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The Filter Space of a Lattice: Its Rôle in General Topology

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Introduction

A <u>filter</u> in a lattice L is a non-void subset F of L for which $x \wedge y \in L$ whenever $x, y \in L$, and $y \in L$ whenever $y \geqslant x$ for some $x \in L$. On the set of all filters F in L one has a naturally arising topology whose basis consists of the sets $\{F \mid a \in F\}$ where $a \in L$; the resulting topological space is the <u>filter space</u> ΦL of the lattice L.

If X is a topological space, with topology $\mathfrak{O}X$, we let $\Phi X = \Phi\mathfrak{O}X$, the filter space of $\mathfrak{O}X$ viewed as a lattice (with set inclusion as its partial order). Each $x \in X$ determines the filter $\mathfrak{O}(x) = \{U \mid x \in U \in \mathfrak{O}X\}$ of its <u>open</u> neighbourhoods, and thus one has the map $X \to \Phi X$ given by $x \rightsquigarrow \mathfrak{O}(x)$. This map is <u>continuous</u> for any X, an <u>embedding</u> for exactly the T_0-spaces X, and in general its image is the <u>reflection</u> of X in the subcategory, of the category of all topological spaces and continuous maps, given by the T_0-spaces. In the following all spaces are taken to be T_0.

The fundamental significance of the embedding $X \to \Phi X$ lies in the fact that a large class of extensions E of a given space X can be realized within ΦX, i.e. are such that the embedding $X \to \Phi X$ can be lifted to an embedding $E \to \Phi X$. The E in question are exactly the <u>strict</u> extensions $E \supseteq X$, i.e. those in which the open sets

$$U^* = \bigcup W \ (X \cap W = U,\ W \in \mathfrak{O}E)$$

form a basis for $\mathfrak{O}E$. This notion goes back to Stone [9]; a detailed account of the rôle of ΦX in this context is given in Banaschewski [1]. The main point about strict extensions of spaces is that many interesting types of extensions (e.g. compactifications, and various of their analogues) are of that kind and hence can be described as, or have actually been explicitly introduced as, suitable subspaces of ΦX.

The use of ΦX in the study of extensions of X has a long history (not to be recalled here); of more recent origin is the result that certain onto maps $E \to X$ can also be realized within ΦX, in such a way that E is embedded into ΦX and the given $E \to X$ corresponds to the operation of taking limits of filter bases in X (Iliadis [7], Banaschewski [2]). This is of importance in the context of projective covers, first introduced for compact and locally compact Hausdorff spaces in Gleason [6].

The purpose of this note is to give an account of the most recent use of the filter spaces ΦX. The notions we are concerned with in this case are the following:

(i) Essential extensions: An extension $E \supseteq X$ of a space X is called essential iff any continuous map $f: E \to Y$ for which $f \mid X$ is an embedding is itself an embedding.

(ii) Injectivity: A space X is called injective (in the category $\mathfrak{T}_0$ of all T_0-spaces and their continuous maps, with respect to embeddings) iff any continuous map f: Y → X lifts to any extension Z ⊇ Y.

(iii) Injective hulls: An essential injective extension of a space X is called an injective hull of X.

These concepts have been investigated, and indeed play an important rôle, in various areas of mathematics. As far as T_0-spaces are concerned, a systematic discussion of injectivity was first given in Scott [8] where the relationship between a particular class of lattices is analyzed in preparation for certain constructions of model theoretic import. Here, we are specifically concerned with the question of the existence of injective hulls and the properties of essential extensions, for which the filter spaces ΦX turn out to provide a natural setting.

The proofs of the results discussed below are given in Banaschewski [3].

1. The Adjointness between Lattices and Spaces.

The correspondence L ⇝ ΦL from lattices to spaces is readily seen to be the object part of a cofunctor (= contravariant functor) from the category $\mathcal{L}$ of all lattices and lattice homomorphisms to the category $\mathcal{T}_0$: For a lattice homomorphism h: L → M, the map Φh: ΦM → ΦL which takes each filter F ⊆ M to the filter $h^{-1}(F)$ is continuous, and the correspondance h ⇝ Φh is functorial. Similarly, one has the "lattice of open sets" cofunctor $\mathfrak{O}$: $\mathcal{T}_0 \to \mathcal{L}$ where $\mathfrak{O}$X, as before, is the topology of X and $\mathfrak{O}f$: $\mathfrak{O}Y \to \mathfrak{O}X$ is again given, for any continuous f: $X \to Y$, by taking inverse images. Φ and $\mathfrak{O}$ are *adjoint on the right*, and the embedding $X \to \Phi X$ introduced above is actually one of the adjunctions. Incidentally, this pair of cofunctors, or some variants of it, provide the starting point for certain studies of duality in Hofmann-Keimel [5]. For the present purpose, the following properties of $\mathfrak{O}$ and Φ are worth noting:

Lemma 1. *A continuous map f is an embedding iff $\mathfrak{O}f$ is onto, and a lattice homomorphism h is onto iff Φh is an embedding*.

By basic categorical principles, an immediate consequence of this is:

Corollary 1. *If a lattice L is projective then its filter space ΦL is injective*.

Now, for lattices one has the following facts: The two-element chain 2 is projective, and every lattice is a homorphic image of a coproduct of two-element chains. It follows from this that the functor Φ produces the corresponding "dual facts". Moreover, the filter space $\Phi 2$ is actually a familiar object, namely the *Sierpinski space* S, i.e. the two-point space with three open sets:

points: 0,1 ; open sets: $\emptyset$, {1}, {0,1}.

Thus one has:

Corollary 2. *S is injective, and every space X can be embedded into a power of S.*

This is well-known (Cech [4], p.485), and can easily enough be proved directly. In the present context it seemed of interest to see how this can be viewed as the counterpart of the rôle of the two-chain among lattices, via the adjointness between $\mathcal{L}$ and $\tilde{\mathfrak{T}}_0$.

As far as the spaces ΦL are concerned, one can actually show much more than the above Corollary 1, but this requires reasoning about specifics rather than general principles. It turns out that the map from the *power set* of a lattice L to ΦL given by generation of filters is continuous if the former is viewed as a power of S; since products and retracts of injectives are injective this proves

Lemma 2. *The filter space of any lattice is injective*.

2. Essential Extensions

Topologies are, of course, *complete* lattices, and for any continuous map $f: X \to Y$ the associated lattice homomorphism $\mathfrak{O}f: \mathfrak{O}Y \to \mathfrak{O}X$ does indeed respect some completeness properties - it preserves arbitrary joins. Thus, the lattice-of-open-sets functor can also be considered as going from $\tilde{\mathfrak{T}}_0$ into the category $\mathcal{JC}\mathcal{L}$ of complete lattices and their *join-complete* homomorphisms. This viewpoint provides a duality for essential embeddings:

Lemma 3. *A continuous map f: X → Y in $\mathfrak{T}_0$ is an essential embedding iff f: Y → X is a coessential onto homomorphism in $\mathcal{JCL}$.*

Here, coessential onto for a homomorphism h: L → M is to mean that h(K) = M iff K = L, for any sublattice K $\subseteq$ L in the sense of $\mathcal{JCL}$, i.e. closed with respect to arbitrary joins in L.

A subspace of the filter space ΦL of a lattice L will be called *separating* iff its members distinguish the elements of L, i.e. for any two distinct elements of L there is a filter in the subspace containing one of them but not the other.

Lemma 4. *For separating subspaces Σ and P ⊇ Σ of a space ΦL, P is an essential extension of Σ iff each F ε P is the join of all G $\subseteq$ F in Σ*.

Putting these lemmas together, one then obtains, with a few additional arguments:

Proposition 1. *For any extension E ⊇ X of a space X*, *the following conditions are equivalent*:

(1) *E is essential*.

(2) *E is strict, and every trace filter of E on X is a join of filters $\mathfrak{O}$(x)*.

(3) *E is superstrict*.

Here, the *trace filters* of E on X are the filters {U ∩ X | U ε $\mathfrak{O}$(y)} for the points y ε E − X, and *superstrict* means that any ring of sets $\mathcal{L} \subseteq \mathfrak{O}E$ which yields a basis for $\mathfrak{O}X$ by restriction to X is itself a basis for $\mathfrak{O}E$.

As a fairly direct consequence one obtains:

Proposition 2. Every space X has a largest essential extension which is unique up to a unique homeomorphism over X, namely the strict extension λX given by the subspace of ΦX consisting of all joins of filters $\mathfrak{O}(x)$.

3. Injective Hulls.

It is clear that the extension λX of a space X is the only possible candidate for being an injective hull of X, and thus X has an injective hull iff λX is injective. More generally, we first consider subspaces Σ of filter spaces ΦL, for an arbitrary lattice L, which are separating and closed with respect to taking joins of filters. Any such determines a kernel operator $k: \Phi L \to \Sigma$ for which kF is the largest $G \in \Sigma$ contained in F. For such Σ and k one then has:

Lemma 5. The following conditions are equivalent:

(1) Σ is injective.

(2) The kernel operator k is continuous.

(3) The kernel operator k preserves updirected joins.

(4) For each $F \in \Sigma$, $F = \bigvee k(F_a)$ $(a \in F)$ where $F_a = \{x \mid x \geqslant a\}$.

A topological criterion for the injectivity of λX which can be derived from this reads as follows:

Proposition 3. _A space X has an injective hull iff, for any $U \in \mathfrak{O}(x)$ $(x \in X)$ there exists a $V \in \mathfrak{O}(x)$ such that $U \cap \Gamma_o V \neq \emptyset$, where $\Gamma_o V = \bigcap \Gamma\{z\}(z \in V)$._

This immediately leads to an "internal" characterization of injectivity itself, and can be used to obtain various further results. For instance: A T_1-space has an injective hull iff it is discrete, and any open subspace of a space which has an injective hull also has an injective hull.

4. Continuous Lattices.

We conclude with some of the results in Scott [8] for which the present setting provides new proofs.

With any partially ordered set S one can associate the space TS whose points are the elements of S and whose topology, the _d-open end topology_, consists of the _ends_ $U \subseteq S$ (i.e. $x \geqslant y$ and $y \in U$ implies $x \in U$) for which $\bigvee A \in U$ implies $A \cap U \neq \emptyset$ for any (up)directed subset $A \subseteq S$.

On the other hand, any space X determines a partially ordered set PX whose elements are the points of X and whose partial order is such that $x \leqslant y$ iff $\mathfrak{O}(x) \subseteq \mathfrak{O}(y)$.

Finally, a partially ordered set S is called a _continuous lattice_ iff S is complete and for any $x \in X$, $x = \bigvee\{\bigwedge U \mid U \in \mathfrak{O}(x)\}$ where $\mathfrak{O} = \mathfrak{O}TS$.

Proposition 4. _(Scott) For any continuous lattice S, TS is an injective space and S = PTS; similarly, for any_ injective space X, PX is a continuous lattice and X = TPX.

It should be added to this that the correspondances S ⇝ TS and X ⇝ PX between continuous lattices and injective spaces can be extended to a category isomorphism, where the maps between the spaces are the continuous maps and the maps between the lattices are those which preserve updirected joins.

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