Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

On the dimensional stability of compact zero-dimensional semilattices

K. H. Hofmann and M. W. Mislove*

Introduction: Let $\underline{Z}$ be the category of compact zero-dimensional semilattice monoids and identity preserving homomorphisms. We consider the question when an object $S \in \underline{Z}$ has the property that each homomorphic image is also in $\underline{Z}$. Equivalently, for which $S \in \underline{Z}$ is $S/R \in \underline{Z}$ for every closed congruence R on S? Lawson [2] has recently considered this question for more general S, and he shows that each finite dimensional locally connected compact semilattice has no dimension raising homomorphisms. However, when applied to objects in $\underline{Z}$, this result only shows that finite objects in $\underline{Z}$ are dimensionally stable.

Our results are not comprehensive, indeed, they are somewhat scattered. However, they do serve our purpose, which is to provide an interesting and informative application of the duality theory developed in [1].

Specifically, we assume that for $S \in \underline{Z}$, $\hat{S} = \underline{Z}(S,2) \in \underline{S}$ (the category of discrete semilattice monoids and identity preserving homomorphisms); dually, that for $S \in \underline{S}$, $\hat{S} = \underline{S}(S,2) \in \underline{Z}$; and that for $S \in \underline{Z}$ or $\underline{S}$, $S \simeq \hat{\hat{S}}$. Moreover, that for $S \in \underline{Z}$, $\hat{S} \simeq (K(S),\vee)$, where $K(S)$ is the set of local minima of S, and, that for $k_1,k_2 \in K(S)$, $k_1 \vee k_2 = \wedge(\uparrow k_1 \cap \uparrow k_2)$, $\uparrow k_i$ being the set of points $s \in S$ with $k_i s = k_i$.

We here wish to record our thanks to A. R. Stralka for several stimulating and informative conversations during the preparation of this material.

*Both authors supported by NSF Grant 28655 A-1.

Definition: An object $S \in \underline{Z}$ is stable if for each closed congruence R on S, $S/R \in \underline{Z}$. Otherwise, S is called instable.

Probably the most natural example of an instable object in $\underline{Z}$ is C, the Canter set in the unit interval I, under min multiplication. Indeed, if $\rho : C \to I$ is the Carathéodory map, then ρ is a continuous surmorphism of semilattices.

Moreover, the property of having I as a semilattice quotient is decisive for instable objects of $\underline{Z}$. Clearly the condition is sufficient. Conversely, if $S \in \underline{Z}$ is instable, then there is a compact semilattice T with a non-degenerate component K and a surmorphism $f : S \to T$. T is a Lawson semilattice since S is ($\underline{Z}(S,2)$ separates the points of S), and so, if $a,b \in K$, $a \neq b$, there is a homomorphism $g : T \to I$ with $g(a) \neq g(b)$. Assuming $g(a) < g(b)$, $g(T) \supseteq [g(a),g(b)]$, and, if $r:I \to [g(a),g(b)]$ is the canonical semilattice retraction, we then have $r \circ g \circ f : S \to [g(a),g(b)]$ is the desired surmorphism. We have proved:

Proposition 1: $S \in \underline{Z}$ is instable if and only if there is a continuous surmorphism $f : S \to I$.

This is a rather simple characterization of the instable objects in $\underline{Z}$; in fact too simple. It sheds little light on the structure of instable objects, and it utilizes an object outside the category $\underline{Z}$ to characterize this notion. We now explore the possibility for a more inherent characterization and we begin by establishing some properties of instable objects.

Proposition 2: If $S \in Z$ is instable, then there is a perfect nondegenerate chain $C \subseteq S$.

Proof. Let S in $\underline{Z}$ be instable. Then, by Proposition 1, there is a surmorphism $f : S \to I$. Define $\tilde{f} : I \to S$ by $\tilde{f}(s) = \wedge f^{-1}(s)$. Clearly f is monotone, (i.e. $t \leq t' \in I$ implies $\tilde{f}(t) \leq \tilde{f}(t')$), $f \circ \tilde{f} = 1_I$, and $\tilde{f}(f(s)) \leq s$ for each $s \in S$. Moreover, if $\{t_\alpha\}_{\alpha\in D} \subseteq I$ with $\{t_\alpha\}_{\alpha\in D} \to t \in I$, and $t_\alpha \leq t$ for each $\alpha \in D$, then

then $\{\tilde{f}(t_\alpha)\}_{\alpha\in D} \to \tilde{f}(t)$ and $\tilde{f}(t_\alpha) \leq \tilde{f}(t)$ for each $\alpha \in D$.

Let $C_o = \tilde{f}(I)^*$. C_o is a compact chain, and if $c \in C_o \setminus \tilde{f}(I)$, then c is not isolated in C_o. If $c = \tilde{f}(t)$ for $t > 0$, then $c = \lim_{n\to\infty} \tilde{f}(t-\frac{1}{n})$, and so c is again not isolated in C_o. Thus $0 = \wedge C_o$ is the only possible isolated point of C_o. We let $C = C_o$ if 0 is not isolated in C_o, and $C = C_o \setminus \{0\}$ otherwise. C is clear by the desired chain.

Corollary: If $S \in \underline{Z}$ is instable, then there is a surmorphism $f \in \underline{S}(\hat{S},D)$ where D is an order dense chain.

Proof: We let C be the chain guaranteed in Proposition 2. Then $i : C \subseteq S$ implies $\hat{i} = f : \hat{S} \twoheadrightarrow \hat{C}$. Since C is a compact perfect chain, $\hat{C} = D$ in an order dense chain.

The question of whether $S \in \underline{Z}$ is instable if and only if S contains a compact perfect nondegenerate chain is settled in the negative by the following.

Lemma: Let X be a set. Then $2^X \in \underline{Z}$ is stable.

Proof: Let $f : 2^X \to I$ be a homomorphism. Since $X = \varinjlim\{F \subseteq X : F$ is finite$\}$, $1 = \lim\{\chi_F : F$ is finite$\}$ Thus, if $t < 1$, there is some finite $F \subseteq X$ with $t < f(\chi_F)$. Now, if $y \in 2^X$ with $f(y) \leq f(\chi_F)$, then $f(\chi_F \cdot y) = f(\chi_F)f(y) = f(y)$. Therefore $f(\chi_F \cdot 2^X) = f(2^X) \cap [0,f(\chi_F)]$ and since F is finite, $\chi_F \cdot 2^X$ is finite. Thus f is not surjective.

Now, let $\mathbb{Q} = \{r \in (0,1] : r$ is rational$\}$. Then $\mathbb{Q}_2 \in \underline{S}$ and $\mathbb{Q}_2 = \widehat{2^{\mathbb{Q}}}$. As we have just seen, $2^{\mathbb{Q}}$ is stable. However, there is a surmorphism $\mathbb{Q}_2 \to \mathbb{Q}$ which extends the identity map on $\mathbb{Q}$, and so, by duality, $2^{\mathbb{Q}}$ contains a compact perfect non-degenerate chain.

Note that for $S \in \underline{Z}$, if there is a surmorphism $f : S \to C$, C the Canter semilattice, then S is instable. Moreover, by duality, this is equivalent to there being a monomorphism $\hat{f} : \mathbb{Q} \hookrightarrow \hat{S}$, i.e. that there is a countable order-dense chain $C_o \subseteq (K(S),\vee)$ with $0 \in C_o$. It is not unreasonable to conjecture that this property characterizes the instable objects in $\underline{Z}$.

As we shall see, this is not the case, but we do have the following.

<u>Theorem 1</u>: Let $S \in \underline{Z}$ and suppose $\hat{S}$ is complete. Then the following are equivalent.

a) S is instable.

b) There is a surmorphism $f \in \underline{Z}(S,C)$. Moreover, if $f : S \to I$ is any surmorphism, then there is a surmorphism $\bar{f} : S \to C$ with $\rho \cdot \bar{f} = f$, $\rho : C \to I$ being the Carathéodory map.

<u>Proof</u>: Clearly b) implies a). Let $f : S \to I$ be a surmorphism. Define $\tilde{f} : I \to S$ as in Proposition 2. Note that, as I is connected, the points of discontinuity of $\tilde{f}$ must be dense in I. Let $0 < t < 1$ be one such point, and set $s = \tilde{f}(t)$. Also, let $u = \wedge\tilde{f}(t,1]$, and $k = \vee\{k' \in K(S) : k' \leq s\}$, where the supremum is taken in $K(S)$. As $s = \vee\{k' \in K(S) : k' \leq s\}$, where this supremum is in S, we have $s \leq k$. But, $s = \tilde{f}(t) = \lim_{t' \uparrow < t} \tilde{f}(t')$ implies $s \notin K(S)$, whence $s < k$.

Note that, for $x \in S$ with $f(x) \geq t$, $f(xs) = f(x)f(s) = t$, and so $s = \tilde{f}(t) \leq xs \leq x$. Hence, if $k' \in K(S)$ with $f(k') > t$, $s \leq k'$, so $k' \geq k''$ for each $k'' \in K(S)$ with $k'' \leq s$. Therefore $k \leq k'$. Now, $f^{-1}(t,1]$ is open in S as f is continuous, and so if $x \in f^{-1}(t,1]$, then $x = \vee(K(S) \cap Sx)$ implies there is $k' \in K(S)$ with $k' \leq x$ and $f(k') > t$. Thus $k \leq k'$, so $k \leq x$. Therefore $k \leq u$.

Let $I_1, I_2 \subseteq I \times 2$ by $I_1 = \{(x,y) : x \leq t$ and $y = 0\}$ and $I_2 = \{(x,y) : t \leq x$ and $y = 1\}$. Define $f_t : S \to I_1 \cup I_2$ by

$$f_t(x) = \begin{cases} (f(x), 1) & \text{if } k \leq x \\ (f(x), 0) & \text{if } k \nleq x \end{cases}.$$ Note that

f_t is a continuous surmorphism of S onto $I_1 \cup I_2$. If $\pi : I_1 \cup I_2$ is a projection on the first coordinate, we clearly have $\pi \circ f_t = f$.

To finish our proof, we now find points $0 < t_1 < t < t_2 < 1$ where f, and hence f_t are discontinuous, and split I_1 and I_2 each into subintervals in analogous fashion to what we just did for I. We continue this process by induction to obtain a system of intervals whose limit is C, the Cantor set. This induces the desired factorization.

To see that the above Theorem is false in general, we construct the following example. Let $\rho : C \to I$ be the Carathéodory map, and let

$$S_o = \{(\rho(x),x) : x \in C\} \subseteq I \times I.$$

For each local minimum $0 \neq k \in C$, choose a sequence $\{p_{kn}\}_{n\in\omega} \subseteq Ck \setminus \{k\}$ with $p_{kn} < p_{kn+1}$ and $\vee p_{kn} = \vee(Ck \setminus \{k\})$. Finally, let $S = S_o \cup \{(\rho(p_{kn}),k) : k \in K(C), n\in\omega\}$. The local minima are precisely the points $(\rho(p_{kn}),k)$. Moreover, if $\pi : S \to I$ is the projection on the first coordinate, there is no factorization of π through C. Indeed, suppose $f : S \to C$ with $\rho \circ f = \pi$. Let $c \in C$ be a local minimum with $c \neq 0$. Then $f^{-1}[c,1]$ is an open-closed subsemilattice of S, and so $p = \wedge f^{-1}[c,1] \in K(S)$. Hence $p = (\rho(p_{kn}),k)$ for some $k \in K(C)$ and $n \in \omega$. If $k' = \vee(Ck \setminus \{k\})$, then $p_{kn} < k'$ and, therefore, $\rho(p_{kn}) < \rho(k') = \rho(k)$. Thus, $\pi(\rho(p_{kn}),k)\pi(\rho(k'),k') = \rho(p_{kn})$. But $\pi(\rho(k'),k') = \rho(k') = \rho(k) = \pi(\rho(k),k)$, whence $f(\rho(k'),k') = f(\rho(k),k)$ or $f(\rho(k),k) \in K(C)$ and $f(\rho(k'),k') = \vee(Cf(\rho(k),k) \setminus \{f(\rho(k),k)\})$. Since $(\rho(p_{kn}),k) < (\rho(k),k)$, we have $t = f(\rho(p_{kn}),k) \leq f(\rho(k),k)$. But $\rho(p_{kn},k) = \wedge f^{-1}\{t\}$ implies $f(\rho(k'),k') \neq f(\rho(k),k)$ since $(\rho(p_{kn}),k) \cdot (\rho(k'),k') \neq (\rho(p_{kn}),k)$. Hence, the only other possibility is $f(\rho(k),k) = t$ and $f(\rho(k'),k') = \vee(Ct \setminus \{t\})$. But, then

$$\begin{aligned}
&\pi((\rho(p_{kn}),k)(\rho(k'),k')) = \rho(f((\rho(p_{kn}),k)(\rho(k'),k'))) \\
&= \rho(f(\rho(p_{kn}),k)f(\rho(k'),k')) = \\
&= \rho(f(\rho(k'),k')) = \pi(\rho(k'),k') = \rho(k') \neq \\
&\neq \pi(\rho(p_{kn}),k)\pi(\rho(k'),k'),
\end{aligned}$$

contradicting that π is a morphism.

We note that this example does not, on the surface, show that an instable object in $\underline{Z}$ must have some map onto the Cantor semilattice. We feel that this property does hold for instable objects, but we are not able to settle the question.

References.

1. Hofmann, K. H., M. W. Mislove, and A. R. Stralka, On the duality of semilattices and its applications to lattice theory, these Proceedings.

2. Lawson, J. D., Dimensionally stable semilattices, Semigroup Forum 5 (1972), 181-185.