text{ (analogous to } \mu\nu a = \nu\mu a) ,$$
$$\mu\nu d = \mu a = b = \nu c = \nu\mu d .$$

Lemma 6: $\mu^n x \le \mu^m x$ and $\nu^n x \le \nu^m x$ for $x \in J_2^4$ if $n \equiv m \pmod 2$ and $n \ge m$.

Proof: The assertion is an immediate consequence of $\mu^2 x \le x$ and $\nu^2 x \le x$.

Lemma 7: Let $n\varepsilon N_o$.

(1) $a\vee\mu^{2n}d = a\vee\mu^{2n+1}d$ (2) $c\vee\nu^{2n}d = c\vee\nu^{2n+1}d$

(3) $d\vee\mu^{2n}a = d\vee\mu^{2n+1}a$ (4) $d\vee\nu^{2n}c = d\vee\nu^{2n+1}c$

Proof: (1): $a\vee\mu^{2n}d=a\vee\mu^{2n}a\vee\mu^{2n}d=a\vee\mu^{2n}(a\vee d)=a\vee\mu^{2n}(a\vee c)=a\vee\mu^{2n}a\vee\mu^{2n}c=a\vee\mu^{2n}c=a\vee\mu^{2n+1}d$; the proofs of (2), (3) and (4) are analogous.

Lemma 8: Let $x\varepsilon J_2^4$, and let $n\varepsilon N_o$.

(1) $x\wedge\mu^{2n}1 = \mu^{2n}x$ (2) $x\wedge\nu^{2n}1 = \nu^{2n}x$

(3) $a\wedge\mu^{2n+1}1 = \mu^{2n+2}a$ (4) $c\wedge\nu^{2n+1}1 = \nu^{2n+2}c$

(5) $d\wedge\mu^{2n+1}1 = \mu^{2n+2}d$ (6) $d\wedge\nu^{2n+1}1 = \nu^{2n+2}d$

(7) $b\wedge\mu^{2n+1}1 = \mu^{2n}b$ (8) $b\wedge\nu^{2n+1}1 = \nu^{2n}b$

(9) $c\wedge\mu^{2n+1}1 = \mu^{2n}c$ (10) $a\wedge\nu^{2n+1}1 = \nu^{2n}a$

Proof: (1): Let $x'\varepsilon J_2^4$ with $x\vee x'=1$. Then $x\wedge\mu^{2n}1=x\wedge(\mu^{2n}x\vee\mu^{2n}x')=\mu^{2n}x\vee(x\wedge\mu^{2n}x')=\mu^{2n}x\vee 0=\mu^{2n}x$. (2): analogous to (1).
(3): $a\wedge\mu^{2n+1}1=a\wedge\mu^{2n+1}(b\vee d)=a\wedge(\mu^{2n+2}a\vee\mu^{2n+1}d)=\mu^{2n+2}a\vee(a\wedge\mu^{2n+1}d)=\mu^{2n+2}a$; (4), (5), (6): analogous to (3).
(7): $b\wedge\mu^{2n+1}1=b\wedge\mu^{2n+1}(a\vee d)=b\wedge(\mu^{2n}b\vee\mu^{2n+1}d)=\mu^{2n}b\vee(b\wedge\mu^{2n+1}d)=\mu^{2n}b$; (8), (9), (10): analogous to (7).

Lemma 9: Let $n\varepsilon N_o$.

(1) $\nu\mu^{n}a = \mu^{n}d\wedge\nu 1$ (2) $\mu\nu^{n}a = \nu^{n}b\wedge\mu 1$

(3) $\nu\mu^{n}b = \mu^{n}c\wedge\nu 1$ (4) $\mu\nu^{n}b = \nu^{n}a\wedge\mu 1$

(5) $\nu\mu^{n}c = \mu^{n}b\wedge\nu 1$ (6) $\mu\nu^{n}c = \nu^{n}d\wedge\mu 1$

(7) $\nu\mu^n d = \mu^n a\wedge\nu 1$ (8) $\mu\nu^n d = \nu^n c\wedge\mu 1$

(9) $\nu\mu^n 1 = \mu^n 1\wedge\nu 1$ (10) $\mu\nu^n 1 = \nu^n 1\wedge\mu 1$

Proof: (1): The case n=0 is proved by $\nu a=d\wedge(a\vee b)=d\wedge\nu 1$. By induction hypothesis, we get for $n>0$: $\nu\mu^n a=\mu\nu\mu^{n-1}a=\mu(\mu^{n-1}d\wedge\nu 1)=\mu^n d\wedge\mu\nu 1=\mu^n d\wedge(a\vee b)\wedge(b\vee c)=\mu^n d\wedge\nu 1\wedge\mu 1=\mu^n d\wedge\nu 1$. The other assertions analogously follow.

<u>Lemma 10:</u> Let $n,i,j\in N_o$.

(1) $\nu^{2n}(\mu^i a\vee\mu^j d)=\nu^{2n}1\wedge(\mu^i a\vee\mu^j d)$ (2) $\mu^{2n}(\nu^i c\vee\nu^j d)=\mu^{2n}1\wedge(\nu^i c\vee\nu^j d)$

(3) $\nu^{2n+1}(\mu^i a\vee\mu^j d)=\nu^{2n+1}1\wedge(\mu^i d\vee\mu^j a)$

(4) $\mu^{2n+1}(\nu^i c\vee\nu^j d)=\mu^{2n+1}1\wedge(\nu^i d\vee\nu^j c)$

Proof: (1): The case n=0 is trivial. The case n=1 is proved by $\nu^2(\mu^i a\vee\mu^j d)=\nu(\nu\mu^i a\vee\nu\mu^j d)=\nu((\mu^i d\wedge\nu 1)\vee(\mu^j a\wedge\nu 1))=\nu((\mu^i d\wedge\nu 1)\vee\mu^j a)=\nu(\nu 1\wedge(\mu^i d\vee\mu^j a))=\nu^2 1\wedge\nu 1\wedge(\mu^i a\vee\mu^j d)=\nu^2 1\wedge(\mu^i a\vee\mu^j d)$. By induction hypothesis, we get for $n>1$: $\nu^{2n}(\mu^i a\vee\mu^j d)=\nu^{2n-2}(\nu^2 1\wedge(\mu^i a\vee\mu^j d))=\nu^{2n}1\wedge\nu^{2n-2}1\wedge(\mu^i a\vee\mu^j d)=\nu^{2n}1\wedge(\mu^i a\vee\mu^j d)$. The other assertions similarly follow.

<u>Lemma 11:</u> $\mu^{2n}x\vee\nu^{2m}x = x$ for $x\in J_2^4$ and $n,m\in N_o$.

Proof: The cases n=0 or m=0 are immediate consequences of Lemma 6. The case n=1, m=1 can be easily checked by Lemma 4. By induction hypothesis, we get for $n+m>2$(w.l.o.g. $m>1$):

$\mu^{2n}x\vee\nu^{2m}x=\mu^{2n}x\vee\nu^{2m}x\vee\mu^{2n}\nu^{2m-2}x=\mu^{2n}x\vee\nu^{2m-2}(\nu^{2}x\vee\mu^{2n}x)=$
$\mu^{2n}x\vee\nu^{2m-2}x=x.$

<u>Lemma 12:</u> Let $n,m\varepsilon N_o$ with $m\leq n$.
(1) $\mu^{2m}a\vee\mu^{2n+1}a=\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)$ (2) $\nu^{2m}c\vee\nu^{2n+1}c=\mu 1\wedge(\nu^{2m}c\vee\nu^{2n}d)$

Proof: (1): The case n=m=0 is proved by $a\vee\mu a=a\vee b=\nu 1=\nu 1\wedge(a\vee d)$.
By induction hypothesis, we get for $n>m=0$: $a\vee\mu^{2n+1}a=$
$a\vee\mu^{2}a\vee\mu^{2n+1}a=a\vee\mu^{2}(a\vee\mu^{2n-1}a)=a\vee\mu^{2}(\nu 1\wedge(a\vee\mu^{2n-2}d))=$
$a\vee(\mu^{2}\nu 1\wedge(\mu^{2}a\vee\mu^{2n}d))=a\vee(\mu^{2}1\wedge\nu 1\wedge(\mu^{2}a\vee\mu^{2n}d))=\nu 1\wedge(a\vee\mu^{2}a\vee\mu^{2n}d)=$
$\nu 1\wedge(a\vee\mu^{2n}d)$. Furthermore it follows for $n\geq m>0$: $\mu^{2m}a\vee\mu^{2n+1}a=$
$\mu^{2m}(a\vee\mu^{2n+1-2m}a)=\mu^{2m}(\nu 1\wedge(a\vee\mu^{2n-2m}d))=\mu^{2m}\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)=$
$\mu^{2m}1\wedge\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)=\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)$.

<u>Lemma 13:</u> Let $n\varepsilon N_o$.

(1) $d\vee\mu^{2n+1}d = d\vee\mu^{2n}a$ (2) $d\vee\nu^{2n+1}d = d\vee\nu^{2n}c$

Proof: (1): The case n=0 is proved by $d\vee\mu d=d\vee c=d\vee a$.
By induction hypothesis we get for $n>0$: $d\vee\mu^{2n+1}d=$
$d\vee\mu^{2}d\vee\mu^{2n+1}d=d\vee\mu^{2}(d\vee\mu^{2n-1}d)=d\vee\mu^{2}(d\vee\mu^{2n-2}a)=d\vee\mu^{2}d\vee\mu^{2n}a=$
$d\vee\mu^{2n}a$. (2): analogous to (1).

<u>Lemma 14:</u> Let $n\varepsilon N_o$.

(1) $\mu^{n}a \leq \nu 1$ (2) $\nu^{n}c \leq \mu 1$

(3) $\mu^{n}b \leq \nu 1$ (4) $\nu^{n}b \leq \mu 1$

Proof: (1): $a \geq \mu^{2n}a$ and $b \geq \mu^{2n+1}a$ implies the assertion. (2), (3), (4): analogous to (1).

In the following we write $(i,j)\sim(k,l)$ for $0 \leq i,j,k,l \leq \infty$, if $m \equiv n \pmod 2$ for all $m,n \in \{i,j,k,l\}\setminus\{\infty\}$; furthermore, let $\mu^{\infty}x = 0 = \nu^{\infty}x$ for all $x \in FM(J_2^4)$.

<u>Lemma 15:</u> Let $0 \leq i,j,k,l \leq \infty$, and let $(i,j)\sim(\infty,\infty)$ and $(k,l)\sim(\infty,\infty)$.

(1) $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) =$

$\mu^{\max\{i,k\}} a \vee \mu^{\max\{j,l\}} d$ if $(i,j)\sim(k,l)$,

$\mu^{\max\{i+1,j+1,k\}} a \vee \mu^{\max\{i+1,j+1,l\}} d$

if $(i,j)\nsim(k,l), \max\{i,j\} \leq \max\{k,l\}$,

$\mu^{\max\{i,k+1,l+1\}} a \vee \mu^{\max\{j,k+1,l+1\}} d$

if $(i,j)\nsim(k,l), \max\{i,j\} > \max\{k,l\}$;

(2) $(\nu^i c \vee \nu^j d) \wedge (\nu^k c \vee \nu^l d) =$

$\nu^{\max\{i,k\}} c \vee \nu^{\max\{j,l\}} d$ if $(i,j)\sim(k,l)$,

$\nu^{\max\{i+1,j+1,k\}} c \vee \nu^{\max\{i+1,j+1,l\}} d$

if $(i,j)\nsim(k,l), \max\{i,j\} \leq \max\{k,l\}$,

$\nu^{\max\{i,k+1,l+1\}} c \vee \nu^{\max\{j,k+1,l+1\}} d$

if $(i,j)\nsim(k,l), \max\{i,j\} > \max\{k,l\}$.

Proof: (1): The proof is divided into the following cases:

1. $(i,j)\sim(k,l)$ 2. $(i,j)\nsim(k,l), \max\{i,j\}\le\max\{k,l\}$

1.1. $i\le k, j\le l$	2.1. $i\le j\le k\le l$	2.7. $j\le i\le k\le l$
1.2. $i\le k, j\ge l$	2.2. $i\le k\le j\le l$	2.8. $j\le k\le i\le l$
1.3. $i\ge k, j\le l$	2.3. $k\le i\le j\le l$	2.9. $k\le j\le i\le l$
1.4. $i\ge k, j\ge l$	2.4. $i\le j\le l\le k$	2.10. $j\le i\le l\le k$
	2.5. $i\le l\le j\le k$	2.11. $j\le l\le i\le k$
	2.6. $l\le i\le j\le k$	2.12. $l\le j\le i\le k$

By symmetry, the case $(i,j)\nsim(k,l), \max\{i,j\}>\max\{k,l\}$ is analogous to 2..

<u>1.1.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=\mu^k a\vee\mu^l d=\mu^{\max\{i,k\}}a\vee\mu^{\max\{j,l\}}d$ (Lemma 6);

<u>1.2.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^i a\wedge\mu^l d)\vee\mu^k a\vee\mu^j d=0\vee\mu^k a\vee\mu^j d=$ $\mu^{\max\{i,k\}}a\vee\mu^{\max\{j,l\}}d$ (Lemma 6);

<u>1.3.:</u> analogous to 1.2.;

<u>1.4.:</u> analogous to 1.1..

<u>2.1.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^i a\vee\mu^{j+1}d)\wedge(\mu^k a\vee\mu^l d)=$ $((\mu^i a\vee\mu^{j+1}d)\wedge\mu^k a)\vee\mu^l d=((\mu^i a\vee\mu^j d)\wedge\nu 1\wedge\mu^k a)\vee\mu^l d=$ $((\mu^i a\vee\mu^{j+1}a)\wedge\mu^k a)\vee\mu^l d=\mu^k a\vee\mu^l d$ (Lemma 7,6,14,12);

<u>2.2.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=((\mu^i a\vee\mu^{j+1}a)\wedge\mu^k a)\vee\mu^l d=$ $(\mu^i a\wedge\mu^k a)\vee\mu^{j+1}a\vee\mu^l d=0\vee\mu^{j+1}a\vee\mu^l d=\mu^{j+1}a\vee\mu^l d$ (2.1., Lemma 6);

<u>2.3.:</u> analogous to 2.2.;

$\underline{2.4.:}$ analogous to 2.1.;

$\underline{2.5.:}$ $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^i a\vee\mu^j d)\wedge(\mu^{k+1} a\vee\mu^l d)=$
$((\mu^i a\vee\mu^j d)\wedge\mu^l d)\vee\mu^{k+1} a=((\mu^i a\vee\mu^{j+1} d)\wedge\mu^l d)\vee\mu^{k+1} a=$
$(\mu^i a\wedge\mu^l d)\vee\mu^{j+1} d\vee\mu^{k+1} a=0\vee\mu^{k+1} a\vee\mu^{j+1} d=\mu^k a\vee\mu^{j+1} d$
(Lemma 7,6);

$\underline{2.6.:}$ analogous to 2.5;

$\underline{2.7.:}$ $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^{i+1} a\vee\mu^j d)\wedge(\mu^k a\vee\mu^{l+1} d)=$
$\mu^k a\vee\mu^{l+1} d=\mu^k a\vee\mu^l d$ (Lemma 7,6);

$\underline{2.8.:}$ $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^{i+1} a\vee\mu^j d)\wedge(\mu^k a\vee\mu^{l+1} d)=$
$(\mu^k a\wedge\mu^j d)\vee\mu^{i+1} a\vee\mu^{l+1} d=0\vee\mu^{i+1} a\vee\mu^{l+1} d=\mu^{i+1} a\vee\mu^l d$
(Lemma 7,6);

$\underline{2.9.:}$ analogous to 2.8.;

$\underline{2.10.:}$ $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^{i+1} a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=$
$((\mu^{i+1} a\vee\mu^j d)\wedge\mu^l d)\vee\mu^k a=((\mu^i a\vee\mu^j d)\wedge\mu^l d)\vee\mu^k a=$
$((\mu^{i+1} d\vee\mu^j d)\wedge\mu^l d)\vee\mu^k a=\mu^k a\vee\mu^l d$ (Lemma 7,6,13);

$\underline{2.11.:}$ $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=((\mu^{i+1} d\vee\mu^j d)\wedge\mu^l d)\vee\mu^k a=$
$(\mu^j d\wedge\mu^l d)\vee\mu^{i+1} d\vee\mu^k a=0\vee\mu^{i+1} d\vee\mu^k a=\mu^k a\vee\mu^{i+1} d$ (2.10.,
Lemma 6);

$\underline{2.12.:}$ analogous to 2.11..

The proof of (2) analogously goes as the proof of (1).

Lemma 16: Let $0 \leq i,j,k,l \leq \infty$, and let $(i,j) \sim (\infty,\infty), (k,l) \sim (\infty,\infty)$ and $\{j,l\} \neq \{\infty\}$.

(1) $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d =$

$\mu^{\min\{i,k\}} a \vee \mu^{\min\{j,l\}} d$ if $(i,j) \sim (k,l)$,

$\mu^{\min\{i-1,j-1,k\}} a \vee \mu^{\min\{i-1,j-1,l\}} d$

if $(i,j) \nsim (k,l), \min\{i,j\} \geq \min\{k,l\}$,

$\mu^{\min\{i,k-1,l-1\}} a \vee \mu^{\min\{j,k-1,l-1\}} d$

if $(i,j) \nsim (k,l), \min\{i,j\} < \min\{k,l\}$;

(2) $\nu^i c \vee \nu^j d \vee \nu^k c \vee \nu^l d =$

$\nu^{\min\{i,k\}} c \vee \nu^{\min\{j,l\}} d$ if $(i,j) \sim (k,l)$,

$\nu^{\min\{i-1,j-1,k\}} c \vee \nu^{\min\{i-1,j-1,l\}} d$

if $(i,j) \nsim (k,l), \min\{i,j\} \geq \min\{k,l\}$,

$\nu^{\min\{i,k-1,l-1\}} c \vee \nu^{\min\{j,k-1,l-1\}} d$

if $(i,j) \nsim (k,l), \min\{i,j\} < \min\{k,l\}$.

Proof: (1): The proof is divided into the following cases:

1. $(i,j) \sim (k,l)$

2. $(i,j) \nsim (k,l), \min\{i,j\} \geq \min\{k,l\}$

2.1. $k \leq l \leq i \leq j$

2.2. $k \leq i \leq l \leq j$ $(l \neq \infty)$

2.3. $k \leq i \leq j \leq l$ $(j \neq \infty)$

2.4. $l \leq k \leq i \leq j$

2.5. $l \leq i \leq k \leq j$

2.6. $l \leq i \leq j \leq k$

2.7. $k \leq l \leq j \leq i$

2.8. $k \leq j \leq l \leq i$

2.9. $k \leq j \leq i \leq l$

2.10. $l \leq k \leq j \leq i$

2.11. $l \leq j \leq k \leq i$

2.12. $l \leq j \leq i \leq k$

By symmetry, the case $(i,j) \nsim (k,l), \min\{i,j\} < \min\{k,l\}$ is analogous to 2..

1. is an immediate consequence of Lemma 6.

2.1.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^{i-1} a \vee \mu^{j-1} d \vee \mu^k a \vee \mu^l d = \mu^k a \vee \mu^l d$ (Lemma 7,6);

2.2.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^i a \vee \mu^{j-1} a \vee \mu^k a \vee \mu^l d = \mu^i a \vee \mu^k a \vee \mu^l d =$
$\mu^i a \vee \mu^k a \vee \mu^{l-1} d = \ldots = \mu^i a \vee \mu^k a \vee \mu^i d = \mu^i a \vee \mu^k a \vee \mu^{i-1} d =$
$\mu^{i-1} a \vee \mu^k a \vee \mu^{i-1} d = \mu^k a \vee \mu^{i-1} d$ (Lemma 13,6,7);

2.3.: analogous to 2.2.;

2.4.: analogous to 2.1.;

2.5.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^{i-1} a \vee \mu^{j-1} d \vee \mu^k a \vee \mu^l d = \mu^{i-1} a \vee \mu^l d$
(Lemma 7,6);

2.6.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^i a \vee \mu^j d \vee \mu^{k-1} a \vee \mu^l d = \mu^i a \vee \mu^j d \vee \mu^l d =$
$\mu^i a \vee \mu^{j+1} d \vee \mu^l d = \mu^i a \vee \mu^l d = \mu^{i-1} a \vee \mu^l d$ (Lemma 7,6);

2.7.: analogous to 2.1.;

2.8.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^{i+1} a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^j d \vee \mu^k a \vee \mu^l d =$
$\mu^{j-1} d \vee \mu^k a \vee \mu^l d = \mu^k a \vee \mu^{j-1} d$ (Lemma 7,6);

2.9.: analogous to 2.8.;

2.10.: analogous to 2.1.;

2.11.: $\mu^i a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^{i+1} a \vee \mu^j d \vee \mu^k a \vee \mu^l d = \mu^j d \vee \mu^k a \vee \mu^l d =$
$\mu^{j-1} a \vee \mu^k a \vee \mu^l d = \mu^{j-1} a \vee \mu^l d$ (Lemma 7,6,13);

2.12.: analogous to 2.11..

The proof of (2) analogously goes as the proof of (1).

Lemma 17: Let $0 \le i,j,k,l < \infty$, and let $N, M \varepsilon 2N_o$ with $N \ge \max\{i,j\}, M \ge \max\{k,l\}$.
$(\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) = \nu^M(\mu^i a \vee \mu^j d) \vee \mu^N(\nu^k c \vee \nu^l d)$

Proof: $(\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) = (\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) \wedge (\mu^N 1 \vee \nu^M 1) =$
$((\mu^i a \vee \mu^j d) \wedge \nu^M 1) \vee ((\nu^k c \vee \nu^l d) \wedge \mu^N 1) = \nu^M(\mu^i a \vee \mu^j d) \vee \mu^N(\nu^k c \vee \nu^l d)$
(Lemma 11,7,6,10).

Lemma 18: $\mu^{2n}x \wedge \nu^{2m}x = \mu^{2n}\nu^{2m}x$ for $x \varepsilon J_2^4$ and $n,m \varepsilon N_o$.

Proof: The cases n=0 or m=0 are immediate consequences of Lemma 6. The case n=1, m=1 can be easily checked by Lemma 4 and Lemma 9. By induction hypothesis, we get for n+m>2 (w.l.o.g.m>1): $\mu^{2n}x \wedge \nu^{2m}x = \mu^{2n}x \wedge \nu^2 x \wedge \nu^{2m}x = \mu^{2n}\nu^2 x \wedge \nu^{2m}x =$
$\nu^2(\mu^{2n}x \wedge \nu^{2m-2}x) = \nu^2\mu^{2n}\mu^{2m-2}x = \mu^{2n}\nu^{2m}x$.

We define that a quadrupel (i,j,k,l) satisfies (*) if one of the following conditions hold for (i,j,k,l):

(1) $i,j,k,l \varepsilon N_o, (i,j) \sim (\infty,\infty), (k,l) \sim (\infty,\infty)$;
(2) $i=\infty, j \varepsilon 2N_o, k=\infty, l \varepsilon 2N_o$;
(3) $i=\infty, j \varepsilon 2N_o+1, k \varepsilon 2N_o, l=\infty$;
(4) $i \varepsilon 2N_o, j=\infty, k=\infty, l \varepsilon 2N_o+1$;
(5) $i \varepsilon 2N_o+1, j=\infty, k \varepsilon 2N_o+1, l=\infty$;
(6) $i=j=k=l=\infty$.

In $FM(J_2^4)$ we define $f(i,j,k,l) := (\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d)$ for $0 \le i,j,k,l \le \infty$.

Proposition 19: $FM(J_2^4)=\{f(i,j,k,l)\mid(i,j,k,l)$ satisfies $(*)\}$
and

(1) $f(i,j,k,l)\wedge f(i',j',k',l') =$

$f(\max\{i,i'\},\max\{j,j'\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\sim(i',j')$, $(k,l)\sim(k',l')$,

$f(\max\{i,i'\},\max\{j,j'\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i,i'\},\max\{j,j'\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\max\{k,l\}>\max\{k',l'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\not\sim(i',j')$, $(k,l)\sim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\not\sim(i',j')$, $(k,l)\sim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\not\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\not\sim(i',j')$ $(k,l)\not\sim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$, $\max\{k,l\}>\max\{k',l'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\not\sim(i',j')$ $(k,l)\not\sim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\not\sim(i',j')$ $(k,l)\not\sim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$, $\max\{k,l\}>\max\{k',l'\}$,

and

(2) $f(i,j,k,l) \vee f(i',j',k',l') =$

$$\begin{aligned}
&f(\min\{i,i'\},\min\{j,j'\},\min\{k,k'\},\min\{l,l'\}) \\
&\quad \text{if } (i,j)\sim(i',j'),(k,l)\sim(k',l') \\
&f(\min\{i,i'\},\min\{j,j'\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\}) \\
&\quad \text{if } (i,j)\sim(i',j'),(k,l)\nsim(k',l'),\min\{k,l\}\geq\min\{k',l'\}\ , \\
&f(\min\{i,i'\},\min\{j,j'\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\}) \\
&\quad \text{if } (i,j)\sim(i',j'),(k,l)\nsim(k',l'),\min\{k,l\}<\min\{k',l'\} \\
&f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k,k'\},\min\{l,l'\}) \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\sim(k',l'),\min\{i,j\}\geq\min\{i',j'\}, \\
&f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k,k'\},\min\{l,l'\} \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\sim(k',l'),\min\{i,j\}<\min\{i',j'\}\ , \\
&f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\}) \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\nsim(k',l'),\min\{i,j\}\geq\min\{i',j'\},\min\{k,l\}\geq\min\{k',l'\}\ , \\
&f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\}) \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\nsim(k',l'),\min\{i,j\}\geq\min\{i',j'\},\min\{k,l\}<\min\{k',l'\}, \\
&f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\}) \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\nsim(k',l'),\min\{i,j\}<\min\{i',j'\},\min\{k,l\}<\min\{k',l'\}. \\
&f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\}) \\
&\quad \text{if } (i,j)\nsim(i',j'),(k,l)\nsim(k',l'),\min\{i,j\}<\min\{i',j'\},\min\{k,l\}\geq\min\{k',l'\}
\end{aligned}$$

Proof: (1) and (2) imply that $\{f(i,j,k,l)\,|\,(i,j,k,l)$ satisfies $(*)\}$ is a sublattice of $FM(J_2^4)$ and, because of $a=f(0,\infty,\infty,1)$, $b=f(1,\infty,1,\infty)$, $c=f(\infty,1,0,\infty)$ and $d=f(\infty,0,\infty,0)$, that it is equal to $FM(J_2^4)$. Thus, what actually has to be proved is (1) and (2).

(1): Because of $((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\wedge((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d)=$ $((\mu^i a\vee\mu^j d)\wedge(\mu^i a\vee\mu^j d))\wedge((\nu^k c\vee\nu^l d)\wedge(\nu^k c\vee\nu^l d))$, the assertion is an immediate consequence of Lemma 15.

(2): The proof is divided into cases s.t. $(1\le s,t\le 6)$, where the case s.t. means that (i,j,k,l) satisfies condition (s) in (*) and (i',j',k',l') satisfies condition (t) in (*). By commutativity of $\vee$, it is sufficient to handle the cases s.t. with $s\le t$. Furthermore, because of $f(\infty,\infty,\infty,\infty)=0$, all the cases s.6 are trivial.

<u>1.1.:</u> Let $M\in 2N_o$ with $M\ge\max\{i,j,k,l,i',j',k',l'\}$.
$((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\vee((\mu^{i'} a\vee\mu^{j'} d)\wedge(\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\nu^M(\mu^{i'} a\vee\mu^{j'} d)\vee\mu^M(\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d\vee\mu^{i'} a\vee\mu^{j'} d)\vee\mu^M(\nu^k c\vee\nu^l d\vee\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^{\min\{i,i'\}} a\vee\mu^{\min\{j,j'\}} d)\vee\mu^M(\nu^{\min\{k,k'\}} c\vee\nu^{\min\{l,l'\}} d)=$
$(\mu^{\min\{i,i'\}} a\vee\mu^{\min\{j,j'\}} d)\wedge(\nu^{\min\{k,k'\}} c\vee\nu^{\min\{l,l'\}} d)$
(Lemma 17,16).

<u>1.2.:</u> Let $M\in 2N_o$ with $M\ge\max\{i,j,k,l,j',l'\}$. Because of $j',l'\in 2N_o$, we have $d=\mu^{M-j'} d\vee\nu^{M-l'} d$ by Lemma 11. Together with Lemma 17 and 18 it follows
$((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\vee(\mu^{j'} d\wedge\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\mu^{j'}\nu^{l'} d=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\mu^M\nu^{l'} d\vee\nu^M\mu^{j'} d=$
$\nu^M(\mu^i a\vee\mu^j d\vee\mu^{j'} d)\vee\mu^M(\nu^k c\vee\nu^l d\vee\nu^{l'} d)$. From this we get the

assertion by Lemma 16 and 17.

<u>1.3.:</u> By Lemma 18, $j'\varepsilon 2N_o+1$ and $k'\varepsilon 2N_o$ imply $\mu^{j'}d\wedge\nu^{k'}c=\mu^{j'-1}c\wedge\nu^{k'}c=\mu^{j'-1}\nu^{k'}c$. Then, the proof is analogous to 1.2..

<u>1.4.:</u> analogous to 1.3..

<u>1.5.:</u> By Lemma 18, $i'\varepsilon 2N_o+1$ and $k'\varepsilon 2N_o+1$ imply $\mu^{i'}a\wedge\nu^{k'}c=\mu^{i'-1}b\wedge\nu^{k'-1}b=\mu^{i'-1}\nu^{k'-1}b$. Then, the proof is analogous to 1.2..

<u>2.2.:</u> If $j\le j'$ and $l>l'$, we get the assertion from $(\mu^j d\wedge\nu^l d)\vee(\mu^{j'}d\wedge\nu^{l'}d)=\mu^j\nu^l d\vee\mu^{j'}\nu^{l'}d=\mu^j\nu^{l'}(\mu^{j'-j}d\vee\nu^{l-l'}d)=\mu^j\nu^{l'}d=\mu^j d\wedge\nu^{l'}d=(\mu^j d\vee\mu^{j'}d)\wedge(\nu^l d\vee\nu^{l'}d)$ by Lemma 16. The other cases are analogous.

<u>2.3.:</u> Let $M\varepsilon 2N_o$ with $M\ge\max\{j,l,j',k'\}$. Using Lemma 6, 11,18,13 and 17, we get $(\mu^j d\wedge\nu^l d)\vee(\mu^{j'}d\wedge\nu^{k'}c)=$
$(d\wedge\mu^j d\wedge\nu^l d)\vee(c\wedge\mu^{j'-1}c\wedge\nu^{k'}c)=$
$((\mu^M d\vee\nu^M d)\wedge\mu^j d\wedge\nu^l d)\vee((\mu^M c\vee\nu^M c)\wedge\mu^{j'-1}c\wedge\nu^{k'}c)=$
$((\mu^M d\wedge\nu^l d)\vee(\mu^j d\wedge\nu^M d))\vee((\mu^M c\wedge\nu^{k'}c)\vee(\mu^{j'-1}c\wedge\nu^M c))=$
$\mu^M\nu^l d\vee\mu^j\nu^M d\vee\mu^M\nu^{k'}c\vee\mu^{j'-1}\nu^M c=$
$\nu^M(\mu^j d\vee\mu^{j'}d)\vee\mu^M(\nu^{k'}c\vee\nu^l d)=(\mu^j d\vee\mu^{j'}d)\wedge(\nu^{k'}c\vee\nu^l d)$.

<u>2.4., 2.5., 3.4.:</u> similar to 2.3..

<u>3.3.:</u> Because of $(\mu^j d\wedge\nu^k c)\vee(\mu^{j'}d\wedge\nu^{k'}c)=$
$(\mu^{j-1}c\wedge\nu^k c)\vee(\mu^{j'-1}c\wedge\nu^{k'}c)$, the proof is analogous to 2.2..

<u>4.4., 5.5.:</u> similar to 3.3..

3.5.: Let $M\in 2N_0$ with $M\geq\max\{j,k,i',k'\}$. Using Lemma 6,11, 18,12,10 and 17, we get in the case $k<k'$ that
$(\mu^j d\wedge\nu^k c)\vee(\mu^{i'} a\wedge\nu^{k'} c)=(c\wedge\mu^{j-1} c\wedge\nu^k c)\vee(b\wedge\mu^{i'-1} b\wedge\nu^{k'-1} b)=$
$((\mu^M c\vee\nu^M c)\wedge\mu^{j-1} c\wedge\nu^k c)\vee((\mu^M b\vee\nu^M b)\wedge\mu^{i'-1} b\wedge\nu^{k'-1} b)=$
$((\mu^M c\wedge\nu^k c)\vee(\mu^{j-1} c\wedge\nu^M c))\vee((\mu^M b\wedge\nu^{k'-1} b)\vee(\mu^{i'-1} b\wedge\nu^M b))=$
$\mu^M\nu^k c\vee\mu^{j-1}\nu^M c\vee\mu^M\nu^{k'-1} b\vee\mu^{i'-1}\nu^M b=\nu^M(\mu^j d\vee\mu^{i'} a)\vee\mu^M(\nu^k c\vee\nu^{k'} c)=$
$\nu^M(\mu^j d\vee\mu^{i'} a)\vee\mu^M(\mu 1\wedge(\nu^k c\vee\nu^{k'-1} d))=$
$\mu(\nu^M(\mu^{j-1} d\vee\mu^{i'-1} a)\vee\mu^M(\nu^k d\vee\nu^{k'-1} c))=$
$\mu((\mu^{j-1} d\vee\mu^{i'-1} a)\wedge(\nu^k d\vee\nu^{k'-1} c))=(\mu^j d\vee\mu^{i'} a)\wedge\mu 1\wedge(\nu^k c\vee\nu^{k'-1} d)=$
$(\mu^{i'} a\vee\mu^j d)\wedge(\nu^k c\vee\nu^{k'-1} d)$. The case $k'<k$ analogously follows.

4.5.: analogous to 3.5..

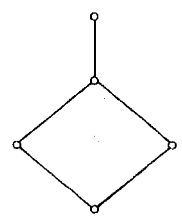
Proposition 20: An isomorphism from $FM(J_2^4)$ onto a subdirect power of $FM(J_1^4)$ is given by $f(i,j,k,l)\longmapsto(e(i,j),e(k,l))$ (the elements $e(i,j)$ of $FM(J_1^4)$ are defined in DAY, HERRMANN, WILLE [2]).

Proof: The assertion is a straightforward consequence of Proposition 19 and of Theorem 4 and 5 in [2].

Proposition 21: Let (i,j,k,l) and (i',j',k',l') satisfy $(*)$. Then $f(i,j,k,l)=f(i',j',k',l')$ if and only if $(i,j,k,l)=(i',j',k',l')$.

Proof: The assertion immediately follows from Proposition 20 and Theorem 5 in [2].

<u>Proposition 22:</u> The congruence lattice of $FM(J_2^4)$ is described by the following diagram :



Proof: By Proposition 20, the intersection of the congruence relations $\Theta(a \vee b, 1)$ and $\Theta(b \vee c, 1)$ is the identity; furthermore, by the Homomorphism Theorem and Corollary 8 in [2], there is only one non-trivial congruence relation greater than $\Theta(a \vee b, 1)$ and $\Theta(b \vee c, 1)$, respectively, namely $\Theta(a \vee b, b \vee c)$. Therefore, the distributivity of the congruence lattice gives us the assertion.

3. $FM(J_4^4)$: In this section we show the existence of epimorphisms α and β from $FM(J_4^4)$ onto $FM(J_2^4)$ (Lemma 23), which separate $FM(J_4^4)$; that is, $\ker\alpha \cap \ker\beta = \omega := \{(X,X) \mid X \in FM(J_4^4)\}$ (Proposition 49). To establish the proof, we define monomorphisms $\underline{\alpha}$ and $\underline{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ (Lemma 26) and meet-morphisms $\bar{\alpha}$ and $\bar{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ (Lemma 43), such that $\underline{\alpha}x \leq \bar{\alpha}x$ and $\underline{\beta}x \leq \bar{\beta}x$ for all $x \in FM(J_2^4)$. We prove that the intervals $[\underline{\alpha}x, \bar{\alpha}x]$ and $[\underline{\beta}x, \bar{\beta}x]$ are the congruence classes of $\ker\alpha$ and $\ker\beta$, respectively (Lemma 48). Thereby, we get $FM(J_4^4)$ as a subdirect power of $FM(J_2^4)$. Both Proposition 22 and Proposition 49 give us the proof of Theorem 3. Furthermore, the word problem can be solved for $FM(J_4^4)$.

For the preparation of the main-assertions, several lemmata have to be proved and are listed in the beginning of part 3. For simplification in the following we choose lower case letters for the generators of J_2^4 and upper case letters for the generators of J_4^4.

Lemma 23: There are epimorphisms α and β of $FM(J_4^4)$ onto $FM(J_2^4)$ such that

(1)	$\alpha 0 = 0$	(2)	$\beta 0 = 0$
	$\alpha A = c$		$\beta A = a$
	$\alpha B = d$		$\beta B = b$

$$\alpha C = a \qquad \beta C = c$$

$$\alpha D = b \qquad \beta D = d$$

$$\alpha 1 = 1 \qquad \beta 1 = 1$$

Proof: Since $FM(J_4^4)$ and $FM(J_2^4)$ are lattices freely generated by J_4^4 and J_2^4, respectively, the homomorphisms from J_4^4 into J_2^4 can be extended to epimorphisms from $FM(J_4^4)$ onto $FM(J_2^4)$.

Lemma 24: There are endomorphisms ϕ and ψ of $FM(J_4^4)$ such that

(1)		(2)	
	$\phi O = O$		$\psi O = O$
	$\phi A = D\wedge(A\vee B)$		$\psi A = B\wedge(A\vee D)$
	$\phi B = C\wedge(A\vee B)$		$\psi B = A\wedge(B\vee C)$
	$\phi C = B\wedge(C\vee D)$		$\psi C = D\wedge(B\vee C)$
	$\phi D = A\wedge(C\vee D)$		$\psi D = C\wedge(A\vee D)$
	$\phi 1 = (A\vee B)\wedge(C\vee D)$		$\psi 1 = (A\vee D)\wedge(B\vee C)$

Proof: It can be easily seen by modularity that (1) and (2) define homomorphisms from J_4^4 into $FM(J_4^4)$. Thus, the freeness of $FM(J_4^4)$ gives as the assertion.

Lemma 25: Let μ and ν the endomorphisms of $FM(J_2^4)$ defined in Lemma 4.

(1) $\phi\psi = \psi\phi$ (2) $\alpha\psi = \mu\alpha$ (3) $\beta\phi = \nu\beta$

(4) $\alpha\phi = \nu\alpha$ (5) $\beta\psi = \mu\beta$

Proof: (1): $\phi\psi A=\phi(B\wedge(A\vee D))=(C\wedge(B\vee D))\wedge((D\wedge(A\vee B))\vee(A\wedge(C\vee D)))$

$=C\wedge(A\vee B)\wedge(A\vee(D\wedge(A\vee B)))\wedge(C\vee D)=C\wedge(A\vee B)\wedge(A\vee D)\wedge(A\vee B)\wedge(C\vee D)$

$=C\wedge(A\vee B)\wedge(A\vee D)\wedge(B\vee C)=(C\wedge(A\vee D))\wedge(A\vee(B\wedge(A\vee D)))\wedge(B\vee C)$

$=(C\wedge(A\vee D))\wedge((A\wedge(B\vee C))\vee(B\wedge(A\vee D)))=\psi(D\wedge(B\vee A))=\psi\phi A$;

$\phi\psi B=\psi\phi B$, $\phi\psi C=\psi\phi C$, $\phi\psi D=\psi\phi D$ (analogous to $\phi\psi A=\psi\phi A$).

(2) $\alpha\psi A=\alpha(B\wedge(A\vee D))=d\wedge(c\vee b)=\mu c=\mu\alpha A$,

$\alpha\psi B=\alpha(A\wedge(B\vee C))=c\wedge(d\vee a)=c=\mu d=\mu\alpha B$,

$\alpha\psi C=\alpha(D\wedge(B\vee C))=b\wedge(d\vee a)=b=\mu a=\mu\alpha C$,

$\alpha\psi D=\alpha(C\wedge(A\vee D))=a\wedge(c\vee b)=\mu b=\mu\alpha D$.

The other assertions analogously follow.

By Lemma 25, the following diagram commutes:

$$\begin{tikzcd} FM(J_4^4) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & FM(J_2^4) \arrow[d, shift right, "\mu"'] \arrow[d, shift left, "\nu"] \\ FM(J_4^4) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & FM(J_2^4) \end{tikzcd}$$

<u>Lemma 26:</u> There are monomorphisms $\underline{\alpha}$ and $\underline{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ such that

(1)	$\underline{\alpha}0 = 0$	(2)	$\underline{\beta}0 = 0$
	$\underline{\alpha}a = C\wedge(A\vee B)$		$\underline{\beta}a = A\wedge(C\vee D)$
	$\underline{\alpha}b = D\wedge(A\vee B)\wedge(B\vee C)$		$\underline{\beta}b = B\wedge(A\vee D)\wedge(C\vee D)$
	$\underline{\alpha}c = A\wedge(B\vee C)$		$\underline{\beta}c = C\wedge(A\vee D)$
	$\underline{\alpha}d = B$		$\underline{\beta}d = D$
	$\underline{\alpha}1 = (A\vee B)\wedge(B\vee C)$		$\underline{\beta}1 = (A\vee D)\wedge(C\vee D)$

Proof: (1) Since $FM(J_2^4)$ is freely generated by J_2^4, it is enough to proof the following statements:

$$\underline{\alpha}a \vee \underline{\alpha}c = (C\wedge(A\vee B))\vee(A\wedge(B\vee C)) = (C\vee(A\wedge(B\vee C)))\wedge(A\vee B)$$

$$= (C\vee A)\wedge(B\vee C)\wedge(A\vee B) = (B\vee C)\wedge(A\vee B) = \underline{\alpha}1$$

$$\underline{\alpha}a \vee \underline{\alpha}d = (C\wedge(A\vee B))\vee B = (B\vee C)\wedge(A\vee B) = \underline{\alpha}1$$

$$\underline{\alpha}b \vee \underline{\alpha}d = (D\wedge(A\vee B)\wedge(B\vee C))\vee B = (B\vee D)\wedge(A\vee B)\wedge(B\vee C) = (A\vee B)\wedge(B\vee C) = \underline{\alpha}1$$

$$\underline{\alpha}c \vee \underline{\alpha}d = (A\wedge(B\vee C))\vee B = (A\vee B)\wedge(B\vee C) = \underline{\alpha}1$$

$$\underline{\alpha}a \wedge \underline{\alpha}b = \underline{\alpha}a\wedge\underline{\alpha}c = \underline{\alpha}a\wedge\underline{\alpha}d = \underline{\alpha}b\wedge\underline{\alpha}c = \underline{\alpha}b\wedge\underline{\alpha}d = \underline{\alpha}c\wedge\underline{\alpha}d = 0 = \underline{\alpha}0.$$

(2): analogous to (1).

<u>Lemma 27:</u>

(1) $\alpha\underline{\alpha} = \beta\underline{\beta} = id_{FM(J_2^4)}$ (2) $\alpha\underline{\beta} = \beta\underline{\alpha} = \mu\nu$

(3) $\underline{\alpha}\mu = \psi\underline{\alpha}$ (4) $\underline{\alpha}\nu = \phi\underline{\alpha}$

(5) $\underline{\beta}\mu = \psi\underline{\beta}$ (6) $\underline{\beta}\nu = \phi\underline{\beta}$

Proof: (1): $\alpha\underline{\alpha}a = \alpha(C\wedge(A\vee B)) = a\wedge(c\vee d) = a = \beta(A\wedge(C\vee D)) = \beta\underline{\beta}a$,

$\alpha\underline{\alpha}b = \alpha(D\wedge(A\vee B)\wedge(B\vee C)) = b\wedge(c\vee d)\wedge(a\vee d) = b =$

$\beta = (b\wedge(c\vee d)\wedge(a\vee d)) = \beta\underline{\beta}b$,

$\alpha\underline{\alpha}c = \alpha(A\wedge(B\vee C)) = c\wedge(a\vee d) = c = \beta(C\wedge(A\vee D)) = \beta\underline{\beta}c$,

$\alpha\underline{\alpha}d = \alpha B = d = \beta D = \beta\underline{\beta}d$.

(2): $\alpha\underline{\beta}a = \alpha(A\wedge(C\vee D)) = c\wedge(a\vee b) = \nu b = \nu\mu a = \mu\nu a = c\wedge(a\vee b)$

$= \beta(C\wedge(A\vee B)) = \beta\underline{\alpha}a$,

$\alpha\underline{\beta}b = \alpha(B\wedge(A\vee D)\wedge(C\vee D)) = d\wedge(c\vee b)\wedge(a\vee b) = \mu(c\wedge(a\vee b))$

$= \mu\nu b = \beta(D\wedge(A\vee B)\wedge(B\vee C) = \beta\underline{\alpha}b$,

$\alpha\underline{\beta}c=\alpha(C\wedge(A\vee D))=a\wedge(c\vee b)=\mu b=\mu\nu c=\beta(A\wedge(C\vee B))=\beta\underline{\alpha}c$,

$\alpha\underline{\beta}d=\alpha D=b=\mu a=\mu\nu d=\beta B=\beta\underline{\alpha}d$.

(3) $\underline{\alpha}\mu a=\underline{\alpha}b=D\wedge(A\vee B)\wedge(B\vee C)=D\wedge(B\vee C)\wedge(A\vee B)\wedge(A\vee D)\wedge(B\vee C)$

$=D\wedge(B\vee C)\wedge(A\vee(B\wedge(A\vee D)))\wedge(B\vee C)=$

$=D\wedge(B\vee C)\wedge((B\wedge(A\vee D))\vee(A\wedge(B\vee C)))=\psi(C\wedge(A\vee B))=\psi\underline{\alpha}a$,

$\underline{\alpha}\mu b=\underline{\alpha}(a\wedge(b\vee c))=C\wedge(A\vee B)\wedge((D\wedge(A\vee B)\wedge(B\vee C))\vee(A\wedge(B\vee C)))$

$=C\wedge(A\vee B)\wedge((A\wedge(B\vee C))\vee(D\wedge(B\vee C)))\wedge(A\vee B)=$

$=C\wedge(A\vee B)\wedge(D\vee(A\vee(B\vee C)))\wedge(A\vee D)$

$=C\wedge(A\vee D)\wedge(A\vee B)\wedge(B\vee C)\wedge(A\vee D)\wedge((D\vee(A\wedge(B\vee C)))\wedge(B\vee C))$

$=\psi(D\wedge(A\vee B)\wedge(B\vee C))=\psi\underline{\alpha}b$,

$\underline{\alpha}\mu c=\psi\underline{\alpha}c$ (analogous to $\underline{\alpha}\mu a=\psi\underline{\alpha}a$),

$\underline{\alpha}\mu d=\underline{\alpha}c=A\wedge(B\vee C)=\psi B=\psi\underline{\alpha}d$.

(4), (5), (6): analogous to (3).

Lemma 28: (1): $\phi^m X\leq\phi^n X$ and $\psi^m X\leq\psi^n X$ for $X\varepsilon J_4^4$ if $m\equiv n(\text{mod } 2)$ and $m\geq n$.

(2): $\phi^m 1\leq\phi^n 1$ and $\psi^m 1\leq\psi^n 1$ if $m\geq n$.

Proof: The assertions are consequences of $\phi^2 X\leq X$, $\psi^2 X\leq X$ and $\phi 1\leq 1$, $\psi 1\leq 1$, respectively.

Lemma 29: (1) $\phi^m X\wedge\phi^n Y=0$ for $X,Y\varepsilon J_4^4\backslash\{1\}$ if (i): $X\not\equiv Y$ and $m\equiv n$ (mod 2) or (ii): $X\varepsilon\{0,A,D\}$ and $Y\varepsilon\{0,B,C\}$ or (iii): $X=Y$ and $m\not\equiv n(\text{mod } 2)$.

(2) $\psi^m X \wedge \psi^n Y = 0$ for $X, Y \in J_4^4 \setminus \{1\}$ if (i): $X \neq Y$ and $m \equiv n \pmod 2$ or (ii): $X \in \{0, A, B\}$ and $Y \in \{0, C, D\}$ or (iii): $X = Y$ and $m \not\equiv n \pmod 2$.

Proof: $X \wedge Y = 0$ for $X, Y \in J_4^4 \setminus \{1\}$ and $X \neq Y$ implies the assertions.

<u>Lemma 30:</u> Let $n \in N$.

(1) $\phi^{2n}A = A \wedge (C \vee \phi^{2n-1}A) = A \wedge (D \vee \phi^{2n-1}B)$

(2) $\phi^{2n-1}A = D \wedge (A \vee \phi^{2n-2}B) = D \wedge (B \vee \phi^{2n-2}A)$

(3) $\phi^{2n}B = B \wedge (C \vee \phi^{2n-1}A) = B \wedge (D \vee \phi^{2n-1}B)$

(4) $\phi^{2n-1}B = C \wedge (A \vee \phi^{2n-2}B) = C \wedge (B \vee \phi^{2n-2}A)$

(5) $\phi^{2n}C = C \wedge (A \vee \phi^{2n-1}C) = C \wedge (B \vee \phi^{2n-1}D)$

(6) $\phi^{2n-1}C = B \wedge (C \vee \phi^{2n-2}D) = B \wedge (D \vee \phi^{2n-2}C)$

(7) $\phi^{2n}D = D \wedge (A \vee \phi^{2n-1}C) = D \wedge (B \vee \phi^{2n-1}D)$

(8) $\phi^{2n-1}D = A \wedge (C \vee \phi^{2n-2}D) = A \wedge (D \vee \phi^{2n-2}C)$

(9) $\psi^{2n}A = A \wedge (C \vee \psi^{2n-1}A) = A \wedge (B \vee \psi^{2n-1}D)$

(10) $\psi^{2n-1}A = B \wedge (A \vee \psi^{2n-2}D) = B \wedge (D \vee \psi^{2n-2}A)$

(11) $\psi^{2n}B = B \wedge (A \vee \psi^{2n-1}C) = B \wedge (D \vee \psi^{2n-1}B)$

(12) $\psi^{2n-1}B = A \wedge (B \vee \psi^{2n-2}C) = A \wedge (C \vee \psi^{2n-2}B)$

(13) $\psi^{2n}C = C \wedge (A \vee \psi^{2n-1}C) = C \wedge (D \vee \psi^{2n-1}B)$

(14) $\psi^{2n-1}C = D \wedge (B \vee \psi^{2n-2}C) = D \wedge (C \vee \psi^{2n-2}B)$

(15) $\psi^{2n}D = D \wedge (C \vee \psi^{2n-1}A) = D \wedge (B \vee \psi^{2n-1}D)$

(16) $\psi^{2n-1}D = C \wedge (A \vee \psi^{2n-2}D) = C \wedge (D \vee \psi^{2n-2}A)$.

(1): The case $n=1$ is proved by

$\phi^2A=\phi(D\wedge(A\vee B))=A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee(C\wedge(A\vee B)))$

$=A\wedge(C\vee D)\wedge(C\vee(D\wedge(A\vee B)))\wedge(A\vee B)=A\wedge(C\vee(D\wedge(A\vee B)))=A\wedge(C\vee\phi A)$ and

$A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee(C\wedge(A\vee B)))=A\wedge(C\vee D)\wedge(D\vee(C\wedge(A\vee B)))\wedge(A\vee B)$

$=A\wedge(D\vee(C\wedge(A\vee B)))=A\wedge(D\vee\phi B)$. By induction hypothesis, we get

for $n>1$: $\phi^{2n}A=\phi^2\phi^{2n-2}A=\phi^2(A\wedge(C\vee\phi^{2n-3}A))=$

$\phi(D\wedge(A\vee B)\wedge((B\wedge(C\vee D))\vee\phi^{2n-2}A)=\phi(D\wedge((B\wedge(C\vee D))\vee(A\wedge(C\vee\phi^{2n-3}A)))$

$=\phi(D\wedge(B\vee(A\wedge(C\vee\phi^{2n-3}A)))\wedge(C\vee D))=\phi(D\wedge(B\vee(A\wedge(C\vee\phi^{2n-3}A))))$

$=\phi(D\wedge(B\vee\phi^{2n-2}A))=A\wedge(C\vee D)\wedge((C\wedge(A\vee B))\vee(D\wedge(B\vee\phi^{2n-2}A)))$

$=A\wedge(C\vee D)\wedge(C\vee(D\wedge(B\vee\phi^{2n-2}A)))\wedge(A\vee B)=A\wedge(C\vee(D\wedge(B\vee\phi^{2n-2}A)))$

$=A\wedge(C\vee\phi(A\wedge(C\vee\phi^{2n-3}A)))=A\wedge(C\vee\phi^{2n-1}A)$ and furthermore

$\phi^{2n}A=\phi^2(A\wedge(D\vee\phi^{2n-3}B))=\phi(D\wedge(A\vee B)\wedge((A\wedge(C\vee D))\vee\phi^{2n-2}B))$

$=\phi(D\wedge((A\wedge(C\vee D))\vee(B\wedge(D\vee\phi^{2n-3}B))))=\phi(D\wedge(A\vee(B\wedge(D\vee\phi^{2n-3}B)))\wedge(C\vee D))$

$=\phi(D\wedge(A\vee\phi^{2n-2}B))=A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee\phi^{2n-1}B)$

$=A\wedge((D\wedge(A\vee B))\vee(\phi^{2n-2}C\wedge\phi^{2n-2}(A\vee B)))$

$=A\wedge(D\vee(\phi^{2n-2}C\wedge\phi^{2n-2}(A\vee B))\wedge(A\vee B)=A\wedge(D\vee\phi^{2n-2}(C\wedge(A\vee B)))$

$=A\wedge(D\vee\phi^{2n-1}B)$ (Lemma 28).

All other cases of Lemma 30 can be proved in a similar way.

<u>Lemma 31:</u> Let $m,n\in N_o$ and $m\leq n$.

(1) $\phi^{2m}A\vee\phi^{2n}C=\phi^{2m}A\vee\phi^{2n-1}C$ (2) $\phi^{2m}B\vee\phi^{2n}D=\phi^{2m}B\vee\phi^{2n-1}D$

(3) $\phi^{2m}C\vee\phi^{2n}A=\phi^{2m}C\vee\phi^{2n-1}A$ (4) $\phi^{2m}D\vee\phi^{2n}B=\phi^{2m}D\vee\phi^{2n-1}B$

(5) $\psi^{2m}A\vee\psi^{2n}C=\psi^{2m}A\vee\psi^{2n-1}C$ (6) $\psi^{2m}B\vee\psi^{2n}D=\psi^{2m}B\vee\psi^{2n-1}D$

(7) $\psi^{2m}C\vee\psi^{2n}A=\psi^{2m}C\vee\psi^{2n-1}A$ (8) $\psi^{2m}D\vee\psi^{2n}B=\psi^{2m}D\vee\psi^{2n-1}B$.

Proof: (1): $\phi^{2m}A\vee\phi^{2n}C=\phi^{2m}(A\vee\phi^{2(n-m)}C)=\phi^{2m}(A\vee(C\wedge(A\vee\phi^{2(n-m)-1}C)))$
$=\phi^{2m}((A\vee C)\wedge(A\vee\phi^{2(n-m)-1}C)=\phi^{2m}A\vee\phi^{2n-1}C$ (Lemma 30).

(2), (3), ..., (8): analogous to (1).

Lemma 32: Let $m,n\in N_o$ and $m\leq n$.

(1) $\phi^{2m}A\vee\phi^{2n+1}B=\phi^{2m}A\vee\phi^{2n}B$ (2) $\phi^{2m}B\vee\phi^{2n+1}A=\phi^{2m}B\vee\phi^{2n}A$

(3) $\phi^{2m}C\vee\phi^{2n+1}D=\phi^{2m}C\vee\phi^{2n}D$ (4) $\phi^{2m}D\vee\phi^{2n+1}C=\phi^{2m}D\vee\phi^{2n}C$

(5) $\psi^{2m}A\vee\psi^{2n+1}D=\psi^{2m}A\vee\psi^{2n}D$ (6) $\psi^{2m}D\vee\psi^{2n+1}A=\psi^{2m}D\vee\psi^{2n}A$

(7) $\psi^{2m}B\vee\psi^{2n+1}C=\psi^{2m}B\vee\psi^{2n}C$ (8) $\psi^{2m}C\vee\psi^{2n+1}B=\psi^{2m}C\vee\psi^{2n}B$

Proof: (1): $\phi^{2m}A\vee\phi^{2n+1}B=\phi^{2m}(A\vee\phi^{2(n-m)+1}B)=\phi^{2m}(A\vee(C\wedge(A\vee\phi^{2(n-m)}B)))$
$=\phi^{2m}((A\vee C)\wedge(A\vee\phi^{2(n-m)}B))=\phi^{2m}A\vee\phi^{2n}B$. (Lemma 30).

(2), (3), ..., (8): analogous to (1).

Lemma 33: Let $m,n\in N_o$.

(1) $\phi^{2m+1}A\vee\psi^{2n+1}C=D$ (2) $\phi^{2m+1}B\vee\psi^{2n+1}D=C$

(3) $\phi^{2m+1}C\vee\psi^{2n+1}A=B$ (4) $\phi^{2m+1}D\vee\psi^{2n+1}B=A$

Proof: The case m=n=0 can be easily checked by Lemma 24.
By induction hypothesis, we get for $m>n=0$: $\phi^{2m+1}A\vee\psi C$
$=\phi^{2m+1}A\vee\psi C\vee\psi\phi^{2}C=\phi^{2}(\phi^{2m-1}A\vee\psi C)\vee\psi C=\phi^{2}D\vee\psi C$
$=(D\wedge(B\vee(A\wedge(C\vee D))))\vee(D\wedge(B\vee C))=((D\wedge(B\vee C))\vee B\vee(A\wedge(C\vee D)))\wedge D$
$=(B\vee C\vee(A\wedge(C\vee D)))\wedge D=(B\vee C\vee D)\wedge D=D$ (Lemma 24). Now let $n>0$.
We get: $\phi^{2m+1}A\vee\phi^{2n+1}C=\phi^{2m+1}A\vee\phi^{2m+1}\psi^{2}A\vee\psi^{2n+1}C$
$=\phi^{2m+1}A\vee\psi^{2}(\phi^{2m+1}A\vee\psi^{2n-1}C)=\phi^{2m+1}A\vee\psi^{2}D=\phi^{2m+1}A\vee\phi^{2m+1}\psi B\vee\psi^{2}D$

$=\phi^{2m+1}A\vee\psi(\phi^{2m+1}B\vee\psi D)=\phi^{2m+1}A\vee\psi C=D$ (Lemma 28);

(2), (3), (4): analogous to (1).

<u>Lemma 34:</u> Let $m,n\varepsilon N_o$.

(1) $\phi^{2m}A\vee\psi^{2n+1}B=A$ (2) $\phi^{2m}B\vee\psi^{2n+1}A=B$

(3) $\phi^{2m}C\vee\psi^{2n+1}D=C$ (4) $\phi^{2m}D\vee\psi^{2n+1}C=D$

(5) $\phi^{2m+1}A\vee\psi^{2n}D=D$ (6) $\phi^{2m+1}B\vee\psi^{2n}C=C$

(7) $\phi^{2m+1}C\vee\psi^{2n}B=B$ (8) $\phi^{2m+1}D\vee\psi^{2n}A=A$

Proof: (1): Case $m=0$ is a consequence of Lemma 28. For $m>0$ we get by Lemma 33:
$A=\phi^{2m-1}D\vee\psi^{2n+1}B\geq\phi^{2m}A\vee\psi^{2n+1}B\geq\phi^{2m+1}D\vee\psi^{2n+1}B=A$. The other assertions similarly follow.

<u>Lemma 35:</u> Let $m,n\varepsilon N_o$.

(1) $\phi^{2m}A\vee\psi^{2n}A=A$ (2) $\phi^{2m}B\vee\psi^{2n}B=B$

(3) $\phi^{2m}C\vee\psi^{2n}C=C$ (4) $\phi^{2m}D\vee\psi^{2n}D=D$

(5) $\phi^{m}1\vee\phi^{n}1=1$

Proof: (1): For $m=0$ or $n=0$ the assertion follows by Lemma 28. If $m,n>0$ we get
$A=\phi^{2m-1}D\vee\psi^{2n-1}B\geq\phi^{2m}A\vee\psi^{2n}A\geq\phi^{2m+1}D\vee\psi^{2n+1}B=A$ by Lemma 33.
(2), (3), (4): analogous to (1). (5) we get by $1=A\vee C=B\vee D$ using Lemma 34 and (1), ..., (4).

In the following let $\phi^{\infty}X=0=\psi^{\infty}X$ for all $X\varepsilon J_4^4$.

<u>Lemma 36:</u> Let $n\varepsilon N_o\cup\{\infty\}$.

(1) $\phi\psi^n A=\phi 1\wedge\psi^n D=(A\vee B)\wedge\psi^n D$ (2) $\phi\psi^n B=\phi 1\wedge\psi^n C=(A\vee B)\wedge\psi^n C$

(3) $\phi\psi^n C=\phi 1\wedge\psi^n B=(C\vee D)\wedge\psi^n B$ (4) $\phi\psi^n D=\phi 1\wedge\psi^n A=(C\vee D)\wedge\psi^n A$

(5) $\psi\phi^n A=\psi 1\wedge\phi^n B=(A\vee D)\wedge\phi^n B$ (6) $\psi\phi^n B=\psi 1\wedge\phi^n A=(B\vee C)\wedge\phi^n A$

(7) $\psi\phi^n C=\psi 1\wedge\phi^n D=(B\vee C)\wedge\phi^n D$ (8) $\psi\phi^n D=\psi 1\wedge\phi^n C=(A\vee D)\wedge\phi^n C$

Proof: (1): The case $n=0$ is trivial. For $n>0$ we get by induction hypothesis and Lemma 25: $\phi\psi^n A=\psi((A\vee B)\wedge\psi^{n-1}D)$ $=((B\wedge(A\vee D))\vee(A\wedge(B\vee C)))\wedge\psi^n D=(A\vee B)\wedge(A\vee D)\wedge(B\vee C)\wedge\psi^n D$ $=(A\vee B)\wedge\psi^n D=(A\vee B)\wedge(C\vee D)\wedge\psi^n D=\phi 1\wedge\psi^n D$. The other assertions analogously follow.

<u>Lemma 37:</u> Let $m,n\varepsilon N_o\cup\{\infty\}$.

(1) $\phi(\psi^m A\vee\psi^n C)=\phi 1\wedge(\psi^m D\vee\psi^n B)$ (2) $\phi(\psi^m A\vee\psi^n D)=\phi 1\wedge(\psi^m D\vee\psi^n A)$

(3) $\phi(\psi^m B\vee\psi^n C)=\phi 1\wedge(\psi^m C\vee\psi^n B)$ (4) $\phi(\psi^m B\vee\psi^n D)=\phi 1\wedge(\psi^m C\vee\psi^n A)$

(5) $\psi(\phi^m A\vee\phi^n B)=\psi 1\wedge(\phi^m B\vee\phi^n A)$ (6) $\psi(\phi^m A\vee\phi^n C)=\psi 1\wedge(\phi^m B\vee\phi^n D)$

(7) $\psi(\phi^m B\vee\phi^n D)=\psi 1\wedge(\phi^m A\vee\phi^n C)$ (8) $\psi(\phi^m C\vee\phi^n D)=\psi 1\wedge(\phi^m D\vee\phi^n C)$

Proof: (1): $\phi(\psi^m A\vee\psi^n C)=((A\vee B)\wedge\psi^m D)\vee((C\vee D)\wedge\psi^n B)$ $=(((A\vee B)\wedge\psi^m D)\vee\psi^n B)\wedge(C\vee D)=(\psi^m D\vee\psi^n B)\wedge(A\vee B)\wedge(C\vee D)$ $=\phi 1\wedge(\psi^k D\vee\psi^l B)$ (Lemma 36). All other cases similarly follow.

Lemma 38: Let $k,l,m \in N_o \cup \{\infty\}$.

(1) $\phi^{2m}(\psi^k A \vee \psi^l C) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^l C)$

(2) $\phi^{2m}(\psi^k A \vee \psi^l D) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^l D)$

(3) $\phi^2 m(\psi^k B \vee \psi^l C) = \phi^{2m} 1 \wedge (\psi^k B \vee \psi^l C)$

(4) $\phi^{2m}(\psi^k B \vee \psi^l D) = \phi^{2m} 1 \wedge (\psi^k B \vee \psi^l D)$

(5) $\psi^{2m}(\phi^k A \vee \phi^l B) = \psi^{2m} 1 \wedge (\phi^k A \vee \phi^l B)$

(6) $\psi^{2m}(\phi^k A \vee \phi^l C) = \phi^{2m} 1 \wedge (\phi^k A \vee \phi^l C)$

(7) $\psi^{2m}(\phi^k B \vee \phi^l D) = \psi^{2m} 1 \wedge (\phi^k B \vee \phi^l D)$

(8) $\psi^{2m}(\phi^k C \vee \phi^l D) = \phi^{2m} 1 \wedge (\phi^k C \vee \phi^l D)$

(9) $\phi^k \psi^l 1 = \phi^k 1 \wedge \psi^l 1$

Proof: (1): The case $m=0$ is trivial. By induction hypothesis we get for $m>0$: $\phi^{2m}(\psi^k A \vee \psi^l C) = \phi^2(\phi^{2m-2} 1 \wedge (\psi^k A \vee \psi^l C))$ $= \phi^{2m} 1 \wedge \phi^2 1 \wedge \phi 1 \wedge (\psi^k A \vee \psi^l C) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^l C)$ (Lemma 37, 28). (2), (3), ..., (8): analogous to (1). (9) immediately follows by (1) with $k=l=0$ and Lemma 37.

Lemma 39: Let $i,j,k,l \in N_o$, and let $M,N \in 2N_o$ with $M \geq \max\{k,l\}$, $N \geq \max\{i,j\}$.

(1) $(\psi^i B \vee \psi^j C) \wedge (\phi^k A \vee \phi^l B) = \phi^M(\psi^i B \vee \psi^j C) \vee \psi^N(\phi^k A \vee \phi^l B)$

(2) $(\psi^i A \vee \psi^j D) \wedge (\phi^k C \vee \phi^l D) = \phi^M(\psi^i A \vee \psi^j D) \vee \psi^N(\phi^k C \vee \phi^l D)$

(3) $(\psi^i A \vee \psi^j C) \wedge (\phi^k B \vee \phi^l D) = \phi^M(\psi^i A \vee \psi^j C) \vee \psi^N(\phi^k B \vee \phi^l D)$

(4) $(\psi^i B \vee \psi^j D) \wedge (\phi^k A \vee \phi^l C) = \phi^M(\psi^i B \vee \psi^j D) \vee \psi^N(\phi^k A \vee \phi^l C)$

Proof: (1): $(\psi^{i}B\vee\psi^{j}C)\wedge(\phi^{k}A\vee\phi^{l}B)$

$=(\psi^{i}B\vee\psi^{j}C)\wedge(\phi^{k}A\vee\phi^{l}B)\wedge(\phi^{M}1\vee\phi^{N}1)$

$=(\psi^{N}1\vee(\phi^{M}1\wedge(\psi^{i}B\vee\psi^{j}C)))\wedge(\phi^{k}A\vee\phi^{l}B)$

$=(\phi^{M}1\wedge(\psi^{i}B\vee\psi^{j}C))\vee(\psi^{M}1\wedge(\phi^{k}A\vee\phi^{l}B))$

$=\phi^{M}(\psi^{i}B\vee\psi^{j}C)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B))$ (Lemma 35, 38).

(2), (3), (4): analogous to (1).

Lemma 40: $(\psi^{i}\phi X\vee\psi^{j}X)\wedge(\phi^{k}\psi X\vee\phi^{l}X)\leq\phi\psi 1$ for all $i,j,k,l\varepsilon N\cup\{\infty\}$ and for all $X\varepsilon J_{4}^{4}$.

Proof: The assertion follows by Lemma 38 and by $\psi 1\geq\psi(\psi^{i-1}\phi X\vee\psi^{j-1}X)$ and $\phi 1\geq\phi(\phi^{k-1}\psi X\vee\phi^{l-1}X)$.

Lemma 41: Let $i,j\varepsilon N_{o}$ with $i\equiv j(\mathrm{mod}\ 2)$; $M\varepsilon 2N$ with $M>\max\{i,j\}$ and let $X\varepsilon J_{4}^{4}$.

(1) $\psi^{i}\phi X\vee\psi^{j}X\geq\psi^{M}\phi 1$ (2) $\phi^{i}\psi X\vee\phi^{j}X\geq\phi^{M}\psi 1$

Proof: (1): $\psi^{i}\phi A\vee\psi^{j}A\geq\psi^{i}\phi A\vee\psi^{j}\phi D=\phi(\psi^{i}A\vee\psi^{j}D)$

$=\phi 1\wedge(\psi^{i}D\vee\psi^{j}A)\geq\phi 1\wedge(\psi^{\max\{i,j\}}D\vee\psi^{\max\{i,j\}}A)\geq\phi 1\wedge\psi^{\max\{i,j\}+1}1$

$\geq\phi 1\wedge\psi^{M}1=\phi\psi^{M}1$ (Lemma 37, 38). The proofs for B,C,D analogously go. For 0 or 1 the assertions are trivial.

(2): analogous to (1).

Lemma 42: Let $i,j,k,l \in N$; $M,N \in 2N$ with $M > \max\{k,l\}$ and $N > \max\{i,j\}$.

(1) $(\psi^i\phi A \vee \psi^j A) \wedge (\phi^k\psi A \vee \phi^l A) = \phi^M(\psi^i D \vee \psi^j A) \vee \psi^N(\phi^k B \vee \phi^l A)$

(2) $(\psi^i\phi B \vee \psi^j B) \wedge (\phi^k\psi B \vee \phi^l B) = \phi^M(\psi^i C \vee \psi^j B) \vee \psi^N(\phi^k A \vee \phi^l B)$

(3) $(\psi^i\phi C \vee \psi^j C) \wedge (\phi^k\psi C \vee \phi^l C) = \phi^M(\psi^i B \vee \psi^j C) \vee \psi^N(\phi^k D \vee \phi^l C)$

(4) $(\psi^i\phi D \vee \psi^j D) \wedge (\phi^k\psi D \vee \phi^l D) = \phi^M(\psi^i A \vee \psi^j D) \vee \psi^N(\phi^k C \vee \phi^l D)$

Proof: (1): $(\psi^i\phi A \vee \psi^j A) \wedge (\phi^k\psi A \vee \phi^l A)$

$= (\psi^i\phi A \vee \psi^j A) \wedge (\phi^k\psi A \vee \phi^l A) \wedge \psi\phi 1$

$= (\psi^i\phi A \vee \psi^j A) \wedge (\phi^k\psi A \vee \phi^l A) \wedge (\psi^N\phi 1 \vee \phi^M\psi 1)$

$= (((\psi^i\phi A \vee \psi^j A) \wedge \phi^M\psi 1) \vee \psi^N\phi 1) \wedge (\phi^k\psi A \vee \phi^l A)$

$= ((\psi^i\phi A \vee \psi^j A) \wedge \phi^M\psi 1) \vee ((\phi^k\psi A \vee \phi^l A) \wedge \psi^N\phi 1)$

$= \psi((\psi^{i-1}\phi A \vee \psi^{j-1}A) \wedge \phi^M 1) \vee \phi((\phi^{k-1}\psi A \vee \phi^{l-1}A) \wedge \psi^N 1)$

$= \psi((((A \vee B) \wedge \psi^{i-1}D) \vee \psi^{j-1}A) \wedge \phi^M 1) \vee \phi((((A \vee D) \wedge \phi^{k-1}B) \vee \phi^{l-1}A) \wedge \psi^M 1)$

$= \psi((\psi^{i-1}D \vee \psi^{j-1}A) \wedge (A \vee B) \wedge \phi^M 1) \vee \phi((\phi^{k-1}B \vee \phi^{l-1}A) \wedge (A \vee D) \wedge \psi^N 1)$

$= \psi((\psi^{i-1}D \vee \psi^{j-1}A) \wedge \phi^M 1) \vee \phi((\phi^{k-1}B \vee \phi^{l-1}A) \wedge \psi^N 1)$

$= \psi\phi^M(\psi^{i-1}D \vee \psi^{j-1}A) \vee \phi\psi^N(\phi^{k-1}B \vee \phi^{l-1}A)$

$= \phi^M(\psi^i D \vee \psi^j A) \vee \psi^N(\phi^k B \vee \phi^l A)$ (Lemma 40, 35, 41, 37, 38).

(2), (3), (4): analogous to (1).

Before we define meet-morphisms, let us recall that the elements of $FM(J_2^4)$ have a representation as quadrupels $f(i,j,k,l) = (\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d)$ with (i,j,k,l) satisfying condition (*) in section 2 (Proposition 19).

Furthermore, let us define for $i\varepsilon N_o$:

$\psi^{-1}A\vee\psi^{i}D:=B\vee\psi^{i}D$, $\psi^{-1}D\vee\psi^{i}A:=C\vee\psi^{i}A$, $\psi^{-1}B\vee\psi^{i}C:=A\vee\psi^{i}C$, $\psi^{-1}C\vee\psi^{i}B:=D\vee\psi^{i}B$, $\psi^{-1}A\vee\psi^{-1}D:=1$, $\psi^{-1}B\vee\psi^{-1}C:=1$, and $\phi^{-1}A\vee\phi^{i}B:=D\vee\phi^{i}B$, $\phi^{-1}B\vee\phi^{i}A:=C\vee\phi^{i}A$, $\phi^{-1}C\vee\phi^{i}D:=B\vee\phi^{i}D$, $\phi^{-1}D\vee\phi^{i}C:=A\vee\phi^{i}B$, $\phi^{-1}A\vee\phi^{-1}B:=1$, $\phi^{-1}C\vee\phi^{-1}D:=1$.

<u>Lemma 43:</u> There are meet-morphisms $\bar{\alpha}$, $\bar{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ such that

(1) $\bar{\alpha}f(i,j,k,l)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$

(2) $\bar{\beta}f(i,j,k,l)=(\psi^{i-1}B\vee\psi^{j-1}C)\wedge(\phi^{k-1}B\vee\phi^{l-1}A)$

Proof: (1): What actually has to be proved is

(i) $\bar{\alpha}f(i,j,k,l)=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(0,0,k,l)$

(ii) $\bar{\alpha}(f(i,j,0,0)\wedge f(r,s,0,0))=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(r,s,0,0)$

(iii) $\bar{\alpha}(f(0,0,k,l)\wedge f(0,0,t,u))=\bar{\alpha}f(0,0,k,l)\wedge\bar{\alpha}f(0,0,t,u)$

since by Proposition 19, (i), (ii) and (iii) we get:

$\bar{\alpha}(f(i,j,k,l)\wedge f(r,s,t,u))=\bar{\alpha}f(w,x,y,z)=\bar{\alpha}f(w,x,0,0)\wedge\bar{\alpha}f(0,0,y,z)$
$=\bar{\alpha}(f(i,j,0,0)\wedge f(r,s,0,0))\wedge\bar{\alpha}(f(0,0,k,l)\wedge f(0,0,t,u))$
$=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(r,s,0,0)\wedge\bar{\alpha}f(0,0,k,l)\wedge\bar{\alpha}f(0,0,t,u)$
$=\bar{\alpha}f(i,j,k,l)\wedge\bar{\alpha}f(r,s,t,u)$, if $0\le i,j,k,l,r,s,t,u<\infty$. If some $n\varepsilon\{i,j,k,l,r,s,t,u\}$ equals ∞, the proof easily can be checked by definition of $\bar{\alpha}$.

(i): $\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(0,0,k,l)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge1\wedge1\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$
$=(\psi^{i-1}D\vee\psi^{j-1})\wedge(\phi^{k-1}D\vee\phi^{l-1}C)=\bar{\alpha}f(i,j,k,l)$

(ii): The proof is divided into the following cases ($(i,j)\sim(r,s)$ means that $m\equiv n \pmod 2$ for all $m,n\in\{i,j,r,s\}\setminus\{\infty\}$ and $i,j,r,s\in N_0\cup\{\infty\}$):

1. $(i,j)\sim(r,s)$	2. $(i,j)\not\sim(r,s), \max\{i,j\}\le\max\{r,s\}$	
1.1. $i\le r,\ j\le s$	2.1. $0<i\le j\le r\le s$	2.8. $0<r\le j\le i\le s$
1.2. $i\le r,\ j\ge s$	2.2. $0<i\le j\le s\le r$	2.9. $0<i\le r\le j\le s$
1.3. $i\ge r,\ j\le s$	2.3. $0<j\le i\le r\le s$	2.10. $0<r\le i\le j\le s$
1.4. $i\ge r,\ j\ge s$	2.4. $0<j\le i\le s\le r$	2.11. $0<j\le s\le i\le r$
	2.5. $0<i\le s\le j\le r$	2.12. $0<s\le j\le i\le r$
	2.6. $0<s\le i\le j\le r$	2.13. $i\cdot j=0$ and $i+j>0$ or $r\cdot s=0$ and $r+s>0$
	2.7. $0<j\le r\le i\le s$	
		2.14. $i=j=0$ or $r=s=0$

By symmetry, the case $(i,j)\not\sim(r,s)$, $\max\{i,j\}>\max\{r,s\}$ is analogous to 2.

1.1. and 1.4. are immediate consequences of part 2, Lemma 6 and Lemma 28.

1.2. $\bar{\alpha}((\mu^i a\vee\mu^j d)\wedge(\mu^r a\vee\mu^s d))=\bar{\alpha}(\mu^r a\vee\mu^j d)=\psi^{r-1}D\vee\psi^{j-1}A$

$=(\psi^{s-1}A\wedge\psi^{i-1}D)\vee\psi^{j-1}A\wedge\psi^{r-1}D=(\psi^{s-1}A\wedge(\psi^{i-1}D\vee\psi^{j-1}A))\vee\psi^{r-1}D$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)=\bar{\alpha}(\mu^i a\vee\mu^j d)\wedge\bar{\alpha}(\mu^r a\vee\mu^s d)$

(Lemma 29,28);

1.3. analogous to 1.2..

2. $\bar{\alpha}((\mu^{i}a\vee\mu^{j}d)\wedge(\mu^{r}a\vee\mu^{s}d))$
$=\bar{\alpha}(\mu^{\max\{i+1,j+1,r\}}a\vee\mu^{\max\{i+1,j+1,s\}}d)$
$=\psi^{\max\{i,j,r-1\}}D\vee\psi^{\max\{i,j,s-1\}}A$

<u>2.1.</u> $\bar{\alpha}(\mu^{i}a\vee\mu^{j}d)\wedge\bar{\alpha}(\mu^{r}a\vee\mu^{s}d)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)$
$(\psi^{i-1}D\vee\psi^{j}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)=((\psi^{i-1}D\vee\psi^{j}A)\wedge\psi^{r-1}D)\vee\psi^{s-1}A$
$=\psi^{i-1}((D\wedge\psi^{j-i+1}A)\wedge C\wedge(D\vee\psi^{r-i-1}A))\vee\psi^{s-1}A$
$=\psi^{i-1}((((\psi^{j-i+1}A\vee D)\wedge\psi^{r-i-1}A)\vee D)\wedge C)\vee\psi^{s-1}A$
$=\psi^{i-1}((((\psi^{j-i}A\vee D)\wedge\psi^{r-i-1}A)\vee D)\wedge C)\vee\psi^{s-1}A$
$=\psi^{i-1}((\psi^{r-i-1}A\vee D)\wedge C)\vee\psi^{s-1}A=\psi^{i-1}\psi^{r-i}D\vee\psi^{s-1}A$
$=\psi^{r-1}D\vee\psi^{s-1}A$ (Lemma 32,28,30);

<u>2.2.,2.3,2.4.:</u> analogous to 2.1.(by symmetry of J_4^4 and commutativity of $\wedge$);

<u>2.5.</u> $\bar{\alpha}(\mu^{i}a\vee\mu^{j}d)\wedge\bar{\alpha}(\mu^{r}a\vee\mu^{s}d)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)$
$=(\psi^{i-1}D\vee\psi^{j}A)\wedge(\psi^{r}D\vee\psi^{s-1}A)=\psi^{r}D\vee(\psi^{s-1}A\wedge(\psi^{i-1}D\vee\psi^{j}A))$
$=\psi^{r}D\vee\psi^{j}A\wedge(\psi^{s-1}A\wedge\psi^{i-1}D)=\psi^{r}D\vee\psi^{j}A=\psi^{r-1}D\vee\psi^{j}A$
(Lemma 32,28,29);

<u>2.6.,2.7.,2.8.:</u> analogous to 2.5.;

<u>2.9.</u> $\bar{\alpha}(\mu^{i}a\vee\mu^{j}d)\wedge\bar{\alpha}(\mu^{r}a\vee\mu^{s}d)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)$
$=((\psi^{i-1}D\vee\psi^{j}A)\wedge\psi^{r-1}D)\vee\psi^{s-1}A=\psi^{i-1}((\psi^{j-i+1}A\vee D)\wedge\psi^{r-i}D)\vee\psi^{s-1}A$
$=\psi^{i-1}((\psi^{j-1}A\vee D)\wedge C)\vee\psi^{s-1}A=\psi^{i-1}\psi^{j-i+1}D\vee\psi^{t-1}A$
$=\psi^{j}D\vee\psi^{t-1}A$ (2.1., Lemma 30,32);

<u>2.10.,2.11.,2.12.:</u> analogous to 2.9.

2.13. Let $0=i<j\leq r\leq s$. $\bar{\alpha}f(0,j,0,0)\wedge\bar{\alpha}f(0,0,r,s)$
$=(C\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)=((C\vee\psi^{j-1}A)\wedge\psi^{r-1}D)\vee\psi^{s}A$
$=((C\vee\psi^{j-1}A)\wedge D\wedge(C\vee\psi^{r-2}A))\vee\psi^{s}A=(D\wedge(C\vee\psi^{r-2}A))\vee\psi^{s}A$
$=\psi^{r-1}D\vee\psi^{s}A=\psi^{r-1}D\vee\psi^{s-1}A$ (Lemma 32).

Corresponding 2.2. until 2.12., all other cases can be easily proved in a similar way.

<u>2.14.:</u> The assertion is trivial because of $\bar{\alpha}f(0,0,0,0)=1$.

(iii): By symmetry of definition of ϕ and ψ , assertion (iii) analogously as (ii) follows.

Since the proof of (2) is analogous to (1), Lemma 43 is completely proved.

<u>Lemma 44:</u> $\alpha\bar{\alpha}=id_{FM(J_2^4)}=\beta\bar{\beta}$

Proof: 1. Let $x=f(i,j,k,l)\varepsilon FM(J_2^4)$ with $i,j,k,l>0$.

$\alpha\bar{\alpha}x=\alpha((\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C))$

$=(\mu^{i-1}\alpha D\vee\mu^{j-1}\alpha A)\wedge(\nu^{k-1}\alpha D\vee\nu^{l-1}\alpha C)$

$=(\mu^{i-1}b\vee\mu^{j-1}c)\wedge(\nu^{k-1}b\vee\nu^{l-1}a)$

$=(\mu^{i}a\vee\mu^{j}d)\wedge(\nu^{k}c\vee\nu^{l}d)=f(i,j,k,l)$ (Lemma 25).

2. Let $x=f(i,j,k,l)\varepsilon FM(J_2^4)$ with some of i,j,k,l equals 0. We have to use the definition for expressions with ψ^{-1} or ϕ^{-1}; then the proof is analogous to 1..

$\beta\bar{\beta}=id_{FM(J_2^4)}$ can analogously be shown.

Lemma 45: $\underline{\alpha}x \le \bar{\alpha}x$ and $\underline{\beta}x \le \bar{\beta}x$ for all $x \varepsilon FM(J_2^4)$

Proof: $\underline{\alpha}f(i,j,k,l) = \underline{\alpha}((\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d))$
$= (\psi^i (C \wedge (A \vee B)) \vee \psi^j B) \wedge (\phi^k (A \wedge (B \vee C)) \vee \phi^l B)$
$\le (\psi^{i-1} D \vee \psi^{j-1} A) \wedge (\phi^{k-1} D \vee \phi^{l-1} C) = \bar{\alpha} f(i,j,k,l)$ (Lemma 37).
$\underline{\beta}x \le \bar{\beta}x$ (analogous to $\underline{\alpha}x \le \bar{\alpha}x$).

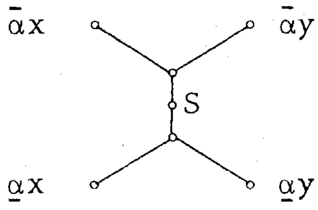
Lemma 46: Let I_α and I_β be the set of all intervals $[\underline{\alpha}x, \bar{\alpha}x]$ and $[\underline{\beta}x, \bar{\beta}x]$ with $x \varepsilon FM(J_2^4)$, respectively.
(1) $\bigcup I_\alpha = FM(J_4^4)$ (2) $\bigcup I_\beta = FM(J_4^4)$

Proof: (1): Since every meet-morphism is isotone, we get $\bar{\alpha}x \le \bar{\alpha}(x \vee y)$, $\bar{\alpha}y \le \bar{\alpha}(x \vee y)$ and this implies $\bar{\alpha}x \vee \bar{\alpha}y \le \bar{\alpha}(x \vee y)$.
Now, let $S \varepsilon [\underline{\alpha}x, \bar{\alpha}x]$ and $T \varepsilon [\underline{\alpha}y, \bar{\alpha}y]$.
By $\underline{\alpha}(x \vee y) = \underline{\alpha}x \vee \underline{\alpha}y \le S \vee T \le \bar{\alpha}x \vee \bar{\alpha}y \le \bar{\alpha}(x \vee y)$ and $\underline{\alpha}(x \wedge y) = \underline{\alpha}x \wedge \underline{\alpha}y \le S \wedge T \le \bar{\alpha}x \wedge \bar{\alpha}y \le \bar{\alpha}(x \wedge y)$, we get $S \vee T \varepsilon [\underline{\alpha}(x \vee y), \bar{\alpha}(x \vee y)]$ and $S \wedge T \varepsilon [\underline{\alpha}(x \wedge y), \bar{\alpha}(x \wedge y)]$. Since $A \varepsilon [A \wedge (B \vee C), A] = [\underline{\alpha}c, \bar{\alpha}c]$, $B \varepsilon [B, B] = [\underline{\alpha}D, \bar{\alpha}D]$, $C \varepsilon [C \wedge (A \vee B), C] = [\underline{\alpha}a, \bar{\alpha}a]$, $D \varepsilon [D \wedge (B \vee A) \wedge (B \vee C), D] = [\underline{\alpha}b, \bar{\alpha}b]$ and since $FM(J_4^4)$ is freely generated by J_4^4, the assertion follows.

(2): analogous to (1).

Lemma 47: I_α and I_β are partitions on $FM(J_4^4)$.

Proof:



$[\underline{\alpha}x,\bar{\alpha}x]\cap[\underline{\alpha}y,\bar{\alpha}y]\neq\emptyset$ implies $\underline{\alpha}x\vee\underline{\alpha}y\leq\bar{\alpha}x\wedge\bar{\alpha}y$. Using Lemma 27 and Lemma 44, we get $x\vee y\leq x\wedge y$ by $\alpha\underline{\alpha}(x\vee y)\leq\alpha\bar{\alpha}(x\wedge y)$. Thus, $x=y$ and the assertion follows by Lemma 46. The assertion for I_β can be proved by the same arguments.

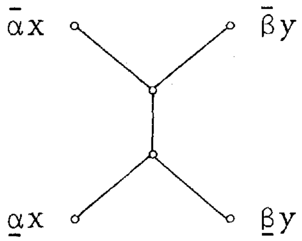
<u>Lemma 48:</u> Let $\ker\alpha_x := \{X\varepsilon FM(J_4^4)\,|\,\alpha X=x\}$ and $\ker\beta_x := \{X\varepsilon FM(J_4^4)\,|\,\beta X=x\}$ for $x\varepsilon FM(J_2^4)$.

(1) $\ker\alpha_x=[\underline{\alpha}x,\bar{\alpha}x]$ (2) $\ker\beta_x=[\underline{\beta}x,\bar{\beta}x]$.

Proof: Lemma 48 is an immediate consequence of Lemma 47 together with Lemma 27 and 44.

<u>Proposition 49:</u> α and β are separating homomorphisms from $FM(J_4^4)$ onto $FM(J_2^4)$.

Proof: By Lemma 48 it is enough to show that the meet of two intervals $[\underline{\alpha}x,\bar{\alpha}x]$ and $[\underline{\beta}y,\bar{\beta}y]$ is empty or consists of only one element. We choose a fixed element $x=f(i,j,k,l)$ of $FM(J_2^4)$. Our first goal is to determine all elements $y\varepsilon FM(J_2^4)$ for which $I_{xy} := [\underline{\alpha}x,\bar{\alpha}x]\cap[\underline{\beta}y,\bar{\beta}y]\neq\emptyset$.



Suppose $I_{xy}\neq\emptyset$. Then $\underline{\alpha}x\vee\underline{\beta}y\le\bar\alpha x\wedge\bar\beta y$ and $\beta\underline{\alpha}x\vee y=\beta\underline{\alpha}x\vee\beta\underline{\beta}y\le\beta\bar\alpha x\wedge\beta\bar\beta y=\beta\bar\alpha x\wedge y$ (Lemma 27,44). Therefore we get $\mu\nu x=\beta\underline{\alpha}x\le y\le\beta\bar\alpha x$ (Lemma 27). By Lemma 4 and Proposition 19, $\mu\nu x=f(j+1,i+1,l+1,k+1)$. For determining $\beta\bar\alpha x$ we have to distinguish in

(i) $i,j,k,l>0$: $\beta\bar\alpha f(i,j,k,l)=\beta((\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C))$
$=(\mu^{i-1}\beta D\vee\mu^{j-1}\beta A)\wedge(\nu^{k-1}\beta D\vee\nu^{l-1}\beta C)$
$=(\mu^{j-1}a\vee\mu^{i-1}d)\wedge(\nu^{l-1}c\vee\nu^{k-1}d)=f(j-1,i-1,l-1,k-1)$
(Lemma 25).

(ii) $i\cdot j=0$ and $i+j>0$ or $k\cdot l=0$ and $k+l>0$: Let w.l.o.g. $0=i<j,k,l$.
$\beta\bar\alpha f(0,j,k,l)=\beta((C\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$
$=(c\vee\mu^{j-1}a)\wedge(\nu^{k-1}d\vee\nu^{l-1}c)=(\mu^{j-1}a\vee\mu d)\wedge(\nu^{l-1}c\vee\nu^{k-1}d)$
$=f(j-1,1,l-1,k-1)$ (Lemma 25).

(iii) $i=j=0$ or $k=l=0$: Let w.l.o.g. $0=i=j<k,l$.
$\beta\bar\alpha f(0,0,k,l)=\beta(1\wedge(\phi^{k-1}D\vee\phi^{l-1}C))=1\wedge(\nu^{l-1}c\vee\nu^{k-1}d)$
$=f(0,0,l-1,k-1)$ (Lemma 25).

For summarizing these results, we define:
$i^*:=j,\ j^*:=i,\ k^*:=l,\ l^*:=k$

and for all $m\in\{i,j,k,l\}$: $$\bar m:=\begin{cases} m-1 & \text{if } m>0\\ 1 & \text{if } m=0 \text{ and } m^*\neq 0\\ 0 & \text{if } m=m^*=0\end{cases}$$

It follows: $I_{xy} \neq \emptyset \iff y \varepsilon [f(j+1,i+1,l+1,k+1), f(\bar{j},\bar{i},\bar{l},\bar{k})]$ $\iff y \varepsilon \{f(\hat{j},\hat{i},\hat{l},\hat{k}) \varepsilon FM(J_2^4) \mid \hat{m} \varepsilon \{\bar{m},m,m+1\}$ and $\hat{m}=m \iff \hat{m}^*=m$ with $m \varepsilon \{i,j,k,l\}\}$ (Proposition 19).

Let us say, y satisfies (**), if y is an element of the set just mentioned.

In the following we have to prove $\bar{\alpha}x \wedge \bar{\beta}y = \underline{\alpha}x \vee \underline{\beta}y$ for a fixed x and all y satisfying condition (**). The proof will be divided in several cases, since we have some possibilities for the choice of y and since the cases with some of i,j,k,l equals 0 or equals ∞ have to be treated separately. By symmetry of J_4^4 and by commutativity of meet and join, it is enough to prove the following cases:

1.1.: $x=f(i,j,k,l)$, $y=f(j+1,i+1,l+1,k+1)$ for $0<i,j,k,l<\infty$

1.2.: $x=f(i,j,k,l)$, $y=f(j-1,i+1,l+1,k+1)$ for $0<i,j,k,l<\infty$

1.3.: $x=f(i,j,k,l)$, $y=f(j-1,i-1,l+1,k+1)$ for $0<i,j,k,l<\infty$

1.4.: $x=f(i,i,k,l)$, $y=f(i,i,l+1,k+1)$ for $0<i,k,l<\infty$

1.5.: $x=f(\infty,2m,\infty,2n)$, $y=f(2m+1,\infty,2n-1,\infty)$ for $0<m,n<\infty$

2.1.: $x=f(0,j,k,l)$, $y=f(j+1,1,l+1,k+1)$ for $0<j,k,l<\infty$

2.2.: $x=f(0,\infty,\infty,2n+1)$, $y=f(\infty,1,2n,\infty)$ for $0\le n<\infty$

3.1.: $x=f(0,0,k,l)$, $y=f(1,1,l+1,k+1)$ for $0<k,l<\infty$

1.1.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{j}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i}C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{i+1}D\vee\psi^{j}B\vee\psi^{j+1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{k+1}D\vee\phi^{l}B\vee\phi^{l+1}C)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)\vee\phi^{M}(\psi^{j+1}A\vee\psi^{i+1}D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j+1}\phi D\vee\psi^{i+1}D)\wedge(\phi^{l-1}\psi D\vee\phi^{k+1}D))$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 39,42).

1.2.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j-2}B\vee\psi^{i}C)\wedge(\phi^{l}B\wedge\phi^{k}A)$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{j-2}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee(\psi^{j-2}B\wedge(\psi^{i-1}D\vee\psi^{j-1}A)))\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee\psi^{j-1}A\vee(\psi^{j-2}B\wedge\psi^{i-1}D))\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee\psi^{j-1}A)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i}C\vee\psi^{j-1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B\vee\psi^{j-1}A\vee\psi^{i+1}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j-1}\phi D\vee\psi^{i+1}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k+1}D)$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 29,39,42).

1.3.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j-2}B\vee\psi^{i-2}C)\wedge(\phi^{l}B\wedge\phi^{k}A)$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i-1}D\vee\psi^{j-1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B\vee\psi^{j-1}\vee\psi^{i-1}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j-1}\phi D\vee\psi^{i-1}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k+1}D))$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 39,42).

1.4.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{i-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{i-1}B\vee\psi^{i-1}C)\wedge(\phi^{l}B\vee\phi^{k-2}A)$

$=\psi^{i-1}((D\vee A)\wedge(B\vee C))\wedge(\phi^{l}B\vee(\phi^{k-2}A\wedge(\phi^{k-1}D\vee\phi^{l-1}C)))$

$=\psi^{i-1}\psi 1\wedge(\phi^{l}B\vee\phi^{k-1}D\vee(\phi^{l-1}D\wedge\phi^{k-2}A)=\psi^{i}1\wedge(\phi^{k-1}D\vee\phi^{l}B)$

$=\phi^{M}\psi^{i}1\vee\psi^{N}(\phi^{k-1}D\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{i}B\vee\psi^{i}A\vee\psi^{i}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k-1}D)$

$=((\psi^{i}\phi B\vee\psi^{i}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{i}\phi D\vee\psi^{i}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k-1}D)$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 29,39,42).

$\underline{1.5.:}$ $\bar{\alpha}x\wedge\bar{\beta}y=\psi^{2m-1}A\wedge\phi^{2n-1}C\wedge\psi^{2m}B\wedge\phi^{2n-2}B=\psi^{2m}B\wedge\phi^{2n-1}C$

$=B\wedge\phi^{2n-1}C\wedge\psi^{2m}B=(\psi^{2m+1}A\vee\phi^{2n}B)\wedge\phi^{2n-1}C\wedge\psi^{2m}B$

$=((\psi^{2m+1}A\wedge\phi^{2n-1}C)\vee\phi^{2n}B)\wedge\psi^{2m}B$

$=((\psi^{2m+1}A\wedge\phi 1\wedge\phi^{2n-1}C\wedge\psi 1)\vee\phi^{2n}B)\wedge\psi^{2m}B$

$=((\psi^{2m+1}\phi D\wedge\phi^{2n-1}\psi D)\vee\phi^{2n}B)\wedge\psi^{2m}B$

$=(\psi^{2m}B\wedge\phi^{2n}B)\vee(\psi^{2m+1}\phi D\wedge\phi^{2n-1}\psi D)=\underline{\alpha}x\vee\underline{\beta}y$ (Lemma 34,37).

$\underline{2.1.:}$ $\bar{\alpha}x\wedge\bar{\beta}y=(C\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j}B\vee C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{j}B\vee C)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(C\vee\psi^{j}B\vee\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((C\vee\psi^{j}B)\wedge(\phi^{k}A\vee\phi^{l}B))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((C\vee\psi^{j}B)\wedge(\phi^{k}A\vee\phi^{l}B)\wedge(B\vee C))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((C\vee\psi^{j}B)\wedge((\phi^{k}A\wedge\psi 1)\vee\phi^{l}B))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j+1}\phi D\vee\psi D)\wedge(\phi^{l+1}\psi D\vee\phi^{k+1}D)$

$=\underline{\alpha}x\vee\underline{\beta}y$ (Lemma 39,37,42).

$\underline{2.2.:}$ $\bar{\alpha}x\wedge\bar{\beta}y=C\wedge\phi^{2n}C\wedge C\wedge\phi^{2n-1}B=\phi^{2n}C=\phi^{2n}(\phi B\vee\psi D)=\phi^{2n+1}B\vee\phi^{2n}\psi D$

$=(\phi B\wedge\phi^{2n+1}B)\vee(\psi D\wedge\phi^{2n}\psi D)=\underline{\alpha}x\vee\underline{\beta}y$ (Lemma 33).

$\underline{3.1.:}$ $\bar{\alpha}x\wedge\bar{\beta}y=1\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(B\vee C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(B\vee C)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(B\vee C)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(B\vee C\vee\psi A\vee\psi D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((B\vee C)\wedge(\phi^{k}A\vee\phi^{l}B))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((B\vee C)\wedge(A\vee B)\wedge(\phi^{k}A\vee\phi^{l}B)\wedge(B\vee C))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

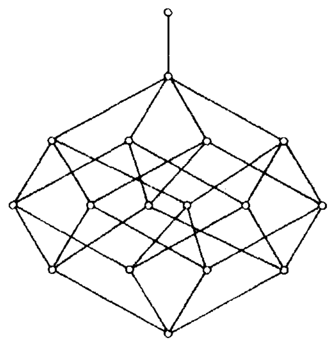
$=((B\vee C)\wedge(A\vee B)\wedge((\phi^{k}A\wedge\psi 1)\vee\phi^{l}B))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\phi B\vee B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi\phi D\vee\psi D)\wedge(\phi^{l+1}D\vee\phi^{k+1}D)$

$=\underline{\alpha}x\vee\underline{\beta}y$ (Lemma 39,30,37,42).

This completes the proof of Proposition 49.

Proposition 50: The congruence lattice of $FM(J_4^4)$ is described by the following diagram :



Proof: By Proposition 49, the intersection of the congruence relations $\ker\alpha$ and $\ker\beta$ is the identity; furthermore by the Homomorphism Theorem and Proposition 22, there is only one coatom greater than $\ker\alpha$ and $\ker\beta$, respectively, namely $\Theta(A\vee B, B\vee C, C\vee D, D\vee A)$. Therefore, Proposition 22 and Proposition 49 together with the distributivity of the congruence lattice gives us the assertion.

Proof of Theorem 3: By the Homomorphism Theorem, the assertions are immediate consequences of Proposition 50.

Another immediate consequence of Proposition 49 is the following Proposition which (together with Proposition 19 and 21) solves the word problem for $FM(J_4^4)$. For stating the Proposition, we define

$g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k}) := \bar{\alpha}f(i,j,k,l) \wedge \bar{\beta}f(\hat{j},\hat{i},\hat{l},\hat{k})$

for $f(i,j,k,l), f(\hat{j},\hat{i},\hat{l},\hat{k}) \in FM(J_2^4)$.

Proposition 51: The elements of $FM(J_4^4)$ can be uniquely represented as octuples $g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$ such that (i,j,k,l) satisfies (*) and $f(\hat{j},\hat{i},\hat{l},\hat{k})$ satisfies (**) with respect to $f(i,j,k,l)$; furthermore, for $\circ=\vee$ and $\circ=\wedge$, respectively, we have

$$g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\circ g(m,n,p,q,\hat{n},\hat{m},\hat{q},\hat{p})=g(r,s,t,u,\hat{s},\hat{r},\hat{u},\hat{t})$$

if and only if

$$f(i,j,k,l)\circ f(m,n,p,q)=f(r,s,t,u) \text{ and}$$

$$f(\hat{j},\hat{i},\hat{l},\hat{k})\circ f(\hat{n},\hat{m},\hat{q},\hat{p})=f(\hat{s},\hat{r},\hat{u},\hat{t}) .$$

Now we can solve the word problems for $FM(J_3^4)$ and $FM(J_{1,1}^4)$.

There exist epimorphisms γ and δ of $FM(J_3^4)$ such that

(1)	$\gamma 0=0$	(2)	$\delta 0=0$
	$\gamma a=d\wedge(a\vee b)$		$\delta a=b$
	$\gamma b=c\wedge(a\vee b)$		$\delta b=a\wedge(b\vee c)$
	$\gamma c=b\wedge(c\vee d)$		$\delta c=d\wedge(b\vee c)$
	$\gamma d=a\wedge(c\vee d)$		$\delta d=c$
	$\gamma 1=(a\vee b)\wedge(c\vee d)$		$\delta 1=b\vee c$

and endomorphisms ρ and σ of $FM(J_{1,1}^4)$ such that

(1)	$\rho 0=0$	(2)	$\sigma 0=0$
	$\rho a=d\wedge(a\vee b)$		$\sigma a=b$
	$\rho b=c\wedge(a\vee b)$		$\sigma b=a$
	$\rho c=b\wedge(c\vee d)$		$\sigma c=d$
	$\rho d=a\wedge(c\vee d)$		$\sigma d=c$
	$\rho 1=(a\vee b)\wedge(c\vee d)$		$\sigma 1=1$

These endomorphisms give us representations of $FM(J_3^4)$ and $FM(J_{1,1}^4)$.

Proposition 52:

$FM(J_3^4)=\{h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})\mid g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\varepsilon FM(J_4^4)\}$ with

$h(k,l,\hat{j},\hat{i},\hat{l},\hat{k}):=(\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)$

for $0\le\hat{i},\hat{j},\hat{k},\hat{l},k,l\le\infty$ and

$\gamma^{-1}x$, $\delta^{-1}x$ analogously defined as in Lemma 43.

Proof: Let τ the epimorphism from $FM(J_4^4)$ onto $FM(J_3^4)$ with $\tau X=x$ for all $X\varepsilon J_4^4$, $x\varepsilon J_3^4$. We get $\tau\psi=\delta\tau$ and $\tau\phi=\gamma\tau$. Since $i-1\le\hat{i}$ and $j-1\le\hat{j}$ it follows $\delta^{i}d\vee\delta^{j}a=\delta^{i-1}c\vee\delta^{j-1}b\ge\delta^{\hat{i}}c\vee\delta^{\hat{j}}b$ and $\tau g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$

$=(\delta^{i-1}d\vee\delta^{j-1}a)\wedge(\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)$

$=(\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)=h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})$

Corollary 53: For $\circ=\vee$ and $\circ=\wedge$, respectively, we have

$h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})\circ h(p,q,\hat{n},\hat{m},\hat{q},\hat{p})=h(t,u,\hat{s},\hat{r},\hat{u},\hat{t})$

if and only if $e(k,l)\circ e(p,q)=e(t,u)$ and

$f(\hat{j},\hat{i},\hat{k},\hat{l})\circ f(\hat{n},\hat{m},\hat{q},\hat{p})=f(\hat{s},\hat{r},\hat{u},\hat{t})$

(The elements $e(i,j)$ of $FM(J_1^4)$ are defined in DAY, HERRMANN, WILLE [2], and $f(i,j,k,l)\varepsilon FM(J_2^4)$).

Proposition 54:

$FM(J^4_{1,1})=\{p(k,l,\hat{l},\hat{k})\mid g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\varepsilon FM(J^4_4)\}$ with
$p(k,l,\hat{l},\hat{k}):=(\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)$
for $0\le k,l,\hat{k},\hat{l}\le\infty$ and $\rho^{-1}x$ analogously defined as in Lemma 43.

Proof: Let η the epimorphism from $FM(J^4_4)$ onto $FM(J^4_{1,1})$ with $\eta X=x$ for all $X\varepsilon J^4_4$ and $x\varepsilon J^4_{1,1}$.
We get $\rho\eta=\eta\phi$ and $\sigma\eta=\eta\psi$. Since
$\sigma^{i-1}d\vee\sigma^{j-1}a=\sigma^{\hat{j}-1}b\vee\sigma^{\hat{i}-1}c=a\vee d=b\vee c=1$ it follows
$\eta g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$
$=(\sigma^{i-1}d\vee\sigma^{j-1}a)\wedge(\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\sigma^{\hat{j}-1}b\vee\sigma^{\hat{i}-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)$
$=(\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)=p(k,l,\hat{l},\hat{k})$

Corollary 55: For $\circ=\vee$ and $\circ=\wedge$, respectively, we have $p(k,l,\hat{l},\hat{k})\circ p(m,n,\hat{n},\hat{m})=p(r,s,\hat{s},\hat{t})$ if and only if $e(k,l)\circ e(m,n)=e(r,s)$ and $e(\hat{l},\hat{k})\circ e(\hat{n},\hat{m})=e(\hat{s},\hat{r})$.

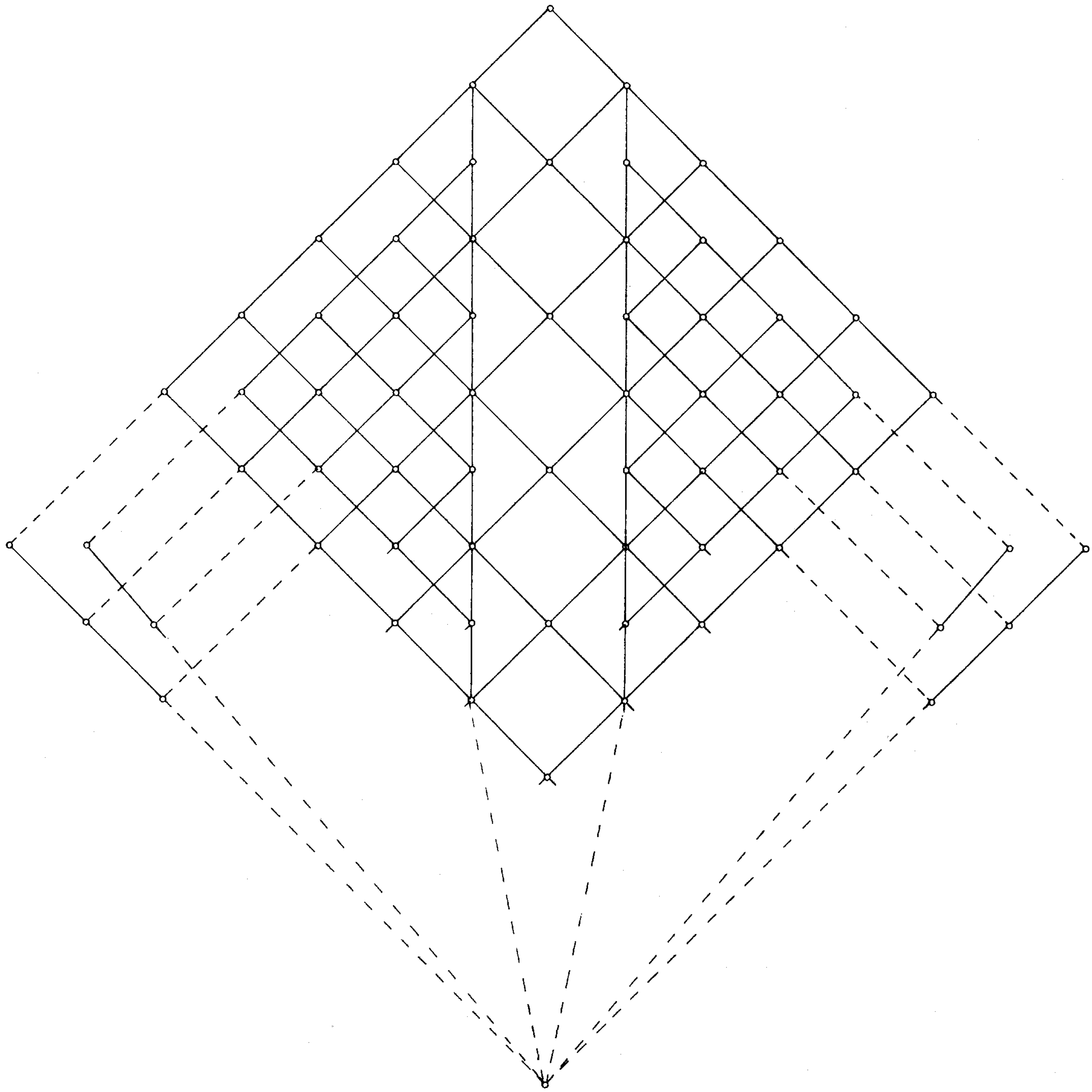
Corollary 56: (1) $FM(J^4_1)$ is a sublattice of $FM(J^4_{1,1})$
(2) $FM(J^4_2)$ is a sublattice of $FM(J^4_4)$.

Proof: (1): There exists a monomorphism λ from $FM(J^4_1)$ into $FM(J^4_{1,1})$ such that

$\lambda 0=0$	$\lambda c=a\wedge(c\vee d)$
$\lambda a=c$	$\lambda d=d$
$\lambda b=b\wedge(c\vee d)$	$\lambda 1=c\vee d$

(2) is an immediate consequence of Lemma 26.

The diagram of $FM(J^4_{1,1})$ is shown below.



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Freie modulare Verbände $FM({}_DM_3)$

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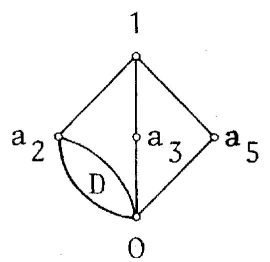
In der vorliegenden Note werden modulare Verbände $FM({}_DM_3)$ untersucht, die bis auf Isomorphie dadurch definiert sind, daß sie in der Klasse aller modularen Verbände von einem partiellen Verband ${}_DM_3$ frei erzeugt werden; die partiellen Verbände ${}_DM_3$ sind dabei folgendermaßen definiert:
Ist M_3 ein fünfelementiger Verband der Länge 2 mit kleinstem Element 0, Atomen a_2, a_3, a_5 sowie größtem Element 1 und ist D ein beschränkter, distributiver Verband mit kleinstem Element 0 und größtem Element a_2, dann ist ${}_DM_3$ der partielle Verband, der D und M_3 als Unterverbände besitzt so, daß $D \cup M_3 = {}_DM_3$ und $D \cap M_3 = \{0, a_2\}$ gilt und daß $d \vee a_3$ bzw. $d \vee a_5$ mit $d \in D$ nur für $d \in D \cap M_3$ existiert.

${}_DM_3$:



Für alle Begriffe, die in dieser Note nicht erklärt werden, sei auf GRÄTZER [1] verwiesen.

Da nicht distributive, modulare Verbände stets Unterverbände enthalten, die zu M_3 isomorph sind, treten in jedem nicht distributiven, modularen Verband partielle Verbände $_DM_3$ als relative Unterverbände auf, wobei häufig die Kenntnis der von solchen relativen Unterverbänden erzeugten Unterverbände wesentliche Einblicke in die Struktur des Verbandes gibt. Aus dieser Tatsache erhält der folgende Satz seine Bedeutung für die Untersuchung nicht distributiver, modularer Verbände.

<u>Satz:</u> M sei ein modularer Verband und $_DM_3$ ein relativer Unterverband von M. Dann sind folgende Aussagen äquivalent:

(1) M wird erzeugt von $_DM_3$.

(2) M ist isomorph zu $FM(_DM_3)$.

(3) M ist isomorph zu der subdirekten Potenz von M_3, die aus allen quasi-eigentlichen, stetigen Abbildungen von dem Stoneschen Raum $S(D)$ in den T_0-Raum M_3 mit der Subbasis $\{[a)|a\epsilon M_3\}$ besteht; dabei heißt eine Abbildung zwischen topologischen Räumen quasi-eigentlich, wenn das Urbild jeder quasi-kompakten, offenen Menge wieder quasi-kompakt ist (s.HOFFMANN&KEIMEL [2; Definition 1.7]).

Der Beweis des Satzes wird durch drei Hilfssätze vorbereitet. Mit einer in WILLE [6] entwickelten Methode wird in Hilfssatz 1 gezeigt, daß $FM({}_DM_3)$ zu einer subdirekten Potenz von M_3 isomorph ist. In Hilfssatz 2 werden Normalformen für die Elemente von $FM({}_DM_3)$ angegeben, an denen sich ablesen läßt, daß die in SCHMIDT [5; Lemma 17.1] konstruierten modularen Verbände M isomorph zu den $FM({}_DM_3)$ sind. Hilfssatz 3 beschreibt dann die Isomorphie der Kongruenzrelationenverbände von D und $FM({}_DM_3)$, die auch schon in SCHMIDT [5] gezeigt ist.

<u>Hilfssatz 1:</u> M sei ein modularer Verband, der von einem relativen Unterverband ${}_DM_3$ erzeugt wird. Dann ist M isomorph zu einer subdirekten Potenz von M_3 .

Beweis: Die Behauptung soll zunächst unter der Voraussetzung bewiesen werden, daß D ein endlicher, distributiver Verband ist. In diesem Fall kann folgendes Lemma aus WILLE [6] angewandt werden.

<u>M_3-Lemma:</u> S sei ein subdirekt irreduzibler, modularer Verband, der von der endlichen Menge $E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1$ erzeugt wird ($E_2, E_3, E_5 \neq \emptyset$); sei ferner $\bar{e}_i := \sup \bigcup (E_j \mid i \text{ teilt } j)$ und $\underline{e}_i := \inf \bigcup (E_j \mid j \text{ teilt } i)$ für $i \in \{2,3,5\}$. Ist $S \neq M_3$ dann gilt:

$$(\bar{e}_2 \wedge \bar{e}_3) \vee (\bar{e}_2 \wedge \bar{e}_5) \vee (\bar{e}_3 \wedge \bar{e}_5) \geq (\underline{e}_2 \vee \underline{e}_3) \wedge (\underline{e}_2 \vee \underline{e}_5) \wedge (\underline{e}_3 \vee \underline{e}_5)$$

Da M isomorph ist zu einem subdirekten Produkt subdirekt irreduzibler Verbände, die von einem homomorphen Bild von ${}_DM_3$ erzeugt werden, ist zu zeigen, daß ein subdirekt irreduzibler, modularer Verband S, der von einem homomorphen Bild $\psi({}_DM_3)$ erzeugt wird und nicht zu M_3 isomorph ist, einelementig sein muß. $|S|=1$ erhält man offenbar, wenn man für alle $c,d\varepsilon D$, für die c oberer Nachbar von d ist, $\psi c=\psi d$ nachweist, da dann $\psi 0=\psi a_2$ und damit $\psi 0=\psi 1$ folgt.

Ist c oberer Nachbar von d in D, dann existiert ein $\vee$-irreduzibles Element $\underline{c}$ in D mit $c=\underline{c}\vee d$; ferner gibt es wegen der Distributivität ein größtes Element $\bar{d}$ in D, das nicht größer oder gleich $\underline{c}$ ist. Für $E_0:=\psi(\bar{d}]$, $E_2:=\psi[\underline{c})$, $E_3:=\{\psi a_3\}$, $E_5:=\{\psi a_5\}$ und $E_1=\emptyset$ liefert nun das M_3-Lemma die Ungleichung

$$(\psi a_2\wedge(\psi\bar{d}\vee\psi a_3))\vee(\psi a_2\wedge(\psi\bar{d}\vee\psi a_5))\vee((\psi\bar{d}\vee\psi a_3)\wedge(\psi\bar{d}\vee\psi a_5)) \geq$$
$$(\psi\underline{c}\vee\psi a_3)\wedge(\psi\underline{c}\vee\psi a_5)\wedge(\psi a_3\vee\psi a_5) =$$
$$(\psi\underline{c}\vee\psi a_3)\wedge(\psi\underline{c}\vee\psi a_5) .$$

Da wegen der Modularität von S die linke Seite der Ungleichung gleich

$$(\psi\bar{d}\vee(\psi a_2\wedge\psi a_3))\vee(\psi\bar{d}\vee(\psi a_2\wedge\psi a_5))\vee((\psi\bar{d}\vee\psi a_3)\wedge(\psi\bar{d}\vee\psi a_5))$$
$$= \psi\bar{d}\vee((\psi\bar{d}\vee\psi a_3)\wedge(\psi\bar{d}\vee\psi a_5)) = (\psi\bar{d}\vee\psi a_3)\wedge(\psi\bar{d}\vee\psi a_5)$$ ist, folgt
$$\psi\bar{d}=\psi a_2\wedge(\psi a_3\vee\psi\bar{d})\wedge(\psi a_5\vee\psi\bar{d})\geq\psi a_2\wedge(\psi\underline{c}\vee\psi a_3)\wedge(\psi\underline{c}\vee\psi a_5)=\psi\underline{c} .$$

Wegen $\underline{c}\wedge\bar{d}=\underline{c}\wedge d$ erhält man $\psi\underline{c}\leq\psi d$, also $\psi c=\psi(\underline{c}\vee d)=\psi d$, womit die Behauptung des Hilfssatzes für endliches D bewiesen ist.

Für einen beliebigen beschränkten, distributiven Verband D wird zunächst gezeigt, daß jede in M_3 geltende Gleichung γ auch in M gilt. Sei E eine endliche Teilmenge von M, die als Bild einer Belegung der Variablenmenge von γ auftritt. Da jedes Element von E in dem Erzeugnis einer endlichen Teilmenge von D und M_3 liegt, gibt es einen endlich erzeugten Unterverband $\underline{D}$ von D mit $0,a_2 \varepsilon \underline{D}$, so daß E im Erzeugnis von $_{\underline{D}}M_3$ enthalten ist. Nun ist bekanntlich ein endlich erzeugter, distributiver Verband endlich (vgl. GRÄTZER [1; Theorem 8.1]). Daher ist nach dem Vorangehenden der von $_{\underline{D}}M_3$ erzeugte Unterverband von M zu einer subdirekten Potenz von M_3 isomorph, woraus folgt, daß γ bei der betrachteten Variablenbelegung gilt. M liegt somit in der kleinsten gleichungsdefinierten Klasse, die M_3 enthält. Nach JÓNSSON [3; Corollary 3.4] ist in dieser Klasse jeder subdirekt irreduzible Verband mit mehr als zwei Elementen isomorph zu M_3. Folglich ist, da es keinen Homomorphismus von $_DM_3$ auf einen zweielementigen Verband gibt, M isomorph zu einer subdirekten Potenz von M_3.

<u>Hilfssatz 2:</u> M sei ein modularer Verband, der von einem relativen Unterverband $_DM_3$ erzeugt wird. Dann besitzt jedes Element a in M eine eindeutige Darstellung
$a=x_a \vee [(y_a \vee a_3) \wedge (z_a \vee a_5)]$ mit $x_a, y_a, z_a \varepsilon D$ und $x_a \wedge y_a = x_a \wedge z_a = y_a \wedge z_a$; speziell gilt $D=[0,a_2]$.

Beweis: Zunächst soll gezeigt werden, daß

$$N:=\{x\vee((y\vee a_3)\wedge(z\vee a_5)) \mid x,y,z\varepsilon D \text{ und } x\wedge y=x\wedge z=y\wedge z\}$$

ein Unterverband von M ist. Dazu wird die Gültigkeit folgender Gleichungen für $x,y,z,x',y',z'\varepsilon D$ mit $x\wedge y=x\wedge z=y\wedge z$ und $x'\wedge y'=x'\wedge z'=y'\wedge z'$ bewiesen:

(∧) $$(x\vee[(y\vee a_3)\wedge(z\vee a_5)])\wedge(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)]) = (x\wedge x')\vee[((y\wedge y')\vee a_3)\wedge((z\wedge z')\vee a_5)]$$

(∨) $$(x\vee[(y\vee a_3)\wedge(z\vee a_5)])\vee(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)]) = x\vee x'\vee((y\vee y')\wedge(z\vee z'))\vee[(y\vee y'\vee((x\vee x')\wedge(z\vee z'))\vee a_3)\wedge \wedge(z\vee z'\vee((x\vee x')\wedge(y\vee y'))\vee a_5)]$$

Nach Hilfssatz 1 ist M zu einer subdirekten Potenz von M_3 isomorph, so daß die Gleichungen (∧) und (∨) genau dann in M gelten, wenn sie bei jeder Belegung von x,y,z,x',y',z' mit 0 oder a_2 in M_3 gelten.

Für den Gültigkeitsbeweis in M_3 kann man o.B.d.A. $x'\leq x$ annehmen, da (∧) und (∨) in x und x' symmetrisch sind. Wegen der Bedingung $x\wedge y=x\wedge z=y\wedge z$ müssen, falls nicht $x=y=z=a_2$ gilt, zwei der Variablen x,y,z mit 0 belegt werden (dasselbe gilt für x',y',z').

Damit erhält man folgende Fallunterscheidung:

Fall 1: $x=y=z=0$

Fall 2: $x=y=0,\ z=a_2$

Fall 3: $y=z=0,\ x=a_2$

Fall 4: $x=y=z=a_2$

Fall 2 impliziert aus Symmetriegründen den Fall $x=z=0, y=a_2$.

Fall 1:

(∧) $0\wedge(0\vee[(y'\vee a_3)\wedge(z'\vee a_5)])=0$

(∨) $0\vee(0\vee[(y'\vee a_3)\wedge(z'\vee a_5)])$

$=(y'\vee(x'\wedge z')\vee a_3)\wedge(z'\vee(x'\wedge y')\vee a_5)$ $\quad (x'\wedge y'=x'\wedge z'=y'\wedge z')$

$=(y'\wedge z')\vee[(y'\vee(x'\wedge z')\vee a_3)\wedge(z'\vee(x'\wedge y')\vee a_5)]$

Fall 2:

(∧) $[a_3\wedge(a_2\vee a_5)]\wedge[(y'\vee a_3)\wedge(z'\vee a_5)]$ $\quad (x'\leq x=0)$

$=a_3\wedge[(y'\vee a_3)\wedge(z'\vee a_5)]$

$=a_3\wedge(z'\vee a_5)$

$=a_3\wedge((a_2\wedge z')\vee a_5)$ $\quad (z'\leq a_2)$

(∨) $[a_3\wedge(a_2\vee a_5)]\vee[(y'\vee a_3)\wedge(z'\vee a_5)]$

$=(y'\vee a_3)\wedge(a_3\vee z'\vee a_5)$

$=(y'\vee a_3)$

$=(y'\wedge(a_2\vee z'))\vee[(y'\vee a_3)\wedge(a_2\vee z'\vee a_5)]$ $\quad (y',z'\leq a_2)$

Fall 3:

(∧) $a_2\wedge(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)])$

$=x'\vee(a_2\wedge(y'\vee a_3)\wedge(z'\vee a_5))$ $\quad (x'\leq a_2)$

$=x'\vee(y'\wedge z')$ $\quad (y',z'\leq a_2)$

$=x'$ $\quad (x'\wedge y'=y'\wedge z')$

$=a_2\wedge x'$

(∨) $a_2\vee(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)])$

$=a_2\vee[(y'\vee z'\vee a_3)\wedge(z'\vee y'\vee a_5)]$ $\quad (x'\leq a_2; y',z'\in\{0,a_2\})$

$=a_2\vee(y'\wedge z')\vee[(y'\vee z'\vee a_3)\wedge(z'\vee y'\vee a_5)]$

Fall 4:

$(\wedge)$ $1\wedge(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)])$

$=(a_2\wedge x')\vee[((a_2\wedge y')\vee a_3)\wedge((a_2\wedge z')\vee a_5)]$ $\quad (x',y',z'\leq a_2)$

$(\vee)$ $1\vee(x'\vee[(y'\vee a_3)\wedge(z'\vee a_5)])$

$=1$

$=a_2\vee[(a_2\vee a_3)\wedge(a_2\vee a_5)$

Da $D\subseteq N$ wegen $d=d\vee[(0\vee a_3)\wedge(0\vee a_5)]$ für $d\varepsilon D$ und $a_3,a_5\varepsilon N$ wegen $a_3=0\vee[(0\vee a_3)\wedge(a_2\vee a_5)]$ sowie $a_5=0\vee[(a_2\vee a_3)\wedge(0\vee a_5)]$ gilt, ist die Erzeugendenmenge ${}_DM_3$ in N enthalten, woraus N=M folgt. Jedes Element a in M hat somit die Darstellung $a=x_a\vee[(y_a\vee a_3)\wedge(z_a\vee a_5)]$. Die Eindeutigkeit dieser Darstellung ergibt sich aus

$x_a=a\wedge a_2$

$(*)$ $y_a=([a\wedge a_5]\vee a_3)\wedge a_2$

$z_a=([a\wedge a_3]\vee a_5)\wedge a_2$.

Die Gleichungen von $(*)$ werden folgendermaßen bewiesen:

$x_a=x_a\vee(y_a\wedge z_a)$ $\quad (x_a\wedge y_a=y_a\wedge z_a)$

$=x_a\vee(a_2\wedge(y_a\vee a_3)\wedge(z_a\vee a_5))$ $\quad (y_a,z_a\leq a_2)$

$=(x_a\vee[(y_a\vee a_3)\wedge(z_a\vee a_5)])\wedge a_2$

$=a\wedge a_2$

$y_a=(y_a\vee a_3)\wedge a_2$ $\quad (y_a\leq a_2)$

$=([(y_a\vee a_3)\wedge a_5]\vee a_3)\wedge a_2$

$=([(x_a\vee[(y_a\vee a_3)\wedge(z_a\vee a_5)])\wedge(0\vee[(a_2\vee a_3)\wedge(0\vee a_5)])]\vee a_3)\wedge a_2$ $\quad (\wedge)$

$=([a\wedge a_5]\vee a_3)\wedge a_2$

$z_a=([a\wedge a_3]\vee a_5)\wedge a_2$ analog

(*) liefert auch unmittelbar $D=[0,a_2]$ in M .

<u>Hilfssatz 3:</u> M sei ein modularer Verband, der von einem relativen Unterverband ${}_D M_3$ erzeugt wird. Ist Θ eine Kongruenzrelation von D, dann ist $\bar{\Theta}:=\{(a,b)\varepsilon M^2 \mid (x_a,x_b),(y_a,y_b),(z_a,z_b)\varepsilon\Theta\}$ eine Kongruenzrelation von M mit $\Theta=\bar{\Theta}\cap D^2$; darüberhinaus wird durch $\Theta\longmapsto\bar{\Theta}$ ein Isomorphismus zwischen den Kongruenzrelationenverbänden von D und M erklärt.

Beweis: Aus $x_{a\wedge c}=x_a\wedge x_c$ und $x_{a\vee c}=x_a\vee x_b\vee[(y_a\vee y_b)\wedge(z_a\vee z_b)]$ sowie den entsprechenden Gleichungen für die anderen Komponenten (siehe ($\wedge$) und ($\vee$)) folgt, daß $\bar{\Theta}$ eine Kongruenzrelation von M ist. $\Theta=\bar{\Theta}\cap D^2$ ist wegen $D=\{a\varepsilon M \mid y_a=z_a=0\}$ klar. Für eine Kongruenzrelation Φ von M erhält man aus (*) $\Phi=\overline{\Phi\cap D^2}$. Demnach hat jede Kongruenzrelation Θ von D genau eine Erweiterung auf M, weshalb $\Theta\longmapsto\bar{\Theta}$ ein Isomorphismus zwischen den Kongruenzrelationenverbänden von D und M sein muß.

<u>Beweis des Satzes:</u> Die Äquivalenz von (1) und (2) folgt unmittelbar aus Hilfssatz 2 mit den Gleichungen ($\wedge$) und ($\vee$). Für den Nachweis der Äquivalenz von (2) und (3) wird gezeigt, daß $FM({}_D M_3)$ isomorph ist zu der subdirekten Potenz

von M_3, die aus allen quasi-eigentlichen, stetigen Abbildungen von $S(D)$ in M_3 besteht. Ist ψ ein Homomorphismus von $FM({}_DM_3)$ auf M_3 mit $\psi a_i = a_i (i \in \{2,3,5\})$, so ist $P := (\psi^{-1}0) \cap D$ wegen $\psi D = \{0, a_2\}$ ein Primideal von D, das nach Hilfssatz 3 ψ eindeutig bestimmt; für ψ soll deshalb auch ψ_P geschrieben werden. Nach Hilfssatz 1 ist dann $FM({}_DM_3)$ isomorph zu der Potenz von M_3, die aus allen Abbildungen $\hat{a}: S(D) \longrightarrow M_3 (a \in FM({}_DM_3))$ mit $\hat{a}(P) = \psi_P a (P \in S(D))$ besteht. Somit ist noch zu zeigen, daß die $\hat{a}$ genau die quasi-eigentlichen, stetigen Abbildungen von $S(D)$ in M_3 sind.

Daß für jedes $a \in FM({}_DM_3)$ die Abbildung $\hat{a}$ quasi-eigentlich und stetig ist, folgt mit Lemma 11.4 in GRÄTZER [1] aus

$$
(**) \quad
\begin{aligned}
&\{P \mid \hat{a}(P) \geq a_2\} = \{P \mid x_a \notin P\} \\
&\{P \mid \hat{a}(P) \geq a_3\} = \{P \mid z_a \notin P\} \\
&\{P \mid \hat{a}(P) \geq a_5\} = \{P \mid y_a \notin P\} \\
&\{P \mid \hat{a}(P) = 1\ \} = \{P \mid x_a, y_a, z_a \notin P\} \quad ,
\end{aligned}
$$

was mit Hilfssatz 2 und (*) folgendermaßen nachgewiesen wird:

$$
\begin{aligned}
\hat{a}(P) \geq a_2 &\Longleftrightarrow \psi_P a \geq a_2 \\
&\Longleftrightarrow a_2 = \psi_P a \wedge a_2 = \psi_P a \wedge \psi_P a_2 = \psi_P (a \wedge a_2) = \psi_P x_a \\
&\Longleftrightarrow x_a \notin P
\end{aligned}
$$

$$\hat{a}(P)\geq a_3 \Longleftrightarrow \psi_P a\geq a_3$$
$$\Longleftrightarrow a_3=\psi_P a\wedge a_3$$
$$\Longleftrightarrow a_2=([\psi_P a\wedge a_3]\vee a_5)\wedge a_2=\psi_P(([a\wedge a_3]\vee a_5)\wedge a_2)=\psi_P z_a$$
$$\Longleftrightarrow z_a\notin P$$

$\hat{a}(P)\geq a_5 \Longleftrightarrow y_a\notin P$ analog

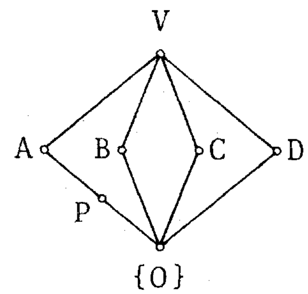
Sei umgekehrt α eine quasi-eigentliche, stetige Abbildung von $S(D)$ in M_3. Dann sind $\alpha^{-1}[a_2)$, $\alpha^{-1}[a_3)$ und $\alpha^{-1}[a_5)$ quasi-kompakte, offene Teilmengen von $S(D)$. Nach Lemma 11.4 in GRÄTZER [1] existieren somit $x,y,z\in D$ mit $\alpha^{-1}[a_2)=\{P\mid x\notin P\}$, $\alpha^{-1}[a_3)=\{P\mid z\notin P\}$ und $\alpha^{-1}[a_5)=\{P\mid y\notin P\}$. Angenommen $x\wedge y\nleq x\wedge z$. Dann gibt es nach Lemma 11.2 in GRÄTZER [1] ein Primideal P von D mit $x\wedge y\notin P$ aber $x\wedge z\in P$, d.h. $x,y\notin P$ aber $z\in P$. Aus $x,y\notin P$ folgt $P\in\alpha^{-1}[a_2)\cap\alpha^{-1}[a_5)=\alpha^{-1}\{1\}\subseteq\alpha^{-1}[a_3)$, was jedoch $z\in P$ widerspricht. Somit war die Annahme falsch, und es gilt $x\wedge y\leq x\wedge z$. Analog zeigt man $x\wedge z\leq y\wedge z$ und $y\wedge z\leq x\wedge y$, womit $x\wedge y=x\wedge z=y\wedge z$ nachgewiesen ist. Setzt man nun $a:= x\vee[(y\vee a_3)\wedge(z\vee a_5)]$, so erhält man nach Hilfssatz 2 $x=x_a$, $y=y_a$ und $z=z_a$, was wegen (**) $\alpha=\hat{a}$ zur Folge hat. Damit ist der Satz vollständig bewiesen.

Es soll noch angemerkt werden, daß nach dem Satz und Hilfssatz 3 wie zwischen den Kategorien der beschränkten, distributiven Verbände und ihren Stoneschen Räumen (vgl. HOFFMANN

& KEIMEL [2; Theorem 5.23]) eine Dualität zwischen der Kategorie der freien modularen Verbände $FM({}_DM_3)$ mit den Homomorphismen, die M_3 identisch auf sich abbilden, und der Kategorie der Stoneschen Räume $\mathcal{S}(D)$ mit den quasi-eigentlichen, stetigen Abbildungen besteht; ferner sind die Kategorien der beschränkten, distributiven Verbände und der freien modularen Verbände $FM({}_DM_3)$ äquivalent, ja sogar isomorph.

Beispiel: Daß für Verbände der Länge 2 mit mehr als fünf Elementen kein entsprechender Satz wie für M_3 gilt, wird an folgendem Beispiel deutlich: Ist V ein Vektorraum über einem Primkörper mit der Basis e_0, e_1, e_2, e_3 und ist $P:=\langle e_0\rangle$, $A:=\langle e_0,e_2\rangle$, $B:=\langle e_1,e_3\rangle$, $C:=\langle e_0+e_1,e_2+e_3\rangle$ und $D:=\langle e_0-e_3,e_1+e_2\rangle$, so bilden die Untervektorräume $\{0\}$, P, A, B, C, D und V einen relativen Unterverband ${}_3M_4$ des Untervektorraumverbandes von V, der durch folgendes Diagramm dargestellt wird:

${}_3M_4$:



Folgende 1-dimensionale Untervektorräume liegen im Erzeugnis von ${}_3M_4$:

$Q_B := (P+C) \cap B = \langle e_1 \rangle$

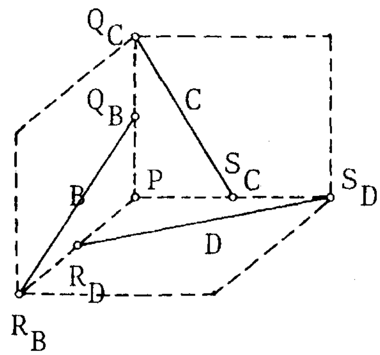
$Q_C := (P+B) \cap C = \langle e_o + e_1 \rangle$

$R_B := (P+D) \cap B = \langle e_3 \rangle$

$R_D := (P+B) \cap D = \langle e_o - e_3 \rangle$

$S_C := (P+D) \cap C = \langle e_o + e_1 + e_2 + e_3 \rangle$

$S_D := (P+C) \cap D = \langle e_o - e_1 - e_2 - e_3 \rangle$



Wegen $V=P+Q_B+R_B+S_C$ und wegen $P+Q_B=P+Q_C=Q_B+Q_C$, $P+R_B=P+R_D=R_B+R_D$, $P+S_C=P+S_D=S_C+S_D$ erzeugt ${}_3M_4$ nach Satz III.2.1 und Satz III.2.4 in MAEDA [4] einen komplementären, einfachen Unterverband der Länge 4, der somit zu einem Untervektorraumverband eines 4-dimensionalen Vektorraumes V' isomorph ist. Da V als Vektorraum über einem Primkörper vorausgesetzt war, muß ${}_3M_4$ folglich den Verband aller Untervektorräume von V erzeugen. Man hat daher unendlich viele Isomorphieklassen subdirekt irreduzibler, modularer Verbände, die von einem homomorphen Bild von ${}_3M_4$ erzeugt werden.

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Some Remarks on Free Orthomodular Lattices

by Günter Bruns and Gudrun Kalmbach

This paper contains some preliminary studies of free orthomodular lattices. An orthomodular lattice (abbreviated: OML) is considered here as a (universal) algebra with basic operations $\vee$, $\wedge$, $'$, 0, 1. All general algebraic notions like subalgebra or homomorphism are to be understood in this way.

We assume the basic notions of the theory of OMLs to be known; the reader can find the necessary information in [1] and [4].

In the first chapter we describe a method to present a finitely generated OML as a direct product of a Boolean algebra and an OML of a special type, which we call tightly generated. We use this to describe certain OMLs which are freely generated by some simple partially ordered sets. In the second chapter we construct a special extension of an OML L. Since it is generated by L and one additional element we call it a one-point extension of L. We use this constructio in the last chapter to prove that the free OML generated by a three-element poset consisting of two comparable elements and an element incomparable with both contains an infinite chain. This answers a question posed by D. Foulis.

1. Some simple free OMLs

As is well known every interval of the form $[0, c]$ in an OML L can be made an OML by defining the orthocomplement a^{*} of an element a in $[0, c]$ by $a^{*} = a' \wedge c$. If c is in the center of L, i. e. if c commutes with every element of L, then the map $x \longrightarrow x \wedge c$ is a homomorphism of L onto $[0, c]$; moreover, the map $x \longrightarrow (x \wedge c', x \wedge c)$ is in this case an isomorphism between L and the direct product $[0, c'] \times [0, c]$.

We start out by describing a simple but useful such splitting of a finitely generated OML. To simplify notation we define for an element a of an OML L: $a^1 = a'$ and $a^0 = a$. We say that an OML L is tightly generated by a finite set G iff it is generated by G and for every map $\delta \in 2^G$ (i.e. $\delta : G \longrightarrow \{0,1\}$) the equation $\bigwedge_{x \in G} x^{\delta(x)} = 0$ holds.

(1.1) Let L be an OML generated by a finite set G and define $c = \bigwedge_{\delta \in 2^G} \bigvee_{x \in G} x^{\delta(x)}$. Then c is in the center of L, the OML $[0,c']$ is Boolean and the OML $[0,c]$ is tightly generated by $\{x \wedge c \mid x \in G\}$. In particular is every finitely generated OML the direct product of a Boolean algebra and a tightly generated OML.

Proof. The element c obviously commutes with every element of G and hence with every element of L, which means that it is in the center of L. To show that $[0,c']$ is Boolean it is enough to show that any two elements

$x \wedge c'$, $y \wedge c'$ with $x, y \in G$ commute in $[0,c']$, i.e. that

$$((x \wedge c') \wedge (y \wedge c')) \vee ((x \wedge c') \wedge (y \wedge c')^{\#}) = x \wedge c'$$

holds, where $(y \wedge c')^{\#}$ is the orthocomplement of $y \wedge c'$ in $[0,c']$. But

$$((x \wedge c') \wedge (y \wedge c')) \vee ((x \wedge c') \wedge (y \wedge c')^{\#}) =$$

$$(x \wedge y \wedge c') \vee ((x \wedge c') \wedge ((y' \vee c) \wedge c')) =$$

$$(x \wedge y \wedge c') \vee (x \wedge y' \wedge c') =$$

$$(x \wedge y \wedge \bigvee_{\delta \in 2^G} \bigwedge_{z \in G} z^{\delta(z)}) \vee (x \wedge y' \wedge \bigvee_{\delta \in 2^G} \bigwedge_{z \in G} z^{\delta(z)}) =$$

$$(\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0 \\ \delta(y)=0}} \bigwedge_{z \in G} z^{\delta(z)}) \vee (\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0 \\ \delta(y)=1}} \bigwedge_{z \in G} z^{\delta(z)}) =$$

$$\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0}} \bigwedge_{z \in G} z^{\delta(z)} = x \wedge c'.$$

In order to show that the OML $[0,c]$ is tightly generated by $\{x \wedge c \mid x \in G\}$ we define for a given $\varepsilon \in 2^G$: $H = \{x \in G \mid \varepsilon(x) = 0\}$ and $J = \{x \in G \mid \varepsilon(x) = 1\}$. We then have to prove that

$$\bigwedge_{x \in H} (x \wedge c) \wedge \bigwedge_{x \in J} ((x \wedge c)' \wedge c) = 0$$

holds, which is shown by the following little calculation:

$$\bigwedge_{x \in H} (x \wedge c) \wedge \bigwedge_{x \in J} (x \wedge c)' \wedge c =$$

$$c \wedge \bigwedge_{x \in H} x \wedge \bigwedge_{x \in J} (x' \vee c') =$$

$$c \wedge \bigwedge_{x \in H} x \wedge (c' \vee \bigwedge_{x \in J} x') =$$

$$c \wedge \bigwedge_{x \in H} x \wedge \bigwedge_{x \in J} x' =$$

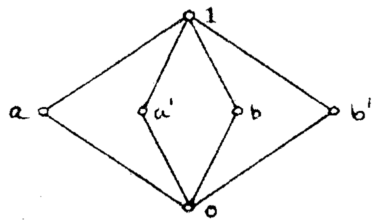
$$\bigwedge_{x \in G} x^{\varepsilon(x)} \wedge \bigwedge_{\delta \in 2^G} \bigvee_{x \in G} x^{\delta(x)} \leq$$

$$\bigwedge_{x \in G} x^{\varepsilon(x)} \wedge \bigvee_{x \in G} x^{1-\varepsilon(x)} = 0,$$

completing the proof.

As a first application of this we characterize the free OML generated by a two-element set. The structure of it is well known, the following simple proof, however, seems to be new.

Let MO2 be the following OML:



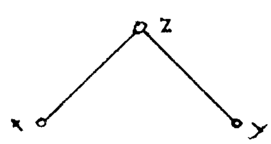
Let p_1, p_2, p_3, p_4 be the atoms of the Boolean algebra 2^4.

(1.2) The OML $2^4 \times MO2$ is freely generated by the set $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b)\}$.

<u>Proof</u>. Let L be an OML generated by the set $\{x,y\}$. With c having the meaning of (1.1), L is isomorphic with the direct product $[0,c'] \times [0,c]$. Since $[0,c']$ is Boolean and is generated by an at most two-element set it has at most 2^4 elements. Since $[0,c]$ is tightly generated by an at most two-element set it is a homomorphic image of MO2 and, hence, has at most six elements. It follows that L has at most $2^4.6 = 96$ elements. But the OML $2^4 \times MO2$ has 96 elements and is generated by $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b)\}$. It follows that it is freely generated by this set.

In a similar fashion one can determine the structure of the OML which is freely generated by the poset

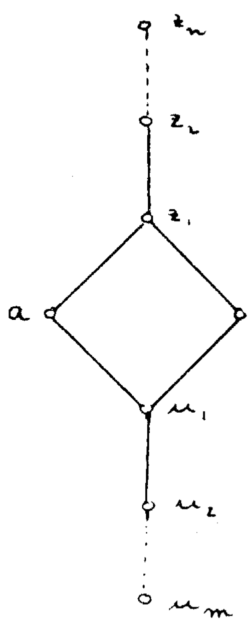
(1)



i.e. by the set $\{x,y,z\}$ with the relations $x \leq z$ and $y \leq z$. If an OML L is generated by a set of this kind and if c is defined as in (1.1) it is easy to see that $[0,c]$ is still tightly generated by the set $\{x \wedge c, y \wedge c\}$ and hence is a homomorphic image of MO2. The Boolean algebra $[0,c']$ is in this case generated by the set $\{x \wedge c', y \wedge c', z \wedge c'\}$ satisfying $x \wedge c', y \wedge c' \leq z\ c'$. From this it follows easily that $[0,c']$ has at most 2^5 elements. We thus obtain that L has at most $2^5.6 = 192$ elements. Again, if p_1, p_2, p_3, p_4, p_5 are the atoms of 2^5 and if a,**b** have the meaning of (1.2), it is easy to see that the 192-element OML $2^5 \times$ MO2 is generated by the set $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b), (p_5', 1)\}$, the elements of which are in the appropriate position. We thus have:

(1.3) The free OML generated by the poset (1) is isomorphic with $2^5 \times$ MO2.

By a slightly more elaborate argument but using the same method it is easy to determine the OML which is freely generated by the poset



We leave this to the reader.

2. The one-point extension of an OML

The problem of determining the structure of an OML L freely generated by a poset P becomes considerably more difficult if P contains elements x,y,z, where x is incomparable with both y and z. We are far from being able to solve it´s word problem. The aim of the rest of this paper is to show that every such OML contains an infinite chain.

As a first step towards this goal we describe a special extension of an orthocomplemented lattice (abbreviated: OCL) which we hope might have other applications than the one given in this paper. We start out with a definition.

Definition. A quasi-ideal in an OCL L is a subset A of L which satisfies tha following conditions:

1. $0 \in A$,
2. if $a \in A$ and $b \leq a$ then $b \in A$,
3. if $a \in A$ then $a' \notin A$,
4. if $M \subseteq A$, if $\bigvee M$ exists and if $\bigvee M \notin A$ then $(\bigvee M)' \in A$,
5. for every $x \in L$: $\bigvee([0,x] \cap A)$ exists.

Note that condition 5 is alwais fulfilled if all chains in L have bounded lenght, the only case we are dealing with in this paper.

We want to construct an OCL E which contains L as a sub-poset, has the same zero and unit as L, the orthocomplementation of which extends the orthocomplementation of L and which is generated by L and one additional element.

We do not know whether our extension can be described by some universal property.

Let A be a quasi-ideal in an OCL L. Define $A' = \{a' \mid a \in A\}$. Let s, s' be arbitrary elements. In order to make our construction set-theoretically sound we have to make the somewhat technical assumption that the sets L, $A \times \{s'\}$ and $A' \times \{s\}$ are pairwise disjoint. We then define the underlying set of our extension to be

$$E = L \cup (A \times \{s'\}) \cup (A' \times \{s\}).$$

To avoid confusion we denote the partial ordering of L by "$\leq_L$" and the join-operation in L by "$\vee_L$". We now define a relation $\leq$ in E by:

$a \leq b$ iff one of the following conditions holds:

1. $a, b \in L$ and $a \leq_L b$,
2. $a \in L$, $b = (x, s')$ and $a \leq_L x$,
3. $a \in A$, $b = (x, s)$ and $a \leq_L x$,
4. $a = (x, s')$, $b \in A'$ and $x \leq_L b$,
5. $a = (x, s')$, $b = (y, s')$ and $x \leq_L y$,
6. $a = (x, s)$, $b \in L$ and $x \leq_L b$,
7. $a = (x, s)$, $b = (y, s)$ and $x \leq_L y$.

It requires some tedious checking that this is indeed a partial ordering of E. It is obvious that this partial ordering extends the partial ordering of L and that the bounds of L are also the bounds of E. We omit the proof that this partial ordering makes E a lattice. For the convenience of the reader we list explicitely all the joins of elements of E. The meets are obtained dually. In the following, x and y are elements of L and a,b are elements of E. The joins are then given by:

$$
\begin{array}{lll}
a \vee b = x \vee_L y & \text{if} & a = x,\ b = y \text{ and } x \vee_L y \in A \text{ or} \\
 & \text{if} & a = x,\ b = y \text{ and } (x \notin A \text{ or } y \notin A) \text{ or} \\
 & \text{if} & a = x,\ b = (y,s') \text{ and } x \vee_L y \in A' \text{ or} \\
 & \text{if} & a = x \in A \text{ and } b = (y,s) \text{ or} \\
 & \text{if} & a = (x,s'),\ b = (y,s') \text{ and } x \vee_L y \notin A \text{ or} \\
 & \text{if} & a = (x,s') \text{ and } b = (y,s), \\
a \vee b = (x \vee_L y, s) & \text{if} & a = x \in A,\ b = y \in A \text{ and } x \vee_L y \notin A \text{ or} \\
 & \text{if} & a = x \in A \text{ and } b = (y,s) \text{ or} \\
 & \text{if} & a = (x,s) \text{ and } b = (y,s), \\
a \vee b = (x \vee_L y, s') & \text{if} & a = x,\ b = (y,s') \text{ and } x \vee_L y \in A \text{ or} \\
 & \text{if} & a = (x,s'),\ b = (y,s') \text{ and } x \vee_L y \in A, \\
a \vee b = \bigwedge([x \vee_L y, 1] \cap A') & \text{if} & a = x,\ b = (y,s') \text{ and } x \vee_L y \notin A, A
\end{array}
$$

It is important to note that the join in L of two elements $x,y \in L$ differs from their join in E iff $x,y \in A$ and $x \vee_L y \notin A$, and dually.

It is now easy to see that the orthocomplementation of L extends to an orthocomplementation of E by the definition:

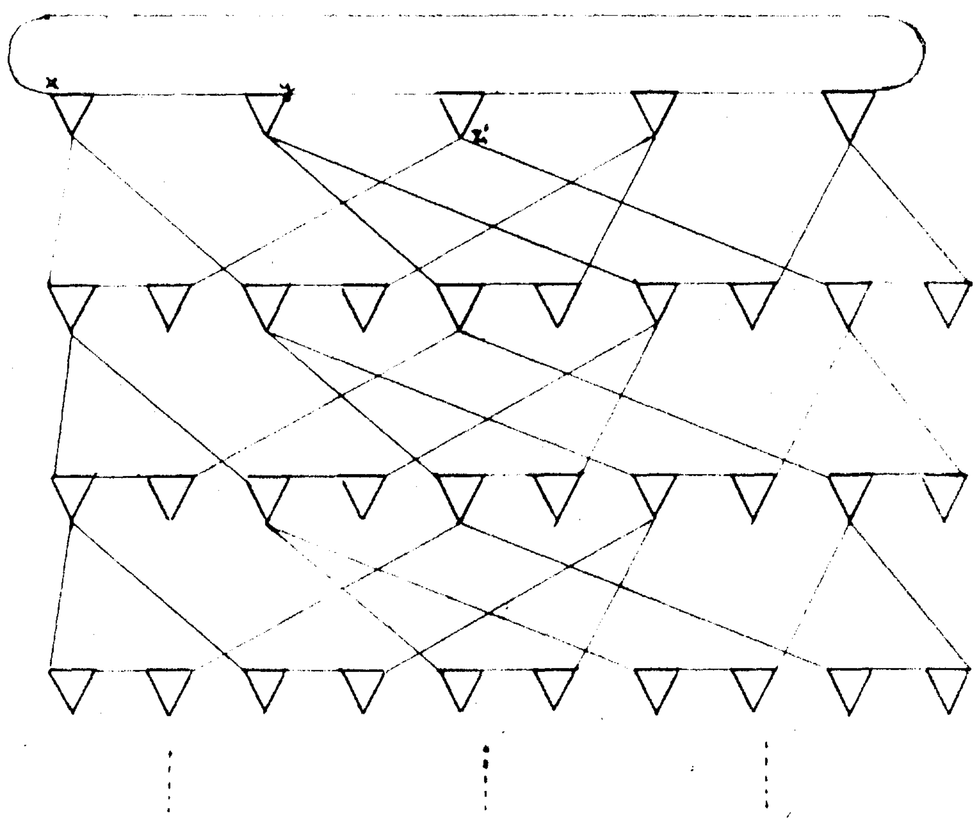
$$(x,s)' = (x',s') \text{ and } (x,s')' = (x',s).$$

Since for every $x \in A$: $x \vee (0,s') = (x,s')$ and dually for every $x \in A'$: $x \wedge (1,s) = (x,s)$ it follows that every element of E is the join of an element of L and the element $(0,s')$ or the complement of such join, in particular, that

E is generated by the set $L \cup \{(1,s)\}$. For the application we have in mind it is finally important to observe that E is an OML if L is an OML. The proof of this is again left to the reader.

3. Existence of infinite chains

In this chapter we sketch a proof of the existence of an OML L which is generated by a three-element poset $P = \{x,y,z\}$ satisfying $y < z$ and which contains an infinite chain. As a first step we construct an infinite OML L generated by such a poset P, in which all maximal chains have four elements. Instead of giving an explicite set-theoretical construction, we modify Greechie´s method [3] for graphical representations of OMLs and simply draw a "graph" of such an OML L. Here it is:



This graph is to be understood in the follwing way. The vertices of each triangle represent the atoms of an eight-element Boolean algebra. The bounds 0,1 of each of these Boolean algebras are "identified" and whenever two vertices of two triangles are connected by a line the atoms represented by the connected vertices are "identified" and so are their complements. Our construction is a special case of "Greechie's paste job" and it follows easily from [3] that our graph if interpreted this way represents indeed an OML. It is finally easy to see that this OML is generated by the elements x,y,z' indicated in the graph and hence also by the elements x,y,z, which are in the appropriate position.

From the graph it is obvious that there exists a countably infinite sequence $b_o, b_1, \ldots b_n, \ldots$ of co-atoms in L which satisfy the following conditions:

(A1) If $0 < a \leq b_i$, $0 < b \leq b_j$ and $i \neq j$ then $a \vee b = 1$,

(A2) if $0 < b \leq b_j$ and $i \neq j$ then $b \vee b_i' = 1$.

From (A2) it follows:

(1) if $i \neq j$ then $b_i' \nleq b_j$,

and from (A1) we obtain:

(2) if $i \neq j$ then $[0,b_i] \cap [0,b_j] = \{0\}$.

Put

$$A_o = [0,b_o] \cup [0,b_1].$$

This is obviously a quasi-ideal. Let

$$L_1 = L \cup (A_0 \times \{s_1'\}) \cup (A_0' \times \{s_1\})$$

be the one-point extension corresponding to it. We now define recursively a sequence $(L_n)_{n<\omega}$ of OMLs and a sequence $(A_n)_{n<\omega}$ where A_n is the quasi-ideal

$$A_n = [0,(1,s_n)]_{L_n} \cup [0,b_{n+1}]_{L_n}$$

in L_n and

$$L_{n+1} = L_n \cup (A_n \times \{s_{n+1}'\}) \cup (A_n' \times \{s_{n+1}\}).$$

It is easy to prove by induction that these sequences have the following properties:

(B1) If $0 <_{L_n} a \leq_{L_n} (1,s_n)$, $0 <_{L_n} b \leq_{L_n} b_j$ and $n+1 \leq j$ then $a \vee_{L_n} b = 1$

(B2) if $0 <_{L_n} b \leq_{L_n} b_j$ and $n+1 \leq j$ then $b \vee_{L_n} (0,s_n') = 1$,

(B3) if $n+1 \leq j$ then $[0,(1,s_n)]_{L_n} \cap [0,b_j]_{L_n} = \{0\}$.

It follows from these properties that for every n, A_n is indeed a quasi-ideal of L_n and that for elements $a,b \in L_i$, $a \vee_{L_n} b \neq a \vee_{L_{n+1}} b$ only holds if $a \vee_{L_n} b = 1$ and dually for meets. This means that every generating set of L is also a generating set of every L_n, in particular that every L_n is generated by P. This then is also true for the direct limit of the family $(L_n)_{n<\omega}$, defined in the obvious fashion. But this direct limit contains the infinite chain $\{(1,s_n) \mid n < \omega\}$, proving that the OML which is freely generated by the poset P contains an infinite chain.

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BREADTH TWO MODULAR LATTICES

by

RALPH FREESE

ABSTRACT: In this paper a characterization of breadth two modular lattices which can be generated by four elements is given. Those which are subdirectly irreducible are listed. An infinite list of coverings in the free modular lattice on four generators is obtained. If V is the variety of lattices generated by all breadth two modular lattices and if L is a lattice freely generated in V by four generators subject to finitely many relations, then the word problem for L is shown to be solvable.

AMS 1970 subject classifications: Primary 06A30; Secondary 06A20, 08A10, 08A15. Key words and phrases: breadth two modular lattice, coverings, splitting modular lattice.

1. INTRODUCTION

In [3] Day, Hermann, and Wille give a list of subdirectly irreducible modular lattices which can be generated by four elements. Their list consists of projective planes and lattices of breadth two. They ask if their list is complete. In this paper we show that it is complete insofar as it contains all the subdirectly irreducible breadth two four-generated modular lattices. This is done by showing that any four-generated breadth two modular lattice is a homomorphic image of an explicit set of lattices. It is shown that corresponding to all but three of the subdirectly irreducible four-generated breadth two modular lattices there is a covering, $u \succ v$, in the free modular lattice on four generators, $FM(4)$, such that if ψ is the unique maximal congruence on $FM(4)$ separating u from v, then $FM(4)/\psi$ is isomorphic to the subdirectly irreducible breadth two lattice. All of these lattices corresponding to coverings in $FM(4)$ are splitting modular lattices in the sense of McKenzie (definitions given below). Let V be the variety (equational class) of lattices generated by all breadth two modular lattices and let $FL(V,4)$ be the free V-lattice on four generators. Then every nontrivial quotient (interval) of $FL(V,4)$ contains a covering. Finally it is shown that the word problem for four-generated lattices in V is solvable.

Section 2 gives some basic definitions and gives the preliminary reductions. Section 3 gives the main result and Section 4 gives the subdirectly irreducibles. Section 5 presents the coverings and other applications mentioned above.

2. PRELIMINARY REDUCTIONS

Let $a \geq b$ in a lattice L. The sublattice $\{x \in L \mid a \geq x \geq b\}$ is denoted a/b and is called a _quotient_ or _quotient sublattice_ or _interval_. We say that a/b _transposes up_ to c/d and c/d _transposes down_ to a/b if $a \vee d = c$ and $a \wedge d = b$. We denote this by $a/b \nearrow c/d$ and $c/d \searrow a/b$. Two quotients connected by a sequence of transposes are called _projective_. If $a > b$ and there is no x such that $a > x > b$, then we say a covers b, and denote this $a \succ b$.

Recall that a lattice has breadth n if the join of any $n + 1$ elements is redundant and there is an irredundant join of n element.

LEMMA 1: _A modular lattice has breadth_ n _if and only if it has a sublattice isomorphic to the lattice of all subsets of an_ n _element set but no sublattice isomorphic to the lattice of all subsets of an_ $n + 1$ _element set_.

PROOF: If the join of the elements $x_1, \ldots, x_n$ is irredundant, then the elements $\overline{x}_i = x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n$, $i = 1, \ldots, n$ generate a sublattice isomorphic to the lattice of subsets of an n element set. The lemma follows easily from this.

LEMMA 2: _Let_ b _cover_ a, $a \prec b$, _in a modular lattice_ L. _Then there exists a unique largest congruence separating_ a

and b, *which is denoted by* $\psi(a,b)$. *Let* $\theta(a,b)$ *denote the smallest congruence of* L *identifying* a *and* b. *Then* $\theta(a,b) \wedge \psi(a,b) = 0$.

PROOF: It follows from Dilworth's basic result on congruences of lattices that $\theta(a,b) \succ 0$ [2], [4], where 0 is the least congruence on L. The lemma follows.

Suppose $u \succ v$ in a free modular lattice F. Let $\psi(u,v)$ be the largest congruence separating u from v and let $K = F/\psi(u,v)$. Now if L is a homomorphic image of F in which the images of u and v are different, then L is a subdirect product of K and a lattice L' which is a homomorphic image of L such that u and v are identified in L'. This is, of course, an immediate corollary to Lemma 2.

For the rest of the paper, L will denote a breadth two modular lattice generated by four distinct generators a, b, c, d and not by any three elements.

LEMMA 3: *Either any three elements of the set* $\{a,b,c,d\}$ *join to the greatest element of* L, $1 = a \vee b \vee c \vee d$, *or* L *is a subdirect product of one or two, two element lattices and a four-generated breadth two modular lattice in which any three of the four generator join to the greatest element.*

PROOF: Suppose the first statement fails, say $b \vee c \vee d < 1$. Since L has breadth two, it follows that $b \vee c \vee d$ is the join of two of the generators, say $c \vee d = b \vee c \vee d$. Now $1 = (a \vee b) \vee c \vee d$ implies that either $c \vee d = 1$ or $a \vee b \vee c = 1$ or $a \vee b \vee d = 1$. The first possibility contradicts $b \vee c \vee d < 1$. Suppose $a \vee b \vee d = 1$, and that $a \vee b \vee c < 1$. Since $c \vee d = b \vee c \vee d$, we have $a \vee c \vee d = 1$.

In the free modular lattice on four generators the join of any three generators is covered by the greatest element. Hence in L, $1 \succ b \vee c \vee d$ and $1 \succ a \vee b \vee c$. Let ψ_1 be the largest congruence on L separating 1 from $b \vee c \vee d$ and ψ_2 the largest congruence separating 1 from $a \vee b \vee c$. Let $\theta_1 = \theta(1, b \vee c \vee d)$ and $\theta_2 = \theta(1, a \vee b \vee c)$ and $\theta = \theta_1 \vee \theta_2$. Since the congruences of lattices distribute, Lemma 2 implies $\theta \wedge \psi_1 \wedge \psi_2 = 0$. Hence L is a subdirect product of L/θ, L/ψ_1 and L/ψ_2. Now $L/\psi_1 \cong L/\psi_2 \cong \underset{\sim}{2}$ the two element lattice. Furthermore, L/θ has the property that any three of its generators join to the greatest element.

If $a \vee b \vee c = 1$, then $\theta_2 = 0$ and $\theta = \theta_1$. In this case L is a subdirect product of L/θ and $L/\psi_1 \cong \underset{\sim}{2}$. As before, L/θ has the desired properties. The remaining cases are handled by symmetry.

Now we impose the additional condition that any three of the four generators of L join to 1 and meet to 0. Since L has breadth two this implies that any three element subset of $\{a,b,c,d\}$ has a two element subset whose elements join to the top. We shall show that all but at most two of the two element subsets of $\{a,b,c,d\}$ join to 1. First we need a lemma.

LEMMA 4: *Let x and y be noncomparable elements in a breadth two modular lattice. Then $x \vee y/x$ and $x \vee y/y$ are both chains.*

PROOF: Suppose $x \leq u$, $v \leq x \vee y$ are noncomparable elements. Then it is not hard to check that the elements u, v, $y \wedge (u \vee v)$ generate a lattice isomorphic to the lattice of subsets of a three element set. Now the lemma follows from Lemma 1.

As remarked above, there is a two element subset of $\{a,b,c\}$ joining to 1; say $a \vee b = 1$. Also, there is a two element subset of $\{a,c,d\}$ joining to 1. If $c \vee d = 1$, then we have two complementary pairs, both of which join to 1. Suppose $a \vee c = 1$. Now consider $\{b,c,d\}$. If either $b \vee d = 1$ or $c \vee d = 1$, then there exists two complementary pairs, both joining to 1. If $b \vee c = 1$, then we have that all pairs not containing d join to 1. In conclusion, either there are two complementary pairs of generators both joining to 1, or there is a generator such that all pairs of generators not including that generator join to 1.

Suppose $a \vee b = 1 = c \vee d$. If a and b were comparable, then one of them would equal 1, contradicting our assumption that L is not generated by three elements. Hence by Lemma 4 $1/a$ is a chain and thus $a \vee c$ and $a \vee d$ must be comparable. By symmetry we may assume $a \vee c \geq a \vee d$. Then $a \vee c = a \vee c \vee d = 1$. Now $1/b$ and $1/d$ are chains by Lemma 4; hence, as above, either $a \vee d = 1$ or $b \vee d = 1$ and either $b \vee c = 1$ or $b \vee d = 1$. Thus either $b \vee d = 1$ or both $b \vee c = 1$ and $a \vee d = 1$. We conclude that if there are two complementary pairs of generators, each pair joining to 1, then at least five of the six pairs of generators join to 1, or four of the six join to 1 and the two pairs that do not join to 1 are complementary.

Let M_5 be the five element length two lattice.

LEMMA 5: *Let L be a breadth two modular lattice generated by a,b,c,d, in which any three of the generators join to 1. Then one of the following must hold.*

(i) *L has the property that at least four of the six pairs of generators join to 1, and if two pairs do not join to 1, they are complementary,*

(ii) *L is a subdirect product of M_5 and a lattice having the property described in (i),*

(iii) *L is a subdirect product of M_5 and a three generated modular lattice.*

PROOF: By symmetry and the remarks preceding Lemma 5 we may assume that $a \vee b = a \vee c = b \vee c = 1$. In order to apply

Lemma 2 we must find elements $v \prec u$ in the free modular lattice on four generators, FM(4), such that if ψ is the maximum congruence separating v from u, then $FM(4)/\psi \cong M_5$. This can easily be done in $FM(x,y,z)$, since it is finite. For example, $x \vee (y \wedge z) \prec x \vee (z \wedge (x \vee y))$ will do.

Let $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ be the generators of FM(4). Then $\underset{\sim}{a} \to x$, $\underset{\sim}{b} \to y$, $\underset{\sim}{c} \to z$, and $\underset{\sim}{d} \to x \wedge y \wedge z$ can be extended to a homomorphism f from FM(4) onto FM(3). It is not difficult to see that if $f(w) = x \vee (y \wedge z)$ then $w \leq \underset{\sim}{a} \vee [(\underset{\sim}{b} \vee \underset{\sim}{d}) \wedge (\underset{\sim}{c} \vee \underset{\sim}{d})]$ and if $f(w) = x \vee (z \wedge (x \vee y))$ then $w \geq \underset{\sim}{a} \vee (\underset{\sim}{c} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b}))$. It follows that in FM(4) $\underset{\sim}{a} \vee [(\underset{\sim}{b} \vee \underset{\sim}{d}) \wedge (\underset{\sim}{c} \vee \underset{\sim}{d})] \prec [\underset{\sim}{a} \vee ((\underset{\sim}{b} \vee \underset{\sim}{d}) \wedge (\underset{\sim}{c} \vee \underset{\sim}{d}))] \vee \underset{\sim}{a} \vee (\underset{\sim}{c} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b})) = \underset{\sim}{a} \vee ((\underset{\sim}{b} \vee \underset{\sim}{d}) \wedge (\underset{\sim}{c} \vee \underset{\sim}{d})) \vee (\underset{\sim}{c} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b}))$ and if ψ is the largest congruence separating these elements then $FM(4)/\psi \cong M_5$.

Hence in L we have

$$a \vee ((b \vee d) \wedge (c \vee d)) \preceq a \vee ((b \vee d) \wedge (c \vee d)) \vee (c \wedge (a \vee b)).$$

Now if we have equality in the above inequality, then

$$a \vee (c \wedge (a \vee b)) \leq a \vee ((b \vee d) \wedge (c \vee d))$$

or

$$(a \vee c) \wedge (a \vee b) \leq a \vee ((b \vee d) \wedge (c \vee d)).$$

Since $a \vee c = a \vee b = 1$ in L the left hand side of this inequality is 1 and hence the right hand side is also. By Lemma 4 either a and $(b \vee d) \wedge (c \vee d)$ are comparable or $b \vee d$ and $c \vee d$ are comparable. If $a \geq (b \vee d) \wedge (c \vee d)$, then $a = a \vee ((b \vee d) \wedge (c \vee d)) = 1$. In this case, L is generated by b, c, and d contrary to our assumption on L. If $(b \vee d) \wedge (c \vee d) \geq a$, then $(b \vee d) \wedge (c \vee d) = 1$ and in this case the conclusion of the lemma holds.

If $b \vee d \geq c \vee d$, then $b \vee d = b \vee c \vee d = 1$. By Lemma 4, $a \vee d$ and $c \vee d$ are comparable, and as above the larger one must be 1. Thus again the conclusion of the lemma holds.

Now we consider the case

$$\begin{aligned} &a \vee ((b \vee d) \wedge (c \vee d)) \\ \prec\ &a \vee ((b \vee d) \wedge (c \vee d) \vee (c \wedge (a \vee b)). \end{aligned}$$

Let θ be the smallest congruence on L identifying these elements and ψ_0 be the unique largest congruence separating these elements. By Lemma 2, L is a subdirect product of L/θ and $L/\psi_0 \cong M_5$. Now arguments just as above show that the conclusions of the lemma hold.

3. MAIN THEOREM

By Lemma 5 we may assume that $a \vee c = a \vee d = b \vee c = b \vee d = 1$. By the dual of Lemma 5 we may assume four of the six pairs of generators meet to 0. We first consider the case $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$. Notice that this situation has a large amount of symmetry. If a relation holds in L, then the relations obtained from it under the permutations (ab), (cd), $(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$ also hold in L. The case when L can be generated by three elements is of course easy. For now we assume that L cannot be generated by any three element. This implies that no two generators can be comparable. If $a \leq c$, for example, then $a \vee c = 1$ implies $c = 1$ contradicting the hypothesis that L is not generated by three elements. If $a \leq b$, then since c is a complement of both a and b, modularity implies $a = b$, again contradicting our assumption. The other cases are handled by symmetry.

Let $a_0 = a^0 = a$, $b_0 = b^0 = b$, $c_0 = c^0 = c$, and $d_0 = d^0 = d$. Define inductively $a_i = a \wedge (c_{i-1} \vee d_{i-1})$, $b_i = b \wedge (c_{i-1} \vee d_{i-1})$, $c_i = c \wedge (a_{i-1} \vee b_{i-1})$, $d_i = d \wedge (a_{i-1} \vee b_{i-1})$ and dually $a^i = a \vee (c^{i-1} \wedge d^{i-1})$, $b^i = b \vee (c^{i-1} \wedge d^{i-1})$, $c^i = c \vee (a^{i-1} \wedge b^{i-1})$, $d^i = d \vee (a^{i-1} \wedge b^{i-1})$. We now derive some formulae concerning these elements

(1) $a_0 = a \geq a_1 \geq a_2 \geq \dots \quad a^0 = a \leq a^1 \leq a^2 \leq \dots$ etc.

(2) $a_i = a_{i-1} \wedge (c_{i-1} \vee d_{i-1})$, $a^i = a^{i-1} \vee (c^{i-1} \wedge d^{i-1})$

(3) $a \wedge d^i = a \wedge c^i = a \wedge b^{i-1} \qquad i \geq 1$

(4) $a \vee d_i = a \vee c_i = a \vee b_{i-1} \qquad i \geq 1$

(5) $a_i \vee d_i = a_i \vee c_i = b_i \vee c_i = b_i \vee d_i =$
$(a_{i-1} \vee b_{i-1}) \wedge (c_{i-1} \vee d_{i-1})$.

For example, (4) can be proved with the aid of (2) and induction:

$$\begin{aligned}
a \vee d_i &= a \vee (d_{i-1} \wedge (a_{i-1} \vee b_{i-1})) \\
&= a \vee (d_{i-1} \wedge [(a \wedge (c_{i-2} \vee d_{i-2})) \vee (b \wedge (c_{i-2} \vee d_{i-2}))]) \\
&= a \vee [d_{i-1} \wedge (c_{i-2} \vee d_{i-2}) \wedge (a \vee (b \wedge (c_{i-2} \vee d_{i-2})))] \\
&= a \vee [d_{i-1} \wedge (a \vee (b \wedge (c_{i-2} \vee d_{i-2})))] \\
&= (a \vee d_{i-1}) \wedge [a \vee (b \wedge (c_{i-2} \vee d_{i-2}))] \\
&= (a \vee b_{i-2}) \wedge (a \vee b_{i-1}) \\
&= a \vee b_{i-1}.
\end{aligned}$$

Note that $a_o = a \geq a_1 \geq a_2 \geq \ldots$ is a descending chain in $a/0$ and $0 = a \wedge d \leq a \wedge d^1 \leq a \wedge d^2 \leq \ldots$ is an ascending chain in $a/0$. By Lemma 4, $a/0$ is a chain, and thus each $a \wedge d^j$ must be comparable with each a_i. Let n be the smallest integer such that $a \wedge b \geq a_{n+1}$, if such an integer exists. Joining both sides of $a_{n+1} \leq a \wedge b$ with c_n we obtain

$$(a \wedge b) \vee c_n \geq [a_n \wedge (c_n \vee d_n)] \vee c_n = (a_n \vee c_n) \wedge (c_n \vee d_n).$$

However, (5) tells us $a_n \vee c_n = (a_{n-1} \vee b_{n-1}) \wedge (c_{n-1} \vee d_{n-1})$.

Thus

$$(a \wedge b) \vee c_n \geq (a_{n-1} \vee b_{n-1}) \wedge (c_n \vee d_n)$$

Hence

$$a \wedge b = (a \wedge b) \vee (c_n \wedge b) = [(a \wedge b) \vee c_n] \wedge b \geq$$
$$b \wedge (c_n \vee d_n) \wedge (a_{n-1} \vee b_{n-1}) = b_{n+1} \wedge (a_{n-1} \vee b_{n-1}) = b_{n+1}.$$

Thus $a \wedge b \geq b_{n+1}$. It follows that n is the smallest integer such that $a \wedge b \geq b_{n+1}$. Now observe

$$a_{n+1} = a \wedge (c_n \vee d_n) \leq a \wedge b \wedge (c_n \vee d_n) = a_{n+1} \wedge b_{n+1}$$

Hence $a_{n+1} = b_{n+1}$. Thus

$$c_{n+2} = c \wedge (a_{n+1} \vee b_{n+1}) = c \wedge a_{n+1} = 0.$$

LEMMA 6: *Let* L *be a breadth two modular lattice generated by four noncomparable generators* a,b,c,d *satisfying* $a \vee c = a \vee d = b \vee c = b \vee d = 1$ *and* $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$. *If* $a_n > a \wedge b \geq a_{n+1}$, *then* $b_n > a \wedge b \geq b_{n+1}$ *and* $a_{n+3} = b_{n+3} = c_{n+2} = d_{n+2} = 0$. *Furthermore*, $c_m > c \wedge d \geq c_{m+1}$ *and* $d_m > c \wedge d \geq d_{m+1}$ *where* m *is either* $n - 1$, n, *or* $n + 1$.

PROOF: If $a_n > a \wedge b \geq a_{n+1}$ then $c_{n+2} = 0$, as shown above. Thus $m \leq n \div 1$. Similarly, $n \leq m + 1$. The rest of the lemma follows easily from the remarks above.

We shall require a few additional observations.

(6) $(a \wedge b^i) \vee d = (a \wedge d^{i+1}) \vee d = d^{i+1}$.

If $a_{i+1} \geq a \wedge b$ then

(7) $a_i/a_{i+1} \nearrow a_i \vee b_i/a_{i+1} \vee b_i \searrow d_{i+1}/d_{i+2}$.

If $d_{i+1} \geq c \wedge d$ then

(8) $d_i/d_{i+1} \nearrow d_i \vee c_i/d_{i+1} \vee c_i \quad a_{i+1}/a_{i+2}$.

(6) easily follows from (3). To see (7), note that since $a_i \geq a \wedge b$, $b_i \geq a \wedge b$ by Lemma 6. From this it follows that $a_i/a_{i+1} \nearrow a_i \vee b_i/a_{i+1} \vee b_i$. Repeatedly using (4) with the poles of a and d interchanged we obtain

$$\begin{aligned} d_{i+1} \vee a_{i+1} \vee b_i &= [d \wedge (a_i \vee b_i)] \vee a_{i+1} \vee b_i \\ &= (d \vee a_{i+1} \vee b_i) \wedge (a_i \vee b_i) \\ &= (d \vee c_{i-1}) \wedge (a_i \vee b_i) \\ &= (d \vee a_i \vee b_i) \wedge (a_i \vee b_i) \\ &= a_i \vee b_i \end{aligned}$$

and

$$\begin{aligned} d_{i+1} \wedge (a_{i+1} \vee b_i) &= d \wedge (a_i \vee b_i) \wedge (a_{i+1} \vee b_i) \\ &= d \wedge (a_{i+1} \vee b_i) \\ &= d \wedge [(a \wedge (c_i \vee d_i)) \vee (b \wedge (c_{i-1} \vee d_{i-1}))] \end{aligned}$$

$$
\begin{aligned}
d_{i+1} \wedge (a_{i+1} \vee b_i) &= d \wedge (c_{i-1} \vee d_{i-1}) \wedge (b \vee a_{i+1}) \\
&= [(d \wedge c_{i-1}) \vee d_{i-1}] \wedge (b \vee d_{i+2}) \\
&= d_{i+2} \vee [b \wedge (d_{i-1} \vee (d \wedge c_{i-1}))] \\
&= d_{i+2}
\end{aligned}
$$

LEMMA 7: _Let_ L _satisfy the hypotheses of Lemma_ 6. _Suppose, also that_

$$(9) \quad a_n > a \wedge b \geq a_{n+1} \quad \underline{\text{and}} \quad d_{n+1} > c \wedge d \geq d_{n+2}.$$

Then

$$(10) \quad a_i \geq a \wedge b^{n-i+1} \geq a \wedge b^{n-i} \geq a_{i+1}$$

$$i \equiv n(\text{mod } 2), \quad i \leq n$$

$$(11) \quad d_k \geq d \wedge c^{n-k+1} \geq d \wedge c^{n-k} \geq d_{k+1}$$

$$k \equiv n + 1(\text{mod } 2), \ k \leq n.$$

Furthermore, the images of $a \wedge b^{n-i+1}$ _and_ $a \wedge b^{n-i}$ _under the projectivity_ (7) _are_ $d \wedge c^{n-i}$ _and_ $d \wedge c^{n-i-1}$. _The images of_ $d \wedge c^{n-k+1}$ _and_ $d \wedge c^{n-k}$ _under the projectivity_ (8) _are_ $a \wedge b^{n-k}$ _and_ $a \wedge b^{n-k-1}$.

PROOF: First we show that

$$(12) \quad a_i \geq a \wedge b^{n-i+1} \geq a \wedge b^{n-i}$$

$$i \equiv n(\text{mod } 2), \ i \leq n$$

and

$$(13) \quad d_k \geq d \wedge c^{n-k+1} \geq d \wedge c^{n-k}$$

$$k \equiv n + 1(\text{mod } 2), \ k \leq n.$$

We prove these inequalities by induction on $n - i$ and $n - k$. First note that the second inequality in both (12) and (13) follows immediately from the monotome nature of the b^j's and c^j's. Now we show that $a_n \geq a \wedge b^1$. Using (3) we have that

$$\begin{aligned} a \wedge [(c \wedge d) \vee (b \wedge a^1)] &= a \wedge [(d \wedge a^1) \vee (b \wedge a^1)] \\ &= a \wedge a^1 \wedge (d \vee (b \wedge a^1)) \\ &= a \wedge (d \vee (b \wedge a^1)) \\ &= a \wedge (d \vee (b \wedge d^2)) \\ &= a \wedge (d \vee b) \wedge d^2 \\ &= a \wedge d^2 \\ &= a \wedge b^1 \end{aligned}$$

Now, since $a^1 = a \vee (c \wedge d) \leq a \vee d_{n+1}$, we have

$$\begin{aligned} a \wedge b^1 &= a \wedge [(c \wedge d) \vee (b \wedge a^1)] \\ &\leq a \wedge [d_{n+1} \vee (b \wedge (a \vee d_{n+1}))] \\ &= a \wedge [d_{n+1} \vee (a \wedge (b \vee d_{n+1}))] \\ &= (a \wedge d_{n+1}) \vee [a \wedge (b \vee d_{n+1})] \\ &= a \wedge (b \vee d_{n+1}) \\ &= a \wedge (b \vee a_n) \\ &= (a \wedge b) \vee a_n \\ &= a_n \end{aligned}$$

Thus, $a \wedge b^1 \leq a_n$.

Now suppose we have shown that $a_i \geq a \wedge b^{n-i+1}$. We shall show that $d_{i-1} \geq d \wedge c^{n-i+2}$. Observe that

$$d \wedge [(a \wedge b^{n-i+1}) \vee (c \wedge d^{n-i+2})] =$$

$$\begin{aligned}
&= d \wedge [(a \wedge d^{n-i+2}) \vee (c \wedge d^{n-i+2})] \\
&= d \wedge d^{n-i+2} \wedge [a \vee (c \vee d^{n-i+2})] \\
&= d \wedge [a \vee (c \wedge a^{n-i+3})] \\
&= d \wedge (a \vee c) \wedge a^{n-i+3} \\
&= d \wedge a^{n-i+3} \\
&= d \wedge c^{n-i+2}.
\end{aligned}$$

Hence, since $d^{n-i+2} = d \vee (a \wedge b^{n-i+1}) \leq d \vee a_i$,

$$\begin{aligned}
d \wedge c^{n-i+2} &= d \wedge [(a \wedge b^{n-i+1}) \vee (c \wedge d^{n-i+2})] \\
&\leq d \wedge [a_i \vee (c \wedge (d \vee a_i))] \\
&= d \wedge [a_i \vee (d \wedge (c \vee a_i))] \\
&= (d \wedge a_i) \vee [d \wedge (c \vee a_i)] \\
&= d \wedge (c \vee a_i) \\
&= d \wedge (c \vee d_{i-1}) \\
&= (c \wedge d) \vee d_{i-1} \\
&= d_{i-1}.
\end{aligned}$$

Thus $d \wedge c^{n-i+2} \leq d_{i-1}$.

Thus if j is either $n - i + 1$ or $n - i$ then $a_i \geq a \wedge b^j$ and $d_{i+1} \geq d \wedge c^{j-1}$. By way of induction suppose that $d_{i+1} \geq d \wedge c^{j-1} \geq d_{i+2}$ for j as above. Then the image of $a \wedge b^j$ under the projectivity (7) is

$$\begin{aligned}
d_{i+1} \wedge [(a \wedge b^j) \vee a_{i+1} \vee b_i] &= d_{i+1} \wedge [(a \wedge (b^j \vee a_{i+1})) \vee b_i] \\
&= d_{i+1} \wedge (a \vee b_i) \wedge (a_{i+1} \vee b^j)
\end{aligned}$$

$$
\begin{aligned}
d_{i+1} \wedge [(a \wedge b^j) \vee a_{i+1} \vee b_i] &= d_{i+1} \wedge (a_{i+1} \vee b^j) \\
&= d_{i+1} \wedge (a_{i+1} \vee b \vee b^j) \\
&= d_{i+1} \wedge (d_{i+2} \vee b^j) \\
&= d_{i+2} \vee (d_{i+1} \wedge d \wedge b^j) \\
&= d_{i+2} \vee (d_{i+1} \wedge d \wedge c^{j-1}) \\
&= d_{i+2} \vee (d \wedge c^{j-1}) \\
&= d \wedge c^{j-1}
\end{aligned}
$$

This shows that $a_i \geq a \wedge b^j \geq a_{i+1}$, which completes the proof of the lemma.

Arguments similar to these prove the following lemma.

LEMMA 8: *Let* L *satisfy the hypotheses of Lemma* 6. *Suppose also that*

(14) $a_n > a \wedge b \geq a_{n+1}$ *and* $d_n > c \wedge d \geq d_{n+1}$

Then

(15) $a_i \geq a \wedge b^{n-i} \geq a_{i+1}$

(16) $d_i \geq d \wedge c^{n-i} \geq d_{i+1}$

Furthermore, the image of $a \wedge b^{n-i}$ *under the projectivity* (7) *is* $d \wedge c^{n-i-1}$. *The image of* $d \wedge c^{n-i}$ *under the projectivity* (8) *is* $a \wedge b^{n-i-1}$.

Let L_n be the modular lattice freely generated by a,b,c,d subject to the relations $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, and $d_n \geq c \wedge d \geq d_{n+1}$. By the above lemma

(17) $a \geq a \wedge b^n \geq a_1 \geq a \wedge b^{n-1} \geq \ldots$

$\geq a_n \geq a \wedge b \geq a_{n+1} \geq a_{n+2} = 0$

(18) $d \geq d \wedge c^n \geq d_1 \geq d \wedge c^{n-1} \geq \ldots$

$\geq d_n \geq c \wedge d \geq d_{n+1} \geq d_{n+2} = 0.$

For notational convenience define

(19) $e_0 = 0, \quad a_i = e_{2n+3-2i} \qquad 0 \leq i \leq n+1$

$a \wedge b^j = e_{2j+2} \qquad 0 \leq j \leq n.$

Then the chain (17) becomes

(20) $e_{2n+3} \geq e_{2n+2} \geq \ldots \geq e_1 \geq e_0 = 0.$

Similarly, using (18) we define h_i, $i = 0, \ldots, 2n+3$. Moreover, we define f_i to be the element obtained from e_i by interchanging a and b, and g_i to be the element obtained from h_i by interchanging c and d. Let U be the following subset of L_n:

$$U = \{e_i \vee f_j \mid 2 \leq i,j \leq 2n+3\} \cup \{g_i \vee h_j \mid 2 \leq i,j \leq 2n+3\}$$
$$\cup \{e_i \vee h_j \mid 0 \leq i,j \leq 2n+3 \text{ and } |i-j| \leq 2\}$$

We shall show that U is closed under joins and meets and hence $U = L_n$. In addition, we shall evaluate all joins and meets of elements of U thereby describing the lattice L_n. First we require a lemma.

LEMMA 9: *The following formulae hold in* L_n.

(21) $a_i \vee b_j = a_i \vee c_{j+1} = a_i \vee d_{j+1} \qquad i \leq j$

(22) $(a \wedge b^i) \vee (b \wedge a^j) = (a \wedge b^i) \vee (c \wedge d^{j-1})$

$= (a \wedge b^i) \vee (d \wedge c^{j-1})$

$n \geq i \geq j$

$$(23)\quad a_i \vee (b \wedge a^j) = a_i \vee (c \wedge d^{j-1}) = a_i \vee (d \wedge c^{j-1})$$

$$i \le n - j, \quad j \le n$$

$$(24)\quad (a \wedge b^i) \vee b_j = (a \wedge b^i) \vee c_{j+1} = (a \wedge b^i) \vee d_{j+1}$$

$$i \ge n + 1 - j$$

The second equality in (21) also holds for $i \le j + 2$ and the second equality in (23) also holds for $i \le n + 1 - j$, $j \le n + 1$.

PROOF: We prove (21) using (4) and induction on i. Thus assume (21) holds when the subscript of a is less than i and assume also that the corresponding formula obtained by interchanging a and d, and b and c holds when the subscript of d is less than i.

$$\begin{aligned} a_i \wedge b_j &= [a \wedge (c_{i-1} \vee d_{i-1})] \vee b_j \\ &= [a \wedge (b_i \vee d_{i-1})] \vee b_j \\ &= (a \vee b_j) \wedge (b_i \vee d_{i-1}) \\ &= (a \vee d_{j+1}) \wedge (a_i \vee d_{i-1}) \\ &= a_i \vee [d_{i-1} \wedge (a \vee d_{j+1})] \\ &= a_i \vee d_{j+1} \end{aligned}$$

To prove (22) note that since $i,j \le n$ we have $(a \wedge b^i) \vee (b \wedge a^j) = a^j \wedge b^i$. Since $b^i \ge b^j \ge d \wedge b^j = d \wedge a^j$,

$$(a \wedge b^i) \vee (d \wedge c^{j-1}) = (a \wedge b^i) \vee (d \wedge a^j)$$
$$= b^i \wedge [a \vee (d \wedge a^j)]$$
$$= b^i \wedge a^j.$$

To prove (23) note that $i \leq n - j$ and $j \leq n$ imply that $b_i \geq b \wedge a^j$ and $a^j \leq a \vee b$. Thus

$$\begin{aligned} a_i \vee (b \wedge a^j) &= [a \wedge (c_{i-1} \vee d_{i-1})] \vee (b \wedge a^j) \\ &= [a \wedge (b_i \vee d_{i-1})] \vee (b \wedge a^j) \\ &= (b_i \vee d_{i-1}) \wedge [a \vee (b \wedge a^j)] \\ &= (b_i \vee d_{i-1}) \wedge a^j \\ &= (c_{i-1} \vee d_{i-1}) \wedge [a \vee (d \wedge a^j)] \\ &= (c_{i-1} \vee d_{i-1}) \wedge [a \vee (d \wedge c^{j-1})] \\ &= a_i \vee (d \wedge c^{j-1}) \end{aligned}$$

Since $i > n + 1 - j$, $b^i \geq d \wedge b^{n+1-j} = d \wedge c^{n-j} \geq d_{j+1}$. Thus

$$\begin{aligned} (a \wedge b^i) \vee b_j &= (a \vee b_j) \wedge b^i \\ &= (a \vee d_{j+1}) \wedge b^i \\ &= (a \wedge b^i) \vee d_{j+1}. \end{aligned}$$

The proof of the last statement of the lemma is similar to above proofs.

The previous lemma can be put into a more compact form.

COROLLARY: <u>The following holds in</u> L_n.

(25) $e_i \vee f_j = e_i \vee g_{j-2} = e_i \vee h_{j-2} \qquad i \geq j$

The joins in U are given by the following.

(26) $(e_i \vee f_j) \vee (e_k \vee f_\ell) = e_p \vee f_g$

$p = \max\{i,k\}, \quad q = \max\{j,\ell\}$

$(g_i \vee h_j) \vee (g_k \vee h_\ell) = g_p \vee h_q$

$(e_i \vee h_j) \vee (e_k \vee h_\ell) = e_p \vee h_q$

If $i \geq j$ and $\ell \geq k$ and $2 \leq i,j,k,\ell \leq 2n+3$ and $r = \max\{\ell+2,j\}$, $s = \max\{i+2,k\}$ then

$$(27) \quad (e_i \vee f_j) \vee (g_k \vee h_\ell) = \begin{cases} e_i \vee h_\ell & \text{if } |i-\ell| \leq 2 \\ e_i \vee f_r & \text{if } i \geq \ell+2 \\ g_s \vee h_\ell & \text{if } \ell \geq i+2 \end{cases}$$

If $j \geq i$ and $\ell \geq k$ then $(e_i \vee f_j) \vee (g_k \vee h_\ell)$ is as above except the roles of e and f are interchanged. The cases $j \geq i$ and $k \geq \ell$, and $i \geq j$ and $k \geq \ell$ are handled similarly.

If $i \geq j$ then

$$(28) \quad e_i \vee f_j \vee e_k \vee h_\ell = \begin{cases} e_p \vee h_{q'} & \text{if } |p-q'| \leq 2 \\ e_p \vee f_{q'+2} & \text{if } p \geq q'+2 \end{cases}$$

where $p = \max\{i,k\}$ and $q' = \max\{j-2,\ell\}$. All other joins in U are similar.

The meet operation is given by

(29) $(e_i \vee f_j) \wedge (e_k \vee f_\ell) = e_r \vee f_s$

$r = \min\{i,k\}, \quad s = \min\{i,\ell\}$

$(g_i \vee h_j) \wedge (g_k \vee h_\ell) = g_r \vee h_s$

$(e_i \vee h_j) \wedge (e_k \vee h_\ell) = e_r \vee h_s$

and

$$(30) \quad (e_i \vee f_j) \wedge (g_k \vee h_\ell) = (e_p \vee f_q) \vee (g_r \vee h_s)$$

where $p = \min\{i,k-2,\ell-2\}$, $q = \min\{j,k-2,\ell-2\}$ $r = \min\{k,i-2,j-2\}$, and $s = \{\ell,i-2,j-2\}$.

If $|k-\ell| \leq 2$ then

$$(31) \quad (e_i \vee f_j) \wedge (e_k \vee h_\ell) = (e_{p'} \vee f_{q'}) \vee (g_{r'} \vee h_{s'})$$

where $p' = \min\{i,k\}$, $q' = \min\{j,k\}$, $r' = s' = \min\{\ell,i-2,j-2\}$.

THEOREM 1: <u>The set</u> U <u>together with the join and meet given in</u> (26) - (31) <u>is the lattice</u> L_n.

<u>PROOF</u>: (26) follows from modularity. The other equations follow easily from the Corollary.

(FIGURE 1)

The lattices L_o, L_1, L_2 are diagrammed in Figure 1. If we let L_n' be the modular lattice generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, and $d_{n+1} \geq c \wedge d \geq d_{n+2}$ then an analysis similar to that of L_n can be carried out. The lattices L_o', L_1', L_2' are diagrammed in Figure 2.

(FIGURE 2)

Now let L_∞ be the modular lattice generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_i \geq a \wedge b$, $i = 0, 1, 2, \ldots.$

It follows that $a_i \geq a \wedge b^j$ and $d_i \geq d \wedge c^j$ for all $i,j \geq 0$.

It remains to consider the case when L is generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$ and all pairs of generators meeting to 0 except for two complementary pairs. By symmetry we may assume $a \wedge b = a \wedge c = b \wedge d = c \wedge d = 0$. Call this lattice L'_∞. We define a_i, b_i, c_i, d_i as before. However we now define $a^i = a \vee (b^{i-1} \wedge c^{i-1})$, $b^i = b \vee (a^{i-1} \wedge d^{i-1})$, $c^i = c \vee (a^{i-1} \wedge d^{i-1})$, $d^i = d \vee (b^{i-1} \wedge c^{i-1})$. We shall show that for all i and j

(32) $a_i \geq a \wedge d^j$, $c_i \geq c \wedge b^j$

We need two equations. The proofs of these are left to the reader.

(33) $a_i = a \wedge (d \vee c_{i-1})$

(34) $c \wedge b^i = c \wedge d^{i+1}$

To prove (32) it is sufficient to prove that $a_i \geq a \wedge d^i$ and $c_i \geq c \wedge b^i$ for all i. This is obvious for $i = 0$. Assume the equations hold for $i = 1, \ldots, n$. Then

$$\begin{aligned} a_{n+1} &= a \wedge (d \vee c_n) \\ &\geq a \wedge (d \vee (c \wedge b^n)) \\ &= a \wedge (d \vee (c \wedge d^{n+1})) \\ &= a \wedge d^{n+1}. \end{aligned}$$

The last step uses that fact that $d^{n+1} \le d \vee c$, which is easily proved by induction. Hence the following chain of elements lies below a.

$$a \ge a_1 \ge a_2 \ge \dots \ge a \wedge d^2 \ge a \wedge d^1 \ge a \wedge d \ge 0$$

With this information an analysis similar to that for L_n can be carried out.

Combining the above information we obtain the following theorem.

THEOREM 2: <u>If L is a breadth two four-generated modular lattice then L is a homomorphic image of a subdirect product of four copies of $\underset{\sim}{2}$, two copies of M_5 and either a three-generated modular lattice or L_n or L'_n for some n,</u> $0 \le n \le \infty$.

Not all four-generated subdirect products of L_n or L'_n with four copies of $\underset{\sim}{2}$ and two copies of M_5 are breadth two. However, it is possible to make a list of lattices such that L is a breadth two four-generated modular lattice if and only if L is a homomorphic image of a lattice from this list. This shall not be done here. In Figure 3 we give an example of a breadth two four-generated modular lattice which is maximal in the sense that it is not a homomorphic image of a properly larger breadth two, four-generated modular lattice.

(FIGURE 3)

4. SUBDIRECTLY IRREDUCIBLES

The utility of Theorem 2 is that the lattices in that theorem have only finitely many homomorphic images. With the aid of this fact we shall now characterize all subdirectly irreducible, four-generated breadth two modular lattices by actually listing them. Let L be such a lattice. Then it follows from Theorem 2 and the distributivity of congruence lattices of lattices that L is either $\underset{\sim}{2}$, M_5, or a homomorphic image of L_n or L_n' for some n, $0 \leq n \leq \infty$. The following lemma shows each L_n and L_n', $1 \leq n < \infty$ is the subdirect product of four subdirectly irreducible lattices.

LEMMA 10: <u>If</u> u/v <u>is a prime quotient in</u> L_n <u>or</u> L_n', $1 \leq n < \infty$, <u>then</u> u/v <u>is projective to a subquotient of</u> a/a_2.

<u>PROOF</u>: Since L_n and L_n' are finite dimensional lattices every prime quotient is projective with a subquotient of either $a/0$ or of $1/a$. Hence it suffices to show that every prime quotient of $a/0$ and of $1/a$ is projective to a subquotient of a/a_2. Suppose u/v is a subquotient for a_i/a_{i+1} with $i \leq n$. By (7) and (8) a_i/a_{i+1} is projective to a_{i-2k}/a_{i-2k+1}, $i = 0, 1, \ldots, \frac{[i]}{2}$. Hence the lemma holds in this case. If u/v is a subquotient of $a/0$ but not of a_i/a_{i+1} for all $i \leq n$ then $u = a_{n+1}$ and $v = 0$. In this case, since $n \geq 1$,

$$u/v \nearrow c_n \vee d_n/c_n \searrow d_n/c \wedge d \nearrow a_{n-1} \vee b_{n-1}/b_{n-1} \vee (c \wedge d) \searrow a_{n-1}/a \wedge b^1$$

Since $a_{n-1}/a \wedge b^1$ is a subquotient of a_{n-1}/a_n, u/v is projective to a subquotient of a/a_2 by the above remarks. By the dual argument every prime subquotient of $1/a$ is projective to a subquotient of a^2/a. Now if $n \geq 2$ then the duals of (7) and (8) tell us that a^2/a is projective to d^3/d^1 which transposes down to $a \wedge d^3/a \wedge d^1 = a \wedge b^2/a \wedge b$. Now we may argue as above. The case $n = 1$ has to be argued separately and is left to the reader. Arguments similar to the above prove the lemma for L'_n.

Lemma 10 has the corollary that L_n and L'_n are each subdirect products of four subdirectly irreducible lattices, $i \leq n < \infty$. More specifically, let $L_{n1} = L_n/\theta(a, a \wedge b^{n-1})$, $L_{n2} = L_n/\theta(a, a_1) \vee \theta(a \wedge b^n, a_2)$, $L_{n3} = L_n/\theta(a, a \wedge b^n) \vee \theta(a_1, a_2)$, $L_{n4} = L_n/\theta(a, a \wedge b^{n-1})$. Since L_n is the modular lattice freely generated by a,b,c,d satisfying the relations $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, $d_n \geq c \wedge d \geq d_{n+1}$, L_{n1} is the modular lattice freely generated by a,b,c,d satisfying the above relations and also satisfying $a = a \wedge b^n = a_1 = a \wedge b^{n-1}$. Similarly, L_{n2} is the modular lattice freely generated by a,b,c,d subject to the relations of L_n and to the additional relations $a = a \wedge b^n = a_1$, $a \wedge b^{n-1} = a_2$, L_{n3} to the additional relations $a = a \wedge b^n$, $a_1 = a \wedge b^{n-1} = a_2$, L_{n4} to the additional relations $a \wedge b^n = a_1 = a \wedge b^{n-1} = a_2$.

Since the permutation (ad)(bc) generates an automorphism of L_n and since $a \wedge b^{n-1}/a_2$ is projective to $d \wedge c^n/a_1$ we have that L_{n1} is isomorphic to L_{n3}. Similarly L_{n2} and L_{n4} are isomorphic. Furthermore, L_{n2} is isomorphic to $L_{n+1,1}$. To see this, one shows that L_{n2} satisfies the defining relations of $L_{n+1,1}$ and vice versa. This can be done with the use of Lemma 8, and is left to the reader. Similar arguments give that L'_n is a subdirect product of L_{n1}, $L_{n+2,1}$, and two copies of $L_{n+1,1}$.

It follows from (17) that L_n has length $4n + 6$. Using Lemma 8 it follows that L_{n1} has length $n + 1$ and L_{n2} has length $n + 2$. Let $S_1 = \underset{\sim}{2}$ $S_2 = M_5$ and $S_{n+1} = L_{n1}$ $n \geq 2$.

In L_∞ $a^2 \geq a^1 \geq a \geq a_1 \geq a_2$ and by (7) and (8) and their duals every prime quotient of L_∞ is projective to a nontrivial subquotient of a^2/a or a/a_2. If we identify a^2 with a and a_1 with a_2 then we get the modular lattice freely generated by a,b,c,d subject to these relations and the relations of L_∞. These relations are equivalent to $a \vee b = a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge b = a \wedge c = a \wedge d = b \wedge c = b \wedge d = c \wedge d = 0$. This is the lattice studied in [3]. We denote it by S_∞. Examining the other congruences on L_∞ yield that L_∞ is a subdirect product of two copies of S_∞ and two copies of S_∞^d, its dual. The same statement holds for L'_∞. These facts together imply that L_n and L'_n, $0 \leq n \leq \infty$, are each a subdirect product of four subdirectly irreducibles chosen from $\{S_n \mid 1 \le n \le \infty\} \vee \{S_\infty^d\}$. It follows from the distributivity of lattice congruences that any subdirectly irreducible, breadth two, four-generated modular lattice is a homomorphic image of one of the S_n or S_∞^d. For $n < \infty$, S_n is finite and hence simple. Thus S_n, $n < \infty$ has no nontrivial homomorphic images. S_∞ and S_∞^d have only one nontrivial homomorphic image: the six element length two lattice, M_6 [3]. Consequently

THEOREM 3: The subdirectly irreducible, breadth two, four-generated modular lattices are precisely the set $\{S_n \mid 1 \le n \le \infty\} \vee \{S_\infty^d, M_6\}$.

In [3] the word problem for S_∞ is solved. If one takes the sublattice K_n of S_∞ generated by $a \vee d_n$, $b \vee d_n$, $c \vee a_n$, $d \vee a_n$ if n is even and by $a \vee d_{n-1}$, $b \vee d_{n-1}$, $c \vee a_{n+1}$, $d \vee a_{n+1}$ if n is odd, then using the above mentioned solution to the word problem in S_∞, one can show that K_n satisfies the relations defining S_n. Since S_n is simple it follows that K_n is isomorphic to S_n. This shows that the lattices of Theorem 3 are precisely the breadth two lattices considered in [3]. See Figures 4 and 5.

(FIGURE 4)

(FIGURE 5)

5. COVERINGS IN $FM(4)$

It is apparent from Lemma 2 that coverings in free modular lattices have important consequences in the study of the structure of modular lattices. Moreover, McKenzie has investigated the connections of coverings in a free lattice to the theory of lattice varieties. In view of these applications we give some examples of coverings in $FM(4)$. In particular, we give an infinite list of covering in $FM(4)$, $u_i \succ v_i$ inequivalent in the strong sense that if $\psi(u_i, v_i)$ is the unique maximal congruence separating u_i from v_i then the $FM(4)/\psi(u_i, v_i)$'s are pairwise nonisomorphic. In fact, there is a covering corresponding to each S_n, $1 \leq n < \infty$.

Let f map $FM(n)$ homomorphically onto L. Then f is called <u>upper bounded</u> if for each $x \in L$ there is an element $u \in FM(4)$ such that $f(u) = x$ and $f(v) = x$ implies $v \leq u$. If the dual property holds then f is <u>lower bounded</u>. If f is both upper and lower bounded then f is <u>bounded</u>. If u is as above we call u the <u>maximal inverse image</u> of x. The minimal inverse image is defined dually. Note that if $f : FM(n) \twoheadrightarrow L$ is bounded and $y \succ x$ in L, and if u is the maximum inverse image of x and v is the minimal inverse image of y, then $u \vee v \succ u$ and $u \wedge v \prec v$ in $FM(n)$. These concepts were defined and studied by R. McKenzie [6]. When L is finite McKenzie gives the following process for deciding if f is bounded. For each $x \in L$ define $M(x)$ to be the family of two elements subsets $\{y, z\}$ of L such that $x \geq y \wedge z$, $x \not\geq y$, $x \not\geq z$ and if $y_0 \geq y$, $z_0 \geq z_0$ and $x \geq y_0 \wedge z_0$ then $y_0 = y$ and $z_0 = z$. Choose $\alpha_0 : L \to FM(n)$ such that α_0 is monotine and $f(\alpha_0(x)) = x$ for all $x \in L$, and such that $\underset{\sim}{a} \leq \alpha_0 f(\underset{\sim}{a})$ for each generator of $FM(n)$. Now define

$$(35) \quad \alpha_i(x) = \alpha_{i-1}(x) \vee \bigvee_{\{x,\, y\} \in M(x)} (\alpha_{i-1}(y) \wedge \alpha_{i-1}(z))$$

Now if $f(u) = x$ then $u \leq \alpha_i(x)$ for some i [6]. Thus f is upper bounded if and only if $\alpha_i = \alpha_{i+1}$ for some i.

Let $FM(4)$ be freely generated by $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$. Let $S_{2n+1} = L_{2n,1}$ be the lattice defined above. Let the generators of S_{2n+1} be a, b, c, d. Let $f : FM(4) \to S_{2n+1}$ be the unique extension of the map $f(\underset{\sim}{a}) = a$, $f(\underset{\sim}{b}) = b$, $f(\underset{\sim}{c}) = c$, $f(\underset{\sim}{d}) = d$. Note that since the maximal inverse image function, when it exists, preserves meets and S_{2n+1} has breadth two we may restrict our attention to the meet irreducibles in S_{2n+1} in calculating the α_i's. The meet irreducibles of S_{2n+1} consist of

$$\begin{array}{l}
a \leq a^1 = a^2 \leq a^3 = a^4 \leq \ldots \leq a^{2n-3} = a^{2n-2} \leq a \vee b \\
b \leq b^1 = b^2 \leq b^3 = b^4 \leq \ldots \leq b^{2n-3} = b^{2n-2} \leq a \vee b \\
\qquad c = c^1 \leq c^2 = c^3 \leq \ldots \leq c^{2n-2} = c^{2n-1} \\
\qquad d = d^1 \leq d^2 = d^3 \leq \ldots \leq d^{2n-2} = d^{2n-1}
\end{array}$$

Now $M(a) = \{\{b, c^1\}, \{b, d^1\}\}$, $M(a^{2i}) = \{\{b^{2i}, c^{2i+1}\}, \{b^{2i}, d^{2i+1}\}, \{c^{2i-1}, d^{2i-1}\}\}$, $i = 1, \ldots, n-1$.

$M(a \vee b) = \{\{c^{2n-1}, d^{2n-1}\}\}$, $M(c^{2i+1}) = \{\{d^{2i+1}, a^{2i+2}\}, \{d^{2i+1}, b^{2i+2}\}, \{a^{2i}, b^{2i}\}\}$, $i = 0, \ldots, n-2$, $M(c^{2n-1}) = \{\{a \vee b, d^{2n-1}\}\}$. The definition of $M(b^{2i})$ is similar to $M(a^{2i})$ and $M(d^{2i+1})$ to $M(c^{2i+1})$.

With these definitions one can choose an appropriate definition of α_0 and compute α_k by (35). For large enough k, $\alpha_k = \alpha_{k+1}$. We shall only give this final function. In $FM(4)$ with generators $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ let

$$(36) \quad \begin{array}{ll} \underset{\sim}{a}^{i} = \underset{\sim}{a} \vee (\underset{\sim}{c}^{i-1} \wedge \underset{\sim}{d}^{i-1}) & \underset{\sim}{b}^{i} = \underset{\sim}{b} \vee (\underset{\sim}{c}^{i-1} \vee \underset{\sim}{d}^{i-1}) \\ \underset{\sim}{c}^{i} = \underset{\sim}{c} \vee (\underset{\sim}{a}^{i-1} \vee \underset{\sim}{b}^{i-1}) & \underset{\sim}{d}^{i} = \underset{\sim}{d} \vee (\underset{\sim}{a}^{i-1} \vee \underset{\sim}{b}^{i-1}) \end{array}$$

Define $g : S_{2n+1} \to FM(4)$ inductively as follows

$$g(a \vee b) = \underset{\sim}{a} \vee \underset{\sim}{b} \vee (\underset{\sim}{c}^{2n-1} \wedge \underset{\sim}{d}^{2n-1})$$
$$g(c^{2n-1}) = \underset{\sim}{c}^{2n-1} \vee (d^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b})) \qquad g(d^{2n-1}) = \underset{\sim}{d}^{2n-1} \vee (\underset{\sim}{c}^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b}))$$
$$g(a^{2i}) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) \qquad g(b^{2i}) = \underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))$$
$$g(c^{2i+1}) = \underset{\sim}{c}^{2i+1} \vee (\underset{\sim}{d}^{2i+1} \wedge g(a^{2i+2})) \qquad g(d^{2i+1}) = \underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))$$

To see that g is the final function we must show that if we let $\alpha_0 = g$ in (35) then $\alpha_1 = g$. The following identities in $FM(4)$ may be proved by induction, starting with $i = n - 1$ and working down.

$$g(a^{2i}) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(d^{2i+1}))$$
$$g(c^{2i-1}) = \underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(a^{2i})) = \underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(b^{2i}))$$

Let $\alpha_0 = g$ we have

$$
\begin{aligned}
\alpha_1(a^{2i}) = {} & \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) \\
& \vee \{[\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{c}^{2i+1} \vee (\underset{\sim}{d}^{2i+1} \wedge g(a^{2i+2}))]\} \\
& \vee \{[\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))]\} \\
& \vee \{[\underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(a^{2i}))] \wedge [\underset{\sim}{d}^{2i-1} \vee (\underset{\sim}{c}^{2i-1} \wedge g(a^{2i}))]\}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\
= {} & [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(d^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\
= {} & [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge (\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\
= {} & [\underset{\sim}{a}^{2i} \wedge g(d^{2i+1})] \vee (\underset{\sim}{b}^{2i} \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))]).
\end{aligned}
$$

With the use of this identity, the modular law and the fact that $\underset{\sim}{c}^{2i-1} \wedge \underset{\sim}{d}^{2i-1} \leq \underset{\sim}{a}^{2i-1}$ it is easy to show that $\alpha_1(a^{2i}) = g(a^{2i})$. Similar argument show that $\alpha_1 = g$. If we extend g to all of S_{2n+1} by letting $g(x \wedge y) = g(x) \wedge g(y)$ then g is well-defined and is the maximum inverse image function. Since S_{2n+1} is isomorphic to its dual we can calculate the minimal inverse limit function h as well. Then since $a^1 \succ a$ in S_{2n+1} we have the following covering in FM(4).

$$
g(a) \vee h(c \wedge d) = g(a) \vee h(a) \vee h(c \wedge d) = g(a) \vee h(a^1) \succ g(a)
$$

Letting $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$ be the elements dual to $\underset{\sim}{a}^i$ and $\underset{\sim}{b}^i$

in $FM(4)$, we have $h(c \wedge d) = \underset{\sim}{c} \wedge \underset{\sim}{d} \wedge (\underset{\sim}{a}_{2n-1} \vee \underset{\sim}{b}_{2n-1})$. Also, $g(a) = \underset{\sim}{a} \vee (\underset{\sim}{b} \wedge (\underset{\sim}{c}^1 \vee (\underset{\sim}{d}^1 \wedge (\underset{\sim}{a}^2 \vee (\underset{\sim}{b}^2 \wedge \ldots$ $(\underset{\sim}{c}^{2n-1} \vee (\underset{\sim}{d}^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b})\ldots)$. Thus we have proved the following theorem.

Theorem 4: <u>For</u> $n = 1, 2, \ldots$ <u>we have the following coverings in</u> $FM(4)$.

$$[\underset{\sim}{c}\wedge\underset{\sim}{d}\wedge(\underset{\sim}{a}_{2n-1}\vee\underset{\sim}{b}_{2n-1})]\vee\underset{\sim}{a}\vee(\underset{\sim}{b}\wedge(\underset{\sim}{c}^1\vee(\underset{\sim}{d}^1\ldots(\underset{\sim}{c}^{2n-1}\vee(\underset{\sim}{d}^{2n-1}\wedge(\underset{\sim}{a}\vee\underset{\sim}{b})\ldots)$$
$$\underset{\sim}{a}\vee(\underset{\sim}{b}\wedge(\underset{\sim}{c}^1\vee(\underset{\sim}{d}^1\ldots\wedge(\underset{\sim}{c}^{2n-1}\vee(\underset{\sim}{d}^{2n-1}\wedge(\underset{\sim}{a}\vee\underset{\sim}{b})\ldots)$$

<u>Furthermore, if</u> ψ_n <u>is the unique maximal congruence separating this covering then</u> $FM(4)/\psi_n \cong S_{2n+1}$.

Similarly one obtains coverings in $FM(4)$ corresponding to each of the S_{2n}'s.

Following McKenzie, call a modular lattice L a <u>splitting modular lattice</u> if there exists an equation ε such that for any variety V of modular lattices either all members of V satisfy ε or $L \in V$. By the above, S_n, $n = 0, 1, 2, \ldots$ is a splitting modular lattice.

COROLLARY: L <u>is a breadth two, four-generated splitting modular lattice if and only if</u> L <u>is isomorphic to</u> S_n <u>for some</u> n, $1 \leq n < \infty$.

<u>PROOF</u>: It was shown in [3] that M_6, S_∞, and S_∞^d are not splitting modular lattices. The corollary follows from the

fact that a splitting modular lattice must be subdirectly irreducible.

Now let V be the variety of modular lattices generated by all breadth two modular lattices and let $FL(V, 4)$ be the free V-lattice on four generators. A lattice L is called _weakly atomic_ if for $x > y$ in L there exists $u, v \in L$ such that $x \geq u \succ v \geq y$.

COROLLARY: $FL(V, 4)$ _is a unique irredundant subdirect product of_ 14 _copies of_ S_1, 14 _copies of_ S_2, _and_ 6 _copies of_ S_n, $n = 3, 4, \ldots$. _Moreover_, $FL(V, 4)$ _is weakly atomic_.

PROOF: In [3] it is shown that V is generated by $\{S_n \mid 1 \leq n < \infty\}$. Hence $FL(V, 4)$ is a subdirect product of S_n, $n = 1, 2, \ldots$. It is easy to check that there are 14 distinct congruence relations ψ on $FL(V, 4)$ such that $FL(V, 4)/\psi \cong S_1$, 14 congruences giving S_2, and 6 congruences giving S_n, $n = 3, 4, \ldots$. With the aid of Lemma 2 and Theorem 4 it can be shown that none of these lattices can be removed from a subdirect representation of $FL(V, 4)$.

If $x > y$ in $FL(V, 4)$ then by the above there exists a homomorphism f from $FL(V, 4)$ onto S_n, for some $n < \infty$, such that $f(x) > f(y)$. Since f is bounded there exists $u, v \in FL(V, 4)$ with $u \succ v$ and $f(x) \geq f(u) > f(v) \geq f(y)$. Now it is easy to see that u/v is projective to a subquotient u'/v' of x/y in two or less steps. By modularity $x \geq u' \succ v' \geq y$, proving the corollary.

The above corollary implies that the word problem for $FL(V, 4)$ is solvable. However, by Jónsson's theorem [5] the four-generated subdirectly irreducible members of V are precisely the lattices listed in Theorem 3 (see also [1]). Hence we have the following corollary.

COROLLARY: *If L is the V-lattice freely generated by four generators subject to finitely many relations, then the word problem for L is solvable.*

With the aid of the results of this paper, C. Herrmann has been able to list all subdirectly irreducible four-generated modular lattices in the class C of all lattices embeddable in a complemented modular lattice. From this it follows that the word problem for four-generated lattices in C is solvable. This contrasts the result of G. Hutchinson that the word problem for nine-generated lattices in C is not solvable. An easy modification of Hutchinson's argument yields that the word problem for seven-generated lattices in C is not solvable.

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FIGURE 1

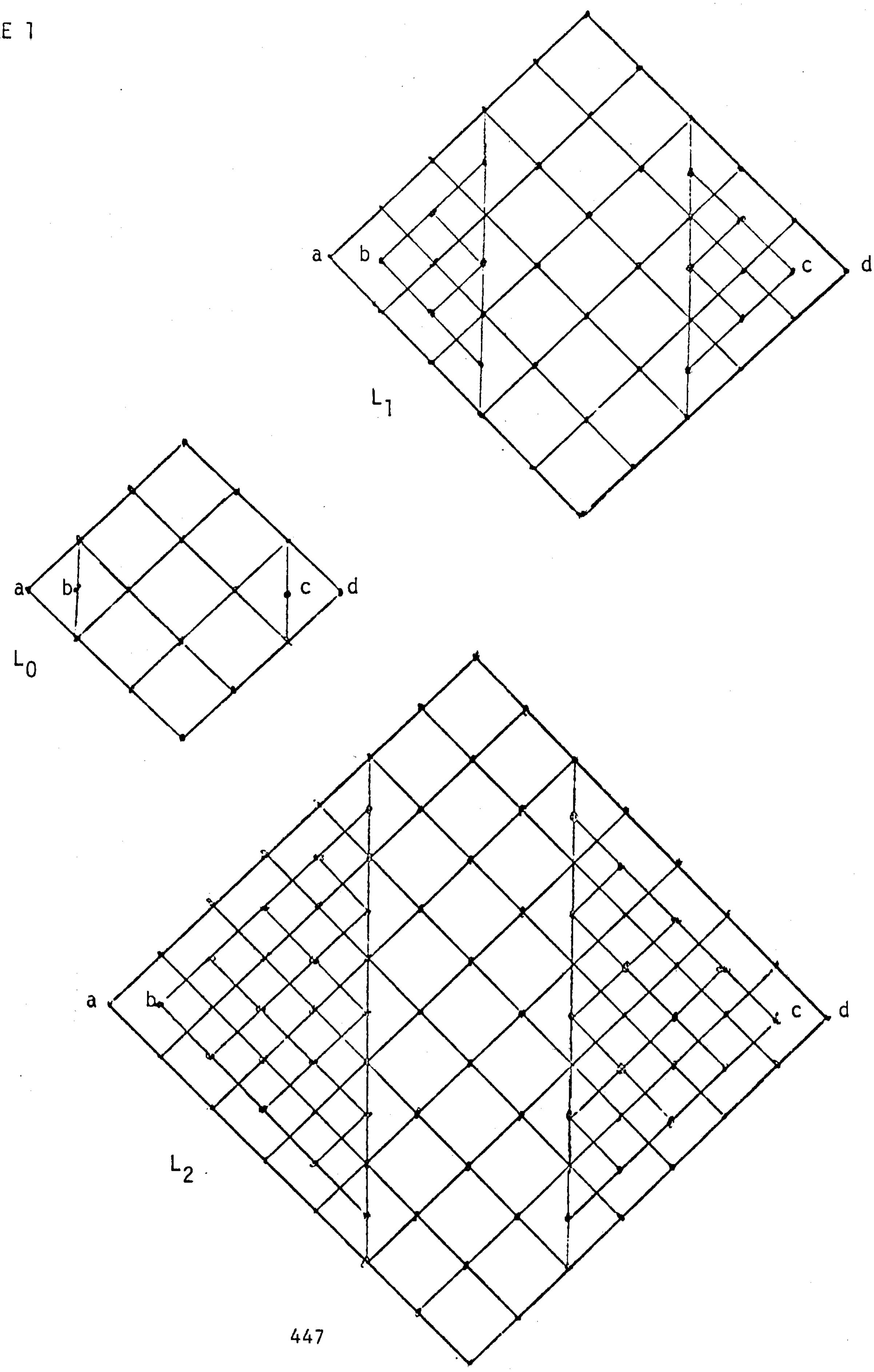


FIGURE 2

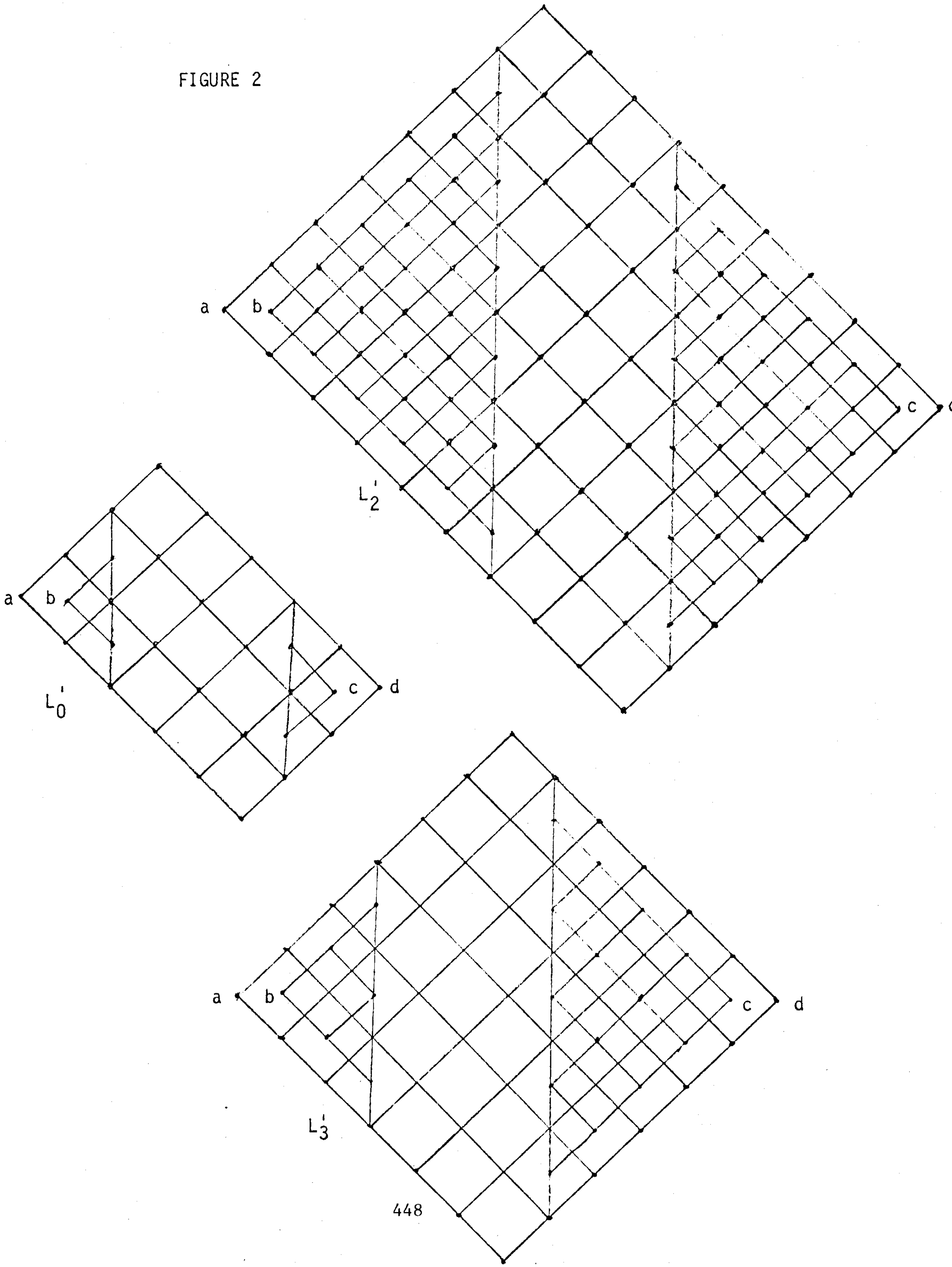


FIGURE 3

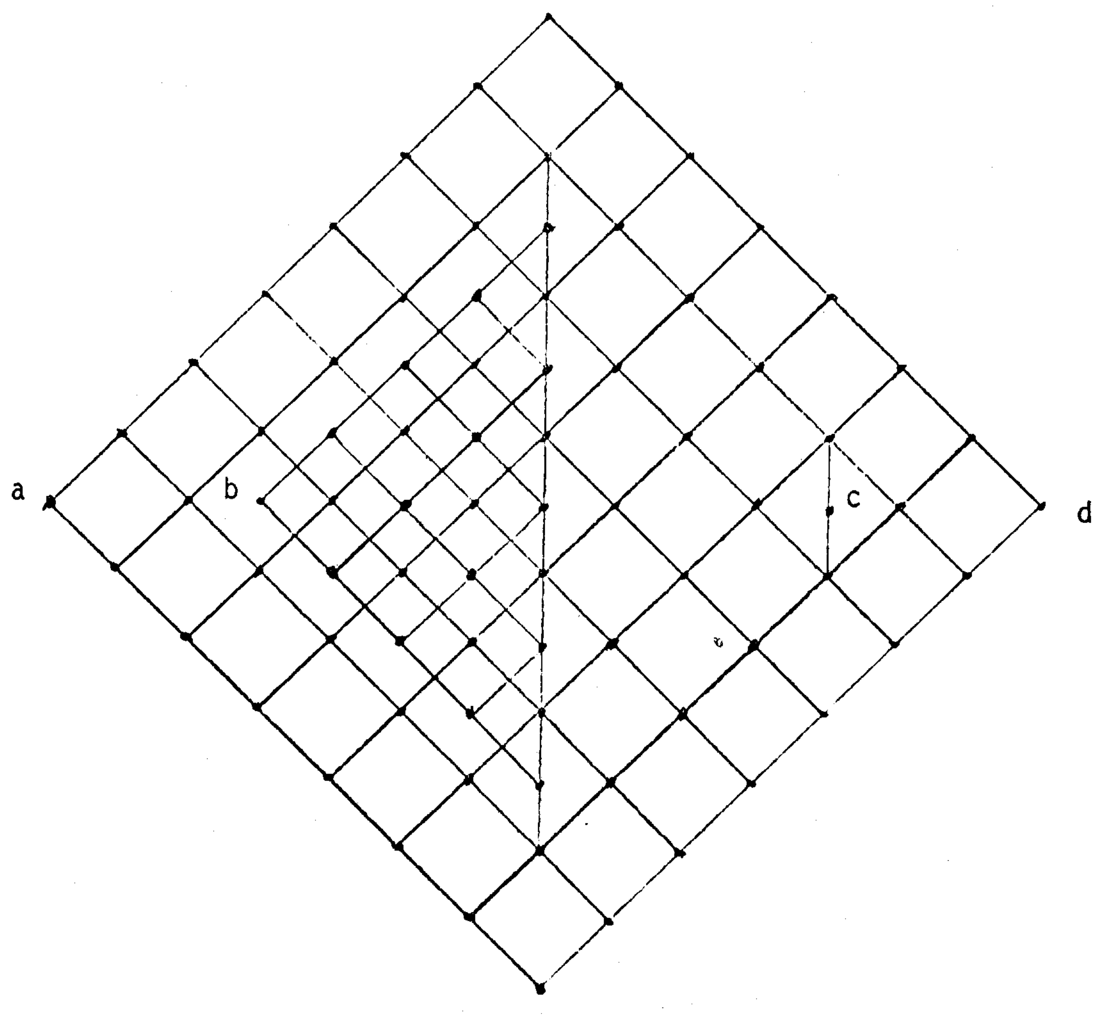
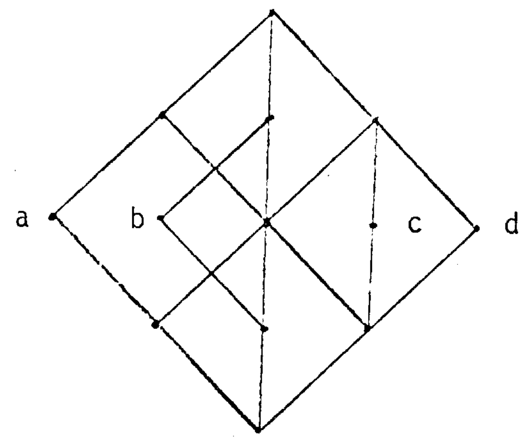
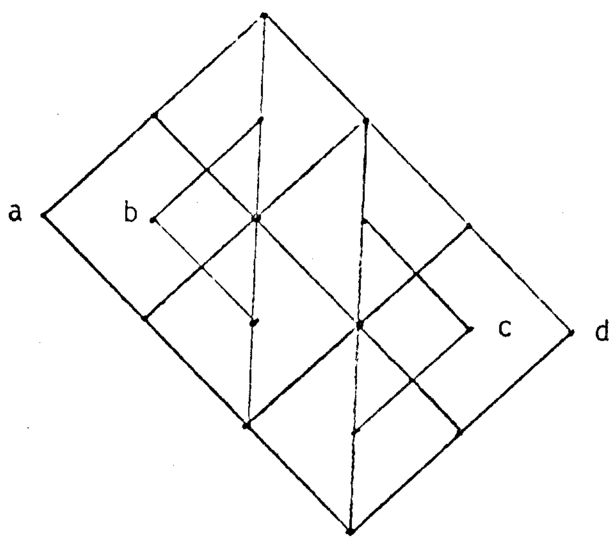


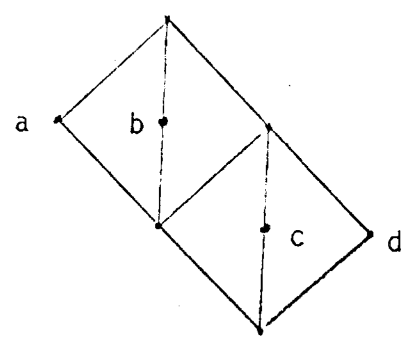
FIGURE 4



S_4

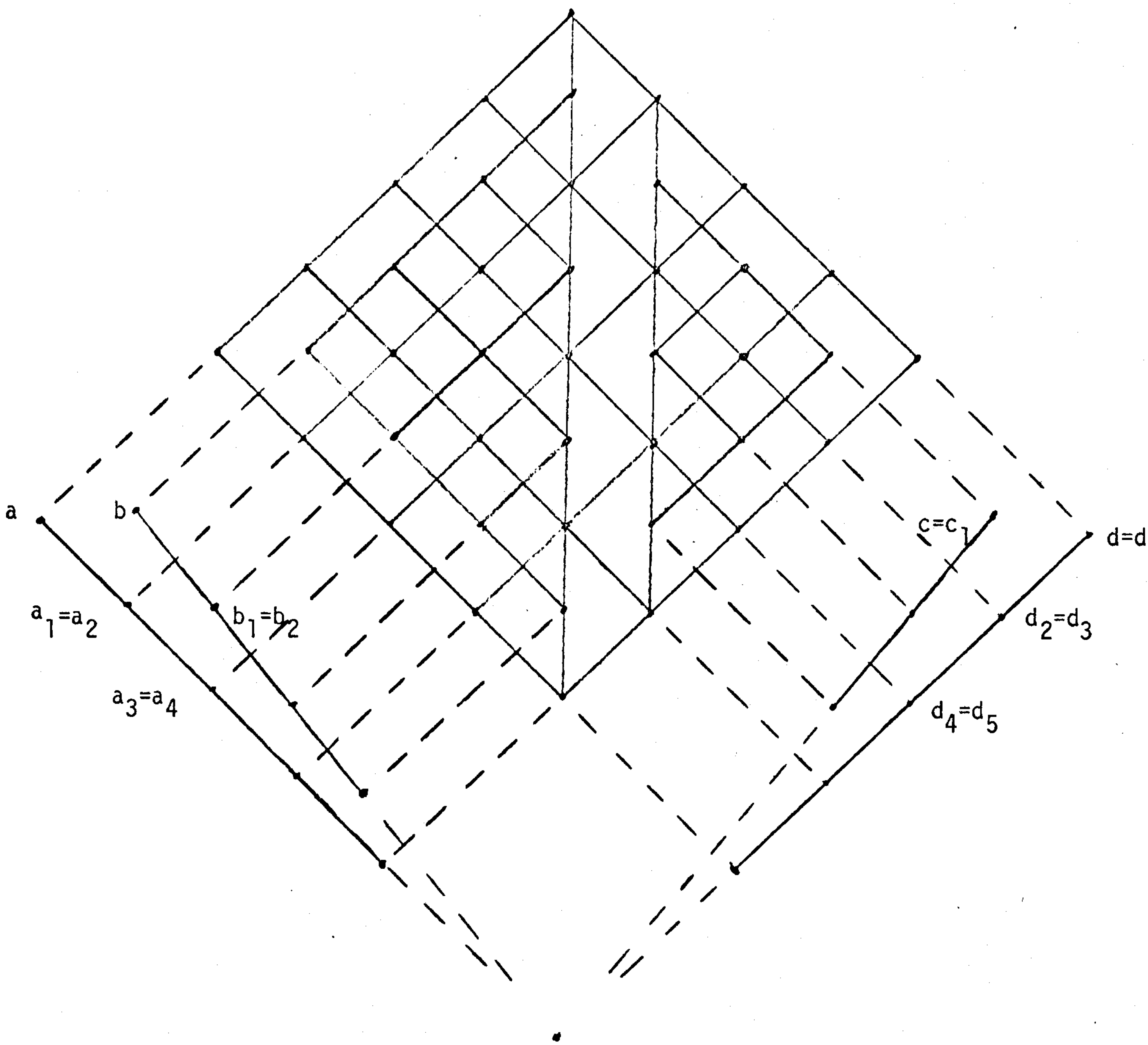


S_5



S_3

FIGURE 5



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STONE DUALITY FOR VARIETIES GENERATED BY QUASI PRIMAL ALGEBRAS

by

K. Keimel and H. Werner

One of the most famous representation theorems is that of M.H. Stone (1937) which says that for every Boolean algebra B there is a Boolean space X such that B is isomorphic to the Boolean algebra of closed-and-open sets of X. This representation theorem leads to a duality between the category of Boolean algebras and the category of Boolean spaces. In 1969 Tah Kai Hu proved that this representation and duality works for any variety generated by a primal algebra. We want to extend this representation and duality to varieties generated by a <u>weakly independent set of quasi primal algebras</u>, i.e. a finite set $\mathfrak{A}$ of finite algebras having a ternary polynomial $d(x, y, z)$ which satisfies $d(x, y, z) = \begin{cases} x & \text{if } y = z \\ z & \text{if } y \neq z \end{cases}$ on every algebra $A \in \mathfrak{A}$. From now on $\mathfrak{A}$ will always denote a weakly independent set of quasi primal algebras. Examples are any finite set of finite fields

or any finite set of finite chains, regarded as lattices with relative pseudocomplementation and dual relative pseudocomplementation. The existence of $d(x, y, z)$ shows that any algebra in the variety $V\mathfrak{A}$ generated by $\mathfrak{A}$ has permutable and distributive congruences and therefor every $R \in V\mathfrak{A}$ is a subdirect product of subalgebras of algebras in $\mathfrak{A}$. For $R \in V\mathfrak{A}$ let $\mathrm{Hom}(R, \mathfrak{A}) := \{\varphi\colon R \to A \mid \varphi \text{ homomorphism}, A \in \mathfrak{A}\}$ and $\mathrm{Spec}(R) := \{\mathrm{Ker}\,\varphi \mid \varphi \in \mathrm{Hom}(R, \mathfrak{A})\}$. The <u>"equalizer topology"</u> on Spec(R) is the topology with the basis $\{E(r, s), D(r, s) \mid r, s \in R\}$, where $D(r, s) = \{\theta \mid (r, s) \notin \theta\}$, $E(r, s) = \{\theta \mid (r, s) \in \theta\}$. Let S be the (finite) discrete space $S = \bigcup \mathfrak{A}$.

THM 1: (1) Spec(R) with the equalizer topology is a Boolean space,

(2) Hom(R, $\mathfrak{A}$) with the topology induced by Hom(R, $\mathfrak{A}$) $\subseteq S^R$ is a Boolean space.

Our first representation and duality will be by means of sheaves, therefor we used the standard construction of sheaves from a subdirect representation of a algebra. Let be $R \subseteq \prod_{x \in X} B_x$ a subdirect product of algebras and let τ be a topology on X such that for r, s $\in$ R the set $\{x \in X \mid r(x) = s(x)\}$ is open, then there is a sheaf $\mathcal{G}$ with base space X and stalks B_x $(x \in X)$ such that $R \subseteq \Gamma\mathcal{G}$, $\Gamma\mathcal{G}$ is the algebra of all global sections of $\mathcal{G}$. This standard construction gives us a sheaf $\mathcal{G}(R)$ for every algebra $R \in V\mathfrak{A}$ with base space Spec(R) and subalgebras of algebras in $\mathfrak{A}$ as stalks.

THM 2: (1^{st} representation theorem) For any $R \in V\mathfrak{A}$ $R \cong \Gamma\mathcal{G}(R)$ holds

THM 3: (1^{st} duality theorem) Let $\mathfrak{G}$ be the category of sheaves $\mathcal{G}$ satisfying

(1) the base space X is Boolean, (2) Every stalk is a subalgebra of some $A \in \mathfrak{A}$, (3) If some $A \in \mathfrak{A}$ contains a 1-element subalgebra, then $\mathcal{G}$ has exactly one one-element stalk.

Then Γ: $\mathfrak{G} \to V\mathfrak{A}$ and $\mathcal{G}$: $V\mathfrak{A} \to \mathfrak{G}$ established a duality. We also can give a representation by continuous functions. Let H be the set of all isomorphisms between subalgebras of algebras in $\mathfrak{A}$. Then H with the relational product is an inverse semigroup and acts as an inverse semigroup of partial homeomorphisms on each Hom(R, $\mathfrak{A}$) and on S. If X, Y are Boolean H-spaces (i.e. Boolean spaces with H acting on them) then we denote by $C_H(X, Y)$ the set of all continuous H-preserving maps φ: $X \to Y$.

THM 4: (2^{nd} representation theorem) For any $R \in V\mathfrak{A}$ $R \cong C_H(\mathrm{Hom}\ (R, \mathfrak{A}), S)$ holds.

THM 5: (2^{nd} duality theorem) Let $\mathfrak{B}_H$ denote the category of Boolean H-spaces.

Then $C_H(\_\_\_, S)$: $\mathfrak{B}_H \to V\mathfrak{A}$ and $\mathrm{Hom}(\_\_\_, \mathfrak{A})$: $V\mathfrak{A} \to \mathfrak{B}_H$ establish a duality

COR. 4.1.: (AHRENS-KAPLANSKI): Let K be a field of characteristic p. For every ring $R \in VK$ there is a Boolean space X together with

(i) a closed subset $\tilde{L}$ of X for every subring L of K

(ii) a homeomorphism $\tilde{\alpha}$: $X \to X$ leaving all $\tilde{L}$ invariant

such that R is isomorphic to the ring of all continuous functions f: $X \to K$ satisfying (1) $f(x) \in L$ for all $x \in \tilde{L}$

(2) $f(\tilde{\alpha} x) = f(x)^p$ for all $x \in X$

COR. 4.2.: (AHRENS-KAPLANSKI): The category of all p-rings is dual to the category of pointed Boolean spaces.

COR. 4.3.: (HU): The variety generated by a primal algebra is dual to the category of Boolean spaces.

COR. 4.4.: (STONE): The category of Boolean algebras is dual to the category of Boolean spaces.

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Isomorphic Embedding of the Lattice of all Subgroups of a Group

by

L. M. Chawla and L. E. Fuller

Let G be any group. Let $\langle G \times G, \circ \rangle$ be the semigroup of all ordered pairs (x, y), $x, y \in G$, the multiplication $\circ$ being defined by

$$(x_1, y_1) \circ (x_2, y_2) = (x_1, y_1x_2y_2) \ .$$

Our main tool to achieve an isomorphic embedding of the lattice of all subgroups of G into the lattice of all sub-semigroups of $\langle G \times G, \circ \rangle$ is the concept of a semigroup table $S(A)$ of a subgroup $A \leq G$, introduced in section 2. $S(A)$ is a certain sub-semigroup of $\langle G \times G, \circ \rangle$ such that the mapping $\alpha : (x,y) = xy \in A$ is an epimorphism of $S(A)$ onto A. We arrive at the concept of a semigroup table $S(A)$ of a subgroup A via its three particular cases $P(A)$, $Q(A)$ and $R(A)$, respectively called the inner, outer and complete tables of a subgroup A. The main results of the paper can now be stated as follows:

(A) The lattice of all subgroups of a group G can be isomorphically embedded into the lattice of all sub-semigroups of $\langle G \times G, \circ \rangle$, the isomorphism being $\psi : A \to R(A)$. (Theorem 3.4)

Using (A) above, Birkhoff's [1] and Whitman's [3] well known theorems, we prove that

(B) Any arbitrary lattice can be isomorphically embedded into the lattice of all sub-semigroups of $\langle G \times G, \circ \rangle$, for a suitable group G. (Theorem 3.5)

(C) For any semigroup table $S(A)$ of a subgroup $A \subseteq G$, the quotient set $S(A) / \operatorname{Ker} \alpha$ is a group ismorphic to A. (Theorem 2.2)

We begin by defining:

$$P(A) = \{(x,y) \mid x, y, xy \in A\} ,$$

$$Q(A) = \{(x,y) \mid x, y \in A', xy \in A\} ,$$

$$R(A) = \{(x, y) \mid x, y \in G, xy \in A\} ,$$

where A' is the complement of A in G.

The following basic Lemma will be required.

Lemma 1.1.

(a) For any $x, y \in G$ and any subgroup A of G, if $xy \in A$, then either $x, y \in A$ or $x, y \in A'$.

(b) If $x \in A'$, $y \in A$, then xy and yx are both in A'.

(c) If $x \in A'$, then $x^{-1} \in A'$.

Let $xy = z$ where $z \in A$. Since $x = zy^{-1}$, it follows that if $y \in A$, $y^{-1} \in A$ and so $x \in A$. But if $y \in A'$, then since $y = x^{-1}z$, it follows that $x \in A'$. This proves (a).

To prove (b) let $x \in A'$, $y \in A$ and $xy = z$ then $z \notin A$, since otherwise $x = zy^{-1} \in A$, a contradiction. By symmetry $yx \in A'$.

Part (c) follows directly from (a).

Because G is a group, there is a natural mapping α of $G \times G$ into G defined by

$$(x,y) \xrightarrow{\alpha} xy \quad \text{or} \quad \alpha(x,y) = xy .$$

This basic mapping has an important property that is given in the next

Lemma 1.2. The restriction of α to P(A), [Q(A), R(A)] is onto A.

Let $a \in A$, then for any fixed $y \in G$, there is a unique $x = ay^{-1} \in G$ so that $xy = a \in A$. Hence if $y \in A$, then $x \in A$ and $(x, y) \in P(A)$. If $y \in A'$, then by Lemma 1.1, $x \in A'$; and so $(x,y) \in Q(A)$. Similarly if $y \in G$, then $x \in G$ and $(x,y) \in R(A)$. This proves the Lemma.

It is easily verified that the set $G \times G$ is a semigroup under each of the two multiplications defined as follows

$$(x_1,y_1) \circ (x_2,y_2) = (x_1,y_1x_2y_2)$$

and

$$(x_1,y_1) \times (x_2,y_2) = (x_1y_1x_2,y_2)$$

Since the two semigroups $\langle G \times G, \circ\rangle$ and $\langle G \times G, \times\rangle$ have similar properties, we shall restrict ourselves to $\langle G \times G, \circ\rangle$.

Lemma 1.3.

(a) Each of the tables $P(A)$, $Q(A)$ and $R(A)$ is a sub-semigroup of $\langle G \times G, \circ\rangle$.

(b) The mapping $(x,y) \xrightarrow{\alpha} xy$ is an epimorphism from each of the sub-semigroups $\langle P(A), \circ\rangle$, $\langle Q(A), \circ\rangle$, $\langle R(A), \circ\rangle$ onto the underlying semigroup of A.

(c) If (x,y) belongs to any of the three sub-semigroups, so does (y^{-1}, x^{-1}) .

To prove (a), let (x_1,y_1) , $(x_2,y_2) \in P(A)$ or $\in Q(A)$ or $\in R(A)$, then it must be shown that $(x_1, y_1) \circ (x_2,y_2) = (x_1,y_1x_2y_2)$ belongs to $P(A)$ or $Q(A)$ or $R(A)$. In all cases $x_1(y_1 x_2y_2) = (x_1y_1)(x_2y_2) \in A$. In the first instance if $x_1, y_1 \in A$ then $y_1(x_2y_2) \in A$ by Lemma 1.1, so $(x_1, y_1x_2x_2) \in P(A)$. In the second case if $x_1, y_1 \in A'$, then $y_1(x_2y_2) \in A'$ by Lemma 1.1, so $(x_1, y_1x_2y_2) \in Q(A)$. Finally, if $x_1, y_1 \in G$, then x_1 and $y_1x_2y_2 \in G$ so $(x_1, y_1x_2y_2) \in R(A)$.

From Lemma 1.2, the mapping $(x,y) \xrightarrow{\alpha} = xy \in A$ is an onto mapping from either of the three sub-semigroups to the set A. That α is an epimorphism from any of the sub-semigroups to the underlying semigroup of A is proved as follows:

Let $\quad \alpha(x_1,y_1) = x_1y_1 = a_1 \in A$

$\qquad \alpha(x_2,y_2) = x_2y_2 = a_2 \in A$

then $\quad \alpha[(x_1,y_1) \circ (x_2,y_2)] = \alpha(x_1,y_1x_2y_2)$

$$= x_1(y_1x_2y_2) = (x_1y_1)\ (x_2y_2)$$

$$= (a_1)\ (a_2) = \alpha(x_1,y_1) \cdot \alpha(x_2,y_2)$$

This proves (b).

To prove (c), let $(x,y) \in P(A)$, then $x, y, xy \in A$, and hence $y^{-1}, x^{-1}, y^{-1}x^{-1} \in A$ and hence $(y^{-1}, x^{-1}) = P(A)$. Let $(x,y) \in Q(A)$, then $x,y \in A'$ and $xy \in A$. Hence by Lemma 1.1, $y^{-1}, x^{-1} \in A'$ and $y^{-1}x^{-1} \in A$ and thus $(y^{-1}, x^{-1}) \in Q(A)$. Similarly $(x,y) \in R(A) \Rightarrow (y^{-1}, x^{-1}) \in R(A)$.

Section 2.

In this section, we generalize the concept of a table of a subgroup so that it includes the above three tables as particular examples. The following definition of a semigroup table of a non-empty subset A of a group G is motivated by Lemmas 1.2 and 1.3 above.

Definition.

A non-empty subset A of a group G is said to have a non-empty subset S(A) of $G \times G$ as its semigroup table if

1) $\alpha(S(A)) = A$,

2) $a \in A$, $(x,y) \in S(A)$ imply $(x, ya) \in S(A)$,

3) $(x,y) \in S(A)$ implies $(y^{-1}, x^{-1}) \in S(A)$.

Note 1. Using 1) and 3), it is easy to see that 2) is also equivalent to

2') $a \in A$, $(x,y) \in S(A)$ imply $(ax, y) \in S(A)$.

Note 2. From 2) and 2'), it is immediate that $(x_1, y_1) \in S(A)$, $(x_2,y_2) \in S(A)$ imply $(x_1,y_1) \circ (x_2,y_2) = (x_1, y_1x_2y_2) \in S(A)$ and $(x_1, y_1) \times (x_2,y_2) = (x_1y_1x_2, y_2) \in S(A)$.

Hence, if A has a semigroup table S(A), then S(A) is a sub-semigroup of $\langle G \times G, \circ \rangle$ as well as that of $\langle G \times G, \times \rangle$. In this paper we restrict ourselves to sub-semigroups of $\langle G \times G, \circ \rangle$. We finally note that each of the three tables P(A), Q(A) and R(A) is a semigroup table of A. However, every semigroup table of A need not coincide with P(A) or Q(A) or R(A) as illustrated below:

Let G be the permutation group P_3 with elements written as $m^i q^j$, where $m^2 = q^3 = 1$ and $qm = mq^2$. If A is the subgroup generated by m, then two semigroup tables for A would be

$$S_1(A) = \{(q, q^2), (mq, mq), (q, mq), (mq, q^2)\}$$
$$S_2(A) = \{(q^2, q), (mq^2, mq^2), (q^2, mq^2), (mq^2, q)\}.$$

It is easy to verify that these satisfy the conditions for S(A) and that their union is Q(A).

We now prove the following theorem:

Theorem 2.1.

A subset A of G has a semigroup table S(A), if and only if A is a subgroup of G.

To prove that the condition is necessary, let S(A) be a semigroup table of A. Let $a_1, a_2 \in A$. Since α is onto A, $a_1 = \alpha(x_1, y_1)$ for some $(x_1, y_1) \in S(A)$. By 2), $(x_1, y_1a_2) \in S(A)$ so that $x_1y_1a_2 = a_1a_2 \in A$. Further by 3), if $(x,y) \in S(A)$ so that $xy = a \in A$, then $(y^{-1}, x^{-1}) \in S(A)$ and $y^{-1}x^{-1} = a^{-1} \in A$. It follows that A is a subgroup of G.

Conversely it was shown in note 2 above that every subgroup has a semigroup table.

Theorem 2.2.

(a) Let $\langle S(A), \circ \rangle$ be any semigroup table of a subgroup A. Then the mapping $(x,y) \xrightarrow{\alpha} xy$ from $S(A)$ onto A is an epimorphism from the semigroup $\langle S(A), \circ \rangle$ onto the underlying semigroup $\langle A, \cdot \rangle$.

(b) Further the quotient set $S(A)/\ker \alpha$ of equivalence classes

$$S_a(A) = \{(x, y) \in S(A) \mid xy = a \in A\}$$

is a group under the induced multiplication $S_a(A) \circ S_b(A) = S_{ab}(A)$. Finally $S(A)/\ker \alpha$ is isomorphic to A.

(a) follows on the same lines as part (b) of Lemma 1.3.

By definition of α, the equivalence classes of the quotient set $S(A)/\mathrm{Ker}\ \alpha$ are determined by

$$S_a(A) = \{(x,y) \in S(A) \mid xy = a \in A\}.$$

Since $S_a(A) \longleftrightarrow a$ is a one-to-one correspondence between the quotient set $S(A)/\ker \alpha$ and the group A, it follows immediately that $S(A)/\ker \alpha$ is a group under the multiplication $S_a(A) \circ S_b(A) = S_{ab}(A)$ and is isomorphic to A.

The above theorem is a generalization of Theorem 2.1 in [1].

Corollary 2.3.

Each of the quotient sets $P(A)/\ker \alpha$, $Q(A)/\ker \alpha$ and $R(A)/\ker \alpha$ arising respectively from the epimorphism α of $\langle P(A), \circ \rangle$, $\langle Q(A), \circ \rangle$ $\langle R(A), \circ \rangle$ onto $\langle A, \cdot \rangle$, is a group isomorphic to the subgroup A.

Section 3.

In this section we establish an isomorphic embedding of the lattice of all subgroups of a group into the lattice of all sub-semigroups of $\langle G \times G, \circ \rangle$. To this end, we need the following concept and the subsequent Lemmas.

If $S(C)$ is any arbitrary semigroup table of a subgroup C and A is any subgroup of C, then let $\overline{S}(A, C)$ be the subset of elements of $S(C)$ which are mapped by α onto the elements of A or explicitly

$$\overline{S}(A, C) = \{(x,y) \mid (x,y) \in S(C), \alpha(x,y) = xy \in A\} .$$

In fact if α_C be the restriction of α to C, then $\overline{S}(A,C) = \alpha_C^{-1}(A)$.

Lemma 3.1.

$\overline{S}(A, C)$ is a sub-semigroup of $S(C)$ and in fact is a semigroup table of A.

Since $\overline{S}(A,C) = \alpha_C^{-1}(A)$, it is the inverse image of a sub-semigroup A of C, and hence is a sub-semigroup of $S(C)$. Further it is easily verified that $\overline{S}(A, C)$ satisfies the three conditions of being a semigroup table of A.

Lemma 3.2.

If A, B are subgroups of G and $S(C)$ is any semigroup table of $C = A \vee B$, then

$$\text{(i)} \quad S(C) = \overline{S}(A, C) \vee \overline{S}(B, C)$$

$$\text{(ii)} \quad \overline{S}(A \cap B, C) = \overline{S}(A, C) \cap \overline{S}(B, C) .$$

Part (ii) follows immediately from the fact that $\alpha_C^{-1}(A \cap B) = \alpha_C^{-1}(A) \cap \alpha_C^{-1}(B)$. For part (i), it follows from Lemma 3.1 that $S(C) \supseteq \overline{S}(A,C) \vee \overline{S}(B, C)$. To prove inclusion in the other order let $(x,y) \in S(C)$. Then $xy \in C$ so there exists some elements $g_1, \ldots, g_r \in G$ such that $xy = g_1 g_2 \cdots g_r$ and that either $g_i \in A$ or $g_i \in B$. Since $xy \in C$, $y^{-1}x^{-1} \in C$ so that by conditions 2) and 2') in the definition of a semigroup table, we have

$$(x, x^{-1}) = (x, y(y^{-1}x^{-1})) \in S(C)$$

$$(y^{-1}, y) = (y^{-1}, x^{-1}xy) \in S(C)$$

Applying again 2) and 2') , the pairs

$$(x, x^{-1}g_1) , (g_2 y^{-1}, y) , (g_3 y^{-1}, y) , \ldots, (g_r y^{-1}, y)$$

belong to $S(C)$. It is easy to see that in fact these pairs belong either to $\overline{S}(A, C)$ or to $\overline{S}(B, C)$. Then by definition of $\overline{S}(A, C) \vee \overline{S}(B, C)$, the product

$$(x, x^{-1}g_1) \circ (g_2y^{-1}, y) \circ (g_3y^{-1}, y) \circ \ldots \circ (g_ry^{-1}, y)$$

$$= (x, x^{-1}g_1 \cdot g_2 \cdot \ldots \cdot g_r) = (x, x^{-1}xy) = (x, y)$$

$$\in \overline{S}(A, C) \vee \overline{S}(B, C) .$$

This completes the proof.

Lemma 3.3.

For any arbitrary subgroups A, B and $C = A \vee B$ of a group G,

(i) $R(C) = \overline{R}(A,C) \vee \overline{R}(B,C) = R(A) \vee R(B)$

(ii) $R(A \cap B) = R(A) \cap R(B)$.

The first equality in (i) follows from Lemma 3.2 (i). For the second equality we have by definition that

$$R(C) = \{(x,y) \mid x, y \in G, xy \in C = A \vee B\}$$

and

$$\overline{R}(A,C) = \{(x,y) \mid (x,y) \in R(C), xy \in A\}$$

$$= \{(x,y) \mid x, y \in G, xy \in A\}$$

$$= R(A) .$$

Part (ii) follows from this result and Lemma 3.2 (ii).

We now prove

Theorem 3.4.

The lattice M of all subgroups of a group G can be isomorphically embedded in the lattice L of all sub-semigroups of the semigroup $\langle G \times G, \circ \rangle$.

Consider the mapping

$$A \xrightarrow{\Psi} R(A) \quad \text{where } A \in M \text{ and } R(A) \in L .$$

By definition of R(A), if $A \neq B$, $R(A) \neq R(B)$ and if $A \subseteq B$, then

$R(A) \subseteq R(B)$. Further, we have

$$A \vee B \overset{\Psi}{\rightarrow} R(A \vee B) = R(A) \vee R(B) ,$$

$$A \cap B \overset{\Psi}{\rightarrow} R(A \cap B) = R(A) \cap R(B) \qquad \text{by Lemma 3.3.}$$

This completes the proof.

Theorem 3.5.

Any arbitrary lattice can be isomorphically embedded in the lattice L of all sub-semigroups of $\langle G \times G, o \rangle$ for a suitable group G.

By Whitman's Theorem [3] every lattice can be isomorphically embedded in the lattice of all equivalence relations on a suitable set. By Birkhoff's Theorem [1], the lattice of all equivalence relations defined on an arbitrarily given set can be isomorphically embedded in the lattice of all subgroups of a suitable group G. But the lattice of all subgroups of any group G can be isomorphically embedded in the lattice L of all sub-semigroups of the semigroup $\langle G \times G, o \rangle$ by our Theorem 3.4 above. This proves the assertation.

Some of the properties of R of Lemma 3.3 are not shared by P or Q. The final theorem indicates this difference.

Theorem 3.6.

For any arbitrary subgroup A and B of a group G

(i) $P(A \vee B) = P(A) \vee P(B)$ if and only if $A \subseteq B$ or $B \subseteq A$.

(ii) $Q(A \vee B) = Q(A) \vee Q(B)$ if and only if $A = B$.

(iii) $\overline{Q}(A, A \vee B) \subseteq Q(A)$

(iv) $\overline{P}(A, A \vee B) \supseteq P(A)$

(v) $Q(A \cap B) = [Q(A) \cap Q(B)] \cup [Q(A) \cap P(B)] \cup [P(A) \cap Q(B)]$

To prove (i) note that the first components of pairs in $P(A) \vee P(B)$ are in $A \cup B$ while for $P(A \vee B)$, they are in $A \vee B$. In both cases the second components are in $A \vee B$. Thus, the equality can occur if and only if $A \cup B = A \vee B$. This in turn is equivalent to having $A \subseteq B$ or $B \subseteq A$.

For part (ii) the first components of elements in $Q(A \vee B)$ are in $(A \vee B)'$ while for $Q(A) \vee Q(B)$, they are in $A' \cup B' = (A \cap B)'$. Hence equality occurs if and only if $A \cap B = A \vee B$. But this is equivalent to $A = B$.

For part (iii) note that by definition

$$\begin{aligned}\bar{Q}(A, A \vee B) &\equiv \{(x,y) \mid x,y \in Q(A \vee B), xy \in A\}\\ &\equiv \{(x,y) \mid x,y \in (A \vee B)', xy \in A\}\\ &\subseteq \{(x,y) \mid x,y \in A', xy \in A\}\\ &\equiv Q(A) .\end{aligned}$$

The last two parts follow directly from the definitions of P and Q and the identity $(A \cap B)' = A' \cup B'$.

The authors are grateful to Dr. M. P. Grillet for her comments on an earlier draft of this paper.

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SPLITTING ALGEBRAS AND A WEAK NOTION OF PROJECTIVITY

by

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1. Introduction:

The classical results in Lattice Theory by Dedekind and Brikhoff that a lattice is modular (distributive) if and only if it does not contain the pentagon, N_5, (resp. N_5 and the 3-diamond, M_3) as a sublattice have been generalized by McKenzie in [13] to the notion of a splitting algebra. That is: a finite subdirectly irreducible algebra is splitting in a variety (= equational class) if there is a largest subvariety of this variety not containing it. In [3], McKenzie characterized the splitting lattices as the bounded homomorphic images of finitely generated free lattices. In [9], Jónsson showed that $M_{3,3}$ is a splitting modular lattice.

As McKenzie noted, his results do not supply necessary and sufficient conditions for a splitting algebra in proper subvarieties of lattices. In this paper, we develop a weak notion of projectivity for a finite algebra in a variety and show that given reasonable restrictions on the variety, every finite subdirectly irreducible satisfying this weak projectivity conditions is a splitting algebra. The reasonable restrictions alluded to are congruence distributivity. Therefore all of the usual lattice-like

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varieties are included (e.g. Lattices, Heyting algebras, Pseudo complemented lattices, Implication semilattices, and Hilbert Algebras).

After developing the general theory, we provide examples in the above varieties and in the last section describe a large class of splitting modular lattices.

We wish to thank Professor R. Wille for his many valuable comments which led to this revised version of these results.

2. Preliminaries

Most of the relevant definitions and results in universal algebra can be found in Grätzer [5]; in lattice theory, Szasz [14] and McKenzie [13].

Let K be a variety of algebras. We will consider (as is usual) K as a category whose maps are all K-homomorphisms. For A and B in K, a surfjective map $f : A \twoheadrightarrow B$ is called a cover (with respect to surjective maps) if for all $g : C \to A$ in K g is surjective if $f \cdot g$ is. Equivalently, $f : A \twoheadrightarrow B$ is a cover if A is the only subalgebra of A whose image under f is B.

$P \in K$ is called projective (with respect to surjective maps) if for any surjective $g : A \twoheadrightarrow B$ and any $f : P \to B$, there exists a lifting $\bar{f} : P \to A$ with $g \cdot \bar{f} = f$. It is well known (or easily seen) that any variety has enough projectives (i.e. every algebra is the homomorphic image of a projective) and that an algebra in K is projective if and only if it is a retract of a K-free algebra.

A cover $f : A \twoheadrightarrow B$ is called a projective (finite) cover according to whether A is projective or finite respectively. If an algebra B in K has a projective cover, then this cover is essentially unique. The general theory of projective covers in an arbitrary category can be found in Banaschewski [2].

We will use the following notations

$A \leq B$: A is a subalgebra of B

$A \xrightarrow{f} B$: f is an injective homomorphism

$A \overset{f}{\twoheadrightarrow} B$: f is a surjective homomorphism.

Also since the precise operations of the algebras considered will play no role, we use upper case Latin letter instead of upper case German letters.

3. Finitely Projective Algebras

Let K be a variety of algebras. An algebra A in K is called finitely projected if for any surjective $f : B \twoheadrightarrow A$ in K, there is a finite subalgebra of B whose image under f is A. Thus a finitely projected algebra is necessarily finite and clearly every homomorphic image of a finite projective algebra in K is finitely projected.

(3.1) Lemma. Let $A \in K$ be finitely projected. Then for any

$B \varepsilon K$ and surjective $f : B \twoheadrightarrow A$ there exists a finite subalgebra $C \leq B$ with $f|C : C \twoheadrightarrow A$ a cover. Moreover if B is projective, so is C.

Proof: Given $f : B \to A$, there is a finite subalgebra $D \leq B$ with $f[D] = A$ since A is finitely projected. Since D is finite, D has only finitely many subalgebras. Therefore we can take C to be minimal in the set $\{E \leq D : f[E] = A\}$.

If B is projective there exists $g : B \quad C$ with $(f|C)\cdot g = f$ therefore $(f|C)\cdot(g|C) = ((f|C)\cdot g)|C = (f|C)$.

Since $f|C$ is a cover, $g|C$ is surjective and since C is finite, $g|C$ is bijective, hence an isomorphism. Therefore C, as a retract of a projective is projective.

Let us note that this lemma shows that the concept of being finitely projected has no content when K is the variety of all groups or all Abelian groups as free (abelian) groups have no subgroups of finite order save the trivial one.

It also gives the following characterization of finitely projected algebras.

(3.2) Theorem: Let K be a variety of algebras; then for any A in K, t.f.a.e.:

(1) A is finitely projected

(2) A has a finite projective cover

(3) A is the homomorphic image of a finite projective algebra in K.

(4) For all projectives P in K and all surjectives $f : P \twoheadrightarrow A$, there exists a finite subalgebra $Q \leq P$ with $f[Q] = A$.

(3.3) Corollary: Every (finite) cover of a finitely projected algebra is finitely projected.

(3.4) Corollary: Every homomorphic image of a finitely projected algebra is finitely projected.

Examples of finitely projected algebras in different varieties will appear in subsequent sections. Our main concern now will be with subdirectly irreducible finitely projected algebras and the role of their projective covers.

(3.5) Lemma: Let P be a projective algebra in a variety K and let $P \overset{\mu}{\rightarrowtail} F_K(X) \overset{\rho}{\twoheadrightarrow} P$ be any retract. Then for any $a,b \in P$ and $\nu : P \to F_K(X)$, $(\nu(a),\gamma(b))$ is in the fully-invariant congruence relation on $F_K(X)$ generated by $(\mu(a),\mu(b))$.

Proof: Consider the endomorphism $\nu\cdot\rho$ of $F_K(X)$. $(\nu\cdot\rho)(\mu(p)) = \nu((\rho\cdot\mu)(p)) = \nu(\dot{p})$ for all $p \in P$. Therefore the statement of the lemma holds.

Although the above lemma is extremely trivial, it has many interesting applications in the determination of conjugate equations for splitting algebras as we shall presently see. Of immediate consequence is the following generalization of Wille [15; Corollary 10].

(3.6) Lemma: Let S be a subdirectly irreducible in a variety K whose least congruence is generated by (u,v). Furthermore assume there is a surjection $f : P \twoheadrightarrow S$ with $a,b \in P$ satisfying

(1) $f(a) = u$ and $f(b) = v$

(2) $\theta_P(a,b)$ is strictly-join-prime.

Then S is a splitting algebra in K. Moreover if $P \xrightarrow{\mu} F_K(X) \overset{\rho}{\twoheadrightarrow} P$ is any retraction, $(\mu(a),\mu(b))$ determine the conjugate equation.

Proof: Take $F_K(X) \overset{\rho}{\twoheadrightarrow} P$ for a suitable set X. Since P is projective, there is indeed a $\mu : P \to F_K(X)$ with $\rho \cdot \mu = 1_P$. We show that S is a splitting algebra by demonstrating that $(\mu(a),\mu(b))$ is indeed its splitting equation.

Since $\theta_P(a,b)$ is strictly-join-prime, and S is subdirectly irreducible, Ker f is strictly-meet-prime and for all congruences θ on P, either $\theta \subseteq \text{Ker } f$ or $\theta_P(a,b) \subseteq \theta$.

Let V be the subvariety of K given by the equation $(\mu(a),\mu(b))$ with $\kappa : F_K(X) \twoheadrightarrow F_V(X)$ the canonical homomorphism. That is, Ker κ is the fully invariant congruence generated by $(\mu(a),\mu(b))$. If S were in V then as $f \cdot \rho : F_K(X) \to S$ is surjective, there would be a surjective morphism $h : F_V(X) \twoheadrightarrow S$. As P is projective in K, there exists $\nu : P \to F_K(X)$ with $f = (h \cdot \kappa) \cdot \nu$. Now $(a,b) \notin \text{Ker } f = \text{Ker}(h \cdot \kappa \cdot \nu)$. But by (3.5) $(\nu(a),\nu(b)) \in \text{Ker } \kappa$ whence $f(a) = h(\kappa(\nu(a))) = h(\kappa(\nu(b))) = f(b)$, a contradiction.

Now if $\mathcal{L}$ is a subvariety of K not containing S and

$\lambda : F_K(X) \twoheadrightarrow F_{\mathcal{L}}(x)$ is the cononical surjection, then if $(\mu(a),\mu(b)) \notin \text{Ker}\,\lambda$ we have $\text{Ker}(\lambda\cdot\mu) \subseteq \text{Ker}\, f$. Therefore by the homomorphism theorem we have: $S \in \underset{\sim}{H}\underset{\sim}{S}(F_{\mathcal{L}}(X))$, a contradiction. Therefore $(\mu(a),\mu(b)) \in \text{Ker}\,\lambda$.

These last two lemmas give us the connection between finitely projected subdirectly irreducibles and splitting algebras in congruence distributive varieties.

(3.7) Theorem: In a congruence distributive variety, every finitely projected subdirectly irreducible algebra is splitting.

Proof: Let $f : P \twoheadrightarrow S$ be the finite projective cover of the subdirectly irreducible S in a congruence distributive variety K. Since S and P are finite, and the congruence lattice of P is distributive, Ker f is (strictly)-meet-prime. Therefore there exists a smallest congruence on P not contained in Ker f which is (strictly)-join-prime and hence principal. That is, there exists $a,b \in P$ such that for all $\theta \in \Theta(P)$, $\theta \subseteq \text{Ker}\, f$ or $\theta_P(a,b) \subseteq \theta$. It is trivial to see that the pair $(f(a),f(b))$ generates the least congruence on S. Therefore (3.6) applies and S is splitting.

(3.8) Corollary: Let K be a congruence distributive variety in which the finitely generated algebras are finite. Then every finite subdirectly irreducible in K is finitely projected hence a splitting algebra. Moreover, the lattice of subvarieties is infinitely distributive.

Proof: The first statement is immediate from (3.2) and the fact that finitely generated K-free algebras are finite.

The second part comes from the first since every subvariety will be generated by its finite subdirectly irreducible members and the variety (or theory) generated by one of these is strictly-join-prime (resp. strictly-meet prime) (see [13] for terminology).

Before proceeding to examples, let us note that if the finite projective cover of a finitely projected subdirectly irreducible can be constructed in a suitable finitely generated free algebra by some algorithmic methods, we can determine a conjugate equation by inspection. This procedure perhaps could be more easily applied than McKenzie's limit tables.

4. Examples:

(A) Heyting Algebras

A Heyting algebra is bounded relatively pseudo-complemented lattice in which relative-pseudo-complementation is taken as an operation. Balbes and Horn, [1], have sufficient algebraic details for what we need.

By Jankov [7], every finite subdirectly irreducible Heyting algebra is a splitting algebra. However from [1], the finite projective Heyting algebras are precisely the finite horizontal sums (see [1] for terminology) of $\underset{\sim}{2}$ and B_2, the four element Boolean algebra, with a copy of $\underset{\sim}{2}$ on top. Homomorphic images of these are just the finite horizontal sums of $\underset{\sim}{2}$ and B and therefore the subdirectly irreducible homomorphic images (1 is join-irreducible) are precisely the projective Heyting algebras again. We have shown:

(4.1) Theorem: The finitely projected Heyting algebras are precisely the finite horizontal sums of copies of $\underset{\sim}{2}$ and B_2. A finite Heyting algebra is projective if and only if it is a finitely-projected subdirectly irreducible.

(B) Implication semi-lattices, Hilbert algebras and Distributive pseduo-complemented lattices

In each of these three varieties, the finitely generated algebras are finite. (See [12], [4] and [11] respectively.) Therefore the finitely projected algebras are precisely the finite ones, every finite subdirectly irreducible is splitting and the lattices of subvarieties are infinitely distributive.

(C) Lattices

While we have no characterization of the finitely projected lattices other than (3.2), we do have the fact that the subdirectly irreducible finitely projected lattices are a proper subclass of the splitting lattices.

(4.2) Theorem: Let L be a finite lattice that has a generating set X of more than two elements which satisfies:

$(*) \emptyset \neq Y, Z \subseteq X$ and $\bigwedge Y \leq \bigvee Z$ imply $Y \cap Z \neq \emptyset$

then L is not finitely projected.

Proof: Take $\phi : FL(X) \twoheadrightarrow L$ extending the identity function on the generators. Then for any subset, $\overline{X}$, of $FL(X)$ given by a choice function on $\Pi(\phi^{-1}[x] : x \in X)$, $\overline{X}$ also satisfies (*). By Jónsson [10: lemma 3], the sublattice of $FL(X)$ generated by $\overline{X}$ satisfies Whitman's first three conditions and since it must also satisfy the fourth, it is isomorphic to $FL(X)$ which is infinite. By (3.1) then, L is not finitely projected.

(4.3) Corollary: There is a splitting lattice which is not finitely projected.

Proof: The lattice Q in diagram (i) is splitting from [13] but its generating set $\{a,b,c\}$ satisfies (*).

5. Finitely Projected Modular Lattices

Let M be the variety of modular lattices. We wish to construct a large class of finitely projected subdirectly irreducible (hence splitting) modular lattices.

By $D(u < a,b,c, < v)$, we mean a <u>non-degenerate</u> 3-diamond as in diagram (ii). T_n $(n \geq 1)$ is the modular lattice $\bigcup_i^{1,n} D(u_i < a_i,b_i,c_i < v_i)$ with $v_i = a_{i+1}$ and $c_i = u_{i+1}$ for each $i = 1,\ldots,n-1$. P_n $(n \geq 1)$ is the modular lattice given by the disjoint union $\bigcup_i^{1,n} D(\bar{u}_i < \bar{a}_i,\bar{b}_i,\bar{c}_i < \bar{v}_i)$ with $\bar{v}_i \wedge \bar{u}_{i+1} = \bar{c}_i$

and $\bar{v}_i \vee \bar{u}_{i+1} = \bar{a}_{i+1}$ for each $i = 1,2,\ldots,n-1$. Thus P_n is obtained from T_n by pulling apart all coincident diamond edges; and there exists a unique surjection $f_n : P_n \to T_n$ by collapsing these pulled-apart edges. (See diagram (iii).)

(5.1) Theorem: For every $n \geq 1$, T_n is a finitely projected simple modular lattice with $f_n : P_n \to T_n$ its projective cover.

(5.2) Corollary: Every T_n, $n \geq 1$, is a splitting modular lattice.

Before proving this theorem we should note that the corollary for $n = 2$ was shown in Jónsson [9]. It must be noted however that Jónsson's result is stronger in that he explicitly described the splitting variety by describing its subdirectly irreducible members. This does not seem to be an easy task for $n \geq 4$, however Hong [6] has some interesting partial results.

Also, explicit descriptions of P_n as a sublattice of FM(n+2) can be obtained via the method of proof and therefore conjugate equations (see [13]) can be obtained.

Proof: Let $S(n)$ be the statement "For any surjective map $g : A \twoheadrightarrow T_n$, there exists a sublattice $C_n \leq A$ with $C_n = \bigcup_i^{1,n} D(p_i < r_i, s_i, t_i < q_i)$ with $p_{i+1} \wedge q_i = t_i$ and $p_{i+1} \vee q_i = r_{i+1}$ for $i = 1,\ldots,n-1$ such that $g(p_i) = u_i$, $g(r_i) = a_i$, $g(s_i) = b_i$, $g(t_i) = c_i$ and $g(q_i) = v_i$ for $i = 1,\ldots,n$."

$S(1)$ is trivially true as $T_1 = M_3 = P_1$ is a finite projective modular lattice. Therefore assume $S(n)$ for $n \geq 1$ and consider a surjective map $g : A \twoheadrightarrow T_{n+1}$.

Since $T_n \leq T_{n+1}$ by considering only the first n diamonds, we have by inductive assumption a sublattice $C_n \leq g^{-1}[T_n]$ satisfying the conditions of $S(n)$. Moreover since g is surjective and $S(1)$ holds, there is a diamond $D_{n+1} = D(p_{n+1} < r_{n+1}, s_{n+1}, t_{n+1} < q_{n+1}) \leq [t_n, \rightarrow)_A$, the sublattice of all elements of A greater than or equal to t_n which is mapped isomorphically onto the $(n+1)^{th}$ diamond of T_{n+1}. We will show that our desired C_{n+1} is a sublattice of the sublattice of A generated by $C_n \cup D_{n+1}$.

Let $z_n = q_n \wedge r_{n+1}$ and $w_n = q_n \wedge p_{n+1}$. Then we have $t_n \leq w_n < z_n \leq q_n$ with the strict in equality holding between w_n and z_n since $g(w_n) = c_n < v_n = g(z_n)$. By Hong's extension ([6; sec. 3.2]) of a result of Jónsson, the sublattice of A generated by $C_n \cup \{z_n, w_n\}$ is of the form

$$\bigcup_i^{1,w} E(p_i, q_i)$$

where each $E(p_i, q_i)$ is a homomorphic image of Q, the lattice in diagram (iv) with the edges $[t_i, q_i]$ and $[p_{i+1}, r_{i+1}]$ transposed and at most $z = q$ and $w = t$. This give a sublattice $\bar{C}_n$ of A satisfying the statement $S(n)$ where $\bar{C}_n = \bigcup_i^{1,n} D(\bar{p}_i < \bar{r}_i, \bar{s}_i, \bar{t}_i < \bar{q}_i)$.

Now consider

$$x_{n+1} = w_n \vee p_{n+1} = \bar{t}_n \vee p_{n+1}$$

$$y_{n+1} = z_n \vee p_{n+1} = \bar{q}_n \vee p_{n+1}$$

Again we have $p_{n+1} \leq x_{n+1} < y_{n+1} \leq r_{n+1}$ and the sublattice of A generated by $D_{n+1} \cup \{x_{n+1}, y_{n+1}\}$ is as described above. It follows easily that $C_{n+1} = \bigcup_{i}^{1,n+1} D(\bar{p}_i < \bar{r}_i, \bar{s}_i, \bar{t}_i < \bar{q}_i)$ satisfies the conditions of the statement $S(n+1)$.

Let us note that if S_n $(n\geq 1)$ is defined by a "snake" of n diamonds and Q_n $(n\geq 1)$ is defined analogously to P_n (see diagram (v)), then the proof is completely analogous and we have:

(5.3) Theorem: For every $n \geq 1$, S_n is a finitely projected simple modular lattice with projective cover $h_n : Q_n \twoheadrightarrow S_n$.

(5.4) Corollary: Every S_n, $n \geq 1$, is a splitting modular lattice.

We should note at this time the existence of non finitely-projected modular lattices. From [3], M_4 is not a splitting modular lattice and therefore is not finitely projected.

Our class of finitely projected modular lattices can be enlarged greatly by a slightly different procedure.

(5.5) Lemma: Every modular lattice that is the subdirect product of two finite chains is finitely projected.

Proof: This is an immediate consequence of the fact that the free modular lattice generated by two chains is both finite and projective.

Now suppose we judiciously insert elements in such a finite lattice to make some of the B_2 boxes into diamonds (see diagram (vi)). It is clear by inspection (and easy to prove) that A and B are finitely projected by firstly pulling back the lattice without the diamond points and then inserting these one at a time. This procedure does not seem to work for C for, having pulled back three diamond points of C along $f : L \twoheadrightarrow C$, we have as a sublattice of L at lattice whose isomorphism type is determined by D in diagram (vii). However in attempting to insert an inverse image point of the last diamond point in the proper B_2 box, we seem to generate a ring around the rosy system of elements on all the other diamond edges, with no conviction as to whether this procedure stops. We conjecture however that given a subdirect product of two finite chains if diamond points are inserted such that there is no sequence of transposes starting at one edge of a diamond and returning to another edge of this diamond <u>without</u> having had to transpose through this diamond then such a modular lattice is finitely projected.

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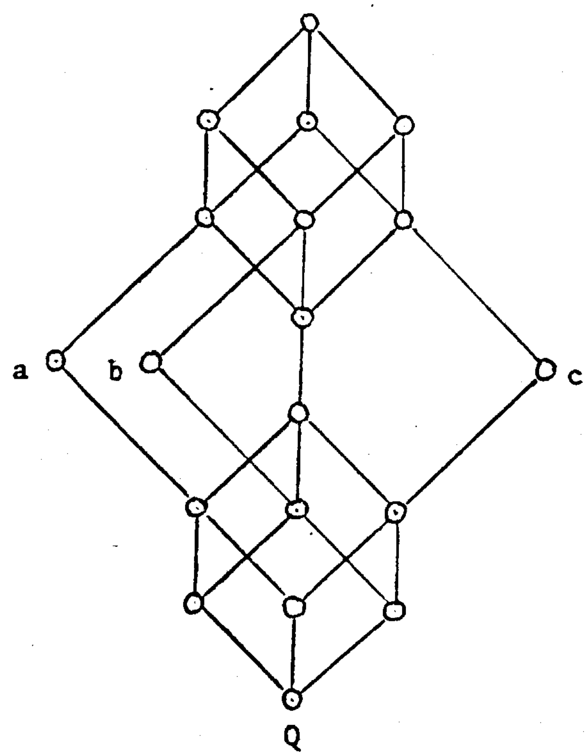
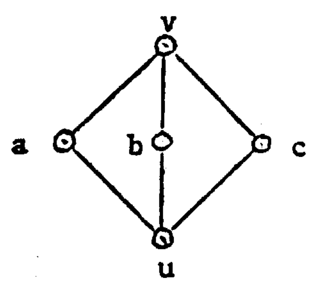
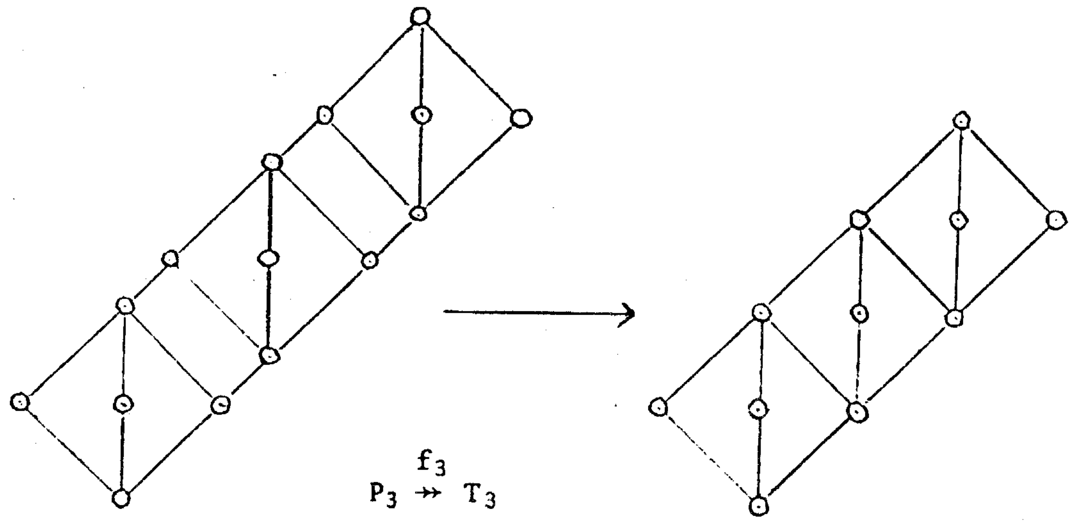


(i)

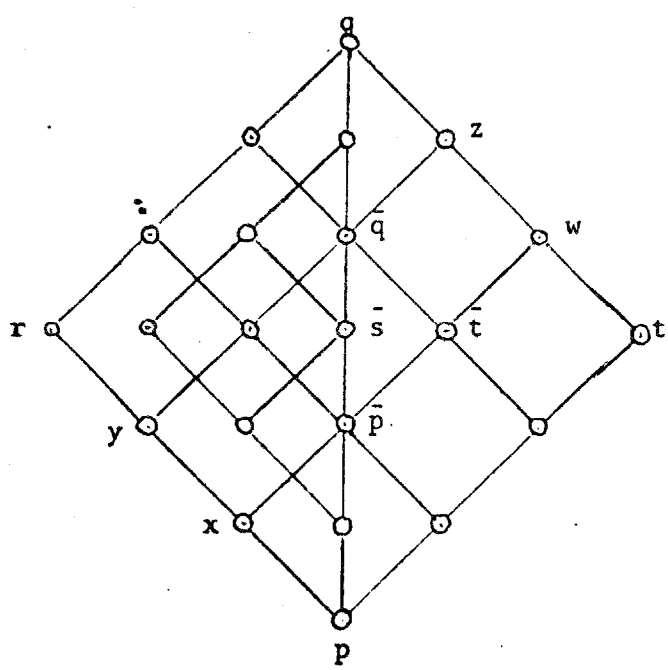


D(u < a,b,c < v)

(ii)


$P_3 \overset{f_3}{\twoheadrightarrow} T_3$

(iii)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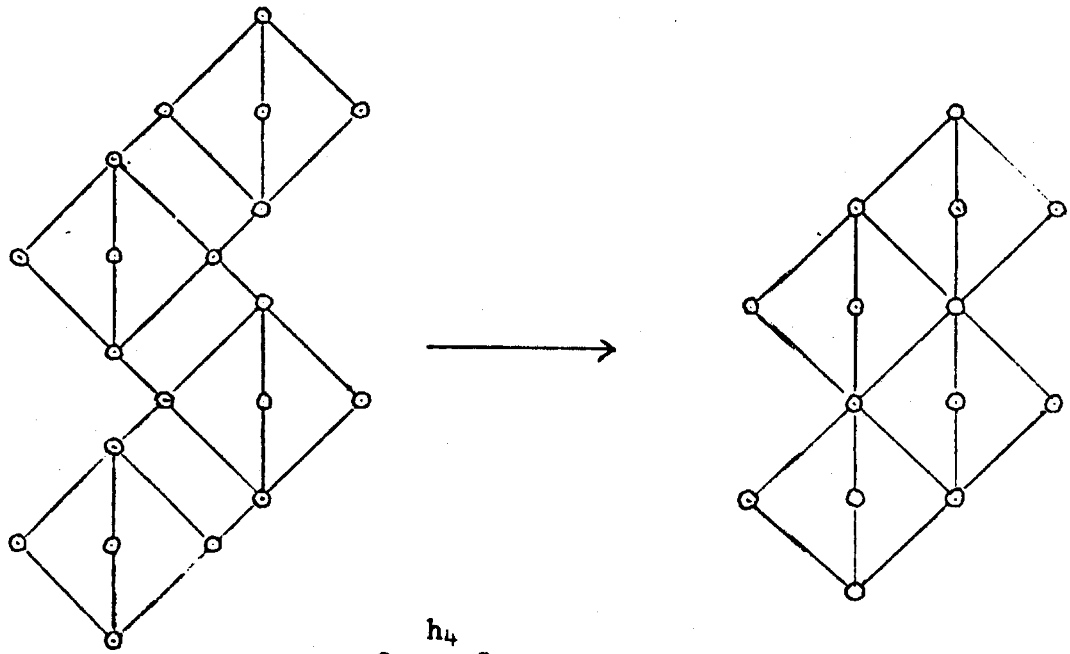
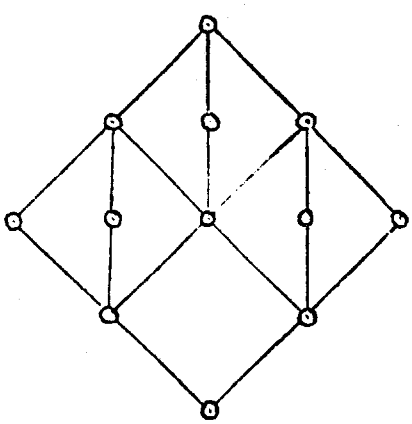
(iv)



$Q_4 \overset{h_4}{\twoheadrightarrow} S_4$

(v)



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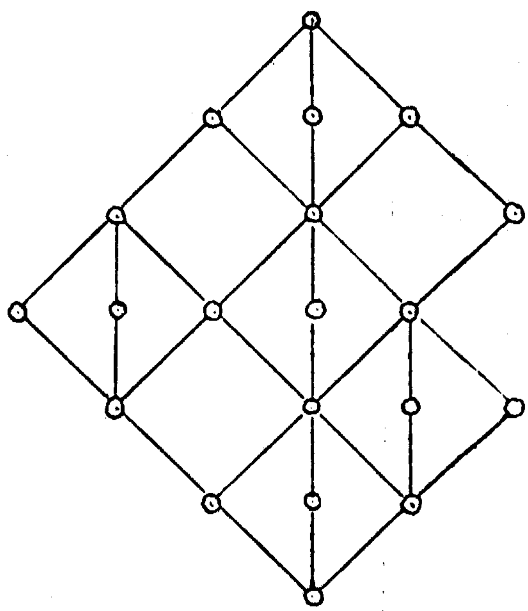

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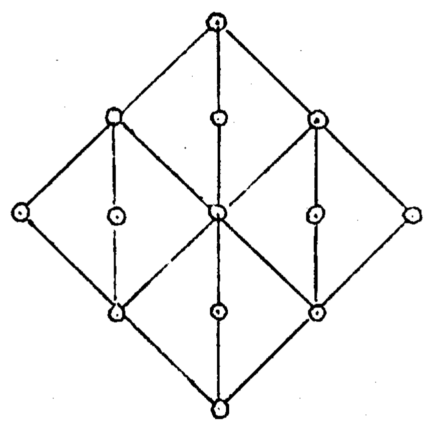

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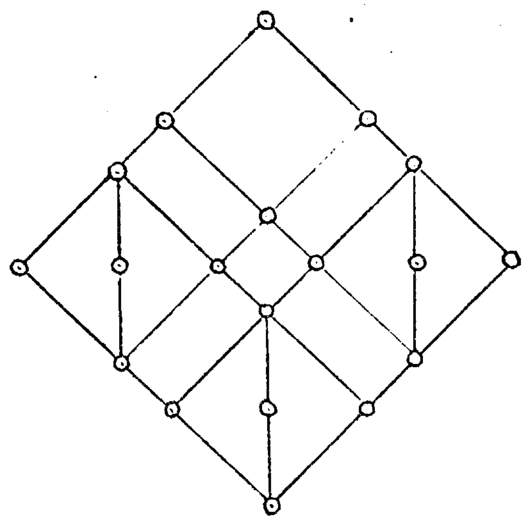

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Equational classes of Foulis semigroups and orthomodular lattices

Donald H. Adams

I. Introduction. This paper follows on from our work exhibiting Baer semigroups as an equational class and investigating the connection between equational classes of lattices and equational classes of Baer semigroups [1]. Here we consider the particular case of Foulis semigroups and orthomodular lattices and, as is usual when one particularizes, we are rewarded with more specific results. This paper does not depend on [1] and may be read separately. However, a fair knowledge of Foulis semigroups and orthomodular lattices is assumed and we refer to Blyth and Janowitz [2] for the basic theory and further references.

In this paper we show that Foulis semigroups from an equational class when they are regarded as algebras of type $<2,1,1,0>$ where the two unary operations are the involution and the focal map. We show that the class of Foulis semigroups coordinatizing the members of an equational class of orthomodular lattices is equational. Conversely, the class of orthomodular lattices coordinatized by the members of an equational class of Foulis semigroups is also equational. We exhibit a homomorphism from the lattice of equational classes of Foulis semigroups onto the lattice of equational classes of orthomodular lattices.

We generally follow the conventions of Grätzer [4], except that we rarely bother to distinguish between an algebra and its base set. We are ambidextrous in the way we write maps: homomorphisms are on the left and residuated maps on the right. We feel that this tends to clarify rather than confuse the situation. We skip over foundational difficulties, especially when dealing with the lattice of equational classes, because all these can be handled by standard tricks - see Grätzer [4] for details.

2. Foulis semigroups. We follow Blyth and Janowitz [2] in using the term *Foulis semigroup* for what was originally called a *Baer *-semigroup* by Foulis [3]. We refer the reader to these two sources for the proofs of any assertions that we leave unproven.

Definition 1. A *Foulis semigroup* is an algebra $\langle F;\cdot,{*},{'},0\rangle$ of type $\langle 2,1,1,0\rangle$ such that

(i) $\langle F;\cdot,0\rangle$ is a semigroup with zero;

(ii) $*$ is an *involution*, i.e. for any $x,y \in F$, $x^{**} = x$ and $(xy)^* = y^*x^*$. If $x = x^*$ then x is *self-adjoint*.

(iii) for each $x \in F$ the element x' is a self-adjoint idempotent or *projection*.

(iv) for each $x \in F$

$$r(x) = \{y \mid xy = 0\} = x'F.$$

In other words, the right annihilator of x is a principal right ideal generated by a projection.

The elements of F of the form x' are called <u>closed projections</u>. The map defined by $x \to x'$ is called the <u>focal map</u>, $1 = 0'$ is an identity in F and $1' = 0$.

<u>Proposition 2</u>. <u>Condition</u> (iv) <u>of Definition 1 is equivalent to</u>

$$\text{(iv)}' \quad x'y(xy)' = y(xy)' \text{ for all } x,y \in F.$$

<u>Proof</u>. Since x' is idempotent, (iv) is equivalent to:

$$xy = 0 \text{ if and only if } x'y = y.$$

Suppose that (iv) holds. Then

$$(xy)'F = \{t \mid xyt = 0\}$$
$$= \{t \mid x'yt = yt\}$$

and since $(xy)' \in (xy)'F$, we get $x'y(xy)' = y(xy)'$.

If, on the other hand, (iv)' holds, then substituting $x = 1$ in the formula gives us the identity $yy' = 0$. Hence, if $x'y = y$, then $xy = xx'y = 0$. If $xy = 0$, then substitution in the formula gives us $x'y = y$. We have shown that $xy = 0$ if and only if $x'y = y$ and so (iv) holds.

Notice that (iv) was the only one of the defining properties of a Foulis semigroup that could not readily be expressed as an identity. Proposition 2 shows us that it can be so expressed and we have the following result.

<u>Corollary 3</u>. <u>The class of all Foulis semigroups is equational</u>.

Foulis semigroups are of interest mainly because, if F is a Foulis semigroup, then the set of closed projections in F, $P'(F)$, is an orthomodular lattice where the operations are given by the formulas:

(1) $e \wedge f = (f'e)'e;$

(2) $e^{\perp} = e' \quad ;$

(3) $0 = 0 \quad .$

We denote the equational class of Foulis semigroups by $\mathcal{B}$ and the equational class of orthomodular lattices by $\mathcal{L}$. From the formulas (1) - (3) the following is clear.

Proposition 4. (i) *If $h: F_1 \to F_2$ is a Foulis semigroup homomorphism, then $h': P'(F_1) \to P'(F_2)$, the restriction of h to the closed projections, is an orthomodular lattice homomorphism. If h is onto, then h' is also onto.*

(ii) *If F_1 is a subalgebra of F_2 where F_1 and F_2 are Foulis semigroups, then $P'(F_1)$ is a subalgebra of $P'(F_2)$ as an orthomodular lattice.*

We denote the direct product of a family of algebras $(A_i | i \varepsilon I)$ by $\Pi(A_i | i \varepsilon I)$. It is straightforward to prove the following result.

Proposition 5. *Let $(F_i | i \varepsilon I)$ be a family of Foulis semigroups. Then $\Pi(P'(F_i) | i \varepsilon I)$ is isomorphic as an orthomodular lattice to $P'(\Pi(F_i | i \varepsilon I))$.*

These two propositions will give us an immediate proof of our first main result. But first we observe that the mapping given by $h \to h'$ in Proposition 4 (i) is a functor from the category of Foulis semigroups to the category of orthomodular lattices. By Proposition 5 it is product-preserving and in this situation one always gets a result of the following type.

Theorem 6. _Let $\mathcal{L}_1$ be an equational class of orthomodular lattices. Then_

$$b(\mathcal{L}_1) = \{F \mid F \in \mathcal{B},\ P'(F) \in \mathcal{L}_1\}$$

is an equational class of Foulis semigroups.

Proof. It follows immediately from Propositions 4 and 5 that $b(\mathcal{L}_1)$ is closed under the taking of homomorphic images, subalgebras and direct products.

Actually, formulas (1)-(3) give us an immediate technique for translating orthomodular lattice identities into Foulis semigroup identities. This means that, given an equational base for the equational class $\mathcal{L}_1$, we can in principle calculate an equational base for the corresponding equational class of Foulis semigroups $b(\mathcal{L}_1)$. If $\mathcal{L}_1$ is the equational class of Boolean lattices, then $b(\mathcal{L}_1)$ is the class of all Foulis semigroups satisfying the identity $x'y' = y'x'$. This follows from the standard result [2, p. 201] that $P'(F)$ is Boolean if and only if $ef = fe$ for all $e,f \in P'(F)$.

Foulis [3] showed that any orthomodular lattice is isomorphic to the lattice of closed projections of the Foulis semigroup $S(L)$ of residuated maps on L (see also [2]). The involution is given by $x\phi^* = (x^{\perp}\phi^{+})^{\perp}$ where ϕ^{+} denotes the residual of ϕ and $x \varepsilon L$. The closed projections are precisely the Sasaki projections defined by $x\phi_y = (x \vee y^{\perp}) \wedge y$ for $x, y \varepsilon L$ and the focal map is given by $\phi' = \phi_g$ where $g = (1\phi)^{\perp}$. We say that a Foulis semigroup F <u>coordinatizes</u> L if L is isomorphic to $P'(F)$.

<u>Proposition 7</u>. <u>Let</u> F <u>be a Foulis semigroup and let</u> $L = P'(F)$. <u>The map</u> $h: F \to S(L)$ <u>defined by</u> $h(x) = \phi_x$ <u>where</u> $\phi_x: P'(F) \to P'(F)$ <u>is defined by</u> $e\phi_x = (ex)''$ <u>for</u> $e \varepsilon L$ <u>is a Foulis semigroup homomorphism of</u> F <u>into</u> $S(L)$. <u>Moreover, if</u> $e, f \varepsilon L$, <u>then</u> $e\phi_f = (e \vee f^{\perp}) \wedge f$.

<u>Proof</u>. Foulis [3] proved all this except the fact that h preserves the local map. Observe that $(\phi_x)' = \phi_g$ where $g = (1\phi_x)' = (1x)''' = x'$ to get $h(x') = [h(x)]'$.

<u>3. Small Foulis semigroups</u>. If F is a Foulis semigroup and if $x \varepsilon F$ is the product of closed projections i.e. $x = e_1 e_2 \ldots e_n$ where $e_i \varepsilon P'(F)$ for $i = 1, 2, \ldots n$, then $x^* = e_n e_{n-1} \ldots e_1$ is also a product of closed projections and x' is a closed projection. Therefore F_o, the subsemigroup of F generated by $P'(F)$, is a subalgebra of F and is a Foulis semigroup coordinatizing $P'(F)$. We call a Foulis semigroup small when it is generated by its

closed projections. Clearly any Foulis semigroup F has a unique small subalgebra F_0 coordinatizing $P'(F)$. If L is an orthomodular lattice we denote by $S_0(L)$ the small Foulis semigroup of products of Sasaki projections on L.

Proposition 8. *$S_0(L)$ is a homomorphic image of any small Foulis semigroup coordinatizing L.*

Proof. The homomorphism in Proposition 7 carries closed projections onto closed projections.

We can now prove some partial converses to Proposition 4.

Proposition 9. *Let $h: L_1 \to L_2$ be an orthomodular lattice homomorphism onto L_2. There exists a unique Foulis semigroup homomorphism $k: S_0(L_1) \to S_0(L_2)$ onto $S_0(L_2)$ such that the restriction of k to the closed projections in $S_0(L_1)$ coincides with h.*

Proof. Let $\phi_1(x)$ denote the Sasaki projection on L_1 generated by $x \in L_1$ and let $\phi_2(y)$ denote the Sasaki projection on L_2 generated by $y \in L_2$. A typical element of $S_0(L_1)$ is of the form $\Pi_{i=1}^{n} \phi_1(x_i)$ and we define the map k by

$$k(\Pi_{i=1}^{n} \phi_1(x_i)) = \Pi_{i=1}^{n} \phi_2(h(x_i)).$$

Suppose that $\Pi_{i=1}^{n} \phi_1(x_i) = \Pi_{j=1}^{m} \phi_1(y_j)$ as maps on L_1. If $u \in L_2$ there exists $u \in L_1$ such that $h(v) = u$ and

$$\begin{aligned} & u\ \Pi_{i=1}^{n} \phi_2(h(x_i)) \\ &= h(v)\ \Pi_{i=1}^{n} \phi_2(h(x_i)) \\ &= h[v\ \Pi_{i=1}^{n} \phi_1(x_i)] \end{aligned}$$

since this expression is just an orthomodular lattice polynomial. This calculation shows that $\Pi_{i=1}^{n} \phi_2(h(x_i)) = \Pi_{j=1}^{m} \phi_2(h(y_j))$ and hence the map k is well-defined.

The map k is clearly a semigroup homomorphism preserving the zero and the involution. To prove that k preserves the focal map remember that $[\Pi_{i=1}^{n} \phi_1(x_i)]' = \phi_1(y)$ where $y = [1 \; \Pi_{i=1}^{n} \phi_1(x_i)]^{\perp}$. Since this expression is an orthomodular lattice polynomial it follows that $h(y) = [1 \; \Pi_{i=1}^{n} \phi_2(h(x_i))]^{\perp}$ and so we get

$$\begin{aligned} k([\Pi_{i=1}^{n} \phi_1(x_i)]') &= k(\phi_1(y)) \\ &= \phi_2(h(y)) \\ &= [k(\Pi_{i=1}^{n} \phi_1(x_i))]' \end{aligned}$$

This completes the proof.

<u>Proposition 10</u>. *Let L_1 be a subalgebra of L_2 as orthomodular lattices. There is a small Foulis semigroup F_1 coordinatizing L_1 and F_1 is a subalgebra of $S_0(L_2)$.*

<u>Proof</u>. If $x \in L_2$ let $\phi(x)$ denote the Sasaki projection on L_2 generated by x. Form the set

$$F_1 = \{\Pi_{i=1}^{n} \phi(x_i) \mid x_i \in L_1, \; n \geq 1\},$$

i.e. F_1 is the set of all products of Sasaki projections on L_2 generated by elements of L_1. F_1 is clearly a subsemigroup of $S_0(L_2)$ closed under involution and containing the zero. If $x_i \in L_1$ for $i = 1,2,\dots,n$, then

$y = 1\ \Pi_{i=1}^{n}\ \phi(x_i) \in L_1$ since it is just an orthomodular lattice polynomial. Since $[\Pi_{i=1}^{n}\ \phi(x_i)]' = \phi(y^{\perp})$ we get that F_1 is closed under the focal map and is hence a subalgebra of $S_0(L)$. Since $[\Pi_{i=1}^{n}\ \phi(x_i)]'' = \phi(y)$, it follows that the closed projections of F_1 are exactly those generated by elements of L_1 and L_1 is isomorphic to $P'(F_1)$ since the lattice operations can be expressed in terms of Foulis semigroup operations.

Note that it is, in general, not true that $S_0(L_1)$ is isomorphic to a subalgebra of $S_0(L_2)$. This is because products of Sasaki projections generated by elements of L_1 may be equal as mappings on L_1 but not when they are regarded as mappings on L_2. However, $S_0(L_1)$ is a homomorphic image of F_1.

4. Equational classes. We are now ready to prove our second main result, the converse to theorem 6.

Theorem 11. *If $\mathcal{B}_1$ is an equational class of Foulis semigroups, then*

$$\ell(\mathcal{B}_1) = \{L \mid L \in \mathcal{L},\ S_0(L) \in \mathcal{B}_1\}$$

is an equational class of orthomodular lattices.

Proof. Proposition 9 implies that $\ell(\mathcal{B}_1)$ is closed under the taking of homomorphic images and Propositions 8 and 10 imply that $\ell(\mathcal{B}_1)$ is closed under the formation of subalgebras. If $(L_i \mid i \in I)$ is a family of members of $\ell(\mathcal{B}_1)$

then Proposition 8 implies that $S_0(\Pi(L_i|i\varepsilon I))$ is a homomorphic image of the unique small subalgebra of $\Pi(S_0(L_i)|i\varepsilon I)$ and therefore $\Pi(L_i|i\varepsilon I)$ is in $\ell(\mathcal{B}_1)$. This proves that $\ell(\mathcal{B}_1)$ is an equational class.

In this case it is not as easy to see how identities may be carried over, but it is again possible in principle. The general idea is to take the Foulis semigroup identity and write it as an orthomodular lattice identity in terms of products of Sasaki projections. For example, the Foulis semigroup identity $xy = yx$ goes over to the orthomodular lattice identity $e\phi_f\phi_g = e\phi_g\phi_f$ (it is equivalent to assume that only two Sasaki projections commute). It is not immediately transparent that this determines the equational class of Boolean lattices.

Note that if $\mathcal{B}_1$ is an equational class of Foulis semigroups, then $S_0(L) \varepsilon \mathcal{B}_1$ if and only if L is coordinatized by F for some $F \varepsilon \mathcal{B}_1$. Therefore

$$\ell(\mathcal{B}_1) = \{L|L \varepsilon \mathcal{L}, \text{ L is isomorphic to } P'(F) \text{ for some } F \varepsilon \mathcal{B}_1\}.$$

Let $\underline{B}$, $\underline{L}$ denote the lattices of equational classes of Foulis semigroups and orthomodular lattices respectively. Then $\ell\colon \underline{B} \to \underline{L}$ and $b\colon \underline{L} \to \underline{B}$ are monotone maps and we can readily prove the following.

<u>Proposition 12</u>. (i) $\ell(b(\mathcal{L}_1)) = \mathcal{L}_1$ for any $\mathcal{L}_1 \varepsilon \underline{L}$;
(ii) $b(\ell(\mathcal{B}_1)) \supseteq \mathcal{B}_1$ for any $\mathcal{B}_1 \varepsilon \underline{B}$.

The inclusion in part (ii) may be strict, i.e. one equational class of orthomodular lattices may be coordinatized by many different equational classes of Foulis semigroups. As an example of this observe that the class of Boolean lattices is coordinatized by each of the following equational classes of Foulis semigroups:

(i) Boolean lattices themselves;

(ii) pseudo-complemented semilattices;

(iii) commutative Foulis semigroups.

Corollary 13. *The map* ℓ *is residuated and* b *is its residual.*

It follows from this that ℓ preserves joins and it is straightforward to check that it also preserves meets since the meet in the lattice of equational classes is intersection (there are some foundational difficulties here).

Theorem 14. *The map* $\ell\colon \underline{B} \to \underline{L}$ *is a complete lattice homomorphism.*

This result illustrates the main point of this paper: that a study of $\underline{B}$ should give information about the structure of $\underline{L}$.

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Orthomodular Logic

by

Gudrun Kalmbach

In this paper I develop an "orthomodular (OM) logic", a propositional logic, the models of which are orthomodular lattices. Since Boolean algebras are special orthomodular lattices, this locic contains the classical propositional calculus as a maximal extension. The OM-logic is incomparable with the other generalizations of classical logic like intuitionistic,Lukasiewicz- and Post-type logics. In fact, the only common extension of each of these with the OM-logic is the classical logic.

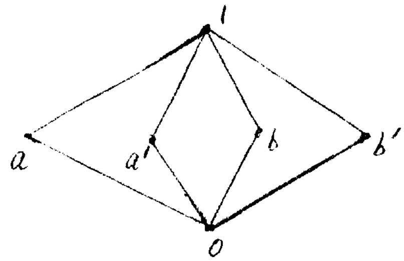
A main concern in this paper is to give a definition of implication which on the one hand has the property, that the modus ponens formulated with it gives a finite axiomatization of the OM-logic and which, on the other hand satisfies the natural requirement that $\alpha \rightarrow \beta$ is a tautology iff $v(\alpha) \leq v(\beta)$ holds for every valuation v in an OM-lattice. Among the five definitions of an implication satisfying the second of these requirements only one of them has the property that it gives the completeness theorem. It is shown that the intermediate OM-logics correspond to the equational classes of OM-lattices and that the deduction theorem does not hold in the OM-logic. This answers a question posed by W. Felscher.

1. We determine all two-variable polynomials p with the property

$$(*)\ p(a,b) = 1 \text{ iff } a \leq b$$

holds in every OML.

Let M02 be the OML



Every element of the free OML $2^4 \times M02$ (see G. Bruns and G. Kalmbach in the same Proceedings) on two generators gives rise in the obvious and well-known fashion to a polynomial in two variables. We introduce a special notation "$\to_i$" $(i = 1,\ldots,5)$ for some of these polynomials and we write "$a \to_i b$" for the value of $\to_i$ at (a,b).

The polynomials $\to_i$ are defined by:

$$a \to_1 b = (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge (a' \vee b))$$

$$a \to_2 b = (a' \wedge b) \vee (a \wedge b) \vee ((a' \vee b) \wedge b')$$

$$a \to_3 b = a' \vee (a \wedge b)$$

$$a \to_4 b = b \vee (a' \wedge b')$$

$$a \to_5 b = (a' \wedge b) \vee (a \wedge b) \vee (a' \wedge b')$$

Theorem: The polynomials p in two variables satisfying (*) are exactly the polynomials $\to_i$ $(i = 1,\ldots,5)$. The polynomials $\to_i$ have the additional property that $(1 \to_i b) = 1$ implies $b = 1$ in every OM-lattice.

2. We define the algebra $\mathfrak{F} = (F;\vee,\wedge,\neg)$ of formulae of our orthomodular (propositional) logic as an algebra with two binary operations $\vee,\wedge$ and

a unary operation $\neg$ which is absolutely freely generated by a countable infinite set V, the propositional variables.

A valuation is a map v of F into some OM-lattice L satisfying for all

$$\alpha,\beta \in F : v(\alpha \vee \beta) = v(\alpha) \vee v(\beta);$$

$$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$$

and

$$v(\neg\alpha) = (v(\alpha))'.$$

A formula $\alpha \in F$ is valid in L iff for every valuation $v: F \to L$ it is $v(\alpha) = 1$. A tautology is a formula $\alpha \in F$ which is valid in every OML. Let T be the set of tautologies.

For notational convenience we introduce a binary operation R in F by

$$\alpha R\beta = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta).$$

It is

$$\alpha R\beta \in T \quad \text{iff} \quad v(\alpha) = v(\beta)$$

for every valuation v.

We also consider the rules of inference

$$R_0: \frac{\begin{array}{c}\alpha \\ \neg\alpha \vee \beta\end{array}}{\beta} \qquad \text{and} \qquad R_i: \frac{\begin{array}{c}\alpha \\ \alpha \to_i \beta\end{array}}{\beta} \qquad (1 \leq i \leq 5)$$

These are "correct" rules of inference in the sense that if α and $\neg\alpha \vee \beta$ $(\alpha \to_i \beta)$ are valid in an OML M then β is valid in M.

For all formulae α,β,γ the following formulae A1 to A13 are tautologies.

$$\text{A1} \quad \neg(\alpha R\beta) \vee (\neg\alpha \vee \beta)$$

A2 $(\alpha R\beta)R(\beta R\alpha)$

A3 $\neg(\alpha R\beta) \vee (\neg(\beta R\gamma) \vee (\alpha R\gamma))$

A4 $\alpha R(\neg(\neg\alpha))$

A5 $(\alpha R\beta)R(\neg\alpha R\neg\beta)$

A6 $\neg(\alpha R\beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma)$

A7 $(\alpha \wedge \beta)R(\beta \wedge \alpha)$

A8 $\neg(\alpha \vee \beta)R(\neg\alpha \wedge \neg\beta)$

A9 $(\alpha \wedge (\alpha \vee \beta))R\alpha$

A10 $(\alpha \wedge (\beta \wedge \gamma))R((\alpha \wedge \beta) \wedge \gamma)$

A11 $(\alpha \vee (\neg\alpha \wedge (\alpha \vee \beta))R(\alpha \vee \beta)$

A12 $(\neg\alpha \wedge \alpha)R((\neg\alpha \wedge \alpha) \wedge\beta)$

A13 $(\neg\alpha \vee \beta) \rightarrow_1 (\alpha \rightarrow_1 (\alpha \rightarrow_1 \beta))$

Let B_0 be the set of tautologies of the form A1 to A13.

Lemma: A set of formulae containing B_0 is closed under R_0 iff it is closed under R_1.

3. Using the well-known technique of Lindenbaum-Tarski algebras we prove that the tautologies A1 to A13 together with the rule of inference R_0 give an axiomatization of our logic.

For an arbitrary set $M \subseteq F$ let $\Gamma_i M$ be the smallest set $S \subseteq F$ containing M and closed under R_i, i.e. satisfying: if $\alpha \in S$ and $\neg\alpha \vee \beta \in S$ resp. $\alpha \rightarrow_i \beta \in S$ then $\beta \in S$.

We define a relation ρ in F by $\alpha\rho\beta$ iff $\alpha R\beta \in \Gamma_0 B_0$.

In the following Lemma α/ρ is the equivalence class of α modulo ρ for $\alpha \in F$.

Lemma: ρ is a congruence relation in $\mathfrak{F}$. The quotient $\mathfrak{F}/\rho$ is an

OML. For every formula $\alpha \in F$ it is

$$\alpha \in \Gamma_0 B_0 \text{ iff } \alpha/\rho = 1.$$

By the Lemma of 2. and the preceeding Lemma we have

Theorem: $\Gamma_0 B_0 = T = \Gamma_1 B_0$

4. For a set A of formulae define mod A to be the class of all orthomodular lattices L in which all formulae $\alpha \in A$ are valid. For a class $\mathfrak{K}$ of OML define form $\mathfrak{K}$ to be the set of all formulae α which are valid in all $L \in \mathfrak{K}$. Then the pair (mod, form) is a Galois-correspondence, i.e. the following rules hold:

$$\text{if } A_1 \subseteq A_2 \text{ then } \operatorname{mod} A_2 \subseteq \operatorname{mod} A_1$$

$$\text{if } \mathfrak{K}_1 \subseteq \mathfrak{K}_2 \text{ then } \operatorname{form} \mathfrak{K}_2 \subseteq \operatorname{form} \mathfrak{K}_1$$

$$A \subseteq \operatorname{form} (\operatorname{mod} A)$$

$$\mathfrak{K} \subseteq \operatorname{mod} (\operatorname{form} \mathfrak{K})$$

We define an intermediate OM-logic to be a set A of formulae which is closed under this Galois-correspondence, i.e. satisfies $A = \operatorname{form}(\operatorname{mod} A)$.

Theorem: A set M of formulae is in intermediate logic iff it contains B_0 and is closed under substitution and the rule R_0 or R_1.

Corollary: For every set A of formulae:

$$\Gamma_0(B_0 \cup \overline{A}) = \operatorname{form}(\operatorname{mod} A) = \Gamma_1(B_0 \cup \overline{A}).$$

Here $\overline{A}$ is the set of formulae β for which there exists a formula

$\alpha \in A$ and an endomorphism φ of F such that $\beta = \varphi(\alpha)$.

To determine the closed classes $\mathfrak{K}$ of orthomodular lattices under the Galois-correspondence above we note first that all classes $\mathfrak{K}$ of the form $\mathfrak{K} = \text{mod } A$ form some set $A \subset F$ are obviously equational classes. Using standard techniques of universal algebra it is not difficult to prove the converse. Thus our theory of intermediate logics is equivalent with the equational theory of orthomodular lattices.

With respect to the other possible rules of inference we can prove

Theorem: There exists a set of formulae A which is closed under substitution, under $\Gamma_i (2 < i \leq 5)$ and which contains T, but for which $A \neq \text{form}(\text{mod } A)$.

The deduction theorem fails to hold in the OM-logic:

Theorem: There exist formulae α, β in F such that $\beta \in \Gamma_1(T \cup \{\alpha\})$, but $(\alpha \rightarrow_1 \beta) \notin T$ and $(\neg\alpha \vee \beta) \notin T$.

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ORDERING UNIFORM COMPLETIONS OF PARTIALLY ORDERED SETS

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We give here a discussion and summary of results whose proofs will appear elsewhere [6].

The familiar construction of the real numbers from the rational numbers via Cauchy sequences was put into its final form by Cantor [2]. In the 1930's, Weil [8] - an alternate view is Tukey's [7] - showed that the essence of this process was a certain similarity between neighbourhood systems of different points. In particular, a Hausdorff uniform space in the sense of [1] can be "completed" by a process which mimics the Cauchy sequence construction mentioned above, of the reals from the rationals.

Both the real numbers and the rational numbers are (additive) groups, and, in fact, the rational numbers are usually considered as a subgroup of the real numbers. It is well known (see, for example, [4]) that this extension of the group operation can be done in the more general case of Hausdorff uniform spaces, provided that the uniform structure and the group structure are sufficiently intertwined. Specifically, on any Hausdorff group with continuous group operations, there are several natural uniformities, at least one of whose completions can always be endowed with a group multiplication which extends the original operation.

Now the real numbers and the rational numbers are not only groups, but totally ordered groups as well. It seems reasonable, therefore, to ask for conditions whereby the order structure of a partially ordered set with

Hausdorff uniformity may be extended to the uniform completion of the set. As in the case of groups, it seems intuitively clear that some sort of connection between the order structure and the uniform structure must be assumed to ensure a satisfactory extension. That connection is provided by the idea of a uniform ordered space. This concept was introduced by Nachbin [5] who used it to investigate the ramifications of complete regularity on ordered sets with uniformities.

To define uniform ordered spaces, we need a little notation and a definition: Let X be a set; as usual, let

$$\Delta(X) = \{(x,x) \in X\times X \mid x \in X\}$$

be the <u>diagonal</u> of X. A <u>semi-uniform structure</u> on X is a filter $\mathcal{J}$ on $X\times X$ such that

(i) for all $V \in \mathcal{J}$, $\Delta(X) \subseteq V$;

(ii) for all $V \in \mathcal{J}$, there exists $U \in \mathcal{J}$ such that $U \circ U \subseteq V$.

Thus a semi-uniform structure is almost a uniformity: it lacks only a symmetric base. This requirement may be added by considering

$$\mathcal{J}^* = \{U \cap V^{-1} \mid U, V \in \mathcal{J}\} .$$

It is easy to see that $\mathcal{J}^*$ is a uniformity, which we call the <u>uniformity generated by</u> $\mathcal{J}$.

Let $(P, \leq)$ be a partially ordered set. Let

$$G(\leq) = \{(x,y) \in P\times P \mid x \leq y\}$$

be the <u>graph</u> of $\leq$. A <u>nearly uniform ordered space</u> is a partially ordered set $(P, \leq)$ with Hausdorff uniformity $\mathcal{U}$ such that there exists a semi-uniform structure $\mathcal{J}$ on P satisfying $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} \supseteq G(\leq)$. A uniform ordered space [5] is a nearly uniform ordered space with a semi-uniform structure $\mathcal{J}$ satisfying the conditions for a nearly uniform ordered space ($\mathcal{J}^* = \mathcal{U}$, $\bigcap \mathcal{J} \supseteq G(\leq)$) and additionally $\bigcap \mathcal{J} \subseteq G(\leq)$.

Every nearly uniform ordered space is locally convex, and there exist nearly uniform ordered spaces which are not uniform ordered spaces. Thus the concepts of partially ordered set with Hausdorff uniformity, nearly uniform ordered space, and uniform ordered space are distinct. In the totally ordered case, however, nearly uniform ordered spaces and uniform ordered spaces are the same.

Since nets are usually easier than filters to use where relations on sets are concerned, we will express our ideas and results in terms of nets rather than filters. This convention will be simplified by restricting ourselves to a single domain as follows: Let $(Y, \mathcal{U})$ be a Hausdorff uniform space. Let $(\tilde{Y}, \tilde{\mathcal{U}})$ be the completion of $(Y, \mathcal{U})$ at $\mathcal{U}$. Let $\tilde{y} \in \tilde{Y}$ and let $\{y_\delta \mid \delta \in \Delta\} \subseteq Y$ be a Cauchy net converging to $\tilde{y}$. If $\mathcal{U}^s$ is the set of symmetric entourages of $\mathcal{U}$ directed downwards, then there exists a Cauchy net $\{x_U \mid U \in \mathcal{U}^s\} \subseteq Y$, with domain $\mathcal{U}^s$, such that $\{x_U\}$ converges to $\tilde{y}$ and such that, as subsets of Y, $\{x_U\} \subseteq \{y_\delta\}$.

Let $(P, \mathcal{U})$ be a nearly uniform ordered space, with completion $(\tilde{P}, \tilde{\mathcal{U}})$ at $\mathcal{U}$. We might reasonably expect the following definition to extend the order on P to $\tilde{P}$: $\tilde{x} \preccurlyeq \tilde{y}$ if and only if for each Cauchy net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, there exists a Cauchy net $\{y_U\} \subseteq P$ converging to $\tilde{y}$ such that $x_U \leq y_U$ for all $U \in \mathcal{U}^s$. However, this relation, which we call the strong order on $\tilde{P}$, does not necessarily extend the original order: the condition that $x_U \leq y_U$ is too restrictive. Thus we are led to weaken this definition as follows: Let $\mathcal{J}$ be a semi-uniform structure for P such that $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} \supseteq G(\leq)$. Then $\tilde{x} \leq_{\mathcal{J}} \tilde{y}$ if and only if for each Cauchy net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, and for each $V \in \mathcal{J}$, there exists a Cauchy net $\{y_U\} \subseteq P$ converging to $\tilde{y}$ such that $(x_U, y_U) \in V$ for all $U \in \mathcal{U}^s$. (We call this relation the <u>$\mathcal{J}$-order</u> on P.) Thus, if $(P, \mathcal{U}, \leq)$ is a uniform ordered space, and if $\mathcal{J}$ is a semi-uniform structure for P such that $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} = G(\leq)$, then $\tilde{x} \leq_{\mathcal{J}} \tilde{y}$ means that for every net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, we can find nets $\{y_U\} \subseteq P$ converging to $\tilde{y}$ which are as close as we want to being (pointwise) greater than or equal to $\{x_U\}$.

For every nearly uniform ordered space, the strong order and the $\mathcal{J}$-orders are partial orders on $\tilde{P}$. Clearly $G(\preccurlyeq) \subseteq G(\leq_{\mathcal{J}})$. Furthermore, if $(P, \mathcal{U}, \leq)$ is a uniform ordered space, then $G(\leq) = G(\leq_{\mathcal{J}}) \cap (P \times P)$ for any $\mathcal{J}$-order $\leq_{\mathcal{J}}$. Also, any $\mathcal{J}$-order makes the uniform completion of a nearly uniform ordered space into a uniform ordered space.

Now the uniform completion of a Hausdorff uniform space $(X, \mathcal{U})$ is "free" in the sense that it satisfies the following universal mapping property [1]: If $(Y, \mathcal{V})$ is any complete Hausdorff uniform space, and if f is any uniformly continuous function from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$, then there exists a unique uniformly continuous function $\tilde{f}$ from the uniform completion $(\tilde{X}, \tilde{\mathcal{U}})$ of $(X, \mathcal{U})$, to $(Y, \mathcal{V})$ such that the diagram

$$\begin{tikzcd}
(X, \mathcal{U}) \arrow[r, "i"] \arrow[dr, "f"'] & (\tilde{X}, \tilde{\mathcal{U}}) \arrow[d, "\tilde{f}"] \\
& (Y, \mathcal{V})
\end{tikzcd}$$

commutes, where i is the canonical embedding of X into $\tilde{X}$. In other words, the uniform completion provides an adjoint to the functor which embeds the category of complete Hausdorff uniform spaces in the category of Hausdorff uniform spaces.

If we hope to achieve a similar property for the ordered case, we must be able to pick out a particular semi-uniform structure with which to define an $\mathcal{J}$-order. We can do this as follows: Let $\mathcal{U}$ be a Hausdorff uniformity on a partially ordered set $(P, \leq)$. Let

$$\mathcal{J}(\mathcal{U}) = \{V \in \mathcal{U} \mid \text{there exist } V_1, V_2, \ldots \in \mathcal{U} \text{ such that } V = V_1, \text{ and for all } n, V_n \supseteq G(\leq) \text{ and } V_{n+1} \circ V_{n+1} \subseteq V_n\}.$$

Thus $\mathcal{J}(\mathcal{U})$ consists of all those entourages which could conceivably be members of a semi-uniform structure which generates $\mathcal{U}$ and whose intersection contains the graph of $\leq$. Then for every (nearly) uniform ordered space, $\mathcal{J}(\mathcal{U})$ is the unique maximal semi-uniform structure satisfying $(\cap\, \mathcal{J}(\mathcal{U}) \supseteq G(\leq))\ \cap\ \mathcal{J}(\mathcal{U}) = G(\leq)$ and $\mathcal{J}(\mathcal{U})^* = \mathcal{U}$. Furthermore, if e is the embedding functor of the category of complete uniform ordered spaces into the category of uniform ordered spaces, both with uniformly continuous order-preserving functions, then the functor $\hat{e}$, which takes a uniform ordered space $(P, \mathcal{U}, \leq)$ to $(\tilde{P}, \tilde{\mathcal{U}}, \leq_{\mathcal{J}(\mathcal{U})})$ and a uniformly continuous function to its unique uniformly continuous extension, is adjoint to e.

The strong order is clearly easier to work with than any of the $\mathcal{J}$-orders; so it is of interest to enquire when the strong order is equivalent to some $\mathcal{J}$-order (and thus to the $\mathcal{J}(\mathcal{U})$-order). It turns out that if

$(P, \mathcal{U}, \leq)$ is a nearly uniform ordered space which satisfies

$$\text{(M) for all } V \in \mathcal{U}, \text{ there exists } W \in \mathcal{U}$$
$$\text{such that } W \circ G(\leq) \subseteq G(\leq) \circ V,$$

then the strong order is equivalent to the $\mathcal{J}(\mathcal{U})$-order. This condition (M) will in fact be satisfied for certain semilattices, lattices, and ℓ-groups.

We would hope that the extension procedure outlined thus far would take lattices to lattices and ℓ-groups to ℓ-groups. As long as the various operations are sufficiently well-behaved with respect to the uniform structure, this does indeed happen. Specifically, we say that a join-semilattice $(L, \vee)$ with Hausdorff uniformity $\mathcal{U}$ is a j-<u>uniform semilattice</u> in case $\vee$ is uniformly continuous with respect to $\mathcal{U}$. A <u>uniform lattice</u> $(L, \vee, \wedge, \mathcal{U})$ is defined similarly, with both operations uniformly continuous. Then a j-uniform semilattice $(L, \vee, \mathcal{U})$ is a uniform ordered space satisfying condition (M), and furthermore, the strong order on its uniform completion $\tilde{L}$ is the unique partial order on $\tilde{L}$ such that $\tilde{L}$ is a join-semilattice, L is a join-subsemilattice of $\tilde{L}$, and the join on $\tilde{L}$ is uniformly continuous. A similar result holds for uniform lattices.

The case for ℓ-groups answers most of a question raised by Conrad in [3]. (It was this question which led the present author to consider the general problem in the first place.) Consider an abelian ℓ-group with Hausdorff group and lattice topology $\mathfrak{J}$. Is the strong order on the completion $\tilde{B}$ of B at the usual uniformity the minimal lattice order on $\tilde{B}$ such that $\tilde{B}$ is an abelian ℓ-group, the positive cone of $\tilde{B}$ contains the positive cone of B, and the lattice operations on $\tilde{B}$ are continuous? With one assumption, some of our previous results may be used to give an affirmative answer in the following more general situation: Let B be an ℓ-group with Hausdorff group and lattice topology $\mathfrak{T}$. Let $\tilde{B}$ be its completion at one of the usual uniformities associated with $\mathfrak{T}$. If at least one of the lattice operations on B is uniformly continuous, then the strong order on $\tilde{B}$ is the minimal lattice order on $\tilde{B}$ whose graph contains that of $\leq$ and whose join is continuous. Furthermore, if $\tilde{B}$ is a group (at least one of whose lattice operations is uniformly continuous), then $(\tilde{B}, \trianglelefteq)$ is an ℓ-group.

The assumption of uniform continuity of at least one of the lattice operations is not at all stringent. In fact, for an ℓ-group B with group and lattice topology $\mathfrak{T}$, the following statements are equivalent:

(i) $\vee$ is uniformly continuous with respect to the right, left or two-sided uniformity;

(ii) $\wedge$ is uniformly continuous with respect to the right, left or two-sided uniformity;

(iii) the topology on B is locally convex.

In the non-locally convex case, it is difficult to see intuitively how to order the completion, and thus a requirement of local convexity does not really restrict the cases that one might expect to consider.

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PLANAR LATTICES

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This paper presents a brief survey of results about finite planar lattices. Unless otherwise stated, all lattices in this paper are finite.

Anyone beginning the study of lattice theory quickly learns that it is important to be able to draw a picture of a lattice; i.e. the Hasse diagram of a lattice. Once he has become skilled in drawing lattice diagrams he soon notices that whenever the diagram is planar he has indeed drawn a lattice. That is, he does not have to check that all l.u.b.'s and g.ℓ.b.'s exist. This heuristic principle can be formalized as a theorem. The following formulation is due to Harry Lakser.

First we give a formal definition of the diagram of a poset. Let $\mathcal{P}$ be a poset on the n element set $\{p_1, \cdots, p_n\}$. A <u>diagram</u> of $\mathcal{P}$ is a set of n points in the (x, y)-plane, $(x_1, y_1), \cdots, (x_n, y_n)$, together with certain arcs between these points such that:

a) If p_i covers p_j then $y_i > y_j$ and there is an arc, a_{ji}, which is the graph of a continuous function of y with domain $[y_j, y_i]$, with $a_{ji}(y_i) = x_i$ and $a_{ji}(y_j) = x_j$ and with no other point, (x_k, y_k), lying on a_{ji}.

b) There are no other arcs than those of condition a).

$\mathcal{P}$ is <u>planar</u> if it has a planar diagram (i.e. any two arcs intersect only at

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an endpoint). Each arc can be thought of as directed from its bottom end point to its top endpoint. A path is an ascending sequence of connected arcs, i.e. a set of arcs forming the graph of a continuous function of y. Thus $p_i < p_j$ if and only if there is a path from (x_i, y_i) to (x_j, y_j).

This definition of the diagram of a poset is a reasonable approximation to the way one draws posets. It has the advantage that the proof of the theorem uses only the intermediate value theorem for continuous functions rather than some version of the Jordan curve theorem. When drawing the diagram of what one hopes is a lattice, there is an obvious condition to be satisfied: avoid dangling points. More precisely, there must be exactly one point which is not the lower endpoint of an arc (the unit) and exactly one point which is not the upper endpoint of an arc (the zero). Lakser's theorem states that if the diagram is planar then this necessary condition is sufficient for the poset to be a lattice.

Theorem: Let $\mathcal{P}$ be a finite poset with a planar diagram; if there is at most one element of $\mathcal{P}$ which has no cover and at most one element which covers no point then $\mathcal{P}$ is a lattice.

Sketch of the proof: Since $\mathcal{P}$ is finite the "at most" in the statement is equivalent to "exactly"; these points are necessarily the unit and zero of the poset. Now let p_1, p_2, p_3, p_4 be four points of $\mathcal{P}$ such that $p_1 < p_3$, $p_1 < p_4$, $p_2 < p_3$, $p_2 < p_4$. To show that $\mathcal{P}$ is a lattice it is sufficient to show that there is a point p_5 of $\mathcal{P}$ with $p_1 \leq p_5$, $p_2 \leq p_5$, $p_5 \leq p_3$, $p_5 \leq p_4$ (and so we may as well assume that p_1 is not comparable to

p_2 and p_3 is not comparable to p_4) . Thus we have paths $\alpha_{1,3}, \alpha_{1,4}, \alpha_{2,3}, \alpha_{2,4}$ (where $a_{i,j}$ goes from (x_i, y_i) to (x_j, y_j)) . We can also find points p_6, p_7 such that $p_6 < p_1, p_6 < p_2, p_3 < p_7, p_4 < p_7$ and such that the paths $\alpha_{6,1}$ and $\alpha_{6,2}$ intersect only at (x_6, y_6) and the paths $\alpha_{3,7}$ and $\alpha_{4,7}$ intersect only at (x_7, y_7) . Using planarity and the intermediate value theorem for continuous functions, a case by case analysis shows that such an element p_5 must exist.

Now let us turn to the problem of characterizing planar lattices. For distributive lattices this characterization is well-known. A distributive lattice is planar iff it is a sublattice of a direct product of two chains iff it does not contain the eight element boolean lattice as a sublattice iff no element covers more than two other elements iff no element is covered by more than two other elements iff it does not contain the poset of figure 1 as a subposet. For modular lattices the following characterization is due to Rudolf Wille [1]. Recall that an element of a lattice is doubly irreducible if it cannot be written as a proper meet or a proper join.

Theorem: A modular lattice $\mathfrak{m}$ is planar iff $\mathfrak{m} - \{d \in \mathfrak{m} | d$ is doubly irreducible$\}$ is a planar distributive lattice iff $\mathfrak{m}$ does not contain any of the posets of figures 1 and 2 as a subposet.

For lattices in general there is no finite set of posets which can be used to test planarity. In fact, planarity for lattices is not a first order property. This result is due to K. Baker, P. Fishburn, and F. Roberts [2] . To see this, consider the fence posets of figure 3 and the crown posets of figure 4. Adding a zero and unit to each turns them into lattices. Notice that the fence lattices are planar but that the crown lattices are not planar. In [2] it is pointed out that an appropriate ultraproduct of fence lattices is isomorphic to an appropriate ultraproduct of crown lattices. Since ultraproducts preserve first order properties, planarity cannot be a first order property. In particular, there is no finite list of posets such that planar lattices are characterized by the absence of these posets as subposets.

Problem 1: Is there a finite list of posets which test planarity in the variety generated by N_5 ?

Problem 2: Is there a finite list of families of posets which tests planarity for all lattices? (The set of crowns would likely be a family in this list).

It seems clear that there ought to be some nice connection between planar lattices and planar graphs. Such a connection has been found by Craig Platt [4]. If $\mathcal{L}$ is a lattice (with 0 as zero and 1 as unit) then $G(\mathcal{L})$, the graph of $\mathcal{L}$, has the same points as $\mathcal{L}$ and has a directed edge from x to y if and only if x is covered by y ; $G^*(\mathcal{L})$, the augmented graph of $\mathcal{L}$, is $G(\mathcal{L})$ together with a directed edge from 1 to 0 .

Theorem (C.R. Platt): $\mathcal{L}$ is a planar lattice iff $G^*(\mathcal{L})$ is a planar graph.

Sketch of the proof: If $\mathcal{L}$ is a planar lattice then clearly $G^*(\mathcal{L})$ is a planar graph. Conversely suppose $G^*(\mathcal{L})$ is a planar graph. Note that we may assume that the edge from 1 to 0 is on the outside of the graph.

Consider the other outside path from 0 to 1 : $0, x_1, \cdots, x_n, 1$. An induction argument proves that the path is directed in the order given; that is, in $\mathcal{L}$ x_1 covers 0 , x_{i+1} covers x_i for $i = 1, \cdots, n-1$, and 1 covers x_n . Another induction argument shows that one of the x_i's is doubly irreducible. Finally an induction on the size of $\mathcal{L}$ is made using the statement: If $|\mathcal{L}| = m$, $G^*(\mathcal{L})$ is planar with $0, x_1, \cdots, x_n, 1$ an outside path from 0 to 1 then $\mathcal{L}$ is planar and can be drawn with straight lines so that $0, x_1, \cdots, x_n, 1$ is an outside path in the diagram of $\mathcal{L}$.

Corollary: Every planar lattice has a planar diagram in which all arcs are straight lines.

Planar lattices form a subclass of the class of dismantlable lattices. A lattice is dismantlable if every sublattice contains a doubly irreducible element. More picturesquely, a lattice is dismantlable if one can keep throwing away doubly irreducible elements until nothing is left. Every planar lattice has a doubly irreducible element (see [2]) and a sublattice of a planar lattice is planar. Hence planar lattices are dismantlable. A recent result of David Kelly and Ivan Rival proves that dismantlable lattices are characterized by the absence of crowns.

Theorem [3]: A lattice is dismantlable if and only if it contains no crown poset as a subposet. A modular lattice is dismantlable if and only if it does not contain the crown of order 3 (i.e. figure 1) as a subposet. A distributive lattice is dismantlable if and only if it is planar.

All the material above refers to finite lattices; however it is often the case that one needs an infinite lattice for a particular problem. Thus it would be useful to have analogs of the above results (especially Lakser's theorem) for infinite planar lattices.

Problem 3: Develop a theory of infinite planar lattices.

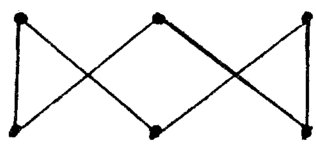

Figure 1

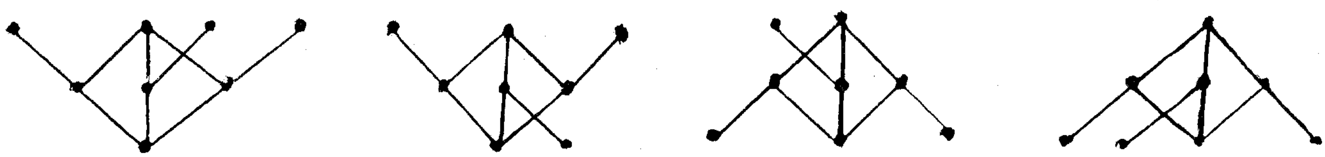

Figure 2

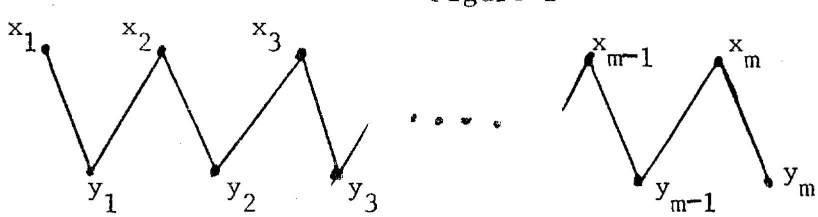

Figure 3: The fence of order m $(m \geq 3)$

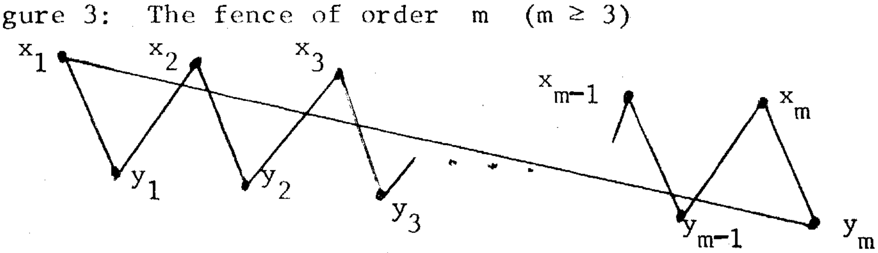

Figure 4: The crown of order m $(m \geq 3)$

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TIGHT RESIDUATED MAPPINGS

by Erik A. Schreiner

1. Introduction. In this note we examine the connection between certain residuated mappings on a complete lattice and the property of complete distributivity. A map $T: L \longrightarrow M$ is residuated with T^+ as its residual if and only if the pair (T, T^+) forms a Galois connection between L and M^*, the dual of M. With this in mind we consider tight residuated mappings which correspond with the tight Galois connections introduced by G. N. Raney [7]. Since Res(L) is a semigroup and a complete lattice, we are able to compose and join tight residuated maps to extend the result of Raney.

In particular, we are able to characterize all complete homomorphisms with completely distributive images. Indeed, for any lattice, we present a method for finding the largest such complete homomorphism.

Tight residuated mappings abound. They are the pointwise join in Res(L,M) of certain basic tight mappings which, by their simplicity, help to illuminate what is occuring. In view of the fact that it is possible to construct a tight residuated map on an atomic Boolean lattice whose image is nonmodular, it is necessary to ask which tight mappings lead to a connection with complete distributivity and which basic tight mappings are the key ones. The answers are, respectively, idempotent tight maps and decreasing basic tight maps.

The image of an idempotent tight residuated map is completely distributive. Conversely, if the image of an arbitrary residuated map is completely distributive then one may find a tight idempotent with the same image. This is used to show that the completely distributive lattices are both injective and projective in the category of complete lattices with residuated mappings. Finally, the consideration of idempotent basic tight maps leads to some simple proofs of certain well-known results concerning atomic Boolean lattices.

2. Basic tight residuated maps. In this paper we consider only complete lattices. A map $T:L \longrightarrow M$ is called residuated if the inverse image of every principal ideal is a principal ideal. With complete lattices, this is equivalent to being a complete join homomorphism. The basic properties and facts concerning residuated maps may be found in the book of Blyth and Janowitz [1]. Let Res(L,M) denote the set of all residuated maps from L to M and Res(L) = Res(L,L). Both are complete lattices under pointwise order while the latter is also a Baer semigroup.

Z. Shmuely [8] has established a one-one correspondence between Res(L,M) and relations $\gamma \subseteq L \times M$ which satisfy:

(1) $(a,b) \in \gamma$, $x \leq a$, $y \geq b$ implies $(x.y) \in \gamma$.

(2) γ is a complete sublattice of $L \times M$.

For such a γ (called a G-relation) the map $T(a) = \wedge\{b \mid (a,b) \in \gamma\}$ is the associated residuated map. For a given $T \in \mathrm{Res}(L,M)$, The set $\sigma(T) = \{(a,b) \mid T(a) \leq b\}$ is the corresponding G-relation.

Let $\theta \subseteq L\times M$ and define
$\theta^{+} = \{(x,y) \mid \text{for each } (a,b) \in \theta,\ x \leq a \text{ or } y \geq b\}$. Then θ^{+} is a G-relation. Raney [7] defined his tight Galois connections in terms of relations of the form θ^{+}. (We have adjusted for the necessary dualization.)

<u>Definition</u> 1. A map $T \in \mathrm{Res}(L,M)$ is <u>tight</u> if there exists a $\theta \subseteq L\times M$ such that $\sigma(T) = \theta^{+}$.

Let $\theta = \{(g,h)\}$. Then the tight map E_h^g obtained from θ^{+} is called a <u>basic tight map</u> and is defined by:

$$E_h^g(x) = \begin{cases} 0 & x \leq g \\ h & \text{otherwise} \end{cases}$$

These maps are either nilpotent (if $h \leq g$) or idempotent (if $h \not\leq g$).

<u>Theorem</u> 2 (Shmuely). $T \in \mathrm{Res}(L,M)$ <u>is tight if and only if</u> $T = \vee\{E_h^g \mid (g,h) \in \theta\}$ <u>for some</u> $\theta \subseteq L\times M$.

Proof: $\sigma(T) = \cap\{\sigma(E_h^g) \mid (g,h) \in \theta\} = \sigma(\vee\{E_h^g \mid (g,h) \in \theta\})$, where T is defined by θ^{+}.

Thus we focus our attention on basic tight maps. The set of basic tight maps. The set of basic tight maps and the set of all tight maps both form two sided ideals of the semigroup Res(L). Hence (ii) and (iii) are equivalent in the following theorem [7, Theorem 4].

Theorem 3 (Raney). *The following conditions are mutually equivalent:*

(i) *L is completely distributive.*

(ii) *The identity map* I_L *in* Res(L) *is tight.*

(iii) *All* T *in* Res(L) *are tight.*

It has come to my attention that D. Mowat [4] in his thesis considered basic tight maps (under the name "two point s.p. maps") and derived a similar result.

3. *Decreasing basic tight maps.* A map $T \in \mathrm{Res}(L)$ is *decreasing* if $T(x) \leq x$ for all $x \in L$. A basic tight map E_b^a is decreasing if and only if the ordered pair (a, b) satisfies the condition: $x \not\leq a$ implies $x \geq b$. Call such a pair (a,b) a *decreasing pair*. Pairs of the form (1,b) and (a,0) are always trivial decreasing pairs. The map $E_b^a = 0$ iff (a,b) is trivial. Let $\beta_2 = \beta_2(L) = \left\{(a,b) \mid (a,b) \text{ a nontrivial decreasing pair on } L.\right\}$ A central role in our study is played by maps of the form:

$$F = \vee\left\{E_b^a \mid (a,b) \in \beta_2\right\}.$$

As usual, if β_2 is empty F is the zero map. Note that F is a tight decreasing map in Res(L).

In terms of our basic tight decreasing maps we have the following critical result of Raney [7, Theorem 5].

<u>Lemma</u> 4. L <u>is completely distributive if and only if</u> $x \not\leq y$ <u>implies there exists a decreasing</u> E_b^a <u>in</u> Res(L) <u>with</u> $E_b^a(x) = b$, $E_b^a(y) = 0$ <u>and</u> $b \not\leq y$.

<u>Theorem</u> 5. L <u>is completely distributive if and only if the map</u> $F = \vee\left\{E_b^a \mid (a,b) \in \beta_2\right\} = I_L$ <u>in</u> Res(L).

Proof: Sufficiency follows from Theorem 3. Necessity is established using Lemma 4.

As we are interested in maps of the form of F above, the relation β_2 may contain surplus pairs. If $\left\{(a_i,b) \mid i \in I\right\} \subseteq \beta_2$ and $a = \wedge a_i$, then $(a,b) \in \beta_2$ and E_b^a dominates the other associated basic maps. Similarly, if $\left\{(a,b_i) \mid i \in I\right\} \subseteq \beta_2$ and $b = \vee b_i$, then $(a,b) \in \beta_2$ with E_b^a again an upper bound. The resulting pair (a,b) in either case may be called a <u>minimax</u> pair. Let $\beta_1 = \beta_1(L) = \left\{(a,b) \in \beta_2 \mid (a,b) \text{ is minimax}\right\}$. Note that $\vee\left\{E_b^a \mid (a,b) \in \beta_1\right\} = \vee\left\{E_b^a \mid (a,b) \in \beta_2\right\}$. The mappings eliminated in our transition from β_2 to β_1 were all nilpotent maps.

We are primarily interested in maps of the form $\vee\left\{E_b^a \mid (a,b) \in \theta \subseteq \beta_1\right\}$ which are idempotent. This will always be the case if all the E_b^a are idempotent. However, the elimination of all nilpotents is too drastic a step in the quest for an idempotent join. For if L is the closed unit interval [0,1] of the real numbers under the usual order, then L is completely distributive. The minimax decreasing pairs are all of the form (b,b) where $b \in (0,1)$. Thus all maps E_b^b are nilpotent yet $\vee\left\{E_b^b \mid b \in (0,1)\right\}$ is the identity map on L.

In order to eliminate the problems presented by isolated nilpotents in our study of idempotent tight maps we make one final adjustment to our relation. Let $F = \vee\left\{E_b^a \mid (a,b) \in \beta_1\right\}$. Define $\beta = \beta(L) = \left\{(a,b) \in \beta_1 \mid F(b) = b\right\}$. Note that if there are no nilpotent decreasing maps E_b^a, then $\beta = \beta_1$ but not conversely. For the remainder of the paper we shall use the notation:

$$E = \vee\left\{ E_b^a \mid (a,b) \in \beta\right\} .$$

We may then restate Theorem 5 as:

L _is_ _completely_ _distributive_ _iff_ E _is_ _the_ _identity_ _in_ Res(L).

4. _Idempotent_ _tight_ _residuated_ _maps_.

Lemma 6. _For_ _any_ _complete_ _lattice_ L, E _is_ _an_ _idempotent_ _decreasing_ _tight_ _map_ _in_ Res(L).

For an arbitrary map $T \in \mathrm{Res}(L)$, the image $T(L) = M$ has the following properties:

(1) $0 \in M$.

(2) $(M,\leq)$ is a complete lattice.

(3) The join in $(M,\leq)$ is the same as the join in L.

In general M is not a sublattice of L due to differing meet operations. Given a subset M of L satisfying the above properties, whether one can always find a $T \in \mathrm{Res}(L)$ with $T(L) = M$ is an open question. The following theorem presents a partial answer. (See also Mowat [4, Theorem 17, page 42] for a related result.)

Theorem 7. *Let M be a subset of L satisfying (1), (2) and (3). Then there exists an idempotent tight map $T \in \mathrm{Res}(L)$ with $T(L) = M$ if and only if $(M,\leq)$ is completely distributive.*

Proof: Given a tight idempotent T, the restriction of T to M is the identity map and may be shown to be tight as a map in Res(M). Thus $(M,\leq)$ is completely distributive by Theorem 3. Conversely, if $(M,\leq)$ is completely distributive, the identity map on M is $E_M = \vee \left\{ E_b^a \,\middle|\, (a,b) \in \beta(M) \right\}$. Since each E_b^a may trivially be extended to a map in Res(L), the desired idempotent in Res(L) is $T = \vee \left\{ E_b^a \,\middle|\, (a,b) \in \beta(M) \right\}$.

To see that the idempotency of T is essential, consider the Boolean lattice $\underline{2}^3$ with atoms a,b,c and respective complements $d = a'$, $e = b'$, $f = c'$. The image of the tight map $T = E_c^f \vee E_d^e \vee E_f^d$ is nonmodular.

In the next theorem we combine the fact that the residual T^+ sets up an isomorphism between the image of T and the image of T^+, appropriate duality and the extension of the identity used in Theorem 7.

Theorem 8. *Let* $T \in \mathrm{Res}(L,M)$ *be an onto map. If* M *is completely distributive, then* $T = S \cdot P$ *where* P *is a tight idempotent in* Res(L) *and* S *is an isomorphism.*

Proof: The following commutative diagram may be obtained:

$$\begin{tikzcd}
L \arrow[r, "T"] \arrow[d, "P"'] & M \\
P(L) = N \arrow[r, "P^+ |_N"'] & W = T^+(M) \arrow[u, "T^+ |_W"']
\end{tikzcd}$$

The restricted maps $P^+ \big|_N$ and $T^+ \big|_W$ are isomorphisms.

Corollary 9 (Crown [2]). *A completely distributive lattice is both injective and projective in the category of complete lattices with residuated maps.*

Proof: It is enough to show that if M is completely distributive, for every monomorphism $P \in \mathrm{Res}(M,L)$ there is a $T \in \mathrm{Res}\ (L,M)$ such that $T \cdot P = I_M$. This follows easily from

Theorem 7. Thus M is injective. Dually, or by use of Theorem 8, M is projective.

Theorem 10. <u>The map</u> E <u>is a complete homomorphism of</u> L <u>onto a completely distributive image</u>. <u>Moreover</u>, E <u>is the largest complete homomorphism with a completely distributive image</u>.

Proof: E is a decreasing idempotent, thus a complete homomorphism by [3, Theorem 3.6]. Since E is also tight, E(L) is completely distributive.

For any complete homomorphism onto a completely distributive image, consider the complete congruence Θ generated and the associated lattice $L/_{\Theta}$. The identity map on $L/_{\Theta}$ is generated from pairs $(\bar{a},\bar{b})$ in $\beta(L/_{\Theta})$. These may be pulled back to obtain pairs (a,b) in $\beta(L)$. The given homomorphism thus was of the form $\vee\left\{ E^a_b \,\middle|\, (a,b) \in \theta \subseteq \beta(L) \right\}$ which is less than E in Res(L).

Thus for each $\theta \subseteq \beta$ such that $\vee\left\{ E^a_b \,\middle|\, (a,b) \in \theta \right\}$ is idempotent we obtain a complete homomorphism with completely distributive image, and all such homomorphisms are of this form. If distinct subrelations of β give rise to distinct maps, as will be the case if there are no nilpotents associated with β, we have a means of enumerating such homomorphisms.

We now turn our attention to idempotent basic decreasing maps. An element b is the image of an idempotent decreasing E_b^a if and only if [b,1] is a completely prime dual ideal, that is, in Raney's terms, b is a completely join-irreducible element. His result [5, Theorem 2] may thus be stated in terms of decreasing maps.

Theorem 11. *L is isomorphic to a complete ring of sets if and only if* $I_L = E$ *and* $\beta(L)$ *contains no nilpotent pairs*.

If L is (dual) semicomplemented, all decreasing E_b^a are idempotents. They may be characterized by the conditions: b is an atom, a is a dual atom, a and b are complements, and a is a distributive element. Combining this with the properties of the map E provides simple proofs of the following well-known results.

Theorem 12. *A (dual) semicomplemented completely distributive lattice is an atomic Boolean lattice*.

Theorem 13. *Any two of the following conditions on a complete lattice imply the third*.

(i) L *is atomistic*.

(ii) L *is completely distributive*.

(iii) L *is a Boolean lattice*.

Finally, we combine these observations with the remark following Theorem 10.

Theorem 14. _Let_ L _be a complete_ (_dual_) _semicomplemented lattice_. _Then every completely distributive complete homomorphic image of_ L _is an atomic Boolean lattice_.

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On Subalgebras of Partial Universal Algebras

by A. A. Iskander

For a partial universal algebra, the projection of a closed subalgebra of its n^{th} direct power onto its k^{th} diredt power $(1 \leq k < n)$ is not always a closed subalgebra. Partial universal algebras for which this projection of a closed subalgebra is always a closed subalgebra are called here (n,k) correct. A similar notion of $(2,1)$ correctness was introduced in [2]. Given any set A, the subalgebra systems of $\langle A;F\rangle^n$ for any set F of partial operations on A was described in [4]. In this note we describe the subalgebra systems for (n,k) correct partial universal algebras. Some of the results were announced in [3].

By algebras we shall mean partial universal algebras. By a subalgebra will always be meant a closed subalgebra.

Let k,n be integers such that $1 \leq k < n$. An algebra $A = \langle A;F\rangle$ is said to be (n,k) correct if for any $f \in F$, $a_1,\ldots,a_m \in A^n$ such that $f(a_{1i},\ldots,a_{mi})$ is defined for all $1 \leq i \leq k$ (a_{si} is the i^{th} component of a_s), there is an m-place polynomial p in F such that $p(a_1,\ldots,a_m)$ is defined, and $p(a_{1i},\ldots,a_{mi}) = f(a_{1i},\ldots,a_{mi})$ for all $1 \leq i \leq k$. An algebra will be called n-correct if it is (n,k) correct for all $1 \leq k < n$.

It is clear that full algebras are n-correct for all n. Every full [1] homomorphic image of an (n,k) correct algebra is also (n,k) correct. The same is true for all quotient algebras. Given an (n,k) correct algebra $\langle A;F\rangle$, if p is an r-place polynomial in F and $a_1, \ldots, a_r \varepsilon A^n$ such that $p(a_{1i}, \ldots, a_{ri})$ is defined for all $1 \leq i \leq k$, then there is an r-place polynomial q in F such that $q(a_1, \ldots, a_r)$ is defined, and moreover $q(a_{1i}, \ldots, a_{ri}) = p(a_{1i}, \ldots, a_{ri})$ for all $1 \leq i \leq k$. An algebra is (n, k) correct iff the projection of any subalgebra of its n^{th} direct power is again a subalgebra; it is also sufficient to consider only finitely generated subalgebras. It is also evident that any subalgebra of an (n, k) correct algebra is also (n, k) correct. An (n, k) correct algebra is also (n, m) correct for all $k < m < n$.

If $1 \leq k < m < n$, then every (n, m) correct algebra is (n, k) correct. In fact if B is a subalgebra of $\langle A;F\rangle^n$ and C is the projection of B onto A^k (first k components) and D is the projection of B onto A^{k+1} (first $k + 1$ components), then $A \times C$ is the projection of $A \times D \times A^{(n-k-2)}$ onto A^{k+i}; $A \times C$ is a subalgebra of $\underline{A}^{k+1}$ iff C is a subalgebra of $\underline{A}^k$.

<u>For every</u> $k > 0$ <u>there is an algebra which is</u> (n, k) <u>correct for all</u> $n > k$ <u>and not</u> $(n, k+1)$ <u>correct for any</u> $n > k + 1$.

Let A be the set of all positive integers. Set $F = \{f, g_1, \ldots, g_{k+1}\}$, where all elements of F are unary and all $g_1, \ldots, g_{k+1}$ are full and the domain of definition is $\{1, \ldots, k+1\} = K$.

$$f(j) = j \quad \text{if} \quad 1 \leq j \leq k$$

$$f(k + 1) = k + 2.$$

Denote by K_i the complement of $\{i\}$ in K.

$$g_i(m) = \begin{cases} f(m) & \text{if } m \in K_i \\ f(\min K_i) & \text{if } m \notin K_i, \end{cases} \qquad 1 \leq i \leq k+1.$$

It is obvious that $\langle A;F\rangle$ is (n,k) correct for all $n > k$. If $\underline{A}$ were $(k+2, k+1)$ correct then there would exist a polynomial $p = h_1 \cdots h_s$ $(h_1, \ldots, h_s \in F)$ with $p((1, 2, \ldots, k, k+1, k+2)) = (1, 2, \ldots, k, k+2, x)$, (since $f((1, 2, \ldots, k, k+1))$ is defined). Since f is not defined at $k+2$ then $h_s = g_i$ for some i. So $(1, 2, \ldots, k, k+2, x) = h_1(h_2(\cdots h_{s-1}(g_i((1, 2, \ldots, k, k+1, k+2)))\cdots))$. But g_i identifies two of the first $k+1$ components of the tuple and the application of $h_{s-1}, \ldots, h_1$ will leave these two components equal. Yet $1, \ldots, k, k+2$ are all distinct. So there is no such p and $\underline{A}$ is not $(k+2, k+1)$ correct.

To say that $\langle A;F\rangle^2$ is (n,k) correct is the same as to say that $\langle A;F\rangle$ is $(2n, 2k)$ correct. Hence direct products of (n,k) correct algebras are not always (n,k) correct.

A^k can be injected into A^n (for $n > k$) in such a way that the image of A^k is always a subalgebra of $\underline{A}^n$ (e.g., $(a_1, \ldots, a_k) \to (a_1, \ldots, a_k, a_k, \ldots, a_k)$ is such an injection). Under such an injection every subalgebra of $\underline{A}^k$ appears as a subalgebra of $\underline{A}^n$. Thus in an (n,k) correct algebra the injection of the projection (onto A^k) of a subalgebra of $\underline{A}^n$ is again a subalgebra of $\underline{A}^n$. Such a condition turns out to be sufficient. This is more precisely made in the following definition and lemma.

If α is a nonvoid subset of $\{1, \ldots, n\}$ and $i = \min \alpha$ define [4]:

$$B\alpha = \{a : a \in A^n,\ a_j = b_j \text{ if } j \notin \alpha,\ a_j = b_i \text{ if } j \in \alpha, \text{ for some } b \in B\} \qquad B \subseteq A^n.$$

If $C \subseteq A^n$ denote by $[C]$ the subalgebra of $\langle A;F\rangle^n$ generated by C.

<u>Lemma</u>: $\langle A;F\rangle$ is (n,k) correct iff $[C]\alpha = [C\alpha]$ for all finite nonvoid $C \subseteq A^n$ and for all $\alpha \subseteq \{1, \ldots, n\}$ with cardinality $n - k + 1$.

If s is a permutation on $\{1, \ldots, n\}$ define [4]:

$$Bs = \{a : a \varepsilon A^n, \; a_i = b_{s^{-1}(i)}, \; 1 \leq i \leq n \text{ for some } b \varepsilon B\}, \quad B \subseteq A^n.$$

If $\underline{A}$ is an algebra $S(\underline{A})$ is the family of all subalgebras of $\underline{A}$.

<u>Theorem</u>: Let $S \subseteq P(A^n)$. $S = S(\langle A;F\rangle^n)$ for some (n,k) correct algebra $\langle A;F\rangle$ iff:

(a) S is an algebraic closure system on A^n

(b) if $B \varepsilon S$, $1 \leq i < j \leq n$ then $B(ij) \varepsilon S$

(d) $[C]_S\{1,2\} \subseteq [C\{1,2\}]_S$ for all finite nonvoid $C \subseteq A^n$

(e) if $\phi \varepsilon S$ then $\phi = \bigcap \{B : B \varepsilon S, \; B \neq \phi\}$

(f) $[C]_S\{1, \ldots, n-k+1\} = [C\{1, \ldots, n-k+1\}]_S$ for all nonvoid finite $C \subseteq A^n$,

where $[C]_S$ is the intersection of all elements of S containing C.

In [4] it was shown that $S = S(\langle A;F\rangle^n)$ for some partial algebra $\langle A;F\rangle$ iff S satisfies conditions (a), (b), (c), (d), (e), where (c) is $\Delta_2 \times A^{n-2} \varepsilon S$ (Δ_2 is the diagonal in A^2). So the necessity of (a), (b), (d), (e) and (f) follows from this result and the lemma. The sufficiency will be established once we show

<u>Claim</u>: Let r be an integer such that $1 < r \leq n$. If S satisfies (a), (b), (d) and

(f') $$[C]_S\{1,\ldots,r\} = [C\{1,\ldots,r\}]_S$$

for all finite nonvoid $C \subseteq A^n$, then $\Delta_2 \times A^{n-2} \varepsilon S$.

By (a) S satisfies (f') for all $C \subseteq A^n$. Thus

$$[\Delta_2 \times A^{n-2}]_S\{2,3,\ldots,r\} \subseteq [(\Delta_2 \times A^{n-2})\{2,\ldots,r\}]_S = [\Delta_r \times A^{n-r}]_S .$$

But

$$[\Delta_r \times A^{n-r}]_S\{1,\ldots,r\} = [(\Delta_r \times A^{n-r})\{1,\ldots,r\}]_S = [\Delta_r \times A^{n-r}]_S$$

by (f'). Also

$$[\Delta_r \times A^{n-r}]_S\{1,\ldots,r\} \subseteq A^n\{1,\ldots,r\} = \Delta_r \times A^{n-r}.$$

Hence

$$\Delta_r \times A^{n-r} \subseteq [\Delta_r \times A^{n-r}]_S \subseteq \Delta_r \times A^{n-2},$$

i.e., $\Delta_r \times A^{n-r} \varepsilon S$. So

$$[\Delta_2 \times A^{n-2}]_S\{2,\ldots,r\} \subseteq [\Delta_r \times A^{n-r}]_S = \Delta_r \times A^{n-r}.$$

From which we deduce $[\Delta_2 \times A^{n-2}]_S \subseteq \Delta_2 \times A^{n-2}$, i.e., $\Delta_2 \times A^{n-2} \varepsilon S$.

<u>Corollary</u>: Let $S \subseteq P(A^n)$. $S = S(\langle A;F\rangle^n)$ for some n-correct algebra $\langle A;F\rangle$ iff S satisfies (a), (b), (e) and

$$[C]_S\{1,2\} = [C\{1,2\}]_S \quad \text{for all nonvoid finite } C \subseteq A^n.$$

This follows from the theorem since n-correctness is equivalent to $(n, n-1)$ correctness and in this case condition (f) implies condition (d).

There are 2-correct partial algebras $\langle A;F\rangle$, for which $S(\langle A;F\rangle^2) \neq S(\langle A;G\rangle^2)$ for any set of full operations G on A. Thus problem 19 of [1] for full universal algebras remains open.

If $S \subseteq P(A^n)$ satisfies (a), (b) and (c), then S satisfies (d) iff S satisfies

(g) if $B \in S$, $B \subseteq \Delta_2 \times A^{n-2}$, then $A \times pr_{2\ldots n}B \in S$,

where $pr_{2\ldots n}B$ is the projection of B onto the last $n-1$ components.

In other words, conditions (a), (b), (c), (e) and (g) give another characterization for $S(\langle A;F\rangle^n)$.

Let S satisfy (d). By (a), $[C]_S\{1,2\} \subseteq [C\{1,2\}]_S$ for all $C \subseteq A^n$. Let now $B \in S$, $B \subseteq \Delta_2 \times A^{n-2}$.

$$[(A \times pr_{2\ldots n}B)(12)]_S\{1,2\} \subseteq [((A \times pr_{2\ldots n}B)(12))\{1,2\}]_S = [B]_S = B.$$

Hence

$$A \times pr_{2\ldots n}B \subseteq [A \times pr_{2\ldots n}B]_S \subseteq A \times pr_{2\ldots n}B$$

i.e.,

$$A \times pr_{2\ldots n}B = [A \times pr_{2\ldots n}B]_S \in S.$$

Conversely, let S satisfy (g) and $\phi \neq C \subseteq A^n$. Then

$$[C]_S\{1,2\} \subseteq \Delta_2 \times A^{n-2} \in S.$$

Hence

$$B = [C\{1,2\}]_S \subseteq \Delta_2 \times A^{n-2}, \quad E = A \times pr_{2\cdots n}B \ \varepsilon\ S \quad \text{and} \quad G = E(12)\ \varepsilon\ S.$$

But $C \subseteq G\ \varepsilon\ S$. Hence $[C]_S \subseteq G$. Thus

$$[C]_S\{1,2\} \subseteq G\{1,2\} = ((A \times pr_{2\cdots n}[C\{1,2\}]_S)(12))\{1,2\} = [C\{1,2\}]_S .$$

A homomorphism h of a join semilattice $\underline{L}$ onto a join semilattice $\underline{L}'$ is said to be correct [2] if for any $a\ \varepsilon\ L$, $b'\ \varepsilon\ L'$, $b' < ah$, there is $b\ \varepsilon\ L$ such that $b \leq a$ and $bh = b'$. h is correct iff h maps ideals of $\underline{L}$ onto ideals of $\underline{L}'$. The mapping $B \to [pr_{1\cdots k}B]$ is a complete semilattice homomorphism of the join semilattices of all subalgebras of $\langle A;F\rangle^n$ onto that of $\langle A;F\rangle^k$. If $\underline{A}$ is (n,k) correct this homomorphism is correct; restricted to finitely generated subalgebras, this mapping remains correct.

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Free products and reduced free products of lattices

by

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1. The purpose of this lecture ** is to direct your attention to a series of papers dealing with the structure of free products of lattices and its applications. Some of the basic ideas go back to P.M. Whitman [17] and R.P. Dilworth [2]. The structure theorem is due to G. Grätzer, H. Lakser, and C.R. Platt [10] and it was to some extent extended by B. Jónsson [14]. Some applications use reduced free products which again go back to R.P. Dilworth [2], and were developed in C.C. Chen and G. Grätzer [1], G. Grätzer [7] and further applied in G. Grätzer and J. Sichler [11] and [12].

In view of the fact that a full proof of the structure theorem has never been given I will state and prove the structure theorem in full detail in §2. Some applications are given without proof in §3. A new approach to reduced free products is given in §4 again with full proofs in view of the fact that the result presented is more general than the one in G. Grätzer [7]. Mostly without proofs, applications are given in §5.

2. For this whole section, let L_i, $i \in I$, be a fixed family of lattices; we assume that L_i and L_j are disjoint for $i, j \in I$, $i \neq j$. We set $Q = \bigcup(L_i, i \in I)$ and we consider Q a poset under the following partial ordering:

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** This lecture is based on Chapter 4 of the forthcoming book [6].

for $a, b \in Q$ let $a \le b$ iff $a, b \in L_i$ for some $i \in I$ and $a \le b$ in L_i.

A free product L of the L_i, $i \in I$, is a free lattice generated by Q, $F(Q)$ ($= F_{\underset{\sim}{L}}(Q)$) (in the sense of Definition 5.2 of [5]). Or, equivalently,

Definition 1. The lattice L is a free product of the lattices L_i, $i \in I$, iff the following conditions are satisfied:

(i) each L_i is a sublattice of L and for $i, j \in I$, $i \neq j$, L_i and L_j are disjoint.

(ii) L is generated by $\bigcup(L_i, i \in I)$.

(iii) for any lattice A, for any family of homomorphisms φ_i: $L_i \to A$, there exists a homomorphism φ: $L \to A$ such that φ on L_i agrees with φ_i for all $i \in I$.

The next definition is a slight adaptation of Definition 4.1 of [5].

Definition 2. Let X be an arbitrary set. The set $\underset{\sim}{P}(X)$ of polynomials in X is the smallest set satisfying (i) and (ii):

(i) $X \subseteq \underset{\sim}{P}(X)$.

(ii) If $p, q \in \underset{\sim}{P}(X)$, then $(p \wedge q), (p \vee q) \in \underset{\sim}{P}(X)$.

The reader should keep in mind that a polynomial is a sequence of symbols and equality means formal equality. As before, parentheses will be dropped whenever there is no danger of confusion.

In what follows, we shall deal with polynomials in $Q = \bigcup(L_i, i \in I)$. Let $a, b, c \in L_i$, $a \vee b = c$. Observe, that as polynomials in Q, $a \vee b$ (which stands for $(a \vee b)$) and c are distinct.

For a lattice A, we define $A^b = A \cup \{0^b, 1^b\}$, where $0^b, 1^b \notin A$; we order A by the rules:

$0^b < x < 1^b$ for all $x \in A$.

$x \leq y$ in A^b iff $x \leq y$ in A, for $x, y \in A$. Thus A^b is a bounded lattice (§6 of [5]). Note, however, that $A^b \neq A$ even if A was itself bounded. It is important to observe that 0^b is meet-irreducible and 1^b is join-irreducible. Thus if $a \wedge b = 0^b$ then either a or b is 0^b, and dually. This will be quite important in subsequent computations.

<u>Definition 3.</u> Let $p \in \underset{\sim}{P}(Q)$ and $i \in I$. The <u>upper</u> i-<u>cover</u> of p, in notation, $p^{(i)}$, is an element of $(L_i)^b$ defined as follows:

(i) for $a \in Q$ we have $a \in L_j$ for exactly one j; if $j = i$, then $a^{(i)} = a$; if $j \neq i$, then $a^{(i)} = 1^b$.

(ii) $(p \wedge q)^{(i)} = p^{(i)} \wedge q^{(i)}$ and $(p \vee q)^{(i)} = p^{(i)} \vee q^{(i)}$ where $\wedge$ and $\vee$ on the right hand side of these equations is to be taken in $(L_i)^b$.

The definition of the <u>lower</u> i-<u>cover</u> of p, in notation, $p_{(i)}$, is analogous, with 0^b replacing 1^b in (i).

An upper cover or a lower cover is <u>proper</u> if it is not 0^b or 1^b. Observe that, however, no upper cover is 0^b and no lower cover is 1^b.

<u>Corollary 4.</u> For any $p \in \underset{\sim}{P}(Q)$ and $i \in I$ we have that

$$p_{(i)} \leq p^{(i)},$$

and if $p_{(i)}$ and $p^{(j)}$ are proper and $p_{(i)} \leq p^{(j)}$, then $i = j$.

<u>Proof.</u> If $p \in X$, then $p = p_{(i)} = p^{(i)}$ so the first statement is true. If the first statement holds for p and q, then

$$(p \wedge q)_{(i)} = p_{(i)} \wedge q_{(i)} \leq p^{(i)} \wedge q^{(i)} = (p \wedge q)^{(i)},$$

and so the first statement holds for $p \wedge q$ and similarly for $p \vee q$. To prove the second statement it is sufficient to verify that if $p_{(i)}$ is proper, then $p^{(j)}$ is not proper for any $j \neq i$. This is obvious for $p \in Q$ by 3(i). If $p = q \wedge r$, and $p_{(i)}$ is proper, then both $q_{(i)}$ and $r_{(i)}$ are proper, hence $q^{(j)} = r^{(j)} = 1^b$, and so $p^{(j)} = 1^b$. Finally, if $p = q \vee r$ and $p_{(i)}$ is proper, then $q_{(i)}$ or $r_{(i)}$ is proper, hence $q^{(j)} = 1^b$ or $r^{(j)} = 1^b$, ensuring $p^{(j)} = q^{(j)} \vee r^{(j)} = 1^b$, completing the proof.

Finally, we introduce a quasi-ordering of $\underset{\sim}{P}(Q)$.

<u>Definition 5.</u> For $p, q \in \underset{\sim}{P}(Q)$, set $p \subseteq q$ iff it follows from rules (i) - (vi) below:

(i) $p = q$.

(ii) For some $i \in I$, $p^{(i)} \leq q_{(i)}$.

(iii) $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$.

(iv) $p = p_0 \vee p_1$ where $p_0 \subseteq q$ and $p_1 \subseteq q$.

(v) $q = q_0 \wedge q_1$ where $p \subseteq q_0$ and $p \subseteq q_1$.

(vi) $q = q_0 \vee q_1$ where $p \subseteq q_0$ or $p \subseteq q_1$.

Definition 5 gives essentially the algorithm we have been looking for. For $p, q \in \underset{\sim}{P}(Q)$, it will be shown that p and q represent the same element of the free product iff $p \subseteq q$ and $q \subseteq p$. We shall show this by actually exhibiting the free product as the set of equivalence classes of $\underset{\sim}{P}(Q)$ under this relation. To be able to do this we have to establish a number of properties of the relation $\subseteq$. All the proofs are by induction and will use the <u>rank</u> of a $p \in \underset{\sim}{P}(Q)$ (see §4 of [5]): for $p \in Q$, $r(p) = 1$; $r(p \wedge q) = r(p \vee q) = r(p) + r(q)$.

<u>Lemma 6.</u> Let $p, q, r \in \underset{\sim}{P}(Q)$ and $i \in I$.

(i) $p \subseteq q$ implies that $p_{(i)} \leq q_{(i)}$ and $p^{(i)} \leq q^{(i)}$.

(ii) $p \subseteq q$ and $q \subseteq r$ implies that $p \subseteq r$.

<u>Proof.</u> Let $p \subseteq q$; we shall prove $p_{(i)} \leq q_{(i)}$ by induction on $r(p) + r(q)$. If $r(p) + r(q) = 2$, then $p, q \in Q$ and so only 5(i) or 5(ii) is applicable to $p \subseteq q$. Hence either $p = q$, in which case $p_{(i)} = q_{(i)}$ or $p^{(j)} \leq q_{(j)}$ for some $j \in I$. This implies that $p^{(j)}$ and $q_{(j)}$ are proper, hence $p = p^{(j)}$, $q = q_{(j)}$, and $p \leq q$. Therefore, $p_{(i)} = p \leq q = q^{(i)}$ if $i = j$, and $p_{(i)} = 0^b \leq 1^b = q^{(i)}$ if $i \neq j$.

Now assume that the implication has been proved for all $p' \subseteq q'$ with $r(p') + r(q') < r(p) + r(q)$.

If $p \subseteq q$ follows from 5(i), then $p = q$, and so $p_{(i)} = q_{(i)}$.

If $p \subseteq q$ follows from 5(ii), then $p^{(j)} \leq q_{(j)}$ for some $j \in I$. If $j = i$, then by Corollary 4 $p_{(i)} \leq p^{(i)} \leq q_{(i)}$, which was to be proved. If $j \neq i$, then by Corollary 4 $p_{(i)} = 0^b$, hence $p_{(i)} \leq q_{(i)}$ is obvious.

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$, say $p_0 \subseteq q$. Thus $(p_0)_{(i)} \leq q_{(i)}$ and so $p_{(i)} = (p_0)_{(i)} \wedge (p_1)_{(i)} \leq (p_0)_{(i)} \leq q_{(i)}$.

If $p \subseteq q$ follows from 5(iv), then $p = p_0 \vee p_1$ where $p_0 \subseteq q$ and $p_1 \subseteq q$. Hence $(p_0)_{(i)} \leq q_{(i)}$ and $(p_1)_{(i)} \leq q_{(i)}$ and so $(p)_{(i)} = (p_0)_{(i)} \wedge (p_1)_{(i)} \leq q_{(i)}$.

If 5(v) or 5(vi) is applicable to $p \subseteq q$, the proof is analogous to the last two cases.

The proof of $p^{(i)} \leq q^{(i)}$ follows by duality.

To prove (ii), let $p \subseteq q$ and $q \subseteq r$. We shall proceed by induction on $a = r(p) + r(q) + r(r)$. If $a = 3$, then $p, q, r \in Q$. If $p = q$ or $q = r$, then $p \subseteq r$ is obvious; otherwise, $p, q, r \in L_i$ for some $i \in I$ and $p \leq r$, so $p \subseteq r$ follows from 5(ii).

Now assume the statement true for sums smaller than a. We can further assume that $p \neq q$ and $q \neq r$.

If $p \subseteq q$ follows from 5(ii), then $p^{(i)} \leq q_{(i)}$ for some $i \in I$. Since $q \subseteq r$, by Corollary 4, $q_{(i)} \leq r_{(i)}$, hence $p^{(i)} \leq r_{(i)}$. Thus $p \subseteq r$, by 5(ii).

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$. Thus, by the induction hypotheses, $p_0 \subseteq r$ or $p_1 \subseteq r$, and so by 5(iii), $p_0 \vee p_1 = p \subseteq r$.

If $p \subseteq q$ follows from 5(iv), then $p = p_0 \vee p_1$, $p_0 \subseteq q$ and $p_1 \subseteq q$, and so again $p_0 \subseteq r$ and $p_1 \subseteq r$, implying $p_0 \vee p_1 = p \subseteq r$ by 5(iv).

If $q \subseteq r$ follows from 5(v) or 5(vi) we can proceed dually (that is, by interchanging $\wedge$ and $\vee$). Only two cases remain; since the second is the dual of the first, we shall state only one:

$q = q_0 \wedge q_1$, 5(v) applies to $p \subseteq q$, and 5(iii) is applicable to $q \subseteq r$ (observe that 5(iv) is not applicable). In this case, 5(v) yields $p \subseteq q_0$ and $p \subseteq q_1$ and 5(iii) yields $q_0 \subseteq r$ or $q_1 \subseteq r$. Hence $p \subseteq q_i \subseteq r$ for $i = 0$ or 1, hence by the induction hypotheses, $p \subseteq r$.

Since by 5(i), $p \subseteq p$ for any $p \in \underset{\sim}{P}(Q)$, the relation $\subseteq$ is a quasi-ordering and so (see Exercise 2.28 of [5]) we can define

$$p \equiv q \text{ iff } p \subseteq q \text{ and } q \subseteq p \quad (p, q \in \underset{\sim}{P}(Q)).$$

$$R(p) = \{q \mid q \in \underset{\sim}{P}(Q) \text{ and } p \equiv q\} \quad (p \in \underset{\sim}{P}(Q)).$$

$$R(Q) = \{R(p) \mid p \in \underset{\sim}{P}(Q)\}.$$

$$R(p) \leq R(q) \text{ if } p \subseteq q.$$

In other words, we split $\underset{\sim}{P}(Q)$ into blocks under the equivalence relation $p \equiv q$; $R(Q)$ is the set of blocks which we partially order under $\leq$.

Lemma 7. $R(Q)$ is a lattice, in fact,

$$R(p) \wedge R(q) = R(p \wedge q) \quad \text{and} \quad R(p \vee q) = R(p) \vee R(q).$$

Furthermore, if $a, b, c, d \in L_i$, $i \in I$, and $a \wedge b = c$, $a \vee b = d$ in L_i, then

$$R(a) \wedge R(b) = R(c) \quad \text{and} \quad R(a) \vee R(b) = R(d).$$

Proof. $p \wedge q \subseteq p$ and $p \wedge q \subseteq q$ by 5(iii). If $r \subseteq p$ and $r \subseteq q$, then $r \subseteq p \wedge q$ by 5(v); this argument and its dual give the first statement.

$c \subseteq a$ and $c \subseteq b$ is obvious by 5(ii), hence $R(c) \leq R(a)$ and $R(c) \leq R(b)$ Now let $R(p) \leq R(a)$ and $R(p) \leq R(b)$ for some $p \in \underset{\sim}{P}(Q)$. Then $p \subseteq a$ and $p \subseteq b$, and so by Lemma 6 $p^{(i)} \leq a^{(i)} = a$ and $p^{(i)} \leq b^{(i)} = b$. Therefore $p^{(i)} \leq c = c_{(i)}$ and thus $p \subseteq c$ by 5(ii). The second part follows by duality.

Let $p, q \in L_i$ $i \in I$ and $R(p) = R(q)$. Then $p \subseteq q$ and $q \subseteq p$. Since only 5(i) and 5(ii) can be applied to these, we easily conclude that $p \leq q$ and $q \leq p$, hence $p = q$. Thus by Lemma 7

$$p \rightarrow R(p) \quad p \in L_i$$

is an embedding of L_i into $R(Q)$. Therefore, by identifying $p \in L_i$ with $R(p)$ we get each L_i as a sublattice of $R(Q)$ and hence $Q \subseteq R(Q)$. It is also obvious that the partial ordering induced by $R(Q)$ on Q agrees with the original partial ordering.

Theorem 8. $R(Q)$ is a free product of the L_i, $i \in I$.

Proof. 1(i) and 1(ii) have already been observed. Let $Q_i: L_i \to A$ be given for all $i \in I$. We define inductively a map

$$\psi: \underset{\sim}{P}(Q) \longrightarrow A$$

as follows: for $p \in Q$ there is exactly one $i \in I$ with $p \in L_i$; set $p\psi = p\varphi_i$; if $p = p_0 \wedge p_1$ or $p = p_0 \vee p_1$, $p_0\psi$ and $p_1\psi$ have already been defined, thus set $p\psi = p_0\psi \wedge p_1\psi$ and $p\psi = p_0\psi \vee p_1\psi$, respectively. Now we prove:

(i) If $p_{(i)}$ is proper, then $p_{(i)}\psi \le p\psi$.

Lemma 9. For $p \in \underset{\sim}{P}(Q)$ and $i \in I$.

(ii) If $p^{(i)}$ is proper, then $p\psi \le p^{(i)}\psi$ for $p \in P(Q)$ and $i \in$

(iii) $p \subseteq q$ implies that $p\psi \le q\psi$ for $p, q \in \underset{\sim}{P}(Q)$.

Proof. (i) If $p \in Q$ and $p_{(i)}$ is proper, then $p \in L_i$, hence $p = p_{(i)}$ and so $p_{(i)}\psi \le p\psi$ is obvious. The induction step is obvious by 3(ii).

(ii) This follows by duality from (i).

(iii) If $p, q \in Q$, then $p, q \in L_i$ for some $i \in I$ and $p \le q$. Therefore, $p\varphi_i \le q\varphi_i$, and so $p\psi \le q\psi$.

If $p \subseteq q$ follows from 5(i), then $p\psi = q\psi$.

If $p \subseteq q$ follows from 5(ii), then, for some $i \in I$, $p^{(i)} \le q_{(i)}$. Thus $p^{(i)}$ and $q_{(i)}$ are proper. Therefore, $p\psi \le p^{(i)}\psi$ by (ii), $p^{(i)}\psi \le q_{(i)}\psi$ because $p^{(i)}$ and $q_{(i)} \in Q$, and $q_{(i)}\psi \le q\psi$ by (i), implying $p\psi \le q\psi$.

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$. Hence $p_0\psi \leq q\psi$ or $p_1\psi \leq q\psi$, therefore $p\psi = p_0\psi \wedge p_1\psi \leq q\psi$.

If $p \subseteq q$ follows from 5(iv) - 5(vi), the proof is analogous to the last one.

Now take a $p \in \underset{\sim}{P}(Q)$ and define

$$R(p)\varphi = p\psi .$$

φ is well-defined since if $R(p) = R(q)$ $(p, q \in \underset{\sim}{P}(Q))$, then $p \subseteq q$ and $q \subseteq p$. Hence by Lemma 9 $p\psi \leq q\psi$ and $q\psi \leq p\psi$, and so $p\psi = q\psi$. Since

$$(R(p) \wedge R(q))\varphi = R(p \wedge q)\varphi = (p \wedge q)\psi = p\psi \wedge q\psi = R(p)\varphi \wedge R(q)\varphi$$

and similarly for $\vee$, we conclude that φ is a homomorphism. Finally, for $p \in L_i$, $i \in I$,

$$R(p)\varphi = p\psi = p\varphi_i$$

by the definition of ψ, hence φ restricted to L_i agrees with φ_i.

Lemma 6(i) implies that if $p \equiv q$ $(p, q \in \underset{\sim}{P}(Q))$, then, for all $i \in I$, $p_{(i)} = q_{(i)}$ and $p^{(i)} = q^{(i)}$. Hence we can define

$$(R(p))_{(i)} = p_{(i)} \quad \text{and} \quad (R(p))^{(i)} = p^{(i)} .$$

All our results will now be summarized. The Structure Theorem of Free Products (G. Grätzer, H. Lakser, and C.R. Platt [10]):

<u>Theorem 10.</u> Let L_i, $i \in I$, be lattices and let L be a free product of the L_i, $i \in I$. Then for every $a \in L$ and $i \in I$ if some element of L_i is contained in a, then there is a largest one with this property, $a_{(i)}$.

If $a = p(a_0, \cdots, a_{n-1})$, where p is a n-ary polynomial and $a_0, \cdots, a_{n-1} \in \bigcup(L_j, j \in I)$, then $a_{(i)}$ can be computed by the algorithm given in Definition 3. Dually, $a^{(i)}$ can be computed. For $a, b \in L$, $a = p(a_0, \cdots, a_{n-1})$, $b = q(b_0, \cdots, b_{m-1})$, $a_0, \cdots, a_{n-1}, b_0, \cdots, b_{m-1} \in \bigcup(L_i, i \in I)$, we can decide whether $a \leq b$ using the algorithm of Definition 5.

3. Using the Structure Theorem of Free Products one can develop a theory which contains most of the known results on free lattices. The normal form theorem of P.M. Whitman [17] stating that the shortest representation of an element of a free lattice is unique up to commutativity and associativity has the following analogue for free products. Let L, L_i, $i \in I$, and Q be as in §2. For $a \in L$ and $p = p(a_0, \cdots, a_{n-1}) \in \underset{\sim}{P}(Q)$ $(a_0, \cdots, a_{n-1} \in Q)$ $a = p$ is a minimal representation of a if $r(p)$ is minimal and we call p a minimal polynomial.

<u>Theorem 1</u> (H. Lakser [15]). Let $p \in \underset{\sim}{P}(Q)$. Then p is a minimal representation iff $p \in Q$, or if $p = p_0 \vee \cdots \vee p_{n-1}$, $n > 1$ where no p_j is a join of more than one polynomial and conditions (i) - (v) below hold, or the dual of the preceding case holds.

(i) Each p_j is minimal, $0 \leq j < n$.

(ii) For each $0 \leq j < n$, $p_j \nleq p_0 \vee \cdots \vee p_{j-1} \vee p_{j+1} \vee \cdots \vee p_{n-1}$.

(iii) If $0 \leq j < r$, $r(p_j) > 1$, $i \in I$, then $(p_j)^{(i)} \nleq p_{(i)}$ in L_i.

(iv) If $p_j = p_j' \wedge p_j''$ $(0 \leq j < n$ and $p_j', p_j'' \in \underset{\sim}{P}(Q))$, then $p_j' \nleq p$.

(v) If $p_j, p_k \in L_i$ $(0 \leq j \leq k < n$ and $i \in I)$, then $j = k$.

Another result of H. Lakser [16] (which is applied in G. Grätzer and J. Sichler [12]) is based on Theorem 1:

<u>Theorem 2.</u> Let M be a sublattice of L, a free product of the L_i, $i \in I$. Assume that $M \cong M_5$ the five-element nondistributive lattice. Then $M \subseteq L_i$ for some i or some L_i has a sublattice isomorphic to $M_5 \times C_2$, where C_2 is the two-element chain.

The most important properties of the free lattice are the following (P.M. Whitman [17] and B. Jónsson [13]):

(W) $x \wedge y \leq u \vee v$ implies that $x \leq u \vee v$ or $y \leq u \vee v$ or $x \wedge y \leq u$ or $x \wedge y \leq v$.

$(SD_\wedge)$ $x \wedge y = x \wedge z = u$ implies that $x \wedge (y \vee z) = u$.

$(SD_\vee)$ is the dual of $(SD_\wedge)$.

The next result is due to G. Grätzer and H. Lakser [9]:

<u>Theorem 3.</u> Let (X) be one of the properties (W), $(SD_\wedge)$, and $(SD_\vee)$. Let A_i be a sublattice of L_i, $i \in I$, and let L be the free product of the L_i, $i \in I$. Let K be a sublattice of L with the property that for all $a \in K$, $a_{(i)}$ and $a^{(i)} \in (A_i)^b$. If all A_i, $i \in I$, satisfy (X), then so does K.

We obtain that the free lattice has (W), $(SD_\wedge)$, and $(SD_\vee)$ by taking $L_i = C_1$ (the one-element chain) and $L = K$.

Naturally, not all results on free lattices have been successfully generalized to free products. As an interesting example I mention the result of F. Galvin and B. Jónsson [4] according to which every chain in a free lattice is countable. A natural generalization of this is the following conjecture:

Let m be a regular cardinal and let L_i, $i \in I$, be lattices with the property that any chain in any of the L_i has cardinality less than m. Then all chains in the free product of the L_i, $i \in I$, have cardinality less than m.

Of course, $m = \aleph_1$ is the most interesting case. The only result relating to the conjecture above is in B. Jónsson [14] in which the general conjecture is reduced to the case $|I| = 2$.

In the same paper, B. Jónsson generalizes some of the results of §2 to K-free products for an arbitrary equational class K of lattices. The problem stated above is completely settled for distributive free product in G. Grätzer and H. Lakser [8].

4. Let L_i, $i \in I$, be bounded lattices and let L be a $\{0, 1\}$-free product of the L_i, $i \in I$. As we shall see, a pair of elements x, y is complementary in L (that is, $x \wedge y = 0$ and $x \vee y = 1$) iff they are complementary in some L_i or if $x_0 \le x \le y_0$, $x_1 \le y \le y_1$ in L_i and $\{x_0, y_0\}$, $\{x_1, y_1\}$ are complementary in L_i. We need a construction in which there are many more complements however we can still keep track of the complements. We call this construction the reduced free product.

In the discussion below let L_i, $i \in I$, be bounded lattices.

Definition 1. A C-relation $\mathcal{C}$ on L_i $i \in I$, is a symmetric binary relation on $\bigcup(L_i, i \in I)$ with the property that if $\{a, b\} \in \mathcal{C}$, $a \in L_i$, $b \in L_j$, then $i \neq j$.

Definition 2. Let $\mathcal{C}$ be a C-relation on L_i, $i \in I$. A lattice L is a $\mathcal{C}$-reduced free product of the L_i, $i \in I$ iff the following conditions hold:

(i) Each L_i, $i \in I$, is a $\{0, 1\}$-sublattice of L and $L = [\bigcup(L_i \mid i \in I)]$.

(ii) If $\{a, b\} \in \mathcal{C}$, then a, b is a complementary pair in L.

(iii) If, for $i \in I$, φ_i is a $\{0, 1\}$-homomorphism of L_i into the bounded lattice A, and $\{a, b\} \in \mathcal{C}$ $(a \in L_i, b \in L_j)$ implies that $a\varphi_i$, $b\varphi_j$ are complementary in A, then there is a homomorphism φ of L into A extending all the φ_i, $i \in I$.

It is obvious that a $\mathcal{C}$-reduced product is unique up to isomorphism. The next result shows that it actually exists, and what is more important we can describe the complementary pairs in it (Theorem 5). Let $Q = \bigcup(L_i, i \in I)$ and define a subset S of $\underset{\sim}{P}(Q)$:

Definition 3. For $p \in \underset{\sim}{P}(Q)$, $p \in S$ is defined by induction on $r(p)$:

(i) $r(p) = 1$, that is, $p \in L_i$ $(i \in I)$ and $p \notin \{0_i, 1_i\}$.

(ii) $p = q \wedge r$ where $q, r \in S$ and the following two conditions hold:

(ii_1) $p \subseteq 0_i$ for no $i \in I$.

(ii_2) $q \subseteq x$ and $r \subseteq y$ for no $\{x, y\} \in \mathcal{C}$.

(iii) $p = q \vee r$ where $q, r \in S$ and the following two conditions hold:

(iii_1) $1_i \subseteq p$ for no $i \in I$.

(iii_2) $x \subseteq q$ and $y \subseteq r$ for no $\{x, y\} \in \mathcal{C}$.

Now we set

$$L = \{0, 1\} \cup \{R(p) \mid p \in S\},$$

and partially order L by

$$0 < R(p) < 1 \quad \text{for} \quad p \in S,$$

$$R(p) \leq R(q) \quad \text{iff} \quad p \subseteq q.$$

If we identify $a \in L_i$ with $R(a)$, then we get the setup we need:

Theorem 4. L is a $\mathcal{C}$-reduced free product of the L_i, $i \in I$.

Proof. L is obviously a poset. To show that L is a lattice we have to find the meet of $R(p)$ and $R(q)$ in L $(p, q \in S)$, and dually. We claim that $R(p) \wedge R(q) = R(p \wedge q)$ if $p \wedge q \in S$ and otherwise $R(p) \wedge R(q) = 0$. This is obvious since if $p \wedge q$ fails (ii_1) or (ii_2), then any $r \subseteq p \wedge q$ will fail (ii_1) or (ii_2).

Now it is obvious that $a \longrightarrow R(a)$ is a $\{0, 1\}$-embedding of L_i into L. So after the identification 2(i) becomes obvious. 2(ii) is clear in view of $3(ii_1)$, $3(ii_2)$, and our description of meet and join in L.

Let K be the free product of the L_i, $i \in I$, as constructed in §1. Then $L - \{0, 1\} \subseteq K$. We define a congruence Θ on K:

$$\Theta = \bigvee(\Theta(x, 0_i) \mid i \in I, x \leq 0_i) \vee \bigvee(\Theta(x, 1_i) \mid i \in I, x \geq 1_i) \vee$$
$$\vee \bigvee(\Theta(x, u \wedge v) \mid x \leq u \wedge v, \{u, v\} \in \mathcal{C}) \vee \bigvee(\Theta(x, u \vee v) \mid x \geq u \vee v, \{u, v\} \in \mathcal{C}).$$

In other words, Θ is the smallest congruence relation under which all 0_i and $u \wedge v$ $(u, v \in \mathcal{C})$ are in the smallest congruence class and dually. We claim that

$$K/\Theta \cong L.$$

To see this, it is sufficient to prove that every congruence class modulo Θ except the two extremal ones contain one and only one element of S.

Let ϵ_i be the identity map as a map of L_i into L. Then there is a map φ extending all ϵ_i, $i \in I$, into a homomorphism of K into L. Let Φ be the congruence induced by φ ($a \equiv b(\Phi)$ iff $a\varphi = b\varphi$). Since L satisfies 2(i) and 2(ii), $\Theta \leq \Phi$. Now if $p, q \in S$, and $R(p)\varphi = R(q)\varphi$, then $R(p) = R(q)$. In other words, $R(p) \equiv R(q)(\Phi)$ implies $R(p) = R(q)$. Therefore, the same holds for Φ. This proves that there is at most one $R(p)$ in the non-extremal congruence classes of Θ. To show "at least one" take a $p \in \underset{\sim}{P}(Q)$ such that $R(p) \not\equiv 0_i(\Theta)$ and $R(p) \equiv 1_i(\Theta)$ (for any/all $i \in I$); we prove that there exists a $q \in S$ such that $R(p) \equiv R(q)(\Theta)$.

Let $p \in L_i$ for some $i \in I$. Then, by assumption, $p \neq 0_i$ and 1_i; hence we can take $q = p$. Let $q = a$, $p = p_0 \wedge p_1$, $R(p_0) \equiv R(q_0)(\Theta)$, $R(p_1) \equiv R(q_1)(\Theta)$ where $q_0, q_1 \in S$. If $q_0 \wedge q_1 \in S$ take $q = q_0 \wedge q_1$. Otherwise, by 3(ii), $q_0 \wedge q_1 \equiv 0_i(\Theta)$, hence $p \equiv 0_i(\Theta)$, contrary to our assumption. The dual argument completes the proof. Thus we have verified that $K/\Theta \cong L$.

Now we are ready to verify 2(iii). For each $i \in I$, let φ_i be a $\{0, 1\}$-homomorphism of L_i into the bounded lattice A. Since K is the free product of the L_i, $i \in I$, there is a homomorphism ψ of K into A extending all the φ_i, $i \in I$. Let Ψ be the congruence induced by ψ (that is, $a \equiv b(\Psi)$ if $a\psi = b\psi$). It obviously follows from the definition of Θ that $\Theta \leq \Psi$. Therefore, by the Second Isomorphism Theorem (see e.g. Lemma 15.8 in [5])

$$[x]\Theta \longrightarrow x\psi$$

is a homomorphism of K/Θ into A. Combining this with the isomorphism $L \cong K/\Theta$ as described above, we get a $\{0, 1\}$-homomorphism φ of L into A extending all the φ_i, $i \in I$.

<u>Theorem 5.</u> Let a, b be a complementary pair in the $\mathcal{C}$-reduced free product L of the L_i, $i \in I$. Then there exist a_0, b_0 and a_1, b_1 such that

$$a_0 \leq a \leq a_1 \quad \text{and} \quad b_0 \leq b \leq b_1$$

such that either $\{a_0, b_0\}, \{a_1, b_1\} \in \mathcal{C}$ or, for some $i \in I$, a_0, b_0 and a_1, b_1 are complementary pairs in L_i, and conversely.

Proof. The converse is, of course, obvious. In either case, by Definition 2, a_0, b_0 and a_1, b_1 are complementary in L, hence

$$a \wedge b \leq a_1 \wedge b_1 = 0 \ , \ a \vee b \geq a_0 \vee b_0 = 1 \ ,$$

and so a, b is complementary in L.

Now to prove the main part of the theorem, take $p, q \in S$ such that $a = R(p)$ and $b = R(q)$ are complementary in L. Then $p \wedge q$ violates $3(ii_1)$ or $3(ii_2)$ and $p \vee q$ violates $3(iii_1)$ or $3(iii_2)$. The four cases will be handled separately.

Case 1. $p \wedge q$ violates $3(ii_1)$ and $p \vee q$ violates $3(iii_1)$. Hence, for some $i, j \in I$, $p \wedge q \subseteq 0_i$ and $1_j \subseteq p \vee q$. Thus in the free product K of the L_i, $i \in I$, $(p \wedge q)^{(i)} = 0_i$ and $(p \vee q)_{(j)} = 1_j$. Note that $q_{(i)}$ is proper, because otherwise $p^{(i)} = 0_i$, that is, $p \subseteq 0_i$ contradicting $p \in S$. Similarly, $q^{(j)}$ is proper. This is a contradiction unless $i = j$, in which case we can put $a_0 = p_{(i)}$, $b_0 = q_{(i)}$, $a_1 = p^{(i)}$, $b_1 = q^{(i)}$ and these obviously satisfy the requirements of the theorem.

Case 2. $p \wedge q$ violates $3(ii_1)$ and $p \vee q$ violates $3(iii_2)$. Hence there exist $i \in I$ and $\{x, y\} \in \mathcal{C}$ such that

$$p \wedge q \subseteq 0_i, \ x \subseteq p, \text{ and } \ y \subseteq q.$$

Let $x \in L_j$ and $y \in L_k$ ($j, k \in I$ and $j \neq k$). Just as in Case 1 we conclude that in K $p^{(i)}$, $q^{(i)}$ are proper, $p^{(i)} \wedge q^{(i)} = 0_i$ $p_{(j)} \geq x$, and $q_{(k)} \geq y$. Hence $i = j$, $i = k$, from which $j = k$ follows, contradicting $j \neq k$.

Case 3. $p \wedge q$ violates $3(ii_2)$ and $p \vee q$ violates $3(iii_1)$. This leads to a contradiction just as Case 2 does.

Case 4. $p \wedge q$ violates $3(ii_2)$ and $p \vee q$ violates $3(iii_2)$. Then there exist $\{a_0, b_0\} \in C$ and $\{a_1, b_1\} \in C$ such that

$$p \subseteq a_1,\ q \subseteq b_1,\ a_0 \subseteq p, \text{ and } b_0 \subseteq q .$$

These obviously satisfy the requirements of the theorem. This completes the proof of Theorem 5.

Theorem 5 is the main result on reduced free products. It is a generalization of the results of G. Grätzer [7], which in turn generalized C.C. Chen and G. Grätzer [1].

5. The simplest application of the results of §4 is to uniquely complemented lattices, that is to lattices in which every element has exactly one complement. A longstanding conjecture of lattice theory was disproved by R.P. Dilworth [2] by showing that not every uniquely complemented lattice is distributive. In fact Dilworth proved that every lattice can be embedded in a uniquely complemented lattice. This result is further sharpened by a theorem of C.C. Chen and G. Grätzer [1]:

Theorem 1. Let L be a bounded lattice in which every element has at most one complement. Then L has a 0 and 1 preserving embedding into a uniquely complemented lattice.

Observe that Theorem 1 implies the Dilworth embedding theorem; indeed, if L is an arbitrary lattice, then by adding a 0 and 1 to L we obtain a lattice L_1 in which every element has at most one complement (in fact if $x \in L_1$, $x \neq 0, 1$, then x has no complement). Apply Theorem 1 to L_1 to get a uniquely complemented lattice containing L as a sublattice.

The proof of Theorem 1 is so simple that we reproduce a sketch of the proof.

If L is complemented, then set $K = L$. Otherwise let $L = L_0$. We define by induction the lattice L_n. If L_{n-1} is defined let I_{n-1} be the set of noncomplemented elements of L_{n-1}. For $i \in I_{n-1}$ let $L_i = \{a_i\}^b$. Define the C-relation C_{n-1} on the family $\{L_{n-1}\} \cup (L_i \mid i \in I_0)$ by the rule

$$\{a, b\} \in C_{n-1} \text{ iff } \{a, b\} = \{i, a_i\} \text{ for some } i \in I_{n-1}.$$

Let L_n be the C_{n-1}-reduced free product. Since

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$

and all these containments are $\{0, 1\}$-embeddings, we can form $K = \bigcup(L_i \mid i \in I)$.

Now consider for $n > 0$ the property

(P_n) if $a_0, b_0, a_1, b_1 \in L_n$, a_0 is a complement of b_0, a_1 is a complement of b_1, $a_0 \leq b_0$, and $a_1 \leq b_1$, then $a_0 = a_1$ and $b_0 = b_1$; if $\{a_0, b_0\}, \{a_1, b_1\} \in \mathcal{C}_n$ and $a_0 \leq a_1$, $b_0 \leq b_1$, then $a_0 = a_1$ and $b_0 = b_1$.

Obviously, (P_0) holds. An easy induction using Theorem 5 of §4 shows that (P_n) holds for all $n \geq 0$. Again by Theorem 5 a is a complement of b in L_n iff the same holds in L_{n-1} or $\{a, b\} \in \mathcal{C}_n$. Therefore we obtain that the direct limit of the L_n is uniquely complemented.

Many variants of Theorem 1 are considered in C.C. Chen and G. Grätzer [1]: Bi-uniquely complemented lattices, lattices in which complementation is a transitive relation, and so on. All these results are based on Theorem 5 of §4.

Another application is to the endomorphism monoid of a bounded lattice. For a bounded lattice L let $\mathrm{End}_{0,1}(L)$ denote the monoid of 0 and 1 preserving endomorphisms of L.

The following result is due to G. Grätzer and J. Sichler [11]:

Theorem 2. Let M be a monoid. Then there exists a bounded lattice L such that

$$M \cong \mathrm{End}_{0,1}(L) .$$

Let $\langle G; R\rangle$ be a graph, that is, a set G with a symmetric binary relation R such that $\langle a, a\rangle \notin R$ for any $a \in G$. We associate with the graph a family of lattices L_a, $a \in G$, where each L_a is a three-element chain $0_a, a, 1_a$. Set $\mathcal{C} = R$; then $\mathcal{C}$ is a C-relation so we can form the $\mathcal{C}$-reduced free product L. We then prove (using Theorem 5 of §4) that every endomorphism extends to a $\{0, 1\}$-endomorphism, and conversely, provided that every element of G lies on a cycle of odd length. We get from the results of Z. Hedrlin and A. Pultr a graph $\langle G; R\rangle$ with $\mathrm{End}(\langle G; R\rangle) \cong M$ satisfying the cycle condition and so we obtain Theorem 2.

The final application I would like to mention concerns hopfian lattices. A lattice L is called <u>hopfian</u> iff $L \cong L/\Theta$ implies that Θ is the trivial congruence relation ω. Equivalently, L is hopfian iff every onto endomorphism is an automorphism.

T. Evans [3] has proved that every finitely presented lattice is hopfian.

Motivated by H. Neumann's results, the question arose whether the free product of two hopfian lattices is hopfian again.

<u>Theorem 3.</u> There exist two bounded hopfian lattices whose bounded free product is not hopfian.

<u>Theorem 4.</u> There exist two hopfian lattices whose free product is not hopfian.

These results are due to G. Grätzer and J. Sichler [12]. Theorem 3 is based on Theorem 2 which reduces Theorem 3 to a graph construction. Theorem 4 is more complicated and it also uses Theorem 2 of §3.

There are many more results on free products and many more results using free products. I hope, however, that this restricted exposition is sufficient to substantiate my claim that the free product is an important construction in lattice theory with which all experts should be familiar.

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SOME UNSOLVED PROBLEMS BETWEEN LATTICE THEORY AND EQUATIONAL LOGIC[1]

Ralph McKenzie

This is a very modest paper. My aim is to have a look at some problems that arise in the regions where lattice theory and equational logic share common ground. The list of problems is selected from my own mathematical experience, and is not intended to be in any way comprehensive or definitive.

Throughout the paper, $L(\Lambda)$ denotes the lattice of equational theories of lattices. Λ is its least element (the set of identities satisfied by every lattice $\langle L,+,\cdot\rangle$), and Ω is its largest element (the set of all lattice identities), while Δ is its one and only maximal element (the equational theory of distributive lattices). If τ is a similarity type of universal algebras, then $L(\tau)$ denotes the lattice constituted by all equational theories of algebras of type τ. If Θ is a member of $L(\tau)$, then $L(\Theta)$ denotes the lattice composed of all $\Theta' \in L(\tau)$ such that $\Theta \leq \Theta'$.

The problems in the first group are the ones of most recent origin. They concern the congruence identities holding in a variety. For an arbitrary equational theory Θ, we can form a theory $\underset{\sim}{Cg}\ \Theta$ belonging to $L(\Lambda)$, which consists of the identities that hold true in the congruence lattice of every algebra belonging to the variety

[1]This is a greatly revised version of the lecture given by the author at the Conference on Lattice Theory, Houston, 1973.

$\mathbf{Var}\ \Theta$ defined by Θ. Everyone is familiar with the theorems which assert that certain properties of congruences in $\mathbf{Var}\ \Theta$ are equivalent to conditions (of "Mal'cev type"), defined directly by reference to Θ itself. For example, the properties "$\mathbf{Cg}\ \theta \geq \Delta$", "$\mathbf{Cg}\ \theta$ includes the modular law" are equivalent to Mal'cev conditions. We recall that each of these properties has very strong implications for the general algebraic theory of $\mathbf{Var}\ \Theta$; see [10] and [9, Cor. 5.5], for instance. Very recently, in [4], it was shown that if Θ is an equational theory of semigroups and if $\mathbf{Cg}\ \Theta \neq \Lambda$, then $\mathbf{Cg}\ \Theta$ includes the modular law. A related paper, [17], revealed that if Θ is any equational theory such that $\mathbf{Cg}\ \Theta$ intersects a certain set of identities which are weaker than modularity then, again, $\mathbf{Cg}\ \Theta$ includes the modular law. Thus it turns out that the set $\{\mathbf{Cg}\ \Theta\colon \Theta$ is an equational theory$\}$ is a proper subset of $L(\Lambda)$--a dramatic development in an area which seemed thoroughly cultivated and not very promising of new results.

I conjecture, very boldly: (1) every theory $\mathbf{Cg}\ \Theta$, distinct from Λ, includes the modular law. And less boldly: (2) $\{\mathbf{Cg}\ \Theta\colon \Theta$ is an equational theory$\}$ is a sublattice of $L(\Lambda)$.

The class of theories known to have a finite base has expanded greatly in the past decade. Many of the proofs of the positive results in this direction have used, at least implicitly, the satisfaction of congruence identities. If we recall some of these results, it will lead us to two conjectures related to (1) above. (I am using two excellent survey articles by S. Oates MacDonald [12] and A. Tarski [19].)

There are equational theories Θ for which every member of $L(\Theta)$ is finitely based, such as the theory of commutative semigroups [18], or of idempotent semigroups [5]. (Note, however, that there exists a 6-element semigroup whose equational theory has no finite base; see [18].) More commonly, the above result does not hold, but the following one does: if $\mathfrak{A}$ is any finite member of $\underset{\sim}{\mathrm{Var}}\ \Theta$, then the identities satisfied by $\mathfrak{A}$ have a finite base. This is known to be the case for the theory of groups (Oates-Powell [13]), for the theory of rings (Kruse [11]), for the theory of lattices (McKenzie [14]), and generalizing the case of lattices, for any theory Θ of a finite similarity type which satisfies $\underset{\sim}{\mathrm{Cg}}\ \Theta \geq \Delta$.[2] Although the Oates-Powell theorem uses the congruence modularity of the variety of groups, it has not yet been generalized in the way my theorem for lattices was generalized by Baker. In fact, the following conjectures are unresolved (they were printed for the first time in [12]): (3) if Θ is any theory of finite type such that $\underset{\sim}{\mathrm{Cg}}\ \Theta$ includes the modular law, then every finite algebra in $\underset{\sim}{\mathrm{Var}}\ \Theta$ has a finite base for its laws; (4) the same conclusion holds if Θ is a theory of finite type and $\underset{\sim}{\mathrm{Cg}}\ \Theta \neq \Lambda$. The first conjecture was made by S. Oates MacDonald and by K. Baker, and S. Burris originated the second. [Burris is said to have some evidence

[2] This result is due to K. Baker [1]; his proof is unpublished. A different proof is given by M. Makkai in a paper soon to appear in Algebra Universalis.

for (4); namely, that for each of the known finite groupoids $\mathcal{G}$ having no finite base of identities, $\mathbf{Cg}\,\boldsymbol{\Theta}\,\boldsymbol{\mathcal{G}} = \Lambda$. Apparently this follows indirectly from the results of [4], since each of these groupoids contains a 2-element subsemigroup that is not a group.]

The second group of problems concerns the free lattice $FL(\omega)$ generated by a denumerable set of freely unrelated elements $\{x_0, x_1, x_2, \ldots\}$. We denote by $FL(n)$, for $1 \leq n < \omega$, the sublattice of $FL(\omega)$ generated by $\{x_0, \ldots, x_{n-1}\}$. The study of these structures shows some analogies, and many differences, to the study of free groups. It is well known for instance that the $FL(\kappa)$ $(\kappa \leq \omega)$, like the free groups, are computable algebras: there is an algorithm (discovered by P. M. Whitman) which tells whether two formal words in the generators define the same element of a free lattice.

Let us denote by $\mathcal{F}$ (respectively, by $\mathcal{F}_0$) the class of all finitely generated (respectively, finite) lattices that are isomorphic to a sublattice of $FL(\omega)$. Then in contrast to the situation for groups, where every subgroup of a free group is itself free, the classes $\mathcal{F}$ and $\mathcal{F}_0$ are highly nontrivial, and there are long-standing open questions about them. Some known facts (from [16]) are the following: $\mathcal{F}$ is precisely the class of finitely generated lattices projective in the category of all lattices, where maps are all the homomorphisms (a result due to A. Kostinsky but proved in [16]); $\mathcal{F}_0$ is a computable class (there is an algorithm for determining whether a finite lattice belongs to $\mathcal{F}_0$); every member of $\mathcal{F}$ has a finite presentation by means of generators and relations, relative to the class of lattices (this was not proved in [16], but is easily demonstrated using the "bounded homomorphisms" discussed there).

B. Jónsson once remarked that every sublattice of a free lattice satisfies three simple conditions (which we formulate below), and over a period of years ([6], [7], and [8]) he has obtained deep results tending to confirm the following conjecture: (5) $\mathscr{F}_0$ is characterized, as a subclass of the class of finite lattices, by the conditions (i)-(iii) below. This conjecture is yet unproved. It appears very plausible that $\mathscr{F}$ is characterized by the same conditions. Jónsson's conditions (for a given lattice L): Let $u_0, u_1, v_0, v_1 \in L$. Then (i) $u_0 \cdot u_1 \leq v_0 + v_1$ implies that either $u_i \leq v_0 + v_1$, or else $u_0 \cdot u_1 \leq v_i$, for some $i = 0,1$; (ii) $u_0 + v_0 = u_0 + v_1$ implies $u_0 + v_0 = u_0 + v_0 \cdot v_1$; (iii) same as (ii) with $+$, $\cdot$ interchanged.

Unlike the situation for groups, it is easily demonstrated that $FL(\kappa)$ and $FL(\lambda)$ (for distinct $\kappa, \lambda \leq \omega$) do not satisfy precisely the same elementary (that is, first-order) sentences. (A famous open problem, due to Tarski, asks whether it is the same with free groups.) For free lattices, as for free groups, the following is open: (6) for each κ ($3 \leq \kappa \leq \omega$), is the elementary theory of $FL(\kappa)$ decidable? Even a very special case of this problem has not been settled. Conjecture: (7) the existential first-order theory of $FL(3)$ is decidable. We should remark that, since $FL(\omega)$ is embeddable into $FL(3)$, all $FL(\kappa)$ for $3 \leq \kappa \leq \omega$ have the same existential theory. The decision problem for this theory is quite different from the so-called "embedding problem for lattices"--the decision problem for the universal theory of lattices--for which Evans [2] gave a positive solution. Also, it is broader in scope than the (solvable) problem of determining membership in $\mathscr{F}_0$.

Here is a concrete elementary sentence which, as shown in [16], has the same truth value in every lattice $FL(\kappa)$ with $3 \leq \kappa < \omega$, and is certainly false in $FL(\omega)$. At present, we have no way of deciding whether it is true or false in FL(3). This would be decided as a particular case by any algorithm giving a positive solution to problem (6).

$$\Phi\colon\ \forall x,y\ \exists u,v\ \forall z.\quad x < y \rightarrow (x \leq u < v \leq y \wedge \neg\, u < z < v).$$

This sentence has importance, independently, from the study of equational theories of lattices. (See [16, Problem 6].) Given a lattice L and two of its members, x and y, we write x/y for the set $\{z\colon x \leq z \leq y\}$. A <u>nontrivial quotient</u> in L is any set of the form x/y having at least two members; an <u>atomic quotient</u> in L is any nontrivial quotient with exactly two members. (So x/y is atomic iff y <u>covers</u> x.) Clearly, L satisfies Φ iff every nontrivial quotient of L contains an atomic quotient of L, in short, iff L is <u>weakly atomic</u>. It turns out that if we identify any $w_0, w_1 \in FL(\omega)$ just in the case that both $w_0/w_0{\cdot}w_1$ and $w_1/w_0{\cdot}w_1$ contain no quotients u/v which are atomic in any FL(n) $(n < \omega)$, then the resulting relation is a fully invariant congruence relation on $FL(\omega)$. This relation is the equational theory of the class of all so-called "splitting lattices." Thus, FL(3) is not weakly atomic just in case there exists a nontrivial lattice identity that holds in each and every splitting lattice.

The third group of problems is a small selection from among those mentioned in [16]. There are many open problems about the abstract structure of $L(\Lambda)$, about particular members of $L(\Lambda)$, and about related properties of equational theories and their models. Among them are the following: (8) Has $L(\Lambda)$ any automorphisms aside from the identity map, and the involution that results from the duality of the two basic operations in lattices? (9) If $\Theta \in L(\Lambda)$, and $L(\Theta)$ is finite, must Θ cover only a finite set of elements of $L(\Lambda)$? (10) Is the equational theory of modular lattices decidable? (This is a long-standing open problem. Only recently, a somewhat richer theory, namely, the universal theory of modular lattices, was proved undecidable by G. Hutchinson.[3] See his article in this volume, and the article by C. Herrmann.) (11) Is it true for every $\Theta \in L(\Lambda)$ that the following are equivalent: (a) $L(\Theta)$ is finite; (b) there is a finite lattice L with $\Theta = \underset{\sim}{\Theta}\,L$? (Compare this with problem (9).)

Finally, I should like to repeat two conjectures about the lattices $L(\tau)$, where τ is an arbitrary similarity type. I proved in [15] basically two results about these lattices: first, that τ is recoverable from the abstract structure of $L(\tau)$; second, that most familiar equational theories--for instance, the theory of groups, of rings, of lattices, or of Boolean algebras--can be singled out

[3] This important result was probably obtained independently and simultaneously by Leonard Lipschitz, and possibly can be found in his doctoral dissertation which was completed about 1972-73 under the direction of Kochen at Princeton.

abstractly in their type lattice and defined as the unique member satisfying a first-order lattice formula $\varphi(x)$, where φ depends of course on the theory to be defined by it. I conjecture: (12) all automorphisms of $L(\tau)$ are basic ones, generated by exchanging operation symbols that have the same rank, and by permuting the "places" of some operation symbols; (13) every member of $L(\tau)$ that is finitely based as a theory, and is a fixed element under all automorphisms of $L(\tau)$, is a first-order definable member of this lattice.

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The Valuation Ring of a Distributive Lattice

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Contents

1. Introduction

Traditionally, the algebraic properties of Boolean algebras are reduced to those of Boolean rings by a well-known construction. A Boolean ring, however, has the double disadvantage of having torsion, and of not being applicable to the richer domain of distributive lattices. In this paper we describe another construction, or functor, called the *valuation ring*, which associates to every distributive lattice L a *torsionless ring* $V(L)$ generated by idempotents. The lattice L can be recovered by giving a suitable order structure to the valuation ring $V(L)$, and thus the entire theory of distributive lattices is reduced to that of a simple class of rings. For example, the representation theory of distributive lattices is subsumed to that of valuation rings, where standard methods of commutative algebra apply.

The applications and further development of the present techniques lie in at least three directions.

First, the valuation ring turns out to be a very simple way of functorially associating a ring to a simplicial complex; we surmise that simplicial homology will benefit from this association.

Second, the theory of pseudo-Boolean functions and programming of Hammer and Rudeanu can be seen to be an informal use of valuation rings; this theory can gain from the rigorous foundation provided by the present ideas.

Third, the notion of quantifier on a Boolean algebra can be transferred to the valuation ring, where it becomes a linear averaging operator; in this way, problems in first-order logic can be translated into problems about commuting sets of averaging operators on commutative rings. The resulting linearization of logic is probably the most promising outcome of the present investigations.

The method of presentation is deliberately informal and discursive. Some of the proofs are barely sketched; we hope to give a thorough presentation elsewhere.

2. The Valuation Ring. The theory of distributive lattices is richer than the better known theory of Boolean algebras; nevertheless it has had an abnormal development, for a variety of reasons of which we shall recall two. First, Stone's representation theorem of 1936 for distributive lattices closely imitated his representation theorem for Boolean algebras, and as a consequence turned out to be too contrived (I have yet to find a person who can state the entire theorem from memory.) Second, a strange prejudice circulated among mathematicians, to the effect that distributive lattices are just Boolean algebra's weak sisters.

More recently, the picture seems to have brightened. The definitive representation theorem for distributive lattices has been proved by H. A. Priestley; it extends at long last to all distributive lattices the duality "distributive lattice - partially ordered sets", first noticed by Birkhoff for finite lattices. Strangely, Nachbin's theory of ordered topological spaces had been available since 1950, but nobody before Priestley had had the idea of taking a totally disconnected ordered topological space as the structure space for distributive lattices.

The second prejudice was more difficult to overcome; it paralleled the criticism of similar prejudices in other branches of mathematics. To stay on comparatively familiar ground, consider what happened in combinatorics. Here, it became clear a short while ago that the notion of set would have to be supplemented by a more pliable notion, which Knuth has called multiset. A multiset is simply a set where every element is assigned a multiplicity, positive negative or zero. Aside from the fact that multisets are found plentifully in nature, they offer a decisive advantage over sets: they form a torsionless ring, where addition and multiplication are defined "elementwise" (Indeed, multisets are functions from a set to the integers.) Sets, on the other hand, have a more rigid algebraic structure: they form a Boolean algebra, or at best a distributive lattice. But it turns out that even for the study of Boolean operations on sets it is preferable to work with the ring of multisets, as was first noted by Whitney; unfortunately, his suggestion went unheeded until recently.

It is this idea that I put forth a few years ago in my paper in the Rado Festschrift (It seems that publishing an idea in a Festschrift is the quickest way to have it forgotten.)

Given a distributive lattice L, can we associate to L a ring V(L) such that, if the lattice L were to be a lattice of sets, then V(L) would "automatically" turn out to be isomorphic to the ring of multisets over the same set?

Such a ring V(L) is easily constructed as follows. To begin with, construct an intermediate ring F(L) consisting of all formal linear combinations of elements of L. Addition is defined formally, and multiplication in two steps: if x and y are elements of L, set $x\,y = x \wedge y$, then extend by bilinearity.

Now, the main fact about the ring F(L) is that the submodule J generated by elements of the form

$$x + y - x \wedge y - x \vee y$$

is an ideal! The verification is easy. For any $z \in L$, we must show that the element

$$z(x + y - x \wedge y - x \vee y) \qquad (*)$$

belongs to J. Expanding this expression we find it equals

$$z \wedge x + z \wedge y - z \wedge (x \wedge y) - z \wedge (x \vee y).$$

We now use various identities satisfied in distributive lattices. The third term equals

$$z \wedge (x \wedge y) = (z \wedge x) \wedge (z \wedge y)$$

using commutative, associative and idempotent laws for the meet operation $\wedge$. The fourth term is simplified by the distributive law:

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) \quad . \qquad (**)$$

Making all these substitutions, we find that (*) equals

$$z \wedge x + z \wedge y - (z \wedge x) \wedge (z \wedge y) - (z \wedge x) \vee (z \wedge y),$$

which clearly belongs to the submodule J, thereby completing the proof that J is an ideal.

Now define the _valuation ring of the distributive lattice L_ to be the quotient ring $V(L) = F(L)/J$.

Before proceeding any further, note the following amusing aside. To define the valuation ring, all we need is a set L, together with two binary operations $\wedge$ and $\vee$, say, such that, (a) the operation $\wedge$ is commutative, associative and idempotent, and (b) the distributive law (**) holds. Nothing else is assumed of the operation $\vee$. Are these identities sufficient to define a distributive lattice?

Now, the construction of the valuation ring is (like every other "construction") a <u>functor</u> from the category of distributive lattices to the category of rings. Thus, every distributive-lattice concept should have an analog for a certain sub-category of rings. For example, an <u>ideal</u> in the lattice-theoretic sense, namely, a subset I of L closed under joins and such that $x \vee y \in I$ for $x \in I$ and $y \in L$, is, when considered as a subset of the valuation ring V(L), an ideal in V(L) in the ring-theoretic sense.

The problem therefore arises of how to recover the lattice L from the valuation ring V(L). Let us consider two special cases. First, suppose that L is the lattice of all subsets of a finite set S. Then the valuation ring V(L) is naturally isomorphic to a ring of multisets on the set S. This non-trivial fact validates our claim that the valuation ring is indeed the algebraic analog of the ring of multisets.

But now take an ordered set P, and let L=L(P) be the lattice of <u>decreasing</u> <u>subsets</u> of P; a subset D of P is decreasing, if $x \in D$ and $y \leq x$ imply that $y \in D$ (decreasing sets are also called order-ideals, but we prefer the former term, recently introduced by Priestley.) Lattice operations are unions and intersections of sets. Then it can be shown that $V(L)$ is isomorphic <u>as a ring</u> to $V(B)$, where B is the Boolean algebra of subsets of P generated by decreasing sets. If P is a finite set, then B is the Boolean algebra of all subsets of P.

In order to strengthen the structure of the valuation ring $V(L)$ we must impose some order structure. We shall do it in the simplest way. A <u>valuation ring</u> $V(L)$ will be a torsion-less commutative ring generated by idempotents, with a distinguished sublattice L of idempotents, such that L generates the ring. In other words, L will be a subset of idempotents closed under products and under the operation $x,y \to x+y-xy$. A morphism of valuation rings ϕ: $V(L) \to V(L')$ is a ring homomorphism which maps L into L'. Every valuation ring $V(L)$ is the valuation ring of the set L considered as a distributive lattice, and the two will be identified.

An element

$$x = \sum_{e \in L} n(e)e \quad , \quad n(e) \geq 0$$

is called <u>monotonic</u>. Monotonic elements are closed under sums and products, in other words they form a cone or semiring. It is possible to characterize a valuation ring in terms of this semiring C, as follows.

A commutative ring R with identity will be called a <u>valuation ring</u> if it is endowed with a distinguished subset, or cone, C, closed under sums and products, and forming a distributive lattice, such that:

(a) The lattice operations in C are compatible with sums and products, that is $f + (g \wedge h) = (f + g) \wedge (f + h)$ similarly with $\vee$, for f, g and h in C, as well as all other identities satisfied in a lattice-ordered commutative ring which can be written without using subtraction;

(b) Every element of C is a (finite) sum of idempotents belonging to C;

(c) Every element of R is the difference of two elements of C.

This intrinsic characterization suggests the extension of the present theory to rings not generated by idempotents; such an extension might give an extension of classical predicate logic (see below.)

The category of valuation rings is equivalent to the category of distributive lattices. It has a generator, namely, the valuation ring of the two-element lattice; we shall see that this fact can be used to obtain a representation theorem for valuation rings. Actually, more is true, but neither category theorists nor first-order logicians have yet invented a precise way of saying it, though the appropriate term was introduced long ago by Birkhoff: the two categories (or first-order theories) are <u>cryptomorphic</u>. In other words, to every fact about one there "naturally" corresponds a fact about the other. The algebraic structure of a valuation ring is richer than that of a ring. It turns out that the linear functional

$$\varepsilon\Big(\sum_{e\varepsilon L} n(e)e\Big) = \sum_{e\varepsilon L} n(e)$$

is an augmentation of the ring, that is, it is a ring homomorphism. Setting

$$f \vee g = \varepsilon(g)f + \varepsilon(f)g - fg, \quad f, g \;\varepsilon\; \mathbf{V(L)}$$

defines a second ring operation on V(L); actually, the same definition works for all augmented algebras.

If L has a minimal element z and a maximal element u, then u acts as an identity and z as an integral (Sweedler) in V(L); that is,

$$fz = \varepsilon(f)z, \quad f \;\varepsilon\; L.$$

In the $\vee$-ring, the roles of u and z are reversed. From now on, we shall assume all valuation rings endowed with u and z, and morphisms to preserve u and z.

The operation of <u>complementation</u> in a valuation ring R is defined as

$$\tau(f) = \varepsilon(f)(u + z) - f, \quad f \in R ,$$

so that in particular

$$\tau(z) = u, \ \tau(u) = z, \ \tau(x) = u + z - x$$

if x is a positive idempotent. Note that the complementation τ is idempotent. Indeed

$$\tau^2(f) = \varepsilon(\tau(f))(u + z) - \tau f =$$

$$= \varepsilon(\varepsilon(f)(u + z) - f)(u + z) - \varepsilon(f)(u + z) + f =$$

$$= (2\,\varepsilon(f) - \varepsilon(f))(u + z) - \varepsilon(f)(u + z) + f = f,$$

as desired. As a further check that the complement τ is indeed a strengthening of the classical lattice-theoretic complement, suppose $R = V(L)$, and let x' be the complement of x in L. Then check that $\tau(x) = x'$. Which identities in distributive lattices carry over to valuation rings? The answer is not hard to guess: all those identities where each variable occurs only once, that is, linearly. For example, the de Morgan law

$$(x \vee y)' = x' \wedge y'$$

carries over to the identity

$$\tau(f \vee g) = \tau(f)\tau(g),$$

but the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

does not, because the variable x occurs twice, or non-linearly, on the right side. It does if one of the entries is idempotent, however.

One of the more interesting identities that carry over to the valuation ring is the <u>inclusion-exclusion principle</u>. It was in fact this identity that originally motivated my definition of the valuation ring. Recall that in the valuation ring, for positive idempotents $x_1, x_2, \ldots, x_n$, one shows that

$$x_1 \vee x_2 \vee \ldots \vee x_n = x_1 + \ldots + x_n - x_1 x_2 - x_1 x_3 - \ldots -$$

$$- x_{n-1} x_n + x_1 x_2 x_3 + \ldots - \ldots + \ldots \pm x_1 x_2 \ldots x_n .$$

For arbitrary elements $f_1, f_2, \ldots, f_n$ one finds

$$f_1 \vee f_2 \vee \ldots \vee f_n = \sum_{i=1}^{n} (-1)^{n-i+1} \sum_{\sigma} \varepsilon(f_{\sigma 1} f_{\sigma 2} \ldots f_{\sigma i}) f_{\sigma(i+1)}$$

$$f_{\sigma(i+2)} \ldots f_{\sigma n} ,$$

where the inner sum ranges over all shuffles σ of the indices $1, 2, \ldots, n$. This identity is valid more generally in any augmented algebra.

3. Canonical Idempotents

Let S be a subset of the monotonic cone of $V(L)$, then the subring generated by S is of the form $V(L')$, where L' is a sublattice of L. Furthermore, if S is finite-dimensional, so is $V(L')$.

Now let L be a finite distributive lattice, and let P be the set of join-irreducibles of L, that is, of those elements $p \varepsilon L$ such that if $p = x \vee y$, then either $p = x$ or $p = y$. Clearly every element of L is the unique irredundant join of join-irreducibles. It is technically preferable not to consider z as a join-irreducible. The join-irreducibles are linearly independent. The Möbius function $\mu(p,q)$ is the integer-valued function on P such that

$$\mu(p,p) = 1$$

$$\mu(p,q) = 0 \quad \text{if } p \not\leq q$$

$$\sum_{p \leq q \leq r} \mu(p,q) = 0 \quad \text{for } p < r.$$

Now set

$$e(p) = \sum_{q \varepsilon P} \mu(q,p) q.$$

It can be shown that the $e(p)$ and z are a set of linearly independent orthogonal idempotents spanning $V(L)$, and that every $x \varepsilon L$ is a **linear combination of the $e(p)$ and z with coefficients 0 or 1;** these properties uniquely characterize the $e(p)$. We shall call them the <u>canonical</u> <u>idempotents</u>. If L is a sublattice of L', then the canonical idempotents of L are sums of those of L', so we may define the canonical idempotents of an arbitrary distributive lattice L as the union of all canonical idempotents of finite sublattices of L. Every linearly independent subset of orthogonal idempotents is then a subset of the set of idempotents of a finite sublattice of L.

The canonical idempotents can be used to derive criteria for the following: when is an $f \varepsilon V(L)$ actually a member of the lattice L, that is, expressible by joins and meets of join-irreducibles? In other words, when is a linear combination

$$f = \sum_{p \varepsilon P} c(p) p$$

actually expressible by the two lattice operations alone? This question is particularly important for free valuation rings (v. below). We shall answer it in two ways.

Expressing f in terms of the canonical idempotents we have

$$f = \sum_{p} a(p)e(p)$$

for some coefficients a, which can be computed in terms of the coefficients **c** .

Now $f \in L$ if and only if

(a) $a(p) = 0$ or 1 for all $p \in P$,

(b) if $a(p) = 1$ and $q \leq p$, then $a(q) = 1$.

In other words, the p for which $a(p) = 1$ form a decreasing set of the set P of join-irreducibles. Since

$$a(q) = \sum_{p \geq q} c(p)$$

this condition can be translated into one in terms of the c-coefficients, which gives the following necessary and sufficient condition for $f \in L$: there exists a <u>decreasing subset</u> A (= **lower** order-ideal: if $p \in A$ and $q \leq p$ then $q \in A$) of join-irreducibles such that

$$(*) \qquad c(q) = \sum_{p \in A} \mu(q,p).$$

For a free valuation ring (v. below) this condition has an elegant topological formulation. The problem whose solution we have just outlined can be restated in purely combinatorial terms: when can a linear combination of idempotents be built up by using only product $x\,y$ and the operation $x + y - x\,y = x \vee y$? There is at least one case when the Möbius function can be explicitly computed and thus the solution can be restated more explicitly, that is the <u>free</u> valuation ring on an ordered set Q. Let Q be a set of commuting idempotents subject to identities $p\,q = p$, which define a partial order $p \leq q$. The monotonic cone generated by sums and products in Q defines the structure of a valuation ring $V(L)$, where L is the distributive lattice freely generated by the ordered set Q. Note that Q is not the set P of join-irreducibles of L; the set P is the set of all distinct products of elements of Q, thus, P is isomorphic to the distributive lattice of increasing sets of Q (order ideals). The Möbius function of P is calculated by the classical inclusion-exclusion principle, and the canonical idempotents are given by the formulas

$$e\Big(\prod_{p\in S} p\Big) = \sum_{B} (-1)^{|B|} \prod_{p\in B} p \quad , \quad S \subseteq Q$$

for every antichain S of Q, the sum ranging over every superset B of S.

Using the canonical idempotents, we can define an order relation in $V(L)$. For any monotonic f, the subring generated by f is of the form $V(L")$ for a finite $L"$. Since every $g \in V(L)$ is of the form $g = f-h$ with monotonic f and h, it follows that $g \in V(L')$ for some finite L'. Hence

$$g = \sum_{p \in P} a(p)e(p) \ ,$$

where P is the set of meet-irreducibles of L', and $e(p)$ the canonical idempotents. Say $g \geq 0$ if $a(p) \geq 0$ for all p. It can be shown that this is an order relation which makes $V(L)$ into a <u>lattice-ordered ring</u>. Note that this is a different order relation from the one defined by the monotonic cone.

The canonical idempotents can be used to systematically solve systems of Boolean equations in a distributive lattice. In fact the notion of <u>pseudo-Boolean function</u> of Hammer and Rudeanu is seen to be a special case of the valuation ring, and much of their theory can be extended to the present context.

4. Representation

If L is finite, then for $x \in L$ we have

$$x = \sum_{p \leq x} e(p) ,$$

and in this way we obtain a representation of every $x \in L$ as the indicator function (characteristic function) of a decreasing subset of the set P of join-irreducibles. The monotonic cone of $V(L)$ is thus represented as the cone of non-increasing functions on P, and $V(L)$ is represented as the ring generated by the indicator functions of increasing subsets of P. We thus obtain a very simple proof of Birkhoff's theorem.

We can extend this result to arbitrary valuation rings. Define $P(L)$ as the set of all prime ideals of the ring $V(L)$ generated by all canonical idempotents. Given any two prime ideals $a,b \in P(L)$, such that $a \not\geq b$ and $a \not\leq b$, we can find two orthogonal idempotents $e,f \in E(L)$ such that $e \in a$ and $f \in b$; now take a finite-dimensional sublattice L' for which $V(L')$ contains both e and f as canonical idempotents; it is then easy to find an increasing element p and a decreasing set q such that $e\,p = e$ and $f\,q = f$.

Now use the canonical idempotents, together with u and z, to define a compact totally *order* disconnected topology on the ordered set P(L). This topology, in view of the above remarks, enjoys the following property: given a,b not comparable, we can find an increasing clopen set p and a decreasing clopen set q such that $a \in p$ and $b \in q$. Such a space is called *totally order disconnected*.

One thus gets the following representation theorem: *every valuation ring is isomorphic to the ring generated by the (monotonic) cone of integer-valued non-increasing continuous functions on a totally order-disconnected compact space*. This representation theorem is easier than the direct representation theorems for lattices, even for Boolean algebras.

Restated in categorical terms, the preceding argument can be made to prove the following. Consider the category *Dis* of distributive lattices having maximal element u and minimal element z, where morphisms are lattice-homomorphisms preserving u and z, as well as the category *Val* of valuation rings,where morphisms are ring homomorphisms preserving u and z and the monotonic cone; finally, the category *Mon* of all rings of continuous integer functions on totally order disconnected compact spaces, endowed with the monotonic cone of all non-increasing functions, and morphisms consisting of all ring homomorphisms preserving the monotonic cone. The three categories are equivalent. (Note that in the category Mon the integral z requires special care.) By this equivalence, a host of questions relating to Boolean algebras and distributive lattices can be simplified.

A variant of the representation theorem replaces prime ideals by morphisms of $V(L)$ into the valuation ring of the two-element distributive lattice. Another variant uses the representation in the finite case and constructs the space $P(L)$ as a categorical limit. This last is perhaps the most satisfactory, though least familiar approach, since it exhibits totally order disconnected spaces as pro-finite ordered sets.

5. Homology

Let P be a finite ordered set. It is well-known that one can associate to P the homology groups of the simplicial complex $\Sigma(P)$ whose faces are all the linearly ordered subsets of P, ordered by inclusion. If P is already a simplicial complex, one obtains ordinary simplicial homology. If P has a unique minimal element z, then the homology of P is trivial. More generally, the rank of the zero-th homology group $H_0(\Sigma(P))$ equals the number of connected components in the Hasse diagram of the ordered set P, but an interpretation of the homology of P in terms of the order of P has not been given.

Now, we can associate to P the valuation ring of the distributive lattice of its decreasing sets, by a (contravariant) functor. This leads to the suspicion that the homology of an ordered set may be defined in an algebraic way by means of the associated valuation ring. It turns out in fact that the Koszul complex construction gives a resolution which is closely related to the simplicial homology of $\Sigma(P)$. Because the technique is not familiar, we briefly describe it here.

Suppose the valuation ring $V(L)$, with set P of join-irreducibles, acts on a module M. The most important case occurs **when** M is a module of integer- **or real-valued** functions on a set S , and the action is obtained by associating to every $p \in P$ the **indicator function of a subset of S , followed by ordinary multiplication. In plain words,** the ordered set P is "represented" by a family of subsets of S , where inclusion of subsets is isomorphic to the order of P . The homology of P thus should be a measure of the complexity of a system of sets, relative to unions and intersections. Note that different modules M can give rise to essentially distinct homologies for the same ordered set P .

For simplicity denote the action of P on M by $(p,m) \to pm$, and list the elements of P , say $p_1, p_2, \ldots, p_n$. Choose anticommutative variables $e_1, \ldots, e_n$ generating an exterior algebra: $e_i e_j = - e_j e_i$. (Note: these are not members of $V(L)$.) Let $E_k(M)$ be the module of all linear combinations of elements of degree k , with coefficients in **M** , that is, of linear combinations of elements of the form

$$m\,(e_{i_1} e_{i_2} \ldots e_{i_k}) , \quad 0 \leq k \leq n , \quad m \in M .$$

Define the boundary **operator** ∂ of such an element by

$$\partial(m\, e_{i_1} e_{i_2} \ldots e_{i_k}) =$$

$$= p_{i_1}\, m\,(e_{i_2} \ldots e_{i_k}) - p_{i_2}\, m\,(e_{i_1} e_{i_3} \ldots e_{i_k}) + \ldots -$$

$$- \ldots \quad + (-1)^{k-1}\, p_{i_k}\, m\,(e_{i_1} e_{i_2} \ldots e_{i_{k-1}}) \;.$$

This is well defined in view of the anticommutativity of the e_i. It is easily verified that $\partial^2 = 0$, so that we obtain a complex associating a **resolution to P and M.**

Our claim is that <u>simplicial homology of an ordered set P can be obtained from the Koszul complex of P considered **as a subset of the valuation ring**</u>.

The following questions may be worth investigating:

(a) Starting with the valuation ring of an infinite distributive lattice L, is it possible to define its homology by approximation by finite sublattices, whose valuation ring is a subring of the valuation ring of L? This might simplify the process of simplicial approximation.

(b) In the finite ordered set P, the submodules M_k generated by $p_1 - p_2 + \ldots + (-1)^{k-1}p_k$ $(p_1 \geq p_2 \geq \ldots \geq p_k)$ generate all of the valuation ring. **Each of these** alternating sums is the indicator of a subset of P; thus we obtain a sequence of increasingly complex subsets of P, whose union is the family of all subsets of P. It is inevitable to conjecture that the dimensions of M_k/M_{k-1} should be related to the Betti numbers of P. This filtration provides a measure of the complexity of a subset of P, which can in turn be used for the study of Boolean functions (see below.)

(c) The Koszul resolution may be expressed in terms of the canonical idempotents, instead of the join-irreducibles. In this way, one obtains an expression for the boundary in terms of the Mobius function. Is it possible in this way to relate the homology to the Mobius function? Judging by the example of geometric lattices, it should be.

(d) It is an open question to construct free resolutions for the valuation ring. Taking the elements of P as generators, one has the relations

$$p\,q = \sum_{r \varepsilon P} c(r) r$$

for suitable coefficients $c(r)$, easily computed in terms of the Mobius function of P. But these relations are not independent, considered as a module over $V(L)$. What are their dependencies? The question is not trivial even in the case of a valuation ring freely generated by an ordered set Q, as considered above, where the only relations are of the form $pq = p$. These relations are not independent; a smaller generating set is obtained by taking only those where q covers p; but even these are not always independent. The question of a free resolution is worth investigating, if only because of the possible connection with the characteristic polynomial of the ordered set P, which, as has been observed, shares some of the properties of the Hilbert polynomial.

In terms of the canonical idempotents, a set of relations is given by the orthogonality relations. However, these are seldom independent; their dependencies depend on linear relations satisfied by the Mobius function.

6. Propositional Calculus

Classical propositional logic is equivalent to the study of the free valuation ring $V(L)$ generated by a sequence of idempotents $x_1, x_2, \ldots$. The elements of this ring will be called Boolean polynomials. The axiomatic of propositional logic amount to an axiomatic for rings generated by idempotents. The constants u and z in the ring $V(L)$ correspond to the propositional constants for truth and falsehood. The implication $p \supset q$ for idempotent p and q turns out to equal $u - p + p\,q$, and the deduction theorem states that if p,q are idempotents and $p \geq q$, then $p \supset q = u$. Verifying that a statement is a tautology amounts to showing that it equals u.

The present context leads to a re-examination of some of the concepts of classical logic, and we shall consider a few by way of example.

Suppose $f,g \in V(L)$ are not idempotents. Is it possible to give a meaning to "f implies g"? For monotonic (or even non-negative) f and g, the natural extension is $f \leq g$.

For a given sequence $f_1, \ldots, f_2$ of Boolean polynomials, not necessarily idempotent, one can define the information of the sequence to be the sublattice L' of L generated by the sequence (with or without taking complements). The complexity of the sequence can then be described by finding a resolution of the set of generators of L', that is the join-irreducibles of L', in the sense of generators and relations. The relations describe, in an intuitive way, the various ways of proving a subset of the f_i from another subset, and the relations between relations give a meaning to the notion "two proofs are equivalent." The Koszul complex built on P or directly on the f_i also gives information on the complexity of Boolean functions. Thus, the study of complexity of Boolean polynomials can be reduced to techniques of commutative algebra.

The duality of classical logic is preserved in the valuation ring: interchanging joins and meets simply interchanges the roles of u and z, and u becomes the integral, whereas z is the unit.

The canonical idempotents of the free valuation ring can be explicitly computed. Any subset A of generators de-defines a join-irreducible

$$x_A = \prod_{x \varepsilon A} x ,$$

and gives for the canonical idempotents $e(A)$ the formula

$$e(A) = \sum_{B \neq \phi} \mu(A,B) x_B .$$

If A is the set $x_1, x_2, \ldots, x_n$, then this can be rewritten as

$$e(A) = x_1 x_2 \ldots x_n (u - x_{n+1})(u - x_{n+2}) \ldots .$$

When is a Boolean polynomial a Boolean function? This question can be interpreted in two ways, according as one admits just meets and joins, or complementation as well. Every Boolean polynomial can be uniquely written as a linear combination of canonical idempotents; it is a Boolean function (including complementation) if **every coefficient** in such an expression is 0 or 1. It is a Boolean function, expressed by **joins** and meets only, if and only if the coefficients which equal 1 form a decreasing set of **P**.

Suppose now that a Boolean polynomial f is given in the form

$$(*) \qquad f = \sum_A c(A) x_A ,$$

where A ranges over a finite set of idempotents. What conditions must the numerical coefficients $c(A)$ satisfy, in order that f be a Boolean function built up out of joins and meets (but not complements)? An elegant answer can be given using the notion of <u>Euler characteristic</u> of a simplicial complex, namely, a family of sets closed under the operation of taking subsets. If Σ is a finite such simplicial complex, and A a member, or "face" of Σ, then the relative simplicial complex (Σ,A) consists of those faces of Σ which contain the face A; let $\chi(\Sigma,A)$ denote the Euler characteristic of the relative simplicial complex (Σ,A). The answer to our question is: a Boolean polynomial (*) is a lattice polynomial if and only if

$$c(A) + 1 = -\chi(\Sigma,A)$$

for some simplicial complex Σ of subsets of the set of join-irreducibles; A ranges through the faces of Σ, and $c(A) = 0$ otherwise.

Now consider the representation of a Boolean polynomial f in terms of joins, meets and complements $\bar{x} = u-x$. In terms of the canonical idempotents a necessary and sufficient condition is that

$$f = \sum_A c(A)e(A) + u \sum_A (|c(A)|-c(A))/2 ,$$

with $c(A) = \pm 1$. Again, this can be turned into a condition in terms of the generators x_i, but we shall not do so. The representation in terms of joins, meets and complements is not unique, as is well-known, and the theory of prime implicants can be developed along present lines. So can the classical theory of Boolean equations.

A (propositional) <u>theory</u> is an <u>ideal</u> in the free valuation ring, generated by <u>Boolean functions</u>, that is, by members of L ; in this case the quotient is again a valuation ring, in general not free; again, the complexity of the axiom system can be investigated by generators and relations, or by finding a suitable basis for the axioms in the valuation ring. Finding the canonical idempotents **explicitly** amounts to solving the decision problem for the theory. We shall illustrate the simplicity of the use of the valuation ring by an example from combinatorics.

Recall that a <u>geometry</u> on a finite set S is a family of n-subsets called **bases** such that if $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ are bases, then for some i, both $(b_i,a_2,\ldots,a_n)$ and $(a_1,b_1,\ldots,\hat{b}_i,\ldots,b_n)$ are bases. A fundamental problem is that of deciding which statements about bases follows from this axiom.

Now one can restate the axiom as an identity in the valuation ring generated by idempotents $(a_1,\ldots,a_n)$ which take the value 1 if the a_i form a basis, and 0 otherwise. The basis axiom then turns into a <u>linear</u> identity, which, simplified by the inclusion-exclusion principle, is

$$(a_1,\ldots,a_n)(b_1,\ldots,b_n) = (b_1,a_2,\ldots,a_n)(a_1,b_2,\ldots,b_n)\vee$$

$$\vee(b_2,a_2,\ldots,a_n)(b_1,a_1,b_3,\ldots,b_n)\vee \ldots \vee(b_n,a_2,\ldots,a_n)$$

$$(b_1,b_2,\ldots,a_1) = \sum_{i=1}^{n} (b_i,a_2,\ldots a_n) \; (a_1,b_1,\ldots,\hat{b}_i,\ldots,b_n) -$$

$$- \sum_{i<j} (b_i,a_2,\ldots,a_n)(b_j,a_2,\ldots,a_n)(a_1,\ldots,\hat{b}_i,\ldots b_n)$$
$$(a_1,\ldots\hat{b}_j,\ldots b_n) + \ldots \quad .$$

This identity can be analyzed by Young's method of standard tableaux. In this way, a decision procedure can be found for combinatorial geometry, and the powerful techniques of representations of the symmetric group can be brought to bear on the problem.

7. Averaging Operators

An averaging operator on a valuation ring $V(L)$ is a linear operator A such that

(1) $A u = u$, $A z = z$.

(2) $A(fAg) = Af\ Ag$.

(3) If f is in the monotonic cone, so is Af.

Sometimes these operators go by the name of Reynolds operators. In probability, they are called <u>conditional expectations</u>. We shall investigate the structure of averaging operators. To this end, it is convenient to consider valuation rings with coefficients in an arbitrary commutative ring R with identity subject to conditions to be specified later, and written $V(L,R)$.

The range of an averaging operator A is a valuation ring of the form $V(L')$, where L' is a sublattice of L . For every $x \varepsilon L$ we have

$$(*) \quad A x = \sum_{e \varepsilon P} c(x,e)e \quad , \quad c(x,e) \varepsilon R ,$$

where P is the set of canonical idempotents of L' other than z , and the sum is finite. We shall characterize an averaging operator by properties of the coefficients $c(x,e)$.

Since $A e = e$ for $e \varepsilon P$ we infer that if $x \wedge e = z$, then $A x \wedge e = z$, or, as we shall say, the support of Ax contains the support of x. Furthermore, we infer

$$(1) \quad c(x \wedge e, e) = c(x,e) .$$

$$(2) \quad c(e,e) = 1 , \quad e \varepsilon P .$$

From the fact that A is linear, or $A(x \vee y) + A(xy) = Ax + Ay$ we add the property

$$(3) \quad c(x \wedge y, e) + c(x \vee y, e) = c(x,e) + c(y,e), \; x,y \varepsilon L,$$

in other words, for fixed e the function c is a valuation on the lattice L. Finally, we have that $A z = z$, so

$$(4) \quad c(z,e) = 0$$

and $A u = u$, whence

$$c(u,e) = 1 .$$

When the lattice L' is finite, and when P is the set of canonical idempotents of L', conditions (1) - (4) on the coefficients c define a unique averaging operator.

When L' is not finite, the right side of (*) is not well-defined; to handle this case, we introduce a seemingly special class of averaging operators. For every finite sublattice π of L', let A_π be an averaging operator whose range is the valuation ring $V(\pi)$, considered as a subring of $V(L)$. If σ is a sublattice of π, we assume that

$$(**) \qquad A_\sigma A_\pi = A_\sigma ,$$

in other words, the operators A_π form a <u>martingale</u> as π runs through all finite sublattices of L'. Now set, for $x \in L$

$$(***) \qquad A\,x = \lim_\pi A_\pi\, x$$

where the limit on the right side means the following: for every $x \in L$ there is a sufficiently large sublattice π of L' such that $A_\pi x = A\,x$, and $A_\sigma x = x$ for all sublattices σ of : L' containing π. We shall say that such an averaging operator is obtained by <u>finite approximation</u>.

Condition (**) implies a condition on the coefficients c, derived as follows. Writing

$$A_\sigma x = \sum_{e\varepsilon P} c(x,e)e; \quad A_\pi x = \sum_{f\varepsilon Q} c(x,f)f,$$

where P and Q are the sets of canonical idempotents of σ and π, we find

$$A_\sigma A_\pi \; x = \sum_e \sum_f c(x,f)c(f,e)e =$$

$$= \sum_e c(x,e)e \quad ,$$

and hence

$$\sum_f c(x,f)c(f,e) = c(x,e) \; .$$

Since σ is a sublattice of π, each canonical idempotent of π is contained in a unique canonical idempotent of σ, and the preceding sum simplifies to

$$\sum_{f \leq e} c(x,f)c(f,e) = c(x,e) \; .$$

Replacing x by xf_o for a fixed canonical idempotent f_o of σ, this gives

$$\sum_{f \leq e} c(xf_o,f)c(f,e) = c(xf_o,e).$$

But $c(xf_o,f) = 0$ unless $f = f_o$, and this sum simplifies to

$$c(xf_o,f_o)c(f_o,e) = c(xf_o,e) ,$$

which in turn can be restated in more elegant form as

$$(5) \qquad c(x,ef)c(f,e) = c(xf,e) .$$

This is the condition for a <u>cocycle</u> in homology. Finally, consider the limit condition (***). If $A\,x$ is given by the right side of (*), and if f is any canonical idempotent of L', then

$$A(xf) = fAx = \sum_e c(x,e)ef = \sum_e c(x,ef)\ ef$$

and thus we have that $c(x,e) = c(x,f)$ for any $f \leq e$; in other words, we require: (6) for every $x \in L$ and every canonical idempotent e of L' such that

$$A\,x = \sum_e c(x,e)e \qquad \text{with } c(x,e) \neq 0$$

one has $c(x,e) = c(x,f)$ for every canonical idempotent f of L' contained in e.

This last condition puts a strong restriction on the sublattice L'. For suppose $f \leq x$; then $c(x,f) = 1$ by (1) and hence $c(x,e) = 1$; thus, if $c(x,e) \neq 1$, then no $f \leq e$ is contained in x. Again, if $f\, x = z$, then $c(x,f) = 0$ by (4); thus, if $c(x,e) \neq 0$, then any canonical idempotent f such that f meets e also meets x, that is, $f \wedge x \neq z$. We conclude that there is a maximal $f \in L'$ contained in x, call it $\forall\mathbf{x}$, and a minimal $e \in L'$ containing x, call it $\exists\mathbf{x}$. The (non-linear !) operators on L

$$x \to \forall\mathbf{x}, \quad x \to \exists\mathbf{x}$$

are quantifiers (universal and existential) in the sense of Halmos, and the sublattice L' must be <u>relatively complete</u> in L.

We thus find that on the right side of (*) one term always is $c(x, \forall\mathbf{x})\forall\mathbf{x}$, with $c(x, \forall\mathbf{x}) = 1$; of the remaining terms, $c(x,e) \neq 0$ only if $e \leq \exists\mathbf{x}$. A function $c\,(x,e)$ defined for $x \in L$ and for all non-zero canonical idempotents $e \in L'$, satisfying condition (1) - (6) is called a <u>fibering</u> of L' by L. We have shown that every averaging operator obtained by finite approximation determines a fibering; conversely, every fibering determines an averaging operator, assuming that L' is relatively complete in L.

Any further statement about the existence of a fibering for a given pair L and L' depends on more delicate measure-theoretic questions. If L and L' are Boolean algebras, the existence of a "universal" fibering can be e established, but this requires a previous classification of subalgebras of a Boolean algebra (Maharam), and cannot be undertaken here. The case of interest in predicate logic is worked out below.

8. <u>Quantifiers</u>.

Every relatively complete Boolean subalgebra L' of a Boolean algebra L defines two quantifiers, the *existential quantifier*

$$\exists x = \inf \{y: y \geq x, y \in L'\}$$

and the universal quantifier

$$\forall x = \sup \{y: y \leq x, y \in L'\}$$

We have seen that every non-trivial averaging operator on the valuation ring $V(L,\mathbf{R})$ defines two such quantifiers. Is it possible to reverse the process? In other words, given L and L', we wish to construct an averaging operator:

$$A x = \sum_e c(x,e)e$$

where e ranges over the canonical idempotents of L', with the following properties:

(a) for $x \in L$, the support of x, that is, $\Sigma\{e: c(x,e)\neq 0\}$, is the idempotent $\exists x$

(b) for $x \in L$, the idempotent $\forall x$ coincides with $\Sigma\{e: c(x,e) = 1\}$.

We shall solve this problem in a special case, which is strong enough to include the quantifiers of predicate logic. It will be simpler to describe the construction in set-theoretic language. Thus, we are given two sets S and T, and on S a Boolean algebra L' of subsets freely generated by elements $y_1, w_1, y_2, w_2, y_3, w_3, \ldots$ such that $y_i \wedge w_i = z$. We identify L' with the Boolean algebra of S-cylinder sets in the product $S \times T$. Now take a Boolean algebra of T-cylinder sets, freely generated by $z_1, z_2, \ldots$.

Set

$$x_i = y_i \vee (z_i \wedge w_i) = y_i + (z_i \wedge w_i) =$$

$$= y_i + t_i .$$

Now let L be the Boolean algebra of subsets of $S \times T$ generated by the x_i, y_i, and w_i. The quantifiers from L to L' can be explicitly described as follows:

(1) If x belongs to the Boolean subalgebra generated by the y_i and w_i, set $\forall x = x$ and $\exists x = x$;

(2) Set $\exists(x \vee y) = \exists x \vee \exists y$

and $\forall(x \wedge y) = \forall x \wedge \forall y$ for all $x,y \in L$;

(3) Set

$$\exists x_i = y_i + w_i, \quad \exists y_i = y_i,$$

$$\exists (x_{i_1} x_{i_2} \ldots x_{i_n}) = (y_{i_1} + w_{i_1}) \ldots (y_{i_2} + w_{i_n}) ,$$

$$\forall x_i = y_i, \quad \forall y_i = y_i ,$$

$$\forall (x_{i_1} \vee \ldots \vee x_{i_n}) = y_{i_1} \vee \ldots \vee y_{i_n},$$

$$\exists \bar{x}_i = \bar{y}_i ,$$

$$\forall \bar{x}_i = \overline{y_i + w_i} .$$

In view of the known properties of quantifiers (Halmos) this gives $\exists x$ for all x in L. We can now construct the averaging operator of L onto L', by choosing a suitable universal ring of coefficients.

Set

$$A\, x_i = y_i + c(x_i, w_i) w_i$$

where the coefficients c belong to an as yet unspecified ring. Define $A(x_{i_1} x_{i_2} \ldots x_{i_n})$ by induction, writing $A(x_1 \ldots x_n)$ for simplicity. Having defined $A(x_1 \ldots x_{n-1})$ we have

$$A(x_1 \ldots x_n) = A(x_1 \ldots x_{n-1}(y_n + t_n)) =$$

$$= y_n A(x_1 \ldots x_{n-1}) + A(x_1 \ldots x_{n-1} t_n) \quad ,$$

so we set

$$A(x_1 \ldots x_{n-1} t_n) = c(x_1 \ldots x_{n-1} x_n, w_1 \ldots w_n) w_1 \ldots w_n \ .$$

Let R be the commutative ring with identity generated by these values of c, together with conditions (1) - (6) of the preceding Section. Condition (6) is made specific by stating that

$$c(x_1 \ldots x_n \ , w_1 \ldots w_n f) = c(x_1 \ldots x_n, w_1 \ldots w_n)$$

for any idempotent f in the range of A. These conditions determine the values of c uniquely, and in fact make it a fibering of L by L'.

To simplify the computation of $A\, x$, write $\overline{x}_i = p_i + q_i$, where $p_i = \overline{y}_i - w_i + t_i$ **is** a cylinder set, and $q_i = w_i - t_i$. Then

$$A\, \overline{x}_i = p_i + c(\overline{x}_i,\, w_i) w_i \ ,$$

as is easily checked.

As an example of computation with the averaging operator A , let us verify that

$$A(x_1 x_2) + A(x_1 \overline{x}_2) = A(x_1) \ .$$

Now,

$$A(x_1\overline{x}_2) = y_1 p_2 + c(x_1, w_1) p_2 w_1 + c(\overline{x}_2, w_2) y_1 w_2 +$$

$$+ c(x_1\overline{x}_2, w_1 w_2) w_1 w_2 \ ,$$

and

$$A(x_1 x_2) = y_1 y_2 + c(x, w_1) y_2 w_1 + c(x_2, w_2) y_1 w_2 +$$

$$+ c(x_1 x_2, w_1 w_2) w_1 w_2 \ .$$

Adding,

$$A(x_1\overline{x}_2) + A(x_1 x_2) = y_1(y_2+p_2) + c(x_1, w_1)(y_2+p_2) w_1 +$$

$$+ y_1 w_2 + c(x_1, w_1 w_2) w_1 w_2 \quad ,$$

where we have used the additivity of e. But $y_2 + p_2 + w_2 = u$, and furthermore $c(x_1, w_1 w_2) = c(x_1, w_1)$ by condition (6) in the definition of a fibering. Simplifying, the right side is seen to equal $y_1 + c(x_1, w_1) w_1$, as desired.

One retrieves the quantifiers from the averaging operator by the following algorithm:

(1) Write $A\,x$ as the sum of multiples of disjoint idempotents, where the multiples are values of c;

(2) To get $\exists x$, replace by 1 all coefficients which are non-zero, and take the sum of the resulting idempotents;

(3) To get $\forall x$, replace by 0 all coefficients which do not equal 1, and take the sum of the remaining idempotents.

We shall informally illustrate the connection with the decision problem for the predicate calculus. Let $x_1, x_2, \ldots$ be predicates in two individual variables: $F_1(x,y), F_2(x,y), \ldots$, and let the y_i and w_i be predicates in one individual variable, such as $G(y)$. In order to analyze the validity or satisfiability of a formula in the predicate calculus quantified in the single variable x, and not necessarily in prenex normal form, reason as follows. Every predicate $F_i(x,y)$ can be decomposed into the disjoint sum of three predicates: $F_{i0}(x,y)$, corresponding to the set of x for which no y

exists for which $F_i(x,y)$ is true, $F_{i1}(x,y)$, corresponding to the set of x for which there exists some y such that $F_i(x,y)$ is true, and $F_{i2}(x,y)$, corresponding to the set of x for which $F_i(x,y)$ is true irrespective of y. Clearly $F_i(x,y) = F_{i0}(x,y) + F_{i1}(x,y) + F_{i2}(x,y)$. Now $F_{i1}(x,y)$ corresponds to t_i, and $F_{i2}(x,y)$ corresponds to y_i. We assign predicates $G_i(x)$ and $H_i(x) = F_{i2}(x,y)$ to t_i and y_i, so that we have

$$A\ F\ (x,y) = c(F,\ G_i)\ G_i\ (x) + H_i(x).$$

By this technique, and its extension to several variables, <u>every formula of the predicate calculus is seen to be equivalent to a formula in a valuation ring endowed with commuting averaging operators</u>. In other words, problems of first-order logic, such as the decision problem, can be shown to be equivalent to algebraic problems for valuation rings with averaging operators.

The case of several commuting quantifiers is technically more complex, but the idea is the same: one considers a Boolean algebra generated by disjoint parellelepipeds of a very special kind in an n-cube; the expression of quantifiers by averaging operator is akin to an Herband expansion, but the linear structure of the valuation ring allows considerable simplifications. We hope to take up these matters elsewhere.

9. Logic and Probability.

In the present context, the algebra of real random variables on a probability space can be viewed as a close analog of a valuation ring, the only difference being that infinite sums of idempotents are allowed. In fact, the passage from predicate logic - i.e., a valuation ring with a set of commuting averaging operators - to probability is achieved by the following steps:

(1) Assign a probability measure μ to the canonical idempotents;

(2) Define an L-space norm on the valuation ring by setting

$$\left|\sum_{e} a(e)e\right| = \Sigma|a(e)|\mu(e) \;;$$

(3) Complete the resulting normed linear space, thereby obtaining an L-space, representable as the space of all integrable functions.

(4) Represent every averaging operator as a conditional expectation operator (in the sense of probability.) Once the restriction that every element of the range be finite-valued is removed, one can show that a conditional expectation operator always exists.

The resulting structure is richer than that of a probability space, because it is endowed in addition with a monotonic **cone of non-decreasing functions.**

By this process certain questions of predicate logic can be seen to be analogous to questions in probability, and new questions in probability are suggested by the analogy - . For example, does the decision problem for averaging operators make sense? Problems of model theory, which can be rephrased and simplified in the context of valuation rings, have analogs for probability spaces. The intriguing possibility arises of handling the decision problem of predicate logic by the techniques of probability.

10. Acknowledgements.

It was L. Solomon who first introduced what we have called canonical idempotents, but his construction remained obscure for several years; he called it the Möbius algebra of an ordered set. A few years later, the present writer introduced the notion of valuation ring of a distributive lattice, quite unaware that it might be related (at least in the finite case) with Solomon's Möbius algebra. It was R. Davis who proved the isomorphism of the two structures; successively, C. Greene made the calculations with canonical idempotents obvious, and used them to systematically derive properties of the Möbius function. It must be pointed out however that the valuation ring is more general than the Möbius algebra, since it does not require any finiteness assumptions.

The valuation ring was later studied by Geissinger in a series of papers; to him is due the existence of an augmentation, the integral, and the elegant duality, which extends to all valuation rings the duality of Boolean algebras.

The representation of distributive lattices in terms of totally order disconnected spaces was recently discovered by Priestley; we have given here the valuation-ring version, which is slightly simpler and tells more. The notion of

quantifier on a Boolean algebra was introduced by Everett and Ulam and extensively studied by Halmos and others, but the precise connection with averaging operators seems to be new, though the analogy had been noted by Wright. Averaging operators on spaces of continuous functions have an extensive literature (Brainerd, Kelley, Wright); in the present context they have not been previously considered.

It seems astonishing that the use of the valuation ring as a technique of proof and as a decision procedure should not have been realized and exploited, even for the propositional calculus. We hope the present paper will contribute to correct this neglect.

The conjectured connection between the homology of an ordered set and the Koszul complex also seems to be new, and we hope its potential usefulness in studies of computational complexity will also be developed.

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PROBLEMS

Problem. Given a partially ordered set P, what subsets of P are images of order-preserving idempotent functions $f: P \longrightarrow P$? (If P is a complete lattice, the answer is: any subset $A \subseteq P$ which is a complete lattice in the induced order.)

Henry Crapo
University of Waterloo

Problems belong to the folklore

1. What are lattices of congruence relations of groupoids (algebras of finite type)? (conjecture: all algebraic lattices)
2. What is the concrete structure of the set of congruence relations for an algebra?

Other problems

1. (See 1. above) Given a complete lattice L, what is the minimum number of operations required to represent L as the congruence lattice of an infinitary algebra? What is the minimum number of operations of rank less than the cordinality of L?
2. If L is a complete (resp., algebraic modular lattice and G is a group, is it always possible to find some infinitary (resp., finitary) algebra A s.t.

   (i) $G \cong \mathrm{Aut}(A)$

   (ii) $L \cong \mathrm{Con}(A)$

   (iii) in $\mathrm{Con}(A)$ $\Theta \vee \Phi = \Theta \Phi \Theta$ for any Θ, Φ?

William A. Lampe
University of Hawaii

1. Problem: Let $V_c(K)$ denote the variety of lattices generated by the congruence lattices of algebras in K, where K is a variety. If $V_c(K)$ is proper, must it be included in the variety of modular lattices?

2. PROBLEM: J. B. Nation has shown (1972) that not every variety of lattices is of the form $V_c(K)$. Characterize $\{V_c(K) : K \text{ a variety}\}$. Is this class a sublattice of the lattice of lattice varieties?

3. PROBLEM: If K is a congruence-modular variety of algebras of finite type, must every finite member of K have a finitely based equational theory?

4. PROBLEM: Are finitely generated free lattices weakly atomic?

5. PROBLEM: If $V(A) \prec V_1$ in the lattice of lattice varieties, and A is a finite lattice, must $V_1 = V(B)$ for some finite B? ($\prec$ means "covered", $V(A)$ is the variety generated by A.)

6. PROBLEM: Is the set of "universal" first order sentences true in the free lattice FL_3 a recursive set?

I make no claim to having originated these problems.

R. McKenzie
University of California
Berkley

If L is a uniquely complemented lattice satisfying a proper lattice identity then L is distributive. (Classically known for modular identity)

R. Padmanabhan
University of Manitoba
Winnipeg

1. Let G be group of automorphisms of a totally ordered set L. Does there exist an integer n such that for every L if G is m-transitive then for every $K \geqslant n$ G is K-transitive.

S. Fajtlowicz
University of Houston

Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

PROBLEMS ON COMPACT SEMILATTICES

1. Let S be a compact, metric, one-dimensional semilattice. Suppose that $\varphi: S \to I$ is an open, monotone, epimorphism. Must φ be an isomorphism?

2. Let S be a compact, metric, finite-dimensional semilattice with small semilattices. Let $x,y \in S$. Does there exist a closed subsemilattice A of S such that $\dim A < \dim S$ and A separates x and y in S?

3. Let S be a compact, connected, finite-dimensional semilattice with small semilattices. Is A the strict projective limit of locally connected semilattices?

4. Let S be a compact, connected, locally connected, one-dimensional semilattice. Is S the strict projective limit of one-dimensional polyhedral semilattices?

5. Let A be a compact space with a closed partial order. Is there a continuous isotone map of A into a compact semilattice S where $\dim A = \dim S$?

6. Consider the class $\mathcal{C}$ of semilattices generated by the min interval by the operations of forming finite products, quotients and closed subsemilattices. Is the class $\mathcal{C}$ precisely the class of compact topological semilattices of finite breadth?

7. Do compact semilattices of finite breadth have the congruence extension property? Does the class $\mathcal{C}$?

8. Let U be an open cover of a compact semilattice S with small semilattices. Does there exist a closed congruence ρ such that the congruence classes of ρ refine U and S/ρ is locally connected and finite-dimensional.

9. Let S be a topological semilattice on an n-cell with boundary B such that $0 \notin B$. Is $B^2 = S$? If $x \in B$ does there exist $y \in B$ such that $xy = 0$?

10. Let S be a one-dimensional, compact, connected similattice with a closed set of end points. Does S have small semilattices?

11. Let S be a topological semilattice on a Peano continuum. Is S an AR?

12. Let S be an n-dimensional semilattice on a Peano continuum. Does S contain an n-cell?

13. Let S be a compact, connected, n-dimensional semilattice. Does there exist $x \in S$ such that $\dim xS = n$?

14. Let S be a locally compact, connected, locally connected semilattice. Is S arcwise connected? Is it acyclic? Suppose that S is not locally connected?

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