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INTRINSIC LATTICE AND SEMILATTICE TOPOLOGIES

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I. Intrinsic Topology Functors.

Lattices and semilattices differ from many other algebraic structures in that there are several rather natural ways to define topologies from the algebraic structure. This chapter is devoted to describing several of these constructions and deriving some of their elementary properties. Some of the proofs that are quite straightforward are omitted.

Definition 1.1. A topology $\mathcal{U}$ on a lattice L is intrinsic if $\mathcal{U}$ is preserved by all automorphisms of L, i.e., if $\alpha \in \operatorname{Aut}(L)$ and $U \in \mathcal{U}$, then $\alpha(U) \in \mathcal{U}$.

Proposition 1.2. The following are equivalent for a topology $\mathcal{U}$ on a lattice L :

(1) $\mathcal{U}$ is intrinsic on L ;

(2) Each automorphism of L is continuous w.r.t. $\mathcal{U}$;

(3) Each automorphism of L is a homeomorphism w.r.t. $\mathcal{U}$.

Proposition 1.3. The intrinsic topologies are a complete sublattice containing 0 and 1 of the lattice of topologies on L .

Proof. It is immediate that the discrete and indiscrete topologies are intrinsic, that the intersection of any collection of intrinsic topologies is again an intrinsic topology, and that the join of two intrinsic topologies is again an intrinsic topology. Hence the proposition follows.

Definition 1.4. Let $\mathcal{L}$ denote the category whose objects are lattices and whose morphisms are lattice homomorphisms. Let $\mathcal{L}_E$ denote the subcategory of $\mathcal{L}$ consisting of the same objects and those morphisms which are isomorphisms. Let $\mathcal{LT}$ denote the category whose objects are pairs $(L,\mathcal{U})$ where L is a lattice and $\mathcal{U}$ is a topology on L and whose morphisms from $(L_1,\mathcal{U})$ to $(L_2,\mathcal{V})$ are lattice homomorphisms which are continuous. An intrinsic topology for lattices is a functor from $\mathcal{L}_E$ to $\mathcal{LT}$ which assigns to an L in $\mathcal{L}_E$ an object $(L,\mathcal{U})$ in $\mathcal{LT}$ and to a morphism $\alpha:L_1 \to L_2$ the morphism defined by α from $(L_1,\mathcal{U})$ to $(L_2,\mathcal{V})$ in $\mathcal{LT}$. Hence isomorphisms in $\mathcal{L}$ must be continuous $\mathcal{LT}$.

A non-empty subset A of a lattice or lower semilattice is lower complete if every non-empty subset B of A has a greatest lower bound which is again in A. Upper complete subsets are defined dually. A non-empty subset which is both upper and lower complete is complete.

There are several natural definitions of convergence in a lattice.

Definition 1.5. Convergence in lattices.

(1) A net $\{x_\alpha\}_{\alpha \in P}$ in a lattice L is said to be ascending if $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$. An ascending net $\{x_\alpha\}$ is said to ascend (or converge) to x if $x = \sup \{x_\alpha : \alpha \in D\}$. The notions of ''descending net'' and ''descend to x'' are defined dually. The notation is $x_\alpha \uparrow x$ ($x_\alpha \downarrow x$) if $\{x_\alpha\}$ ascends (descends) to x.

(2) A net $\{x_\alpha\}_{\alpha \in D}$ in a lattice L is said to order converge to x (denoted $x_\alpha \to x$) if there exists nets $\{y_\alpha\}_{\alpha \in D}$, $\{z_\alpha\}_{\alpha \in D}$ such that $z_\alpha \leq x_\alpha \leq y_\alpha$ for all $\alpha \in D$ and $z_\alpha \uparrow x$ and $y_\alpha \downarrow x$.

(3) A net $\{x_\alpha\}_{\alpha \in D}$ is said to order-star converge to x if every subnet of $\{x_\alpha\}$ has a subnet of it which order converges to x. Order-star convergence is denoted $x_\alpha \overset{*}{\to} x$.

(4) For a net $\{x_\alpha\}_{\alpha \in D}$ in a complete lattice L, by definition,

$$\liminf x_\alpha = \bigvee_\alpha \bigwedge_{\beta \geq \alpha} x_\beta ,$$

$$\limsup x_\alpha = \bigwedge_\alpha \bigvee_{\beta \geq \alpha} x_\beta .$$

A net $\{x_\alpha\}$ is said to lower star converge to x if every subnet of $\{x_\alpha\}$ has a subnet of it which has x as its lim inf. Upper star convergence is defined dually.

A non-empty subset A of a lattice or semilattice is *lower Dedekind complete* if every descending net in A descends to some element of A. *Upper Dedekind completeness* and *Dedekind completeness* are defined in the predictable way.

Proposition 1.6. Let L be a lattice.

(1) If $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$, then $x_\alpha \to x$.

(2) If $x_\alpha \to x$, then so does any subnet.

Hence $x_\alpha \to x$ implies $x_\alpha \overset{*}{\to} x$.

(3) If L is complete, then $x_\alpha \to x$ if and only if

$$x = \limsup x_\alpha = \liminf x_\alpha .$$

Proof. Parts (1) and (2) are straightforward. See [6, p. 244] for part (3).

There are two basic methods of defining intrinsic topology functors. The first of these is declaring a set closed if it contains all of its limit points with respect to some convergence criterion. If the convergence criterion satisfies the condition that any convergent net still converges to the same limit point if the domain of the net is restricted to a cofinal subset, then the closed sets defined in this way actually form a topology of closed sets. The four convergence criteria given in Definition 1.5 all satisfy this condition.

Definition 1.7.

(a) The Dedekind topology (D) .

(b) The order topology (O) .

(c) The lower star topology (L_*) .

A subset A of a lattice L is closed in the Dedekind resp. order resp. lower star topology if whenever $\{x_\alpha\}$ is a net in A which ascends or descends resp. order converges resp. lower star converges to x, then $x \in A$.

(d) The chain topology (χ) . A subset A of a lattice L is closed in the chain topology if for all chains C in A, A also contains $\sup C$ and $\inf C$ if they exist.

A second method of defining intrinsic topologies in lattices is by declaring a certain collection of sets defined in order theoretic terms to be a subbasis for the closed (or open) sets.

First however, we need to introduce certain terminology. If A is a subset of a lattice L,

$$L(A) = \{x \in L: x \leq a \quad \text{for some} \quad a \in A\}$$
$$M(A) = \{y \in L: a \leq y \quad \text{for some} \quad a \in A\} .$$

The non-empty subset A is a semi-ideal if $L(A) = A$ and an ideal if it is both a semi-ideal and a sublattice. A proper ideal I is completely irreducible if whenever I

is the intersection of a collection of ideals, then I is in the collection. Equivalently, I is an ideal which is maximal with respect to not containing some element $x \in L$, $x \neq 0$.

Definition 1.8 (continued).

(e) The interval topology (I). A subbase of closed sets are all sets of the form $L(x)$ and $M(x)$, $x \in L$.

(f) The complete topology (K). A subbase of closed sets is defined by taking as a subbase for the closed sets all sets which contain all inf's and sup's which exist of its non-empty subsets. In complete lattices, these are precisely the complete subsets.

(g) The lower complete topology (LK). A subbase of closed sets is defined by taking all Dedekind closed sets which are lower subsemilattices.

(h) The semi-ideal topology (Σ). A subbase for the closed sets is given by all Dedekind closed semi-ideals together with sets satisfying the dual condition, i.e., $M(A) = A$ and A is Dedekind closed.

(i) The ideal topology (Δ). A subbase for the open sets consists of all completely irreducible ideals and sets which satisfy the dual conditions.

Before defining the last intrinsic topologies, we need to define another notion of convergence.

Definition 1.9. If A is a subset of a lattice L, let $A^{\wedge}$ denote the smallest Dedekind closed lower semilattice containing A. $A^{\vee}$ is defined dually. The net $\{x_\alpha\}$ weakly order converges to x (denoted $x_\alpha \rightarrowtail x$) if

(i) $x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^{\wedge} \subset L(x)$ and dually

(ii) $x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^{\vee} \subset M(x)$.

The net $\{x_\alpha\}$ weakly order star converges to x (denoted $x_\alpha \overset{*}{\rightarrowtail} x$) if every subnet has a subset which weakly order converges to x.

The net $\{x_\alpha\}$ weakly lower star converges to x (denoted $x_\alpha \overset{*}{\hookrightarrow} x$) if condition (i) is satisfied for some subnet of every subnet of $\{x_\alpha\}$. Weak upper star convergence is defined dually.

Definition 1.10.

(j) The weak order topology (WO).

(k) The weak lower star topology (WL_*).

A set A is closed in the weak order resp. weak lower star topology if whenever $\{x_\alpha\}$ is a net in A which weak order star resp. weak lower star converges to x, then $x \in A$.

Proposition 1.11. All the topologies (a)-(k) define intrinsic topology functors.

Definition 1.12. If $(L,\leq)$ is a lattice, the dual lattice is $(L,\geq)$. There is a factor δ on $\mathcal{L}$ which assigns to a lattice the dual lattice and to a morphism the morphism between the dual lattices which as a function is the same as the original function. The functor δ restricted to $\mathcal{L}_E$ is still a functor. A functor δ_T corresponding to δ can be defined on $\mathcal{LT}$ by assigning to $(L,\leq,\mathcal{U})$ the triple $(L,\geq,\mathcal{U})$. Given an intrinsic topology functor Γ , one can define a dual functor Γ' by $\Gamma' = \delta_T\Gamma\delta$. The functor Γ is self-dual if $\Gamma = \Gamma'$.

Proposition 1.13. Let Γ be an intrinsic topology functor which is self-dual. Then any anti-isomorphism between two lattices is continuous.

Proof. Let $\alpha:(L,\leq) \to (L',\leq)$ be an anti-isomorphism. Then $\alpha:(L,\geq) \to (L',\leq)$ is an isomorphism and hence is continuous. Since the topology Γ assigns to $(L,\leq)$ and $(L,\geq)$ are the same, $\alpha:(L,\leq) \to (L',\leq)$ is continuous.

Proposition 1.14. Of the intrinsic topologies (a)-(k), only L_*, LK and WL_* fail to be self-dual. We denote their duals by U_* , UK and WU_* resp.

Definition 1.15. The intrinsic topology functor Γ is finer than the intrinsic topology functor Λ if for each lattice L the topology Γ assigns to L is finer (i.e., has more open sets) than the topology Λ assigns to L . We write

$\Lambda \leq \Gamma$. The relation of "finer than" is a partial order on any subset of intrinsic topology functors.

Definition 1.16. An intrinsic topology functor Γ is linear if for any lattice L and any maximal chain M in L , the topology which Γ assigns to L restricted to M is the same as the topology generated by taking the open intervals of M as a basis.

Proposition 1.17. All topologies (a) - (k) except Δ are linear. If Γ is a linear intrinsic topology functor, then $\Gamma \leq X$.

Definition 1.18. A topology $\mathcal{U}$ on a lattice L is order compatible if it contains the interval topology and is contained in the Dedekind topology. An intrinsic topology functor Γ is order compatible if $I \leq \Gamma \leq D$.

Proposition 1.19. The intrinsic topology functors (a)-(k) are all order compatible except for Δ and X .

Proposition 1.20. Let L be a lattice, and let $\{x_\alpha\}$ be an ascending net in L .

(a) If $\mathcal{U}$ is a topology on L courser than the Dedekind topology and if $x_\alpha \uparrow x$, then $\{x_\alpha\}$ converges to x in the topology of $\mathcal{U}$.

(b) If $\mathcal{U}$ is a topology on L finer than the interval topology and $\{x_\alpha\}$ clusters to x in $\mathcal{U}$, then $x_\alpha \uparrow x$.

Proof. (a) First we show that $\{x_\alpha\}$ converges to x in the Dedekind topology. Let U be an open set in the Dedekind topology which contains x. If $\{x_\alpha\}$ is not residually in U, then cofinally it lies in the complement of U. This cofinal collection of $\{x_\alpha\}$ also ascends to x, and since the complement of U is Dedekind closed, $x \in L\backslash U$, a contradiction. Hence $\{x_\alpha\}$ is residually in U. Thus $\{x_\alpha\}$ converges to x in the Dedekind topology, and hence in any coarser topology.

(b) Since $\mathcal{U}$ is finer than the interval topology $M(x_\alpha)$ is closed for each α. If $x \notin M(x_\alpha)$, then the complement of $M(x_\alpha)$ would be an open set containing x such that $\{x_\alpha\}$ is residually not in this open set, an impossibility. Hence $x_\alpha \leq x$ for all α. Now suppose $x_\alpha \leq y$ for all α. Since $L(y)$ is closed and $x_\alpha \in L(y)$ for all α, it follows that $x \in L(y)$, i.e., $x \leq y$. Thus x is the least upper bound, i.e., $x_\alpha \uparrow x$.

Corollary 1.21. Let L be a lattice, $\{x_\alpha\}$ an ascending net in L, $x \in L$, and $\mathcal{U}$ an order compatible topology on L. The following are equivalent.

(1) $x_\alpha \uparrow x$;

(2) $\{x_\alpha\}$ converges to x;

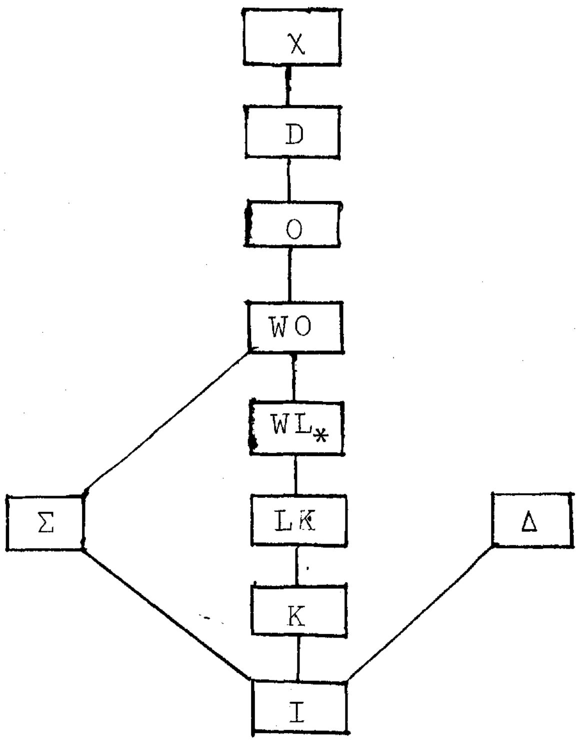
(3) $\{x_\alpha\}$ clusters to x.

Proof. (1) ⇒ (2) . Proposition 19(a).

(2) ⇒ (3) . Immediate.

(3) ⇒ (1) . Proposition 19(b).

Proposition 1.22. The following is a Hasse diagram of the intrinsic topology functors considered for arbitrary lattices (since L_* is defined for complete lattices, it is omitted).



II. Convexity.

Definition 2.1. The convexity functors c, c', σ are functors from $\mathcal{LT}$, the category of lattices with topologies and continuous homomorphisms, back into $\mathcal{LT}$. The functors are defined as follows:

(a) The functor c assigns to an object $(L,\mathcal{U})$ the object $(L,c(\mathcal{U}))$ where $c(\mathcal{U})$ is the topology generated by the open, convex elements of $\mathcal{U}$;

(b) The functor c' assigns to an object $(L,\mathcal{U})$ the object $(L,c(\mathcal{U}))$ where $c'(\mathcal{U})$ is the topology whose closed sets are those generated by the closed, convex sets in $(L,\mathcal{U})$.

(c) The functor σ assigns to an object $(L,\mathcal{U})$ the object $(L,\sigma(\mathcal{U}))$ where $\sigma(\mathcal{U})$ is generated by those open sets of $\mathcal{U}$ which are increasing or decreasing, i.e., those $U \in \mathcal{U}$ such that $M(U) = U$ or $L(U) = U$.

Proposition 2.2. The functor c resp. c' resp. σ is a reflection (categorically) from $\mathcal{LT}$ into the full subcategory of lattices with a basis of open, convex sets, resp. with topology generated by closed, convex sets resp. with topology generated by open, increasing and open, decreasing sets. (For the functor c this means that the following triangle can always be filled in uniquely to be a commutative diagram for any morphism into a locally convex lattice $(M,\mathcal{V})$

$$\begin{tikzcd} (L, c(\mathcal{U})) \arrow[dr, dashed] & \\ (L,\mathcal{U}) \arrow[u] \arrow[r] & (M,\mathcal{V}) \end{tikzcd} .$$

Similar statements hold for c' and σ .)

<u>Definition 2.3</u>. An intrinsic topology functor Γ is <u>convex</u> resp. <u>closed-convex</u> resp. <u>fully convex</u> if Γ composed with c resp. c' resp. σ is again Γ .

<u>Proposition 2.4</u>. The topology $\sigma(\mathcal{U})$ is coarser than $c(\mathcal{U})$ and $c'(\mathcal{U})$; furthermore $c(\sigma(\mathcal{U})) = c'(\sigma(\mathcal{U})) = \sigma(\mathcal{U})$. Hence if an intrinsic topology functor is fully convex, it is both convex and closed-convex.

<u>Proof</u>. Since increasing and decreasing sets are convex, every member of $\sigma(\mathcal{U})$ will be a member of $c(\mathcal{U})$. The complements of the open increasing or open decreasing sets are closed decreasing or closed increasing sets. Hence the closed sets are generated by closed convex sets, and hence every element of $\sigma(\mathcal{U})$ is one of $c'(\mathcal{U})$. The rest of the proposition follows easily.

<u>Proposition 2.5</u>. The intrinsic topology functors Σ, I, and Δ are all fully convex, hence convex and closed-convex.

<u>Proposition 2.6</u>. Let Γ be an order compatible intrinsic

topology functor. If Γ is convex, then $I \leq \Gamma \leq c(D)$. If Γ is fully convex, then $I \leq \Gamma \leq \Sigma$.

Proof. If $\Gamma \leq D$ and Γ is convex, then $\Gamma = c(\Gamma) \leq c(D)$. Similarly if Γ is fully convex, then $\Gamma \leq \sigma(D)$. By Proposition 1.22 $\Sigma \leq D$; hence $\sigma(D) \geq \sigma(\Sigma) = \Sigma$. On the other hand a subbasic closed set in $\sigma(D)$ is an increasing or decreasing set which is Dedekind closed, and hence in Σ . Thus $\sigma(D) \leq \Sigma$. Hence $\Sigma = \sigma(D)$ and $\Gamma \leq \Sigma$.

III. Complete Lattices.

The main purpose of this paper is to study intrinsic topologies in complete lattices and in compact topological lattices and semilattices. A basic and non-trivial result is the following result of Rennie ([19] or [20]).

Theorem 3.1. For a complete lattice L , $c(\chi) \leq O$. (Note: Rennie denotes the topology $c(\chi)$ by L ; we shall call it the convex order topology. Rennie actually proved this theorem for conditionally complete lattices.)

Corollary 3.2. For a complete lattice L , $c(O) = c(D) = c(\chi)$.

Proof. Since c is a functor on $\mathcal{LT}$, it follows from Proposition 1.22 that $c(O) \leq c(D) \leq c(\chi)$. But since

$c(\chi) \leq 0$ and c is a reflection, $c(\chi) \leq c(0)$.

Diagram 3.3. The following is a Hasse diagram for the principal intrinsic topology functors which we have considered for the category of complete lattices. M. Stroble is preparing a master's thesis which contains a much more exhaustive account of relationships between intrinsic topologies. All dominations in the diagram are fairly easy to establish either by straightforward arguments or using earlier results.

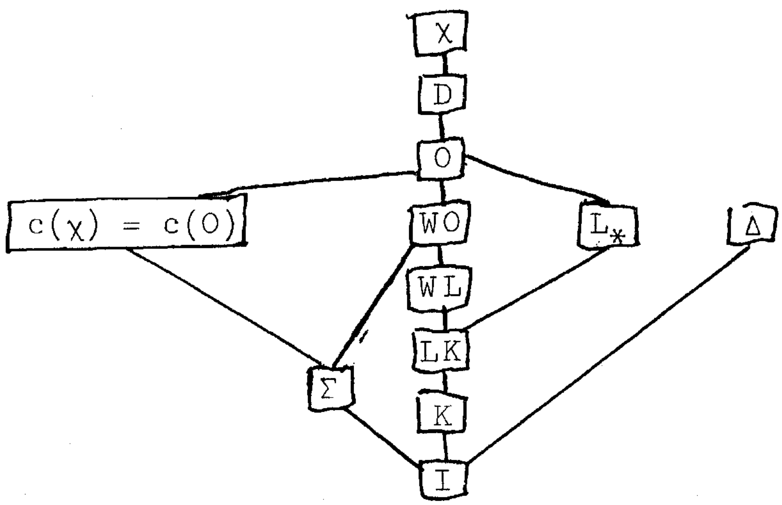


Diagram 3.3

Proposition 3.4. For complete lattices,

$$\Sigma = \sigma(\chi) = \sigma(D) = \sigma(O) = \sigma(WO) = \sigma(WL_*) = \sigma(c(\chi)) .$$

Proof. By Proposition 2.6 we have $\sigma(D) \leq \Sigma$. Since σ is a reflection and Σ is fully convex

$\Sigma = \sigma(\Sigma) \leq \sigma(O), \sigma(WO), \sigma(WL_*), \sigma(c(\chi)) \leq \sigma(D) \leq \Sigma$, which shows all equalities except $\Sigma = \sigma(\chi)$. Now $\Sigma = \sigma(\Sigma) \leq \sigma(\chi) = \sigma\sigma(\chi) \leq \sigma c(\chi) \leq \Sigma$; hence $\Sigma = \sigma(\chi)$.

Because of the extensive collapsing that takes place at $c(O)$ and Σ , these two intrinsic topology functors are of special interest. They are the finest linear topologies that are convex and fully convex resp.

We turn now to consideration of the behavior of these topologies with respect to subspaces, products, and homomorphic images.

A. Subspaces. Most intrinsic topologies of Diagram 3.3 are hereditary for complete sublattices.

<u>Proposition 3.5</u>. Let L be a complete lattice and let M be a complete subset of L . Then for the functors χ, D, O, WO, WL_*, LK, K, L_*, and I the topology assigned to M agrees with the one assigned to L restricted to M . For Σ and $c(O)$ the identity function on M is continuous from the topology assigned to M to the subspace topology and vice-versa for Δ .

<u>Proof</u>. All the verifications are quite straight-forward. For Δ note that an ideal P in M maximal with respect to missing $x \in M$ can be extended to an ideal in L maximal with respect

to missing x whose intersection with M is P .

Note that a complete sublattice is closed in the K topology and hence in any finer topology on L .

B. Products. Proposition 3.6. Let $\{L_\alpha : \alpha \in A\}$ be a collection of complete lattices. For all intrinsic topology functors of Diagram 3.3 the identity function from ΠL_α with the topology assigned to it by the functor to the product topology of the topologies assigned to each coordinate is continuous. The interval topology I is productive.

C. Homomorphic images. Continuity in intrinsic topologies we are considering is closely related to the preservation of limits of increasing and decreasing nets.

Definition 3.7. Let L and M be complete lattices. An order-preserving function f from L into M is linearly complete if for any chain C in L $f(\inf C) = \inf f(C)$ and $f(\sup C) = \sup (f(C))$; f is complete if for any $x_\alpha \uparrow x$ and $y_\beta \downarrow y$ we have $f(x_\alpha) \uparrow f(x)$ and $f(y_\beta) \downarrow f(y)$. Note that for the case f is a homomorphism, f is complete if and only if f preserves arbitrary joins and meets.

Proposition 3.8. Let f be an order-preserving function from L to M which is continuous from the Dedekind (chain) topology on L to the interval topology on M . Then f is complete (linearly complete).

Proof. Let $x_\alpha \uparrow x$. By Proposition 1.20(a) $\{x_\alpha\}$ converges to x in the Dedekind topology. Hence $\{f(x_\alpha)\}$ converges to $f(x)$ in the interval topology on M . By 1.20(b) $f(x_\alpha) \uparrow f(x)$. Similarly $x_\alpha \downarrow x$ implies $f(x_\alpha) \downarrow f(x)$. Hence f is complete.

The linearly complete case is analogous.

Proposition 3.9. Let L and M be complete lattices and let f be an order-preserving function from L into M . The following are equivalent:

(1) f is continuous from L into M for the intrinsic topology Γ where $\Gamma = \chi$, D, $c(\chi) = c(D)$, or $\Sigma = \sigma(\chi) = \sigma(D)$;

(2) f is complete;

(3) f is linearly complete.

Proof. Suppose f is continuous for $c(D)$. Then $(L,D) \to (L,c(D)) \xrightarrow{f} (M,c(D)) \to (M,I)$ is continuous. Hence by 3.8 f is complete.

Conversely suppose f is complete. Let U be a basic convex open set in M which has Dedekind closed complement. Then $f^{-1}(U)$ is convex and it follows easily since f is complete $L\setminus f^{-1}(U) = f^{-1}(M\setminus U)$ is Dedekind closed since

$M \setminus U$ is. Hence $f^{-1}(U)$ is open in L . Thus f is continuous from $(L,c(D))$ to $(M,c(D))$. Hence if $\Gamma = c(D)$, (1) is equivalent to (2).

In a strictly analogous manner (1) is equivalent to (3) if $\Gamma = c(\chi)$. But since for complete lattices, $c(\chi) = c(D)$, we have (2) is equivalent to (3).

The proofs that (1) is equivalent to (2) for $\Gamma = D$ and (1) is equivalent to (3) for $\Gamma = \chi$ and $\sigma(\chi) = \Sigma$ follow the pattern of the previous proofs.

<u>Proposition 3.10</u>. Let L and M be complete lattices and f a lower homomorphism (i.e., $f(x \wedge y) = f(x) \wedge f(y)$) from L into M . Then f is complete if and only if f is continuous for the intrinsic topology Γ where $\Gamma = L_*$, WL_* , or LK .

<u>Proof</u>. That f is complete if f is continuous for Γ follows from 3.8 in a fashion analogous to the use of 3.8 in the proof of 3.9.

Conversely, suppose f is complete. For any non-empty subset A of L , let $a = \inf A$. We show $f(a) = \inf f(A)$. Now the set $\{a_1 \wedge \dots \wedge a_n : n \in \omega , a_i \in A \text{ for } i = 1,\cdots,n\}$ directed by itself descends down to a . Then the image net $\{f(a_1) \wedge \dots \wedge f(a_n) : n \in \omega , a_i \in A \text{ for } i = 1,\cdots,n\}$ descends to $f(a)$ (by completeness) and to $\inf (f(A))$ (by its definition).

Since f is complete and we have just seen that f preserves arbitrary inf's , it follows easily that f is continuous for L_* since $f(\bigvee_\alpha \bigwedge_{\beta\geq\alpha} x_\beta) = \bigvee_\alpha(f(\bigwedge_{\beta\geq\alpha} x_\beta)) =$

$= \bigvee_\alpha \bigwedge_{\beta\geq\alpha} f(x_\beta)$.

Let B be a lower subsemilattice of Y which is Dedekind closed. Since f is complete $f^{-1}(B)$ is Dedekind closed and since f is a lower homomorphism $f^{-1}(B)$ is a subsemilattice. Hence f is continuous for LK .

To show continuity for WL_* , we first note that the inverse image of a point is a Dedekind closed lower subsemilattice and hence contains its inf . Suppose that the net $\{x_\alpha\}$ weakly lower order converges to x , i.e.,

$$x \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}^\wedge \subset L(x) \ .$$

Let B be a Dedekind closed lower semilattice containing residually many of the set $\{f(x_\alpha)\}$. Then $f^{-1}(B)$ contains residually many of the set $\{x_\alpha\}$ and we have just seen that $f^{-1}(B)$ is a Dedekind closed lower subsemilattice. Hence it must be the case $x \in f^{-1}(B)$, and hence $f(x) \in B$. Thus

$$f(x) \in \bigcap_\alpha \{f(x_\beta) : \beta \geq \alpha\}^\wedge \ .$$

Let A be a Dedekind closed lower semilattice containing residually many of $\{x_\alpha\}$. Since f is a lower homomorphism, $f(A)$ is a lower semilattice. Let $\{y_\gamma\}$ be an ascending resp. descending net in $f(A)$ converging to y . Then $f^{-1}(y_\gamma) \cap A$ is a Dedekind closed lower semilattice, and hence has a least element a_γ . If $y_\gamma \leq y_\delta$, then $f(a_\gamma \wedge a_\delta) = f(a_\gamma) \wedge f(a_\delta) = y_\gamma \wedge y_\delta = y_\gamma$; hence $a_\gamma = a_\gamma \wedge a_\delta$, i.e., $a_\gamma \leq a_\delta$. Thus $\{a_\gamma\}$ is an ascending resp. descending net in A . Since A is Dedekind closed, the **limit** a of $\{a_\gamma\}$ is in A . Since f is complete $y = f(a) \in f(A)$. Thus $f(A)$ is Dedekind closed.

Now let $b \in \bigcap_\alpha \{f(x_\beta) : \beta \geq \alpha\}^\wedge$.

Then for any α , $f(\{x_\beta : \beta \geq \alpha\}^\wedge)$ is a Dedekind closed lower subsemilattice of M and hence contains $\{f(x_\beta) : \beta \geq \alpha\}^\wedge$, and in particular contains b . Let t_α be the least element of $f^{-1}(b) \cap \{x_\beta : \beta \geq \alpha\}^\wedge$. Then $\{t_\alpha\}$ is an increasing net and increases to some element t . Since $f^{-1}(b)$ is Dedekind closed, $t \in f^{-1}(b)$. Since $\{t_\alpha\}$ is eventually in any set $\{x_\beta : \beta \geq \alpha\}^\wedge$, then $t \in \bigcap_\alpha \{x_\beta : \beta \geq \alpha\}$. Thus $t \leq x$. Hence $b = f(t) \leq f(x)$. Thus $\{f(x_\alpha)\}$ weakly lower order converges to $f(x)$. From this fact it follows easily that f is continuous for WL_* .

<u>Proposition 3.11</u>. Let L be a complete lattice and f a homomorphism from L into M . Then f is complete if and only if f is continuous for the intrinsic topology Γ where $\Gamma = O$, K , or I .

Proof. That f continuous implies f is complete follows as in 3.9 and 3.10 . We saw in 3.10 that a complete lower homomorphism preserves arbitrary meets. Hence f preserves arbitrary joins and meets. It follows easily that the inverse of a closed set is closed with respect to the order topology; hence f is continuous for O .

We also saw in 3.9 that the inverse of a Dedekind closed set is Dedekind closed, and since f is a homomorphism, the inverse of a lattice is a lattice. Hence f is continuous with respect to K .

If $L(y)$ is a subbasic closed set in M with the interval topology, then $f^{-1}(L(y))$ is a Dedekind closed sublattice, and hence has a largest element x . Then $f^{-1}(L(y)) = L(x)$ and hence is closed. Dually $f^{-1}(M(y))$ is closed. Thus f is continuous.

Proposition 3.12. Let f, L and M as in 3.11. If f is a complete homomorphism, then f is continuous for the intrinsic topology Δ .

Proof. Let U in M be a subbasic open set, an ideal maximal with respect to missing y . Then $f^{-1}(U)$ is an ideal maximal with respect to missing x , the least element of $f^{-1}(y)$. Hence f is continuous.

Propositions 3.9 through 3.12 allow one to consider the

IV. Complete Semilattices.

A meet semilattice S is said to be complete if every non-empty subset has a greatest lower bound and if every ascending net ascends to some element of S. For complete semilattices S if $x \in S$ then $L(x)$ is a complete lattice (if $\emptyset \neq A \subset L(x)$, then $\sup A = \inf \{b : a \in A \text{ implies } a \leq b\}$). Hence if S has a 1, S is a complete lattice.

Many of the intrinsic topologies for complete lattices together with their properties transfer to complete semilattices. As a matter of fact the ones which are not self-dual were motivated by the semilattice case. Also the functors c, c', and σ can be defined for the category of complete semilattices.

Proposition 4.1. Let S be a complete semilattice. Then $c(D) = c(\chi)$ on S.

Proof. Since $D \leq \chi$ we have $c(D) \leq c(\chi)$. We show $c(\chi) \leq D$. It will then follow that $c(\chi) = c(c(\chi)) \leq c(D)$, completing the proof.

Let U be a basic convex open set in $c(\chi)$. We show U is open in D by showing that its complement is closed. Suppose $\{x_\alpha\}$ is a net in $S \setminus U$ and $x_\alpha \downarrow x$ where $x \in U$. Let A be a maximal descending i.e., downward directed family containing the set $\{x_\alpha\}$ in $M(x) \setminus U$.

intrinsic topologies within a larger framework. They can be viewed as functors from the category of complete lattices with morphisms complete homomorphisms to the category of complete lattices with a topology and morphisms continuous homomorphisms. We summarize some of the results of this section.

Proposition 3.13. Let f be a homomorphism from a complete lattice L into a complete lattice M.

The following are equivalent:

(1) f is continuous for any intrinsic topology Γ of Diagram 3.3 except Δ;

(2) f is complete;

(3) f is linearly complete;

(4) the inverse of a point has a least and greatest element.

Proof. That (1) and (2) are equivalent follows from 3.9, 3.10, and 3.11. That (2) and (3) are equivalent follows from 3.9. From the proof of 3.10 it follows that (2) implies (4). From the proof that (2) $\Rightarrow$ (1) for the interval topology I, all that was needed was that f satisfy (4). Hence (4) implies (1) for $\Gamma = I$.

Problem 3.14. Given the hypotheses of Proposition 3.13, for which intrinsic topologies is f a closed map?

We note first that $M(A) = A$. For if A is descending, then $M(A)$ is descending. Also if $b \geq a$ for some $a \in A$, then we have $b \geq a \geq x$. Since $a \notin U$ and U is convex, we have $b \notin U$. Hence $M(A)$ is a descending family in $M(x)\backslash U$ which contains A . Since A is maximal, $M(A) = A$.

Secondly we note A is a subsemilattice. For if $a, b \in A$, then since A is descending there exists $c \in A$ such that $a \geq c$ and $b \geq c$. Thus $a \wedge b \geq c$. Since $A = M(A)$, $a \wedge b \in A$.

Thirdly we note that if $p \in M(x)$, then $p \in A$ if $p \wedge a \notin U$ for all $a \in A$. For in this case $(p \wedge A) \cup A$ is a descending set missing U and containing A , and hence must be A by maximality of A .

Now let P be a maximal chain in A , and let $p = \inf P$. Since $A \cap U = \emptyset$ and U is open in $c(\chi)$ and hence χ , we have $p \notin U$. Let $a \in A$. Then by the second note $a \wedge P \subset A$. Since $p = \inf P$, we have $a \wedge p = \inf (a \wedge P)$. But again since U is χ open, $a \wedge p \notin U$. Hence by the third note $p \in A$. Hence by the second note if $b \in A$, then $b \wedge p \in A$. But $b \wedge p \cup M$ is then a chain; thus $b \wedge p \in M$ by maximality of M in A . Thus $b \wedge p = p$ since $p = \inf M$. Hence $p = \inf A$. But $x = \inf \{x_\alpha\} \geq \inf A = p$. Since $p \in M(x)$, $x \leq p$. Hence $x = p$. This is impossible since $x \in U$ and $p \notin U$. Thus if $x_\alpha \downarrow x$, $\{x_\alpha\} \subset S\backslash U$, then $x \in S\backslash U$.

If $x_\alpha \uparrow x$ where $\{x_\alpha\} \subset S \backslash U$, then applying the techniques of the preceding part of the proof to the complete lattice $L(x)$ and the open set $U \cap L(x)$, we obtain that $x \notin U$. Hence U is open in D , which is the needed result to complete the proof.

<u>Diagram 4.2</u>. The following is a diagram of intrinsic topology functors for complete semilattices.

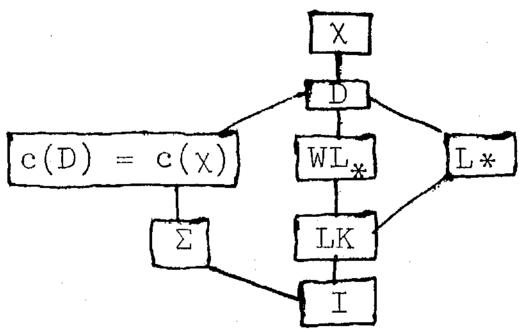


Diagram 4.2

All of these topologies were considered for complete lattices in section 3. Analogous results remain valid for complete semilattices and the proofs require only minor modification. The following two propositions are examples.

<u>Proposition 4.3</u>. Let S and T be complete semilattices and let f be a homomorphism from S into T . The following are equivalent:

(1) f is complete;

(2) f is linearly complete;

(3) f is continuous for the intrinsic topologies Γ assigns to S and T where Γ is any intrinsic topology of

Diagram 4.2 except I .

Proof. The proofs that f being complete is equivalent to f being continuous for D , WL_* , L_* , LK, $c(D)$ or Σ are the same as in section 3; also the same proof holds to show f being linearly complete is equivalent to f being continuous for χ or $c(\chi)$. Since by 4.1 $c(\chi) = c(D)$, we have (2) is equivalent to (1).

Proposition 4.4. Let S be a complete semilattice and T a complete subsemilattice. Then the topology that the intrinsic topology functor Γ assigns to T agrees with the one restricted to T that Γ assigns to S for $\Gamma = WL_*, L_*, LK, D$ and χ .

Proof. Straightforward.

We now define a functor from the category of complete semilattices and complete (semilattice) homomorphisms to the category of complete distributive lattices and complete (lattice) homomorphisms.

For a complete semilattice S let $\mu(S)$ be the set of all non-empty semi-ideals that are Dedekind closed ordered by inclusion. Since the finite union and arbitrary intersection of Dedekind closed semi-ideals is another such, $\mu(S)$ is a complete distributive lattice. If S and T are complete semilattices and f is a complete homomorphism from S into T, define $\mu(f):\mu(S) \to \mu(T)$ by $\mu(f)(A) = L(f(A))$.

Proposition 4.5. The μ defined in the preceding paragraph is indeed a functor from the category of complete semilattices and complete morphisms to the category of complete distributive lattices and complete morphisms.

Before the proof of the theorem, we first establish two lemmas.

Lemma 1. If A and B are semi-ideals in S, then $A \wedge B = A \cap B$.

Proof. Since $A \wedge B \subset A$ and $A \wedge B \subset B$, we have $A \wedge B \subset A \cap B$. Conversely if $x \in A \cap B$, then $x = x \wedge x \in A \wedge B$.

Lemma 2. If A is a Dedekind complete subsemilattice of a semilattice S, then $L(A)$ is Dedekind closed.

Proof. The set $L(A)$ clearly contains limits of descending nets. Let $\{x_\alpha\}$ be an ascending net in $L(A)$, $x_\alpha \uparrow x$. Since A is a subsemilattice and Dedekind complete, $M(x_\alpha) \cap A$ has a least element a_α for each α. Then $\{a_\alpha\}$ is an ascending net which ascends to $a \in A$, since A is Dedekind complete. Then $x \leq a$, and hence $x \in L(A)$.

Proof (of Proposition 4.5). We have seen already that $\mu(S)$ is is a complete distributive lattice if S is a complete semilattice. Let $f:S \to T$ be a complete homomorphism of complete semilattices. Let A be a Dedekind closed semi-ideal in S. Then $f(A)$ is a subsemilattice of T. Since a complete semilattice homomorphism preserves arbitrary meets, $f(A)$ is lower complete. Let $\{y_\alpha\}$ be a net in $f(A)$, $y_\alpha \uparrow y$. For each α, $\exists a_\alpha \in A$ such that $f(a_\alpha) = y_\alpha$. Since A is a semi-ideal the least element b_α of $f^{-1}(y_\alpha)$ is also in A. The net $\{b_\alpha\}$ is increasing, and hence increases to $b \in A$. Since f is complete $y = f(b) \in f(A)$. Thus $f(A)$ is Dedekind closed (and thus Dedekind complete). Hence by Lemma 2 $L(f(A))$ is Dedekind closed. Thus $\mu(f)$ is indeed a function from $\mu(S)$ to $\mu(T)$.

Let A and B be Dedekind closed ideals in S. Then

$$L(f(A \cup B)) = L(f(A) \cup f(B)) = L(f(A)) \cup L(f(B))$$

and using Lemma 1

$$L(f(A \cap B)) = L(f(A \wedge B)) = L(f(A) \wedge f(B)) = L(f(A)) \wedge L(f(B))$$
$$= L(f(A)) \cap L(f(B)) \text{ ; hence } \mu(f) \text{ is a homomorphism.}$$

Let $\{A_\alpha\}$ be a descending family of Dedekind closed semi-ideals. Then $A_\alpha \downarrow A$ where $A = \bigcap_\alpha A_\alpha$. We have easily that

$$L(f(A)) = L(f(\bigcap_\alpha A_\alpha)) \subset L(\bigcap_\alpha f(A_\alpha)) \subset \bigcap_\alpha L(f(A_\alpha))$$

Conversely let $y \in \bigcap_\alpha L(f(A_\alpha))$. For each α, let a_α

be the least element of A_α such that $y \leq f(a_\alpha)$. For indices α, β, then $f(a_\alpha \wedge a_\beta) = f(a_\alpha) \wedge f(a_\beta) \geq y$. Hence since $a_\alpha \wedge a_\beta \in A_\alpha \wedge A_\beta = A_\alpha \cap A_\beta$, we have $a_\alpha \wedge a_\beta = a_\alpha = a_\beta$. Hence $\{a_\alpha\}$ is a constant net $a \in \bigcap_\alpha A_\alpha = A$. Thus $y \in L(f(A))$.

Now let $\{A_\alpha\}$ be a net increasing to A. Then for all α, $A_\alpha \subset A$ implies $L(f(A_\alpha)) \subset L(f(A))$. Hence $L(f(A))$ is an upper bound. Suppose B is a Dedekind closed semi-ideal in T containing all $L(f(A_\alpha))$. Since f is complete, $f^{-1}(B)$ is Dedekind closed and a semi-ideal which contains A_α for all α. Thus $f^{-1}(B) \supset A$, and hence $B \supset L(f(A))$. Thus $L(f(A))$ is the join of the set $\{L(f(A_\alpha))\}$.

Thus $\mu(f)$ is a complete homomorphism. The other functorial properties for μ follow easily.

V. Algebraically Continuous Operations.

Definition 5.1. Let S be a complete semilattice. Then the meet operation is said to be algebraically continuous if for any $x_\alpha \uparrow x$ and any $y \in S$, then $x_\alpha \wedge y \uparrow x \wedge y$. In this case S is said to be meet-continuous.

One has always that if $x_\alpha \downarrow x$, then $x_\alpha \wedge y \downarrow x \wedge y$, or more generally, if $x_\alpha \downarrow x$ and $y_\beta \downarrow y$, then $x_\alpha \wedge y_\beta \downarrow x \wedge y$.

Proposition 5.2. The meet operation in a complete semilattice S is algebraically continuous if and only if $x_\alpha \uparrow x$ and $y_\beta \uparrow y$ implies $x_\alpha \wedge y_\beta \uparrow x \wedge y$.

Proof. The second condition easily implies the first by taking the constant net consisting of the element y . Conversely, let $x_\alpha \uparrow x$ and $y_\beta \uparrow y$. Then $x \wedge y \geq x_\alpha \wedge y_\beta$ for all choices of α and β . Suppose $z \geq x_\alpha \wedge y_\beta$ for all α , β . If α is fixed, $x_\alpha \wedge y_\beta \uparrow x_\alpha \wedge y$. Hence $z \geq x_\alpha \wedge y$ for all α . But $x_\alpha \wedge y \uparrow x \wedge y$; hence $z \geq x \wedge y$, i.e., $x \wedge y$ is the join of $\{x_\alpha \wedge y_\beta\}$.

Proposition 5.3. The meet operation in a complete lattice is algebraically continuous if and only if $\{x_\alpha\}$ order converges to x and $\{y_\beta\}$ order converges to y implies $\{x_\alpha \wedge y_\beta\}$ order converges to $x \wedge y$.

Proof. See [6, p. 248].

Proposition 5.4. In a complete semilattice (lattice) S the following are equivalent:

(1) S is meet continuous;

(2) For $y \in Y$, the function from S into S which sends x to $x \wedge y$ is continuous for the intrinsic topology Γ ;

(3) The meet operation is continuous from $S \times S$ with the Γ topology to S with the Γ topology for the intrinsic topology functor Γ .

For the semilattice case Γ may be chosen to be any topology of Diagram 4.2 except I , and for the lattice case any topology of Diagram 3.3 except K, I or Δ .

Proof. Since the function $x \to x \wedge y$ for a fixed y is a semilattice homomorphism and since one has always $x_\alpha \downarrow x$ implies $x_\alpha \wedge y \downarrow x \wedge y$, the function is a complete homomorphism for every y if and only if S is meet continuous. The equivalence of (1) and (2) now follows from Proposition 4.3 for the semilattice case, and Propositions 3.9 and 3.10 cover all the lattice cases except 0. Proposition 5.3 shows that if the lattice S is meet continuous then translation by y is a continuous function in the order topology for each y (show the inverse of a closed set is closed). If translation by y is continuous in the order topology for each y then Proposition 3.8 implies each translation is complete, and hence that S is meet continuous.

The meet operation from $S \times S$ to S is a semilattice homomorphism which satisfies if $(x_\alpha, y_\alpha) \downarrow (x,y)$, then $x_\alpha \wedge y_\alpha \downarrow x \wedge y$. Hence by Proposition 5.2 the meet operation is complete if and only if S is meet continuous. The proof that (1) and (3) are equivalent now parallels the proof that (1) and (2) were equivalent.

Lemma 5.5. Let S be a semilattice endowed with a topology $\mathcal{U}$ for which the functions $x \to x \wedge y$ are continuous for every $y \in S$. If $U \in \mathcal{U}$, then $M(U) \in \mathcal{U}$.

Proof. $M(U) = \bigcup_{y \in U} \{x : x \wedge y \in U\}$; each set in the union is open since translation by y is continuous.

Proposition 5.6. Let L be a complete lattice which is both meet- and join-continuous. Then on L the $c(D)$ and Σ topologies.

Proof. Let U be open and convex in the $c(D)$ topology. By 5.4 the translation functions $x \to x \wedge y$ are continuous for the $c(D)$ topology. Hence by Lemma 5.5 and its dual, $M(U)$ and $L(U)$ are open in $c(D)$. Hence since $\Sigma = \sigma(D) = \sigma(c(D))$, $L(U)$ and $M(U)$ are open in Σ. Since U is convex, $U = L(U) \cap M(U)$ is open Σ. Since continuity always holds in the reverse direction, the proposition is established.

VI. Topological Semilattices and Lattices.

The central and most difficult results of the paper lie in this and the last section. They concern the problem of starting with a compact topology on a semilattice or lattice and trying to identify it as an intrinsic topology.

Definition 6.1. Let S be a semilattice endowed with a topology $\mathcal{U}$. If the function from S into S defined by $x \to x \wedge y$ is continuous for each $y \in S$, then S is a semitopological semilattice. If the meet operation from $S \times S$ with the product topology into S is continuous and S is Hausdorff, then S is a topological semilattice. A lattice

L endowed with a topology $\mathcal{U}$ is a <u>semitopological</u> (<u>topological</u>) <u>lattice</u> if L is a semitopological (topological) semilattice with respect to both the meet and the join operations.

<u>Proposition 6.2</u>. Let $(S,\mathcal{U})$ be a compact Hausdorff semitopological semilattice. Then S is complete and $\mathcal{U}$ is order compatible.

<u>Proof</u>. Let $x \in S$. Then $L(x) = S \wedge \{x\}$ is compact since S is compact and translation is continuous; thus L(x) is closed since S is Hausdorff. Now $M(x) = \{y : y \wedge x = x\}$ is closed since $\{x\}$ is closed and translation by x is continuous. Thus we have the identity function from $(S,\mathcal{U}) \to (S,I)$ is continuous.

Now let $\{x_\alpha\}$ be an increasing net in S. Then $\{x_\alpha\}$ clusters to x for some $x \in S$ since S is compact. By Proposition 1.20(b), $x_\alpha \uparrow x$. A similar result holds for decreasing nets. Hence S is Dedekind complete (and hence complete) and the net $\{x_\alpha\}$ must converge to its least upper bound if increasing and greatest lower bound if decreasing. This implies $(S,D) \to (S,\mathcal{U})$ is continuous, i.e., $(S,\mathcal{U})$ is order compatible.

<u>Proposition 6.3</u>. Let S be a compact Hausdorff first countable semitopological semilattice. If a sequence $\{x_n\}_{n=1}^{\infty}$ clusters to x, then there exists a subsequence for which x is the

lim inf of both the subsequence and any subsequence of the subsequence.

Proof. Let $\{W_i\}$ be a countable base at x. Set $V_0 = W_1$. Pick V_1, an open set, such that $x \in V_1 \subset V_1^* \subset W_1$ and $x \wedge V_1 \subset W_1$. Pick an open set O such that $x \wedge O \subset V_1$. Pick $y_1 = x_{n_1} \in O \cap V_1$. Suppose $\{V_i\}_{i=0}^{k-1}$ and $\{y_i = x_{n_i}\}_{i=1}^{k-1}$ have been chosen satisfying for all $i=1,\cdots,k-1$:

(1) V_i is open;

(2) $x \in V_i \subset V_i^* \subset W_i \cap V_{i-1}$;

(3) $V_i \wedge y_{i-1} \subset V_{i-1}$;

(4) $y_i \in V_i$ and $x \wedge y_i \in V_i$.

Then by regularity there exists an open set U such that $x \in U \subset U^* \subset W_k \cap V_{k-1}$. Since $x \wedge y_{k-1} \in V_{k-1}$, there exists an open set $V_k \subset U$ such that $x \in V_k$ and $V_k \wedge y_{k-1} \subset V_{k-1}$. Pick an open set P such that $P \subset V_k$ and $x \wedge P \subset V_k$. Pick $y_k = x_{n_k} \in P$. Continuing the process inductively one gets a sequence of open sets $\{V_i\}_{i=0}^{\infty}$ and a subsequence $\{y_i\}_{i=1}^{\infty}$ of $\{x_n\}$ satisfying (1)-(4).

Now for positive integers n and k,

$$\begin{aligned} y_n \wedge y_{n+1} \wedge \cdots \wedge y_{n+k} &\in y_n \wedge \cdots \wedge y_{n+k-2} \wedge (y_{n+k-1} \wedge V_{n+k}) \\ &\subset y_n \wedge \cdots \wedge y_{n+k-2} \wedge V_{n+k-1} \\ &\subset y_n \wedge \cdots \wedge V_{n+k-2} \\ &\ \ \vdots \\ &\subset V_n \end{aligned}$$

For a fixed n, $z_{n,k} = y_n \wedge \dots \wedge y_{n+k}$ is a decreasing sequence. Hence by Proposition 6.2 we have $z_{n,k} \downarrow z_n$ and the sequence $\{z_{n,k}\}$ converges to z_n. Since each $z_{n,k} \in V_n$, we have $z_n \in V_n^* \subset W_n$. If $n \leq m$, then $z_n \leq z_m$ (since $z_n = \bigwedge_{j=n}^{\infty} y_j$ and $z_m = \bigwedge_{j=m}^{\infty} y_j$). Hence $\{z_n\}$ increases to some z, and hence converges to z. Since the sequence is eventually in V_n^* for each n, we have $z \in \bigcap_n V_n^* \subset \bigcap_n W_n$. Hence $z = x$. This shows x is the lim inf of the subsequence $\{y_n\}$. By techniques analogous to those already employed, one shows that any subsequence of $\{y_n\}$ has lim inf in $\cap W_n$; hence the lim inf must be x.

<u>Definition 6.4</u>. If S is a semilattice, the <u>graph of the partial order on</u> S is the set

$$\mathrm{Gr}(\leq) = \{(x,y) \in S \times S : x \leq y\} .$$

A basic fact concerning topological semilattices is the following well-known result.

<u>Proposition 6.5</u>. A topological semilattice has closed graph in the product topology.

<u>Proof</u>. Let S be a topological semilattice. Then S is is Hausdorff; so the diagonal Δ of $S \times S$ is a closed set. Define a continuous function $f: S \times S \to S \times S$ by

$f(x,y) = (x, x\wedge y)$; then $Gr(\leq) = f^{-1}(\Delta)$ and hence is closed.

Theorem 6.6. Let S be a compact Hausdorff semitopological semilattice. Then S is a topological semilattice (and hence $Gr(\leq)$ closed).

Proof. First we assume S is in addition metrizable. In this case we wish to show that $Gr(\leq)$ is closed. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $Gr(\leq)$ which converges to (x,y) in the product topology of $S \times S$. By Proposition 6.3 there exists a subsequence $\{x_{n_i}\}$ such that

$$x = \bigvee_i \bigwedge_{j\geq i} x_{n_j} .$$

For the subsequence of the $\{y_n\}$ corresponding to the one chosen for $\{x_n\}$, there exists by 6.3 again a subsequence of this subsequence with y as the lim inf . Denote this sub-sub-sequence by $\{y_n'\}$ and the corresponding one for $\{x_n\}$ by $\{x_n'\}$ (by 6.3 the latter still has x as its lim inf). Then

$$x = \bigvee_n \bigwedge_{m\geq n} x_n' \leq \bigvee_n \bigwedge_{m\geq n} y_n' = y .$$

Thus $(x,y) \in Gr(\leq)$. Now by Proposition 7.4 of [16] a compact Hausdorff semitopological semilattice with closed graph is a topological semilattice. This concludes the proof for the case S is metrizable.

The non-metrizable case follows from a reduction to the metric case. The reduction is analogous to that given in Theorem 5.1 of [16]. I am currently preparing for future publication a general reduction technique which will include both cases.

Proposition 6.7. Let S be a complete semilattice (lattice). Then the graph of the partial order is closed in the topology Γ assigns to $S \times S$ for $\Gamma = \chi$, D, L_*, WL_* , and LK($\Gamma = \chi$, D, O, WO, L_*, WL_* , and LK) .

Proof. Most of the proofs follow easily from the definition of the topology. Assume $\{(x_\alpha, y_\alpha)\}$ is a net in $Gr(\leq)$ which weakly lower converges to (x,y) . Then $x \in \bigcap_\alpha \{x_\beta: \beta \geq \alpha\}^\wedge \subset \bigcap_\alpha L(\{y_\beta: \beta \geq \alpha\}^\wedge)$; the latter containment holds because $L(\{y_\beta: \beta \geq \alpha\}^\wedge)$ is lower complete and Dedekind closed (as we saw in Lemma 2 of Proposition 4.5) and contains $\{x_\beta: \beta \geq \alpha\}$. Let z_α be the least element of $\{y_\beta: \beta \geq \alpha\}^\wedge$ which is larger than x . Then $\{z_\alpha\}$ is an increasing net which is eventually in each $\{y_\beta: \beta \geq \alpha\}^\wedge$; thus if $z \in S$ is the point such that $z_\alpha \uparrow z$, then

$$x \leq z \in \bigcap_\alpha \{y_\beta: \beta \geq \alpha\}^\wedge \subset L(y) .$$

Hence $x \leq y$. The case for weak lower star convergence follows from the above by taking subnets.

Note that $Gr(\leq)$ is lower complete, and hence closed in the LK topology.

If $Gr(\leq)$ is closed, then $Gr(\geq)$ is closed for an intrinsic topology since the coordinate reversing function is an automorphism. Hence $\Delta = Gr(\leq) \cap Gr(\geq)$ is closed. If the intrinsic topology is also productive, then of necessity it must be Hausdorff. Since E. E. Floyd [8] has given an example of a non-Hausdorff complete lattice with respect to every linear topology, it follows that any lattice topology in Proposition 6.7 is not productive for this lattice (that the order is productive is incorrectly stated in [9]).

The next proposition is a key tool in identifying a topology as an intrinsic topology. First, however, we introduce some additional notation. If A is a subset of a complete semilattice S, then

$$A^{+} = \{x: \text{ there is a net } \{x_\alpha\} \text{ in } A \text{ with } x_\alpha \uparrow x\}$$

and A^{-} is defined dually.

<u>Proposition 6.8</u>. Let S be a compact topological semilattice and let T be a subsemilattice of S. Then $T^{*} = T^{-+-+}$. If T is a semi-ideal then $T^{*} = T^{++}$.

Proof. Since T^* is closed, it is Dedekind closed by Proposition 6.2. Hence $T^* \supset T^{-+-+}$.

Conversely, let $x \in T^*$. Choose by continuity of multiplication a sequence $\{W_n : n \in \omega\}$ satisfying for all n,

(i) $x \in W_n^\circ$, $W_n = W_n^*$

(ii) $W_n \wedge W_n \subset W_{n-1}^\circ$.

Choose for each n an element $x_n \in W_n \cap T$. By techniques analogous to those in 6.3 we have $z_n = \bigwedge_{i=n}^{\infty} x_i \in W_n$. Hence $z = \bigvee_n z_n = \bigvee_n \bigwedge_{m \geq n} x_n$ is in $\bigcap_{n \in \omega} W_n$. Since T is a subsemilattice, we have $z_n \in T^-$. Hence $z \in T^{-+}$. Thus $(T^{-+}) \cap (\bigcap_{n \in \omega} W_n) \neq \emptyset$.

Now T^{-+} is a subsemilattice since $a_\alpha \downarrow a$, $b_\alpha \downarrow b$ implies $a_\alpha \wedge b_\alpha \downarrow a \wedge b$ (always) and $a_\alpha \uparrow a$ and $b_\alpha \uparrow b$ implies $a_\alpha \wedge b_\alpha \uparrow a \wedge b$ (by joint continuity). Also by condition (ii) $\bigcap_{n \in \omega} W_n$ is a compact semilattice. Hence w, the meet of $(T^{-+}) \cap (\bigcap_{n \in \omega} W_n) \neq \emptyset$ is the limit of a descending net in T^{-+} (and hence is in T^{-+-}) and is in $\bigcap_{n \in \omega} W_n$.

We show that w, the meet of $(T^{-+}) \cap (\bigcap_{n \in \omega} W_n)$, is less than or equal to x. If not, by closed graph, there exists an open set A containing w and an open set B containing x such that if $a \in A$ and $b \in B$, then $a \not\leq b$.

However, when one was choosing all the $\{W_n\}$ in the earlier part of the proof, they could have been chosen so that $W_n \subset B$ for all n. If z was again the lim inf of $\{x_n\}$, each $x_n \in W_n$, then $z \in B$ and $z \in T^{-+}$. Hence $w \leq z$, a contradiction.

Now let D be the set of all sequences $\{W_n : n \in \omega\}$ satisfying (i) and (ii). If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geq \{V_n\}$ if $W_n \subset V_n$ for all n. It is straightforward to verify that $(D,\leq)$ is a directed set. For each $\{W_n\}$, choose w, the meet of $(T^{-+}) \cap (\bigcap_{n\in\omega} W_n)$. This defines an ascending net with all elements in the net below x. Since any closed neighborhood of x can be chosen as a W_1 for some sequence in the net, and the w chosen for this sequence will be in W_1, then eventually the net is in any open set around x. Thus it ascends to x. Hence $x \in T^{-+-+}$. Thus $T^* = T^{-+-+}$.

If T is a semi-ideal, then $T^- = T$. To finish the proof, we show $T^{+-} = T^+$. We actually show T^+ is a semi-ideal. Let $a \in T^+$ and let $x \leq a$. Then there exists an ascending net $\{a_\alpha\}$ in T such that $a_\alpha \uparrow a$. By continuity of the meet operations $a_\alpha \wedge x \uparrow a \wedge x = x$. Since T is a semi-ideal, $a_\alpha \wedge x \in T$ for all α. Hence $x \in T^+$. This concludes the proof.

Theorem 6.9. Let $(S,\mathcal{U})$ be a compact topological semilattice

If $\{x_\alpha\}$ converges to x in $\mathcal{U}$, then $\{x_\alpha\}$ weakly lower converges to x. Conversely if $\{x_\alpha\}$ weakly lower star converges to x, then $\{x_\alpha\}$ converges to x in $\mathcal{U}$. Hence the topology $\mathcal{U}$ is the WL_* topology.

Proof. Suppose $\{x_\alpha\}$ converges to x in $\mathcal{U}$. By Proposition 6.8 for any α, the set $\{x_\beta: \beta \geq \alpha\}^\wedge$ is closed in $\mathcal{U}$ (since it is a Dedekind closed semilattice). Hence $x \in \bigcap_\alpha \{x_\beta: \beta \geq \alpha\}^\wedge$.

Suppose $y \in \bigcap_\alpha \{x_\beta: \beta \geq \alpha\}^\wedge$. If $y \not\leq x$, then there exist by closed graph open sets A and B such that $y \in A$, $x \in B$ and if $a \in A$, $b \in B$, then $a \not\leq b$. There exists an index γ such that if $\alpha \geq \gamma$, then $x_\alpha \in V$, where $x \in V^\circ \subset V^* \subset B$. Then $P = \{x_\alpha: \alpha \geq \gamma\}^* \subset B$. Now P is closed, hence compact. Then $L(P)$ is closed [17, p. 44], and hence Dedekind closed. $L(P)$ is also a subsemilattice. If $t \in L(P)$, then there exists $b \in B$ such that $t \leq b$. Hence $t \notin A$. Thus $\bigcap_\alpha \{x_\beta: \beta \geq \alpha\}^\wedge \subset \{x_\beta: \beta \geq \gamma\}^\wedge \subset L(P)$. But $y \notin L(P)$, a contradiction. Thus $\bigcap_\alpha \{x_\beta: \beta \geq \alpha\}^\wedge \subset L(x)$. Hence $\{x_\alpha\}$ weakly lower converges to x.

Conversely, let $\{x_\alpha\}$ weakly lower star converge to x. If $\{x_\alpha\}$ fails to converge to x, then there exists $y \in S$, $y \neq x$ such that $\{x_\alpha\}$ clusters to y. Then a subnet of the $\{x_\alpha\}$ converges to x. Since $\{x_\alpha\}$ weakly lower star converges to x , a subnet of this subnet weakly lower converges to x . But this sub-subnet still converges to y , and hence by the first part of the proof weakly lower converges to y . Since a net can weakly lower converge to a most one point, $x = y$, a contradiction. Thus $\{x_\alpha\}$ converges to x .

<u>Theorem 6.10</u>. Let $(L,\mathcal{U})$ be a compact topological lattice. Then $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ weakly order converges to x . The $c(O)$, WO, WL_* , and Σ topologies agree and are equal to $\mathcal{U}$.

<u>Proof</u>. If $\{x_\alpha\}$ converges to x , then by Theorem 6.9 and its dual it weakly lower converges and weakly upper converges to x . Hence $\{x_\alpha\}$ weakly order converges to x .

Conversely let $\{x_\alpha\}$ weakly order converge to x . Then if $\{x_\alpha\}$ fails to converge to x , some subnet converges to $y \neq x$. Then the subnet weakly order converges to y by the first part of the proof. Hence

$$y \in \bigcap_{\alpha_j} \{x_{\beta_i} : \beta_i \geq \alpha_j\}^\wedge \subset \bigcap_{\alpha_j} \{x_\beta : \beta \geq \alpha_j\}^\wedge \subset L(x) ;$$

similarly $y \in M(x)$. Thus $y = x$, a contradiction.

It now follows immediately that $\mathcal{U} = WO$. By Theorem 6.9, $\mathcal{U} = WL_*$. By Diagram 3.3, $(L,WO) \to (L,\Sigma)$ is continuous. By Proposition 5.6, $\Sigma = c(O)$. By [17, p. 48], since L has closed graph, the closed semi-ideals of L and the closed dual semi-ideals form a subbase for the closed sets. Hence $(L,\Sigma) \to (L,\mathcal{U})$ is continuous, i.e., they agree.

An alternate proof that $\mathcal{U} = c(O)$ may be found in [15].

<u>Problem 6.11</u>. Must $\mathcal{U}$ in 6.10 also be the O-topology?

The characterizations in this section are quite useful in the study of topological semilattices and lattices. They reduce the study to certain algebraic categories with continuous homomorphisms corresponding to complete homomorphisms.

VII. <u>Small Semilattices and Lattices</u>.

An important class of topological semilattices (lattices) are those which possess a basis of neighborhoods at each point which are subsemilattices (sublattices). We say such semilattices (lattices) have small semilattices (lattices). Some of the basic properties of semilattices with small semilattices may be found in [13].

<u>Proposition 7.1</u>. Let $(S,\mathcal{U})$ be a compact topological semilattice with small semilattices. Then the L_*, WL_* and LK topologies agree and are all equal to $\mathcal{U}$. Furthermore a net $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ lower star converges to x.

Proof. We begin with the last assertion first. Let $\{x_\alpha\}$ converge to x in $\mathcal{U}$. Then for any fixed α, let $y_\alpha = \bigwedge_{\beta \geq \alpha} x_\beta$. Since $\{(y_\alpha, x_\beta)\colon \beta \geq \alpha\}$ is a subset of $Gr(\leq)$ for any fixed α and $Gr(\leq)$ is closed, we have $y_\alpha \leq x$ for all α. Given any neighborhood N of x, there exists a neighborhood M of x such that $M^* \subset N$ and M is a subsemilattice. There exists an index γ such that $x_\alpha \in M$ for $\alpha \geq \gamma$. Since M is a subsemilattice all finite meets of the set $\{x_\alpha\colon \alpha \geq \gamma\}$ are again in M; hence $y_\beta \in M^*$ for $\beta \geq \gamma$. Then $\{y_\alpha\}$ is eventually in any open set around x, and so must ascend to x. Thus x is the lim inf of the net $\{x_\alpha\}$.

Conversely, let $\{x_\alpha\}$ lower star converge to x. If $\{x_\alpha\}$ clusters to y, then there is a subnet which converges. By the first of this proof any subnet of this subnet must have y for its lim inf. Hence $y = x$, and thus $\{x_\alpha\}$ converges to x.

It follows easily from what we have just shown that $\mathcal{U} = L_*$. By 6.9 $\mathcal{U} = WL_*$.

We show $\mathcal{U} = LK$ by showing LK is Hausdorff; this will be sufficient since $(S,\mathcal{U})$ is compact and $(S,\mathcal{U}) = (S,L_*) \to (S,LK)$ is continuous (Diagram 4.2).

Let $x,y \in S$, $x \neq y$. We may assume $x \nleq y$. Since S has small semilattices, there exists $z \leq x$, $z \in S\backslash L(y)$, such that $x \in M(z)^{\circ}$. Then $M(z)$ and $(S \setminus M(z))^*$ are lower complete sets. Their complements are open sets in LK separating x and y .

Proposition 7.2. Let $(L,\mathcal{U})$ be a compact topological lattice such that each point has a basis of neighborhoods which are sublattices. Then $\{x_\alpha\}$ converges to x in $\mathcal{U}$ if and only if $\{x_\alpha\}$ order converges to x . Furthermore $\mathcal{U} = O = I$ and all topologies in between O and I in Diagram 3.3.

Proof. If $\{x_\alpha\}$ converges to x in $\mathcal{U}$, then by the first part of the proof of 7.1 and its dual, $\{x_\alpha\}$ order converges to x .

Conversely suppose $\{x_\alpha\}$ order converges to x . Then $\{x_\alpha\}$ converges to x in the order topology and hence converges to x in the WO topology (Diagram 3.3) which is the $\mathcal{U}$ topology (6.10). Hence it follows that $\mathcal{U}$ is the order topology.

Again we complete the proof by showing I is Hausdorff. Suppose $\{x_\alpha\}$ is a net in L . Then since L is compact Hausdorff, some subnet converges to some x , and hence by the first part of the proof order converges to x . By a result of K . Atsumi [4, Theorem 3] L with the interval topology is Hausdorff.

We now turn our attention to the converse problem. We wish to postulate algebraic conditions which will be sufficient to insure that a semilattice admits a topology with small semilattices. First, however, we give a preliminary result concerning compactness. This generalizes results of Frink, who showed the interval topology was compact in a complete lattice [9], and Insel, who showed the complete topology was compact in a complete lattice [11].

<u>Proposition 7.3</u>. Let S be a complete semilattice. Then S with the LK topology is compact.

<u>Proof</u>. Let $\{A_\alpha\}$ be a collection of Dedekind closed lower subsemilattices of S with the finite intersection property. For each finite subset $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ pick the least element of $\bigcap_1^n A_{\alpha_i}$. With the finite subsets ordered by inclusion, these least elements form an ascending net and hence ascend to some element a. Since each A_α is Dedekind closed, $a \in \bigcap_\alpha A_\alpha$. Since the Dedekind closed lower subsemilattices form a subbase for the closed sets of S, by Alexander's lemma, L is compact.

We are now ready for a converse to Proposition 7.1.

<u>Proposition 7.4</u>. Let S be a complete semilattice in which the meet operation is algebraically continuous and the

LK topology is Hausdorff. Then S with the LK topology is a compact topological semilattice with small semilattices.

<u>Proof</u>. By 7.3 S is compact. By 5.4 the meet operation is separately continuous for LK. Hence by 6.6 S is a topological semilattice.

We show now that S has small semilattices. We first consider the case that S has a largest element 1, and show S has small semilattices at 1. Let U be an open set, $1 \in U$. Since $Gr(\leq)$ is closed, by a result of Nachbin [17], there is a convex open set V with $1 \in V \subset U$. Then $A = S \setminus V$ is compact and decreasing.

Since the LK topology is Hausdorff, for each $a \in A$, there exist basic open sets P_a and Q_a in the LK topology with $a \in P_a$, $1 \in Q_a$, and $P_a \cap Q_a = \emptyset$. P_a is the complement of finitely many complete subsemilattices. Finitely many of the $\{P_a : a \in A\}$ cover A, say $A \subset \bigcup_1^n P_i$. Let $Q = \bigcap_1^n Q_i$. For each P_i, let $S \setminus P_i = S_{i,1} \cup \ldots \cup S_{i,m_i}$ be the representation of the complement of P_i in terms of complete subsemilattices. Consider all possible sets of the form $S_{1,j_1} \cap \ldots \cap S_{n,j_n}$ where $1 \leq j_i \leq m_i$ for each i. Each such intersection is a subset of V and the union of all such intersections contain Q. Since there are only finitely many such intersections and each such is closed, some such intersection, call it T, must have an interior. Since T is a complete subsemilattice T has a least element t.

By continuity of the meet operation, $M(T^\circ)$ is an open set containing 1. Note that $M(t) \supset M(T^\circ)$; hence $M(t)$ is a neighborhood of 1. Since A is decreasing, $M(t) \subset V$. Since $M(t)$ is a subsemilattice, S has small semilattices at 1.

Now let $x \in S$. It follows easily that the LK topology restricted to $L(x)$ agrees with the LK topology on $L(x)$. Since x is the largest element of $L(x)$, it follows from the first part that $L(x)$ has small semilattices at x. By [13] this implies S has small semilattices at x.

Proposition 7.5. Let L be a complete lattice in which the meet and join operations are algebraically continuous and the complete topology K is Hausdorff. Then L equipped with the complete topology is a compact topological lattice with a basis of sublattices.

Proof. Since LK is compact, K is Hausdorff, and $(L, LK) \to (L, K)$ is continuous, the K and LK topologies agree. Hence by 7.4 and its dual L is a compact topological lattice with a basis of subsemilattices with respect to each operation.

Let $x \in L$, and U an open neighborhood of x. Let V be an open, convex set such that $x \in V \subset U$. Then there exists a lower subsemilattice T such that $x \in T^{\circ} \subset T \subset T^{*} \subset V$. Then T^{*} will be a compact lower subsemilattice and hence have a least element t. Let P be an upper subsemilattice such that $x \in P^{\circ} \subset P^{*} \subset T^{\circ}$. Let p be the greatest element of P^{*}. Then $x \in P^{\circ} = P^{\circ} \cap T^{\circ} \subset L(p) \cap M(t) = [t,p] \subset V$, the last inclusion holding since V is convex. Then $[t,p]$ is a sublattice, a neighborhood of x, and a subset of U. Hence L has a basis of sublattices.

Propositions 7.4 and 7.5 would be significantly improved if it were possible to find a ''reasonable'' algebraic condition to replace the hypothesis that LK or K be Hausdorff.

VIII. Comments - Historical and Otherwise

Intrinsic topologies in lattices first appeared with G. Birkhoff's definition of the order topology in the late 1930's [5]. Shortly thereafter O. Frink [9] introduced the interval topology and studied basic properties of the order and interval topologies.

Interest revived in intrinsic topologies in the middle 50's with the work of B. C. Rennie [19, 20], Frink's introduction of the ideal topology [10], the work of A. J. Ward [23] and E. E. Floyd's examples of lattices with pathological intrinsic topologies [8]. Rennie's work contains germs of several of the developments pursued here.

About this time A. D. Wallace initiated interest in topological lattices [22] and early investigations in this area were carried out by L. W. Anderson [1, 2, 3] in the late 50's. During this same period E. S. Wolk introduced the concept of order compatible topologies [24], and T. Naito gave a necessary and sufficient condition for all such topologies to be identical in a complete lattice [18].

In the 60's A. J. Insel introduced and studied the complete topology [11, 12]. D. Strauss [21] appears to be the first to investigate intrinsic topologies in compact topological lattices. Some additions were given by T. H. Choe in [7]. Recently I had shown that any compact topological lattice has the $c(0)$ topology [15]. An implicit algebraic characterization of the topology of a compact topological semilattice is also included.

A problem of recurring interest in intrinsic topologies relates to the Hausdorffness of certain topologies, in particular the interval topology. Ward [23] and K. Atsumi [4] for instance treat this latter problem. Floyd's example [8] shows that the order topology may fail to be Hausdorff. Insel [12] gave necessary and sufficient conditions for the complete topology to be Hausdorff. Strauss [21] characterized those compact topological lattices in which the interval topology is Hausdorff. Propositions 7.2 and 7.5 are essentially her results. Recently I published an example of a compact distributive topological lattice in which the interval topology is not Hausdorff [14].

The preceding is by no means an exhaustive account of the work in intrinsic topologies, but rather should be considered as a background out of which this paper grew.

There are several directions for future investigation. The Hasse diagram of the relation between the various intrinsic topologies needs to be rigorously verified for the following classes; complete lattices and semilattices, complete algebraically continuous lattices and semilattices and compact topological semilattices and lattices. A complete list of counter-examples even for Diagram 3.3 to show it is the best possible is not known. Other interesting classes in which to study intrinsic topologies might be vector lattices and equationally compact semilattices and lattices. A complete semilattice S inherits many topologies as a subspace of the complete lattice $\mu(S)$ (see Section 4). How do these relate to the topologies already given to S directly?

The ideal topology has been frequently ignored in the considerations of this paper. In particular, what can be said about it in compact topological lattices?

The considerations of this paper may be generalized to arbitrary lattices in a variety of ways. Many of the functors considered are already defined for all lattices. Another method of extension of an intrinsic topology functor defined on complete lattices is to take the completion by cuts of an arbitrary lattice and give the lattice the subspace topology. Alternately, one may declare a set open if and only if its intersection with each complete sublattice is open with respect to some intrinsic topology functor Γ. Most of even the basic properties of these extensions remain unexplored.

Finally, the definition of intrinsic topology given here (automorphisms are continuous) is somewhat artificial. A precise definition of intrinsic topologies in terms of generating a topology from the algebra needs to be given in logic and set language and basic properties of such topologies explored.

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