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SPLITTING ALGEBRAS AND A WEAK NOTION OF PROJECTIVITY

by

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1. Introduction:

The classical results in Lattice Theory by Dedekind and Brikhoff that a lattice is modular (distributive) if and only if it does not contain the pentagon, N_5, (resp. N_5 and the 3-diamond, M_3) as a sublattice have been generalized by McKenzie in [13] to the notion of a splitting algebra. That is: a finite subdirectly irreducible algebra is splitting in a variety (= equational class) if there is a largest subvariety of this variety not containing it. In [3], McKenzie characterized the splitting lattices as the bounded homomorphic images of finitely generated free lattices. In [9], Jónsson showed that $M_{3,3}$ is a splitting modular lattice.

As McKenzie noted, his results do not supply necessary and sufficient conditions for a splitting algebra in proper subvarieties of lattices. In this paper, we develop a weak notion of projectivity for a finite algebra in a variety and show that given reasonable restrictions on the variety, every finite subdirectly irreducible satisfying this weak projectivity conditions is a splitting algebra. The reasonable restrictions alluded to are congruence distributivity. Therefore all of the usual lattice-like

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varieties are included (e.g. Lattices, Heyting algebras, Pseudo complemented lattices, Implication semilattices, and Hilbert Algebras).

After developing the general theory, we provide examples in the above varieties and in the last section describe a large class of splitting modular lattices.

We wish to thank Professor R. Wille for his many valuable comments which led to this revised version of these results.

2. Preliminaries

Most of the relevant definitions and results in universal algebra can be found in Grätzer [5]; in lattice theory, Szasz [14] and McKenzie [13].

Let K be a variety of algebras. We will consider (as is usual) K as a category whose maps are all K-homomorphisms. For A and B in K, a surfjective map $f : A \twoheadrightarrow B$ is called a cover (with respect to surjective maps) if for all $g : C \to A$ in K g is surjective if $f \cdot g$ is. Equivalently, $f : A \twoheadrightarrow B$ is a cover if A is the only subalgebra of A whose image under f is B.

$P \in K$ is called projective (with respect to surjective maps) if for any surjective $g : A \twoheadrightarrow B$ and any $f : P \to B$, there exists a lifting $\bar{f} : P \to A$ with $g \cdot \bar{f} = f$. It is well known (or easily seen) that any variety has enough projectives (i.e. every algebra is the homomorphic image of a projective) and that an algebra in K is projective if and only if it is a retract of a K-free algebra.

A cover $f : A \twoheadrightarrow B$ is called a projective (finite) cover according to whether A is projective or finite respectively. If an algebra B in K has a projective cover, then this cover is essentially unique. The general theory of projective covers in an arbitrary category can be found in Banaschewski [2].

We will use the following notations

$A \leq B$: A is a subalgebra of B

$A \xrightarrow{f} B$: f is an injective homomorphism

$A \xrightarrow{f} B$: f is a surjective homomorphism.

Also since the precise operations of the algebras considered will play no role, we use upper case Latin letter instead of upper case German letters.

3. Finitely Projective Algebras

Let K be a variety of algebras. An algebra A in K is called finitely projected if for any surjective $f : B \twoheadrightarrow A$ in K, there is a finite subalgebra of B whose image under f is A. Thus a finitely projected algebra is necessarily finite and clearly every homomorphic image of a finite projective algebra in K is finitely projected.

(3.1) Lemma. Let $A \in K$ be finitely projected. Then for any

$B \in K$ and surjective $f : B \twoheadrightarrow A$ there exists a finite subalgebra $C \leq B$ with $f|C : C \twoheadrightarrow A$ a cover. Moreover if B is projective, so is C.

Proof: Given $f : B \to A$, there is a finite subalgebra $D \leq B$ with $f[D] = A$ since A is finitely projected. Since D is finite, D has only finitely many subalgebras. Therefore we can take C to be minimal in the set $\{E \leq D : f[E] = A\}$.

If B is projective there exists $g : B \quad C$ with $(f|C)\cdot g = f$ therefore $(f|C)\cdot(g|C) = ((f|C)\cdot g)|C = (f|C)$.

Since $f|C$ is a cover, $g|C$ is surjective and since C is finite, $g|C$ is bijective, hence an isomorphism. Therefore C, as a retract of a projective is projective.

Let us note that this lemma shows that the concept of being finitely projected has no content when K is the variety of all groups or all Abelian groups as free (abelian) groups have no subgroups of finite order save the trivial one.

It also gives the following characterization of finitely projected algebras.

(3.2) Theorem: Let K be a variety of algebras; then for any A in K, t.f.a.e.:

(1) A is finitely projected

(2) A has a finite projective cover

(3) A is the homomorphic image of a finite projective algebra in K.

(4) For all projectives P in K and all surjectives $f : P \twoheadrightarrow A$, there exists a finite subalgebra $Q \leq P$ with $f[Q] = A$.

(3.3) Corollary: Every (finite) cover of a finitely projected algebra is finitely projected.

(3.4) Corollary: Every homomorphic image of a finitely projected algebra is finitely projected.

Examples of finitely projected algebras in different varieties will appear in subsequent sections. Our main concern now will be with subdirectly irreducible finitely projected algebras and the role of their projective covers.

(3.5) Lemma: Let P be a projective algebra in a variety K and let $P \overset{\mu}{\rightarrowtail} F_K(X) \overset{\rho}{\twoheadrightarrow} P$ be any retract. Then for any $a,b \in P$ and $\nu : P \to F_K(X)$, $(\nu(a),\gamma(b))$ is in the fully-invariant congruence relation on $F_K(X)$ generated by $(\mu(a),\mu(b))$.

Proof: Consider the endomorphism $\nu \cdot \rho$ of $F_K(X)$. $(\nu \cdot \rho)(\mu(p)) = \nu((\rho \cdot \mu)(p)) = \nu(\dot{p})$ for all $p \in P$. Therefore the statement of the lemma holds.

Although the above lemma is extremely trivial, it has many interesting applications in the determination of conjugate equations for splitting algebras as we shall presently see. Of immediate consequence is the following generalization of Wille [15; Corollary 10].

(3.6) Lemma: Let S be a subdirectly irreducible in a variety K whose least congruence is generated by (u,v). Furthermore assume there is a surjection $f : P \twoheadrightarrow S$ with $a,b \in P$ satisfying

(1) $f(a) = u$ and $f(b) = v$

(2) $\theta_P(a,b)$ is strictly-join-prime.

Then S is a splitting algebra in K. Moreover if $P \xrightarrow{\mu} F_K(X) \overset{\rho}{\twoheadrightarrow} P$ is any retraction, $(\mu(a),\mu(b))$ determine the conjugate equation.

Proof: Take $F_K(X) \overset{\rho}{\twoheadrightarrow} P$ for a suitable set X. Since P is projective, there is indeed a $\mu : P \to F_K(X)$ with $\rho \cdot \mu = 1_P$. We show that S is a splitting algebra by demonstrating that $(\mu(a),\mu(b))$ is indeed its splitting equation.

Since $\theta_P(a,b)$ is strictly-join-prime, and S is subdirectly irreducible, Ker f is strictly-meet-prime and for all congruences θ on P, either $\theta \subseteq \text{Ker } f$ or $\theta_P(a,b) \subseteq \theta$.

Let V be the subvariety of K given by the equation $(\mu(a),\mu(b))$ with $\kappa : F_K(X) \twoheadrightarrow F_V(X)$ the canonical homomorphism. That is, Ker κ is the fully invariant congruence generated by $(\mu(a),\mu(b))$. If S were in V then as $f\cdot\rho : F_K(X) \to S$ is surjective, there would be a surjective morphism $h : F_V(X) \twoheadrightarrow S$. As P is projective in K, there exists $\nu : P \to F_K(X)$ with $f = (h\cdot\kappa)\cdot\nu$. Now $(a,b) \notin \text{Ker } f = \text{Ker}(h\cdot\kappa\cdot\nu)$. But by (3.5) $(\nu(a),\nu(b)) \in \text{Ker } \kappa$ whence $f(a) = h(\kappa(\nu(a))) = h(\kappa(\nu(b))) = f(b)$, a contradiction.

Now if $\mathcal{L}$ is a subvariety of K not containing S and

$\lambda : F_K(X) \twoheadrightarrow F_{\mathcal{L}}(x)$ is the cononical surjection, then if $(\mu(a),\mu(b)) \notin \operatorname{Ker} \lambda$ we have $\operatorname{Ker}(\lambda\cdot\mu) \subseteq \operatorname{Ker} f$. Therefore by the homomorphism theorem we have: $S \in \underset{\sim}{H}\underset{\sim}{S}(F_{\mathcal{L}}(X))$, a contradiction. Therefore $(\mu(a),\mu(b)) \in \operatorname{Ker} \lambda$.

These last two lemmas give us the connection between finitely projected subdirectly irreducibles and splitting algebras in congruence distributive varieties.

(3.7) Theorem: In a congruence distributive variety, every finitely projected subdirectly irreducible algebra is splitting.

Proof: Let $f : P \twoheadrightarrow S$ be the finite projective cover of the subdirectly irreducible S in a congruence distributive variety K. Since S and P are finite, and the congruence lattice of P is distributive, Ker f is (strictly)-meet-prime. Therefore there exists a smallest congruence on P not contained in Ker f which is (strictly)-join-prime and hence principal. That is, there exists $a,b \in P$ such that for all $\theta \in \Theta(P)$, $\theta \subseteq \operatorname{Ker} f$ or $\theta_P(a,b) \subseteq \theta$. It is trivial to see that the pair $(f(a),f(b))$ generates the least congruence on S. Therefore (3.6) applies and S is splitting.

(3.8) Corollary: Let K be a congruence distributive variety in which the finitely generated algebras are finite. Then every finite subdirectly irreducible in K is finitely projected hence a splitting algebra. Moreover, the lattice of subvarieties is infinitely distributive.

Proof: The first statement is immediate from (3.2) and the fact that finitely generated K-free algebras are finite.

The second part comes from the first since every subvariety will be generated by its finite subdirectly irreducible members and the variety (or theory) generated by one of these is strictly-join-prime (resp. strictly-meet prime) (see [13] for terminology).

Before proceeding to examples, let us note that if the finite projective cover of a finitely projected subdirectly irreducible can be constructed in a suitable finitely generated free algebra by some algorithmic methods, we can determine a conjugate equation by inspection. This procedure perhaps could be more easily applied than McKenzie's limit tables.

4. Examples:

(A) Heyting Algebras

A Heyting algebra is bounded relatively pseudo-complemented lattice in which relative-pseudo-complementation is taken as an operation. Balbes and Horn, [1], have sufficient algebraic details for what we need.

By Jankov [7], every finite subdirectly irreducible Heyting algebra is a splitting algebra. However from [1], the finite projective Heyting algebras are precisely the finite horizontal sums (see [1] for terminology) of $\underset{\sim}{2}$ and B_2, the four element Boolean algebra, with a copy of $\underset{\sim}{2}$ on top. Homomorphic images of these are just the finite horizontal sums of $\underset{\sim}{2}$ and B and therefore the subdirectly irreducible homomorphic images (1 is join-irreducible) are precisely the projective Heyting algebras again. We have shown:

(4.1) Theorem: The finitely projected Heyting algebras are precisely the finite horizontal sums of copies of $\underset{\sim}{2}$ and B_2. A finite Heyting algebra is projective if and only if it is a finitely-projected subdirectly irreducible.

(B) Implication semi-lattices, Hilbert algebras and Distributive pseduo-complemented lattices

In each of these three varieties, the finitely generated algebras are finite. (See [12], [4] and [11] respectively.) Therefore the finitely projected algebras are precisely the finite ones, every finite subdirectly irreducible is splitting and the lattices of subvarieties are infinitely distributive.

(C) Lattices

While we have no characterization of the finitely projected lattices other than (3.2), we do have the fact that the subdirectly irreducible finitely projected lattices are a proper subclass of the splitting lattices.

(4.2) Theorem: Let L be a finite lattice that has a generating set X of more than two elements which satisfies:

$(*)\ \emptyset \neq Y, Z \subseteq X$ and $\bigwedge Y \leq \bigvee Z$ imply $Y \cap Z \neq \emptyset$

then L is not finitely projected.

Proof: Take $\phi : FL(X) \twoheadrightarrow L$ extending the identity function on the generators. Then for any subset, $\overline{X}$, of $FL(X)$ given by a choice function on $\Pi(\phi^{-1}[x] : x \in X)$, $\overline{X}$ also satisfies $(*)$. By Jónsson [10: lemma 3], the sublattice of $FL(X)$ generated by $\overline{X}$ satisfies Whitman's first three conditions and since it must also satisfy the fourth, it is isomorphic to $FL(X)$ which is infinite. By (3.1) then, L is not finitely projected.

(4.3) Corollary: There is a splitting lattice which is not finitely projected.

Proof: The lattice Q in diagram (i) is splitting from [13] but its generating set $\{a,b,c\}$ satisfies $(*)$.

5. Finitely Projected Modular Lattices

Let M be the variety of modular lattices. We wish to construct a large class of finitely projected subdirectly irreducible (hence splitting) modular lattices.

By $D(u < a,b,c, < v)$, we mean a <u>non-degenerate</u> 3-diamond as in diagram (ii). T_n $(n \geq 1)$ is the modular lattice $\bigcup_i^{1,n} D(u_i < a_i,b_i,c_i < v_i)$ with $v_i = a_{i+1}$ and $c_i = u_{i+1}$ for each $i = 1,\ldots,n-1$. P_n $(n \geq 1)$ is the modular lattice given by the disjoint union $\bigcup_i^{1,n} D(\bar{u}_i < \bar{a}_i,\bar{b}_i,\bar{c}_i < \bar{v}_i)$ with $\bar{v}_i \wedge \bar{u}_{i+1} = \bar{c}_i$

and $\bar{v}_i \vee \bar{u}_{i+1} = \bar{a}_{i+1}$ for each $i = 1,2,\ldots,n-1$. Thus P_n is obtained from T_n by pulling apart all coincident diamond edges; and there exists a unique surjection $f_n : P_n \to T_n$ by collapsing these pulled-apart edges. (See diagram (iii).)

(5.1) Theorem: For every $n \geq 1$, T_n is a finitely projected simple modular lattice with $f_n : P_n \to T_n$ its projective cover.

(5.2) Corollary: Every T_n, $n \geq 1$, is a splitting modular lattice.

Before proving this theorem we should note that the corollary for $n = 2$ was shown in Jónsson [9]. It must be noted however that Jónsson's result is stronger in that he explicitly described the splitting variety by describing its subdirectly irreducible members. This does not seem to be an easy task for $n \geq 4$, however Hong [6] has some interesting partial results.

Also, explicit descriptions of P_n as a sublattice of $FM(n+2)$ can be obtained via the method of proof and therefore conjugate equations (see [13]) can be obtained.

Proof: Let $S(n)$ be the statement "For any surjective map $g : A \twoheadrightarrow T_n$, there exists a sublattice $C_n \leq A$ with $C_n = \bigcup_i^{1,n} D(p_i < r_i, s_i, t_i < q_i)$ with $p_{i+1} \wedge q_i = t_i$ and $p_{i+1} \vee q_i = r_{i+1}$ for $i = 1,\ldots,n-1$ such that $g(p_i) = u_i$, $g(r_i) = a_i$, $g(s_i) = b_i$, $g(t_i) = c_i$ and $g(q_i) = v_i$ for $i = 1,\ldots,n$."

$S(1)$ is trivially true as $T_1 = M_3 = P_1$ is a finite projective modular lattice. Therefore assume $S(n)$ for $n \geq 1$ and consider a surjective map $g : A \twoheadrightarrow T_{n+1}$.

Since $T_n \leq T_{n+1}$ by considering only the first n diamonds, we have by inductive assumption a sublattice $C_n \leq g^{-1}[T_n]$ satisfying the conditions of $S(n)$. Moreover since g is surjective and $S(1)$ holds, there is a diamond $D_{n+1} = D(p_{n+1} < r_{n+1}, s_{n+1}, t_{n+1} < q_{n+1}) \leq [t_n, \rightarrow)_A$, the sublattice of all elements of A greater than or equal to t_n which is mapped isomorphically onto the $(n+1)^{th}$ diamond of T_{n+1}. We will show that our desired C_{n+1} is a sublattice of the sublattice of A generated by $C_n \cup D_{n+1}$.

Let $z_n = q_n \wedge r_{n+1}$ and $w_n = q_n \wedge p_{n+1}$. Then we have $t_n \leq w_n < z_n \leq q_n$ with the strict in equality holding between w_n and z_n since $g(w_n) = c_n < v_n = g(z_n)$. By Hong's extension ([6; sec. 3.2]) of a result of Jónsson, the sublattice of A generated by $C_n \cup \{z_n, w_n\}$ is of the form

$$\bigcup_{i}^{1,w} E(p_i, q_i)$$

where each $E(p_i, q_i)$ is a homomorphic image of Q, the lattice in diagram (iv) with the edges $[t_i, q_i]$ and $[p_{i+1}, r_{i+1}]$ transposed and at most $z = q$ and $w = t$. This give a sublattice $\overline{C}_n$ of A satisfying the statement $S(n)$ where
$\overline{C}_n = \bigcup_{i}^{1,n} D(\bar{p}_i < \bar{r}_i, \bar{s}_i, \bar{t}_i < \bar{q}_i)$.

Now consider

$$x_{n+1} = w_n \vee p_{n+1} = \bar{t}_n \vee p_{n+1}$$

$$y_{n+1} = z_n \vee p_{n+1} = \bar{q}_n \vee p_{n+1}$$

Again we have $p_{n+1} \leq x_{n+1} < y_{n+1} \leq r_{n+1}$ and the sublattice of A generated by $D_{n+1} \cup \{x_{n+1}, y_{n+1}\}$ is as described above. It follows easily that $C_{n+1} = \bigcup_{i}^{1,n+1} D(\bar{p}_i < \bar{r}_i, \bar{s}_i, \bar{t}_i < \bar{q}_i)$ satisfies the conditions of the statement $S(n+1)$.

Let us note that if S_n $(n \geq 1)$ is defined by a "snake" of n diamonds and Q_n $(n \geq 1)$ is defined analogously to P_n (see diagram (v)), then the proof is completely analogous and we have:

(5.3) Theorem: For every $n \geq 1$, S_n is a finitely projected simple modular lattice with projective cover $h_n : Q_n \twoheadrightarrow S_n$.

(5.4) Corollary: Every S_n, $n \geq 1$, is a splitting modular lattice.

We should note at this time the existence of non finitely-projected modular lattices. From [3], M_4 is not a splitting modular lattice and therefore is not finitely projected.

Our class of finitely projected modular lattices can be enlarged greatly by a slightly different procedure.

(5.5) Lemma: Every modular lattice that is the subdirect product of two finite chains is finitely projected.

Proof: This is an immediate consequence of the fact that the free modular lattice generated by two chains is both finite and projective.

Now suppose we judiciously insert elements in such a finite lattice to make some of the B_2 boxes into diamonds (see diagram (vi)). It is clear by inspection (and easy to prove) that A and B are finitely projected by firstly pulling back the lattice without the diamond points and then inserting these one at a time. This procedure does not seem to work for C for, having pulled back three diamond points of C along $f : L \twoheadrightarrow C$, we have as a sublattice of L at lattice whose isomorphism type is determined by D in diagram (vii). However in attempting to insert an inverse image point of the last diamond point in the proper B_2 box, we seem to generate a ring around the rosy system of elements on all the other diamond edges, with no conviction as to whether this procedure stops. We conjecture however that given a subdirect product of two finite chains if diamond points are inserted such that there is no sequence of transposes starting at one edge of a diamond and returning to another edge of this diamond without having had to transpose through this diamond then such a modular lattice is finitely projected.

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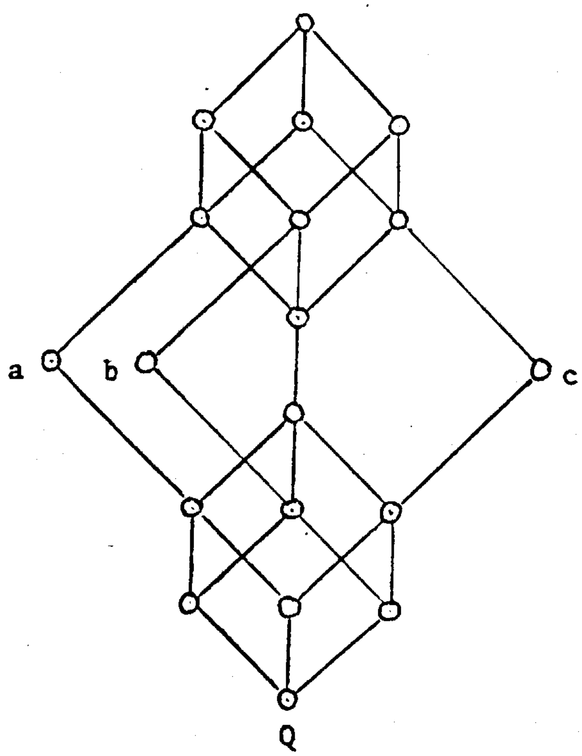
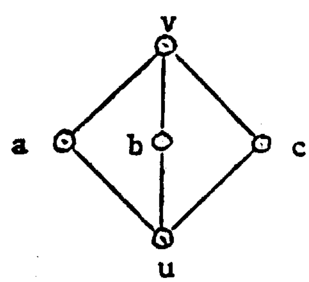
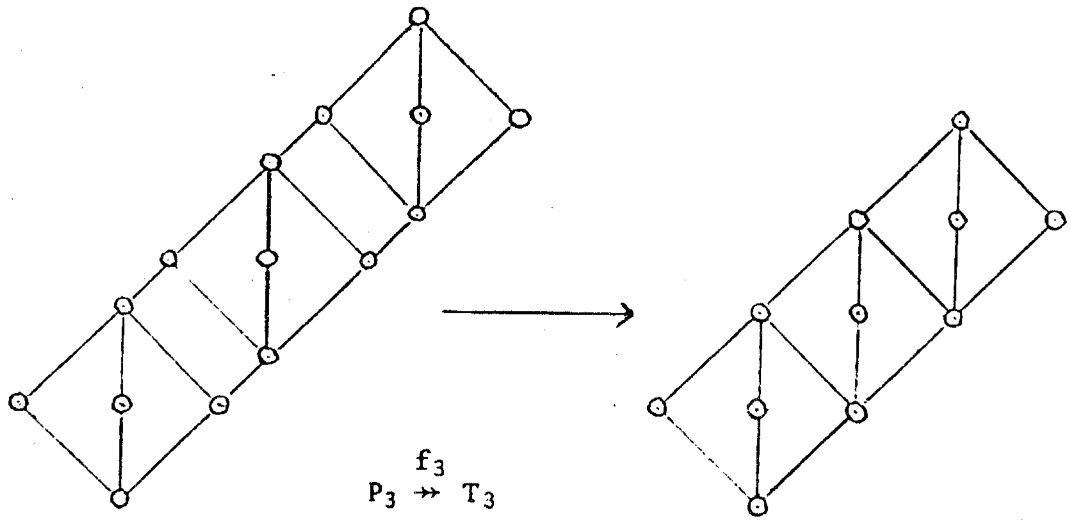


(i)



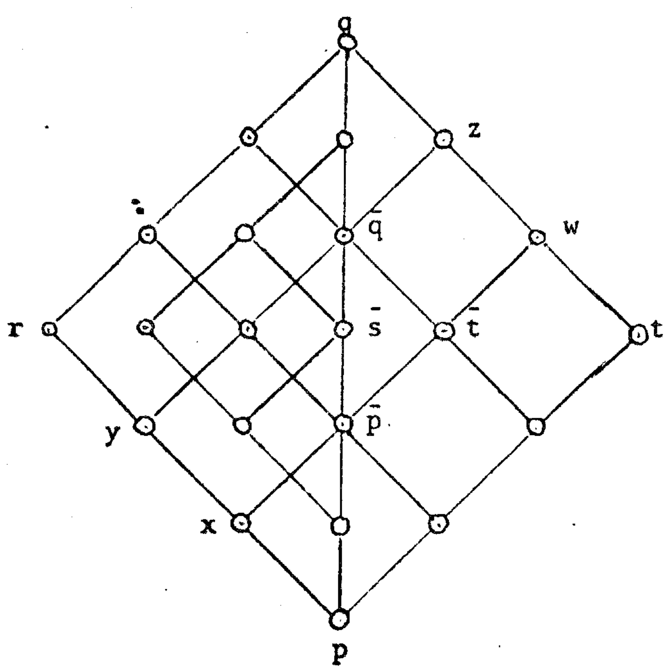
D(u < a,b,c < v)

(ii)

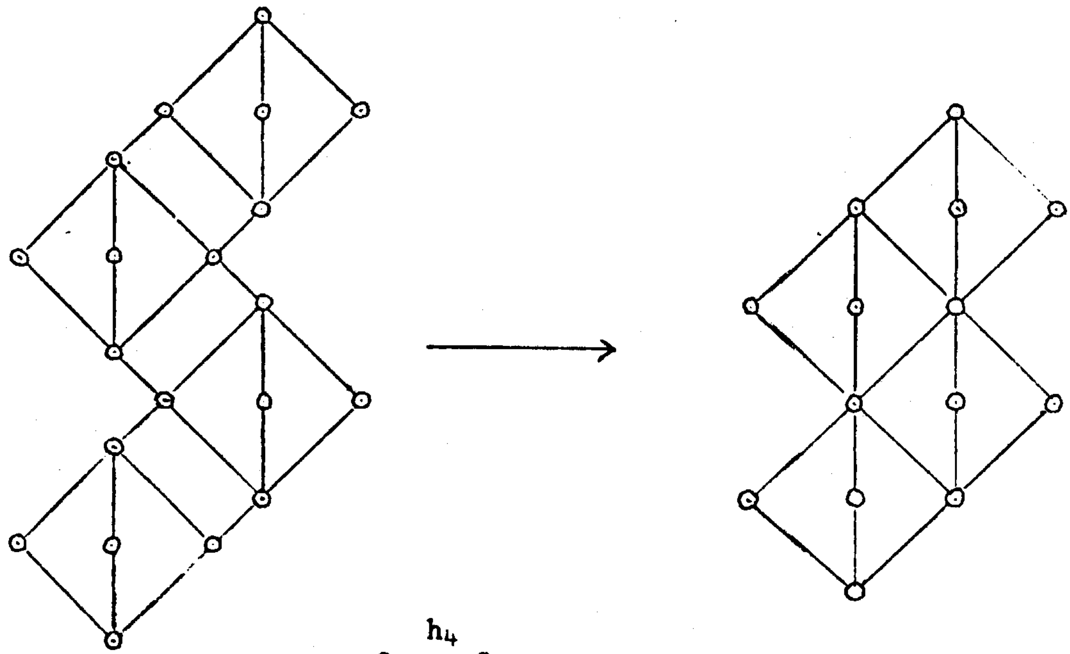


$$P_3 \overset{f_3}{\twoheadrightarrow} T_3$$

(iii)

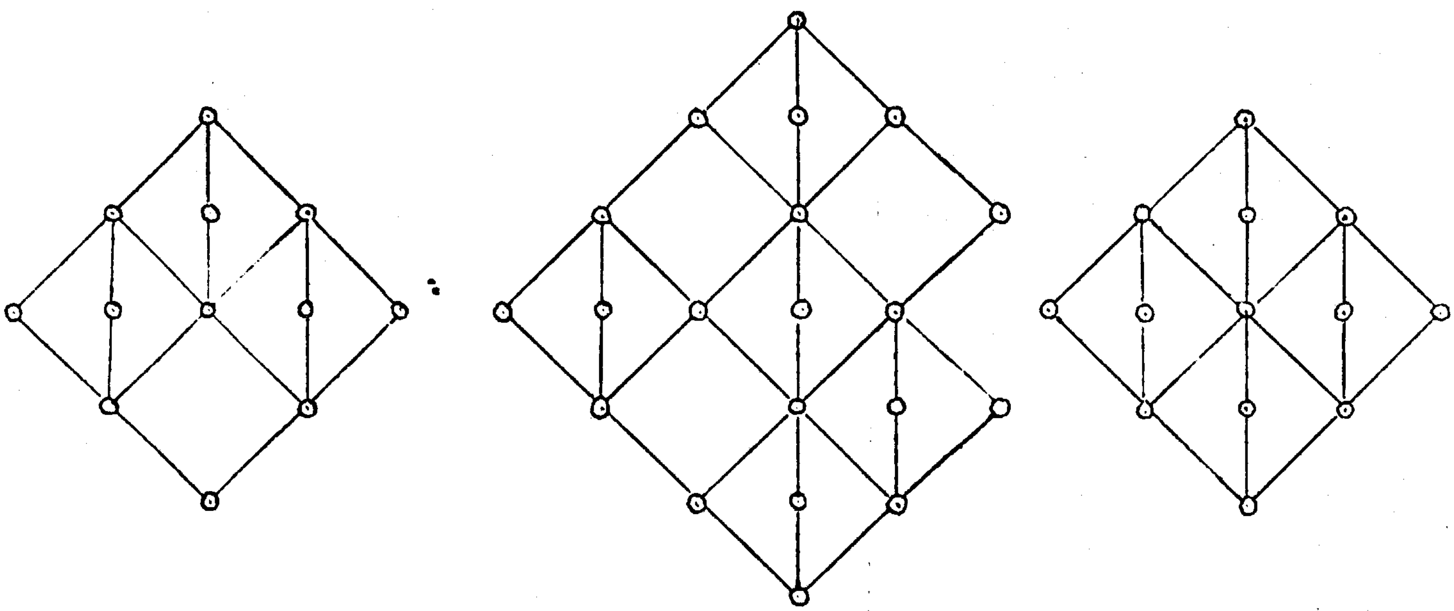


(iv)



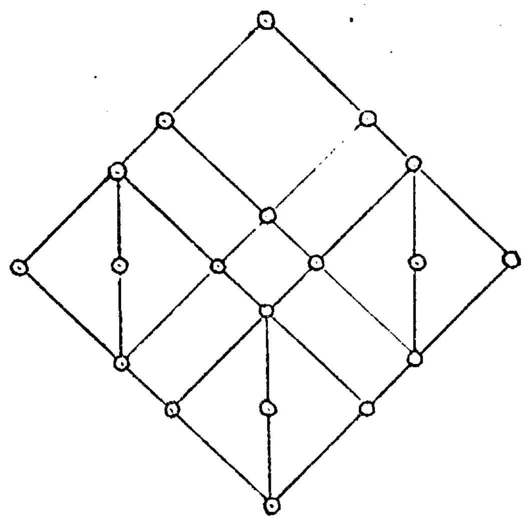
$$Q_4 \overset{h_4}{\twoheadrightarrow} S_4$$

(v)



A B C

(vi)



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(vii)