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PLANAR LATTICES

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This paper presents a brief survey of results about finite planar lattices. Unless otherwise stated, all lattices in this paper are finite.

Anyone beginning the study of lattice theory quickly learns that it is important to be able to draw a picture of a lattice; i.e. the Hasse diagram of a lattice. Once he has become skilled in drawing lattice diagrams he soon notices that whenever the diagram is planar he has indeed drawn a lattice. That is, he does not have to check that all l.u.b.'s and g.ℓ.b.'s exist. This heuristic principle can be formalized as a theorem. The following formulation is due to Harry Lakser.

First we give a formal definition of the diagram of a poset. Let $\mathcal{P}$ be a poset on the n element set $\{p_1, \cdots, p_n\}$. A <u>diagram</u> of $\mathcal{P}$ is a set of n points in the (x, y)-plane, $(x_1, y_1), \cdots, (x_n, y_n)$, together with certain arcs between these points such that:

a) If p_i covers p_j then $y_i > y_j$ and there is an arc, a_{ji}, which is the graph of a continuous function of y with domain $[y_j, y_i]$, with $a_{ji}(y_i) = x_i$ and $a_{ji}(y_j) = x_j$ and with no other point, (x_k, y_k), lying on a_{ji}.

b) There are no other arcs than those of condition a).

$\mathcal{P}$ is <u>planar</u> if it has a planar diagram (i.e. any two arcs intersect only at

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an endpoint). Each arc can be thought of as directed from its bottom end point to its top endpoint. A path is an ascending sequence of connected arcs, i.e. a set of arcs forming the graph of a continuous function of y. Thus $p_i < p_j$ if and only if there is a path from (x_i, y_i) to (x_j, y_j).

This definition of the diagram of a poset is a reasonable approximation to the way one draws posets. It has the advantage that the proof of the theorem uses only the intermediate value theorem for continuous functions rather than some version of the Jordan curve theorem. When drawing the diagram of what one hopes is a lattice, there is an obvious condition to be satisfied: avoid dangling points. More precisely, there must be exactly one point which is not the lower endpoint of an arc (the unit) and exactly one point which is not the upper endpoint of an arc (the zero). Lakser's theorem states that if the diagram is planar then this necessary condition is sufficient for the poset to be a lattice.

Theorem: Let $\mathcal{P}$ be a finite poset with a planar diagram; if there is at most one element of $\mathcal{P}$ which has no cover and at most one element which covers no point then $\mathcal{P}$ is a lattice.

Sketch of the proof: Since $\mathcal{P}$ is finite the "at most" in the statement is equivalent to "exactly"; these points are necessarily the unit and zero of the poset. Now let p_1, p_2, p_3, p_4 be four points of $\mathcal{P}$ such that $p_1 < p_3$, $p_1 < p_4$, $p_2 < p_3$, $p_2 < p_4$. To show that $\mathcal{P}$ is a lattice it is sufficient to show that there is a point p_5 of $\mathcal{P}$ with $p_1 \leq p_5$, $p_2 \leq p_5$, $p_5 \leq p_3$, $p_5 \leq p_4$ (and so we may as well assume that p_1 is not comparable to

p_2 and p_3 is not comparable to p_4). Thus we have paths $\alpha_{1,3}, \alpha_{1,4}, \alpha_{2,3}, \alpha_{2,4}$ (where $a_{i,j}$ goes from (x_i, y_i) to (x_j, y_j)). We can also find points p_6, p_7 such that $p_6 < p_1, p_6 < p_2, p_3 < p_7, p_4 < p_7$ and such that the paths $\alpha_{6,1}$ and $\alpha_{6,2}$ intersect only at (x_6, y_6) and the paths $\alpha_{3,7}$ and $\alpha_{4,7}$ intersect only at (x_7, y_7). Using planarity and the intermediate value theorem for continuous functions, a case by case analysis shows that such an element p_5 must exist.

Now let us turn to the problem of characterizing planar lattices. For distributive lattices this characterization is well-known. A distributive lattice is planar iff it is a sublattice of a direct product of two chains iff it does not contain the eight element boolean lattice as a sublattice iff no element covers more than two other elements iff no element is covered by more than two other elements iff it does not contain the poset of figure 1 as a subposet. For modular lattices the following characterization is due to Rudolf Wille [1]. Recall that an element of a lattice is doubly irreducible if it cannot be written as a proper meet or a proper join.

Theorem: A modular lattice $\mathfrak{m}$ is planar iff $\mathfrak{m} - \{d \in \mathfrak{m} | d$ is doubly irreducible$\}$ is a planar distributive lattice iff $\mathfrak{m}$ does not contain any of the posets of figures 1 and 2 as a subposet.

For lattices in general there is no finite set of posets which can be used to test planarity. In fact, planarity for lattices is not a first order property. This result is due to K. Baker, P. Fishburn, and F. Roberts [2]. To see this, consider the fence posets of figure 3 and the crown posets of figure 4. Adding a zero and unit to each turns them into lattices. Notice that the fence lattices are planar but that the crown lattices are not planar. In [2] it is pointed out that an appropriate ultraproduct of fence lattices is isomorphic to an appropriate ultraproduct of crown lattices. Since ultraproducts preserve first order properties, planarity cannot be a first order property. In particular, there is no finite list of posets such that planar lattices are characterized by the absence of these posets as subposets.

Problem 1: Is there a finite list of posets which test planarity in the variety generated by N_5 ?

Problem 2: Is there a finite list of families of posets which tests planarity for all lattices? (The set of crowns would likely be a family in this list).

It seems clear that there ought to be some nice connection between planar lattices and planar graphs. Such a connection has been found by Craig Platt [4]. If $\mathcal{L}$ is a lattice (with 0 as zero and 1 as unit) then $G(\mathcal{L})$, the graph of $\mathcal{L}$, has the same points as $\mathcal{L}$ and has a directed edge from x to y if and only if x is covered by y ; $G^*(\mathcal{L})$, the augmented graph of $\mathcal{L}$, is $G(\mathcal{L})$ together with a directed edge from 1 to 0 .

Theorem (C.R. Platt): $\mathcal{L}$ is a planar lattice iff $G^*(\mathcal{L})$ is a planar graph.

Sketch of the proof: If $\mathcal{L}$ is a planar lattice then clearly $G^*(\mathcal{L})$ is a planar graph. Conversely suppose $G^*(\mathcal{L})$ is a planar graph. Note that we may assume that the edge from 1 to 0 is on the outside of the graph.

Consider the other outside path from 0 to 1 : $0, x_1, \cdots, x_n, 1$. An induction argument proves that the path is directed in the order given; that is, in $\mathcal{L}$ x_1 covers 0 , x_{i+1} covers x_i for $i = 1, \cdots, n-1$, and 1 covers x_n . Another induction argument shows that one of the x_i's is doubly irreducible. Finally an induction on the size of $\mathcal{L}$ is made using the statement: If $|\mathcal{L}| = m$, $G^*(\mathcal{L})$ is planar with $0, x_1, \cdots, x_n, 1$ an outside path from 0 to 1 then $\mathcal{L}$ is planar and can be drawn with straight lines so that $0, x_1, \cdots, x_n, 1$ is an outside path in the diagram of $\mathcal{L}$.

Corollary: Every planar lattice has a planar diagram in which all arcs are straight lines.

Planar lattices form a subclass of the class of dismantlable lattices. A lattice is dismantlable if every sublattice contains a doubly irreducible element. More picturesquely, a lattice is dismantlable if one can keep throwing away doubly irreducible elements until nothing is left. Every planar lattice has a doubly irreducible element (see [2]) and a sublattice of a planar lattice is planar. Hence planar lattices are dismantlable. A recent result of David Kelly and Ivan Rival proves that dismantlable lattices are characterized by the absence of crowns.

Theorem [3]: A lattice is dismantlable if and only if it contains no crown poset as a subposet. A modular lattice is dismantlable if and only if it does not contain the crown of order 3 (i.e. figure 1) as a subposet. A distributive lattice is dismantlable if and only if it is planar.

All the material above refers to finite lattices; however it is often the case that one needs an infinite lattice for a particular problem. Thus it would be useful to have analogs of the above results (especially Lakser's theorem) for infinite planar lattices.

Problem 3: Develop a theory of infinite planar lattices.

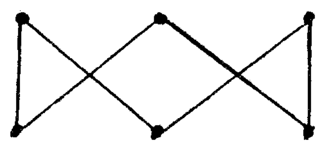

Figure 1

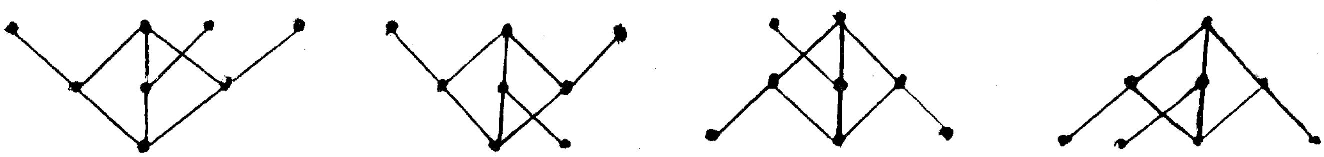

Figure 2

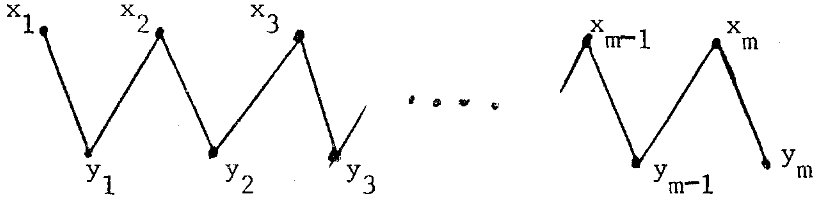

Figure 3: The fence of order m $(m \geq 3)$

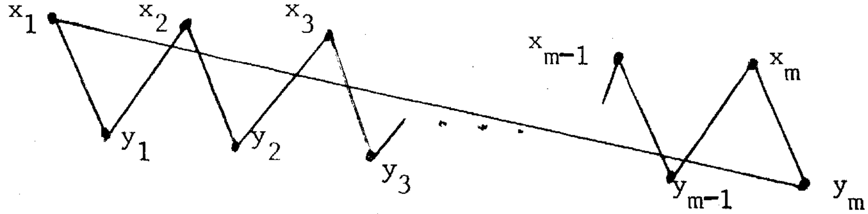

Figure 4: The crown of order m $(m \geq 3)$

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