ERRATUM TO "THE ZARISKI TOPOLOGY ON THE PRIME SPECTRUM OF A MODULE" (HOUSTON J. MATH., 25(3), 1999, 417-432)

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1. Introduction

Proposition 5.2(3) and its result Proposition 6.3 in the published version [1] are incorrect. The original Proposition 5.2(3) in [1] states: For a module M over a ring R, let P be an element of $X = \mathrm{Spec}(M)$. Then the set $\{P\}$ is closed in X for the Zariski topology $\iff$ (i) P is a maximal submodule of M, and (ii) $|\mathrm{Spec}_p(M)| = 1$ where $p = (P{:}M)$.

The statement is correct if M is finitely generated. However, in general, it is not so as the example of the Z-module $M = Q,\ P = (0)$, shows. Since (0) is the unique prime submodule of Q, the Zariski topology on $\mathrm{Spec}(Q)$ is the trivial topology by ([1] p.420, Example 1(a)). Accordingly, $\{(0)\}$ is closed in $\mathrm{Spec}(Q)$. However (0) is not a maximal submodule of Q, hence (i) is not true.

A topological space is a T_1-space if and only if every singleton subset is closed. Based on this fact, in [1], Proposition 6.3 was deduced from Proposition 5.2(3) as follows: For an R-module M, $\mathrm{Spec}(M)$ is a T_1-space $\iff$ $Max(M) = \mathrm{Spec}(M)$, where $Max(M)$ is the set of all maximal submodules of M. However, the Z-module Q is also a counterexample to Proposition 6.3 because $\mathrm{Spec}(Q) = \{(0)\}$ is a T_1-space with $Max(Q) \neq \mathrm{Spec}(Q)$.

Here we give, respectively, a simple correction of [1] Proposition 5.2(3) and that of [1] Proposition 6.3 in Proposition 1 and Proposition 2 of §2 below.

2. Correction

For an R-module M with $\mathrm{Spec}(M) = X$, let Ψ be a subset of $\mathrm{Spec}(R)$ defined by $\Psi = \{(P{:}M) | P \in X\}$. p is a maximal element of Ψ whenever $p \subseteq q$, where $q \in \Psi$, implies that $p = q$.

Proposition 1. *For an R-module M, let P be an element of X. Then the set $\{P\}$ is closed in X*

$\iff$ (i) $p = (P{:}M)$ is a maximal element of Ψ, and

(ii) $Spec_p(M) = \{P\}$, that is, $|Spec_p(M)| = 1$.

Proof. From ([1], p.425, Proposition 5.2(1)), we know that $\{P\}$ is closed $\iff$ $\{P\} = V(P)$. Let $q \in \Psi$ such that $p \subseteq q$. We show that $q = p$. If $Q \in X$ is a q-prime submodule, then $Q \in V(P) = \{P\}$ so that $Q = P$ and $q = p$, which proves (i). If P' is any member of $\mathrm{Spec}_p(M)$, then $P' \in V(P) = \{P\}$ whence $P' = P$ and (ii) follows. Conversely we assume (i) and (ii), and show that $V(P) \subseteq \{P\}$. If $Q \in V(P)$, then $q = (Q{:}M) \supseteq (P{:}M) = p$. Hence (i) implies $q = p$ and consequently

(ii) implies $Q = P$, so that $V(P) \subseteq \{P\}$. Since the other inclusion is trivially true, we have $\{P\} = V(P)$, namely, $\{P\}$ is closed in X. $\square$

Proposition 2. *Let M be an R-module. Then*
$Spec(M)$ is a T_1-space
$\Longleftrightarrow$ *(i) $p = (P{:}M)$ is a maximal element of Ψ for every $P \in X$, and*
(ii) $|Spec_p(M)| \leq 1$ for every $p \in Spec(R)$.

Proof. Note that (ii) is equivalent to that $|\mathrm{Spec}_p(M)| = 1$ for every $p \in \Psi$. Thus, in view of Proposition 1 above, (i) and (ii) are equivalent to that the singleton set $\{P\}$ is closed in X for every $P \in X$, that is, X is a T_1-space. $\square$

References

[1] C. P. Lu, *The Zariski topology on the prime spectrum of a module*, Houston J. Math., 25(3) (1999), 417-432.

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