Spectral-Fourier Method for Axisymmetric Problems

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Abstract

This paper is a short presentation of a joint work with Mejdi Azaïez, Christine Bernardi and Yvon Maday, which will be published in a book entitled "Spectral Methods for Axisymmetric Domains". Here we focus on the solution of a simple boundary value problem: the Dirichlet problem associated with the Laplace operator.

Key words: axisymmetric, spectral, Dirichlet, Laplace.

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1 Introduction

1.1 The geometry

As a prototype for axisymmetric boundary value problems, we consider the Dirichlet problem for the Laplacian Δ in a three-dimensional axisymmetric domain $\breve{\Omega}$. Let us denote by $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ the meridian domain of $\breve{\Omega}$.

If (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and (r, z, θ) are the corresponding cylindrical coordinates in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/_{2\pi\mathbb{Z}}$, we have

$$(x, y, z) \in \breve{\Omega} \iff (r, z) \in \Omega \quad \text{and} \quad \theta \in \mathbb{T}.$$

The rotation axis is $r = 0$.

Our geometrical assumptions are the following:

- Ω is a polygonal domain with sides and corners,
- $\overline{\Omega} \cap \{r = 0\} =: \Gamma_0$ is a full side of Ω.

We denote by Γ the part of the boundary of Ω which is not contained in Γ_0 and we have, for the boundary of $\breve{\Omega}$

$$(x, y, z) \in \partial\breve{\Omega} \iff (r, z) \in \Gamma \quad \text{and} \quad \theta \in \mathbb{T}.$$

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1.2 The Dirichlet problem

In Cartesian coordinates (x, y, z), the Dirichlet problem writes:

$$(1) \qquad \begin{cases} -\Delta\breve{u} = \breve{f} & \text{in } \breve{\Omega}, \\ \breve{u} = \breve{g} & \text{on } \partial\breve{\Omega}. \end{cases}$$

In cylindrical coordinates problem (1) becomes

$$(2) \qquad \begin{cases} -\partial_r^2\tilde{u} - \frac{1}{r}\,\partial_r\tilde{u} - \partial_z^2\tilde{u} - \frac{1}{r^2}\,\partial_\theta^2\tilde{u} = \tilde{f} & \text{in } \Omega \times \mathbb{T}, \\ \tilde{u} = \tilde{g} & \text{on } \Gamma \times \mathbb{T}. \end{cases}$$

Thus, the coefficients are independent from the angular variable θ and a natural method for this problem is the angular Fourier decomposition.

To this respect, let us recall that the Fourier decomposition, associated with a cut-off frequency, is known as the Fourier version of spectral methods.

1.3 Fourier decomposition

For $\breve{v}$ defined on $\breve{\Omega}$, the Fourier coefficients are

$$v^k(r, z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \tilde{v}(r, z, \theta)\, e^{-ik\theta}\, d\theta.$$

The Dirichlet problem (1) on $\breve{\Omega}$ is equivalent to the sequence of Dirichlet problems on Ω: $\forall k \in \mathbb{Z}$

$$(3) \qquad \begin{cases} -\partial_r^2 u^k - \frac{1}{r}\,\partial_r u^k - \partial_z^2 u^k + \frac{k^2}{r^2}\, u^k = f^k & \text{in } \Omega, \\ u^k = g^k & \text{on } \Gamma. \end{cases}$$

We propose a numerical analysis of each problem (3), combined with a cut-off frequency in the Fourier parameter k, which yields a numerical approach to the solution of the three-dimensional problem (1) by a finite number of discrete two-dimensional problems.

In the case when the right hand side $(\breve{f}, \breve{g})$ is invariant by rotation, i.e. if its only non zero Fourier coefficient is (f^0, g^0), $\breve{u}$ has only its coefficient u^0 non zero and problem (1) is equivalent to the only problem (3) with $k = 0$. A numerical analysis by Finite Element Method of this fully axisymmetric situation was performed in [9].

A complete coupling between the Fourier decomposition and Finite Element Method for general data is studied more recently in [8].

After the presentation of an abstract framework for the Fourier analysis coupled with an approximation of the problems (3), we construct a spectral method adapted to the different problems (3) and satisfying the convenient estimates with respect to k.

2 Variational formulations

2.1 The spaces

The pivot space on $\breve{\Omega}$ is $L^2(\breve{\Omega})$ and the variational space on $\breve{\Omega}$ is $H^1(\breve{\Omega})$.

We give hereafter their characterization by Fourier coefficients: let v be a function defined on $\breve{\Omega}$ and let v^k be its Fourier coefficients.

Concerning the L^2 norm, we have the following equivalence

$$\|\breve{v}\|^2_{L^2(\breve{\Omega})} = \sum_{k\in\mathbb{Z}} \|v^k\|^2_{L^2_1(\Omega)}$$

where the norm in $L^2_1(\Omega)$ is defined as

$$\|w\|^2_{L^2_1(\Omega)} = \int_\Omega |w(r,z)|^2\, r\, dr\, dz.$$

The space $L^2_1(\Omega)$ is the L^2 space on Ω associated with the measure $r\,dr\,dz$.

Concerning the H^1 norm on $\breve{\Omega}$ we have

$$\|\breve{v}\|^2_{H^1(\breve{\Omega})} = \sum_{k\in\mathbb{Z}} \|v^k\|^2_{H^1_{(k)}(\Omega)}$$

where the norm with parameter $\|\cdot\|_{H^1_{(k)}(\Omega)}$ is defined as

$$\|w\|^2_{H^1_{(k)}(\Omega)} = \int_\Omega \left(|\partial_r w|^2 + |\partial_z w|^2 + \frac{k^2}{r^2}|w|^2\right) r\,dr\,dz$$

Thus, for any $k\in\mathbb{Z}$, the pivot space of problem (3) is $L^2_1(\Omega)$, whereas, as can be seen by the definition of the norm $\|\cdot\|_{H^1_{(k)}(\Omega)}$, the variational spaces are different according as k is 0 or not: these are

$$\begin{cases} H^1_1(\Omega) := \{w \mid \partial_r w, \partial_z w, w \in L^2_1(\Omega)\} & \text{if } k=0,\\ V^1_1(\Omega) := \{w \mid \partial_r w, \partial_z w, \frac{w}{r} \in L^2_1(\Omega)\} & \text{if } k\neq 0.\end{cases}$$

2.2 The problems

In order to introduce the variational formulation for the continuous problems (3), we need the subspaces of the variational spaces with zero Dirichlet traces:

$$\begin{cases} H^1_{1\diamond}(\Omega) = \{w \in H^1_1(\Omega),\ w=0 \text{ on } \Gamma\} & \text{if } k=0,\\ V^1_{1\diamond}(\Omega) = \{w \in V^1_1(\Omega),\ w=0 \text{ on } \Gamma\} & \text{if } k\neq 0.\end{cases}$$

We introduce the product in $L^2_1(\Omega)$

$$\langle f, v\rangle = \int_\Omega f(r,z)\,\overline{v}(r,z)\, r\,dr\,dz.$$

For $k=0$, the variational formulation of problem (3) is

$$(4)\left[\begin{array}{l} \textit{find } u^0 \in H^1_1(\Omega), \textit{ with } u^0 - g^0 \in H^1_{1\diamond}(\Omega)\ , \ s.t.\\ \qquad\qquad \forall v \in H^1_{1\diamond}(\Omega),\quad a_0(u^0,v) = \langle f^0, v\rangle\end{array}\right.$$

whereas for $k\neq 0$, this is

$$(5)\left[\begin{array}{l} \textit{find } u^k \in V^1_1(\Omega), \textit{ with } u^k - g^k \in V^1_{1\diamond}(\Omega)\ , \ s.t.\\ \qquad\qquad \forall v \in V^1_{1\diamond}(\Omega),\quad a_k(u^k,v) = \langle f^k, v\rangle\end{array}\right.$$

with, for any $k\in\mathbb{Z}$:

$$a_k(u,v) = \int_\Omega \left(\partial_r u\,\partial_r \overline{v} + \partial_z u\,\partial_z\overline{v} + \frac{k^2}{r^2}u\,\overline{v}\right) r\,dr\,dz$$

The integrodifferential forms a_0, resp. a_k for $k\neq 0$, are continuous and coercive on $H^1_{1\diamond}(\Omega)$, resp. on $V^1_{1\diamond}(\Omega)$:

$$|a_k(u,u)| = \|u\|^2_{H^1_{(k)}(\Omega)}.$$

3 X - Fourier method

We introduce an abstract method of approximation "X" for each problem (4) and (5) and thus define a discrete method for the three-dimensional problem (1) via the introduction of a *cut-off frequency*.

3.1 Spaces and forms

Let $K\in\mathbb{N}$ denote the cut-off frequency and let δ denote the parameter of discretization for the problems (4) and (5). For the Finite Element Method, the size of the mesh h plays the role of δ, for the $p-$version δ equals p, the degree of polynomials, and for the spectral method, we use to denote by N the degree of the polynomials and it will be taken as δ.

The approximate spaces are essentially generated by two families of finite dimensional subspaces of $L^2_1(\Omega)$:

$$\big(\mathbb{X}_\delta(\Omega)\big)_\delta \text{ family of approximate spaces for } H^1_1(\Omega)$$

$$\big(\mathbb{X}^*_\delta(\Omega)\big)_\delta \text{ family of approximate spaces for } V^1_1(\Omega).$$

These two families generate a new family of spaces $\big(\mathbb{S}_{K\delta}(\breve{\Omega})\big)_{K\delta}$ approximating $H^1(\breve{\Omega})$, defined by

$$\mathbb{S}_{K\delta}(\breve{\Omega}) = \Big\{\breve{v}_{K\delta} = \sum_{k=-K}^{K} v^k e^{ik\theta};\quad v^0 \in \mathbb{X}_\delta(\Omega),$$
$$\text{and for } k\neq 0,\quad v^k \in \mathbb{X}^*_\delta(\Omega)\Big\}.$$

We can also take into account a quadrature formula and introduce for any $k \in \mathbb{Z}$ the sesquilinear forms $a_{k,\delta}$ which approach the forms a_k and we also need an approximate scalar product $\langle \cdot,\cdot \rangle_\delta$ for $\langle \cdot,\cdot \rangle$.

3.2 Discretized problems

The exact solution $\breve{u}$ of problem (1) is approached by $\breve{u}_{K\delta}$ belonging to $\mathbb{S}_{K\delta}(\breve{\Omega})$:

$$\breve{u}_{K\delta} = \sum_{k=-K}^{K} u_\delta^k \, e^{ik\theta} \tag{6}$$

with u_δ^0 solution of the discretized variational problem for $k=0$:

$$\left[\begin{array}{l} \text{find } u_\delta^0 \in \mathbb{X}_\delta(\Omega), \text{ with } u_\delta^0 - g_\delta^0 \in \mathbb{X}_\delta^\diamond(\Omega), \text{ s.t.} \\ \qquad \forall v_\delta \in \mathbb{X}_\delta^\diamond(\Omega), \quad a_{0,\delta}(u_\delta^0, v_\delta) = \langle f^0, v_\delta \rangle_\delta \end{array}\right. \tag{7}$$

and u_δ^k solution of the discretized variational problem for $k \neq 0$:

$$\left[\begin{array}{l} \text{find } u_\delta^k \in \mathbb{X}_\delta^*(\Omega), \text{ with } u_\delta^k - g_\delta^k \in \mathbb{X}_\delta^0(\Omega), \text{ s.t.} \\ \qquad \forall v_\delta \in \mathbb{X}_\delta^0(\Omega), \quad a_{k,\delta}(u_\delta^k, v_\delta) = \langle f^k, v_\delta \rangle_\delta \end{array}\right. \tag{8}$$

Here $\mathbb{X}_\delta^\diamond(\Omega)$, resp. $\mathbb{X}_\delta^0(\Omega)$, denote the subspace of $\mathbb{X}_\delta(\Omega)$, resp. $\mathbb{X}_\delta^*(\Omega)$, with null traces on Γ. Moreover, the functions g_δ^k are approximates of g^k in $\mathbb{X}_\delta(\Omega)$, resp. $\mathbb{X}_\delta^*(\Omega)$ if $k \neq 0$.

3.3 Fourier error estimates

Let us introduce the *exact truncated Fourier series*

$$\breve{u}_{[K]} = \sum_{k=-K}^{K} u^k \, e^{ik\theta}.$$

We obviously have

$$\begin{aligned} \|\breve{u} - \breve{u}_{K\delta}\|^2_{H^1(\breve{\Omega})} &= \|\breve{u} - \breve{u}_{[K]}\|^2_{H^1(\breve{\Omega})} \\ &\quad + \sum_{k=-K}^{K} \|u^k - u_\delta^k\|^2_{H^1_{(k)}(\Omega)} \end{aligned} \tag{9}$$

Lemma 3.1 *For any $s \geq 0$, if $\breve{u}$ belongs to the Sobolev space $H^{s+1}(\breve{\Omega})$, there holds:*

$$\|\breve{u} - \breve{u}_{[K]}\|_{H^1(\breve{\Omega})} \leq c\, K^{-s} \, \|\breve{u}\|_{H^{s+1}(\breve{\Omega})}.$$

Let $\breve{f} \in H^{s-1}(\breve{\Omega})$ and $\breve{g} \in H^{s+1}(\breve{\Omega})$. If $\breve{\Omega}$ has a smooth boundary, then $\breve{u}$ belongs $H^{s+1}(\breve{\Omega})$. But, with our geometrical assumptions, $\breve{\Omega}$ has conical points and edges in a generic way. Thus, as proven in [7] and [4] for instance, $\breve{u}$ does not belong to $H^{s+1}(\breve{\Omega})$ in general: indeed, $\breve{u}$ has singular parts near the conical points and near the edges. Nevertheless, we prove that the singular parts near the conical points involve only a finite number of Fourier coefficients and moreover that the singular parts near the edges are regular with respect to the angular variable θ, and as a consequence we can state:

Theorem 3.1 *For any $s \geq 0$, if $\breve{f} \in H^{s-1}(\breve{\Omega})$ and $\breve{g} \in H^{s+1}(\breve{\Omega})$, there holds for the solution $\breve{u}$ of problem* (1)*:*

$$\|\breve{u} - \breve{u}_{[K]}\|_{H^1(\breve{\Omega})} \leq c\, K^{-s} \left(\|\breve{f}\|_{H^{s-1}(\breve{\Omega})} + \|\breve{g}\|_{H^{s+1}(\breve{\Omega})} \right).$$

3.4 Abstract error estimates

We characterize for any $s \geq 0$, the Sobolev space $H^s(\breve{\Omega})$ by the Fourier coefficients: for any $k \in \mathbb{Z}$, there exist spaces $H^s_{(k)}(\Omega)$ endowed with a norm $\|\cdot\|_{H^s_{(k)}(\Omega)}$ such that

$$\|\breve{w}\|^2_{H^s(\breve{\Omega})} = \sum_{k \in \mathbb{Z}} \|w^k\|^2_{H^s_{(k)}(\Omega)}.$$

With the help of different functional tools (tensorization of the variables r and z, Taylor decomposition in $r=0$ of the functions of r, Hardy's inequalities, weighted Sobolev spaces) we prove that $H^s_{(k)}(\Omega)$ is a subspace of the Sobolev space $H^s_1(\Omega)$ of exponent s associated to the measure $r\,dr\,dz$: For $|k| > s-1$

$$\|w\|^2_{H^s_{(k)}(\Omega)} = \|w\|^2_{H^s_1(\Omega)} + k^{2s} \, \|r^{-s} \, w\|^2_{L^2_1(\Omega)}$$

and for $|k| \leq s-1$, $H^s_{(k)}(\Omega)$ is a subspace of $H^s_1(\Omega)$ characterized by the nullity of a certain set of traces on the rotation axis $r=0$. For instance, for $k=0$, all traces of odd rank are 0, and for a larger value of $|k|$, this set of traces increases: for $|k| > s-1$ all existing traces are 0.

Combining Theorem 3.1 with (9), we obtain

Theorem 3.2 *We assume that for any $k \in \mathbb{Z}$, we have the estimate between the solutions of problems* (4) *and* (7)*, resp.* (5) *and* (8) *if $k \neq 0$*

$$\|u^k - u_\delta^k\|_{H^1_{(k)}(\Omega)} \leq X(k,\delta) \Big(\|f^k\|_{H^{s-1}_{(k)}(\Omega)} + \|g^k\|_{H^{s+1}_{(k)}(\Omega)} \Big)$$

for a constant $X(k,\delta) > 0$.

Then, for the approximate solution $\breve{u}_{K\delta}$ defined in (6) *we have the error estimate*

$$\begin{aligned} \|\breve{u} - \breve{u}_{K\delta}\|_{H^1(\breve{\Omega})} &\leq \Big(K^{-s} + \sup_{k=-K}^{K} X(k,\delta) \Big) \\ &\quad \times \left(\|\breve{f}\|_{H^{s-1}(\breve{\Omega})} + \|\breve{g}\|_{H^{s+1}(\breve{\Omega})} \right). \end{aligned} \tag{10}$$

4 Spectral element method

4.1 Domain decomposition

Since Ω is a polygonal domain, it can be covered by a "disjoint union" of quadrilaterals Ω_ℓ, i.e. satisfying

$$\Omega_\ell \cap \Omega_{\ell'} = \emptyset \text{ if } \ell \neq \ell' \quad \text{and} \quad \overline{\Omega} = \cup_\ell \overline{\Omega}_\ell.$$

Since we have assumed that $\overline{\Omega} \cap \{r = 0\} = \Gamma_0$ is a full side of Ω, it is possible to choose a decomposition of Ω satisfying:

$$\overline{\Omega}_\ell \cap \Gamma_0 = \begin{cases} ① & \text{a whole edge of } \Omega_\ell \\ ② & \text{or } \emptyset. \end{cases}$$

The model one dimensional sub-element is the interval

$$\Lambda = (-1, +1)$$

and the model two dimensional element is the square

$$\Sigma = \Lambda \times \Lambda \ni (\zeta, \xi).$$

There exist local maps $\mathcal{F}_\ell : \Sigma$ onto Ω_ℓ. According to the situation of Ω_ℓ with respect to the rotation axis Γ_0, there are two models:
① $\mathcal{F}_\ell$ maps the side $\zeta = -1$ onto $\overline{\Omega}_\ell \cap \Gamma_0$,
② "ordinary" Cartesian situation.

4.2 Discrete spaces

The parameter of discretization in Ω is taken as the degree N of the polynomials on which are based the discrete spaces

$$\delta = N, \quad N \longrightarrow +\infty.$$

The basic spaces of polynomial functions are defined on the interval Λ:

$$\mathbb{P}_N(\Lambda) = \{w \text{ polynomial}, \deg w \leq N\}$$
$$\mathbb{P}_N^*(\Lambda) = \{w \in \mathbb{P}_N(\Lambda),\ w(-1) = 0\},$$

from which are constructed the spaces on the square Σ by tensorization

$$\mathbb{P}_N(\Sigma) = \mathbb{P}_N(\Lambda) \otimes \mathbb{P}_N(\Lambda)$$
$$\mathbb{P}_N^*(\Sigma) = \mathbb{P}_N^*(\Lambda) \otimes \mathbb{P}_N(\Lambda).$$

The spaces $X_\delta(\Omega)$ and $\mathbb{X}_\delta^*(\Omega)$ of the abstract theory are now denoted $X_N(\Omega)$ and $\mathbb{X}_N^*(\Omega)$ respectively, and defined as

$$\mathbb{X}_N(\Omega) = \left\{v_N \in \mathcal{C}^0(\overline{\Omega}),\quad \forall \ell, \left(v_N\big|_{\Omega_\ell}\right) \circ \mathcal{F}_\ell \in \mathbb{P}_N(\Sigma)\right\}$$

and

$$\mathbb{X}_N^*(\Omega) = \left\{v_N \in \mathbb{X}_N(\Omega),\quad v_N\big|_{\Gamma_0} = 0\right\}.$$

We have

$$v_N \in \mathbb{X}_N^*(\Omega) \Longrightarrow \begin{cases} ① & \left(v_N\big|_{\Omega_\ell}\right) \circ \mathcal{F}_\ell \in \mathbb{P}_N^*(\Sigma), \\ ② & \left(v_N\big|_{\Omega_\ell}\right) \circ \mathcal{F}_\ell \in \mathbb{P}_N(\Sigma). \end{cases}$$

4.3 Quadrature formulas

One of the specificities of the spectral methods is the quadrature formulas whose nodes are taken as the roots of certain families of orthogonal polynomials (which are the orthonormal bases of eigenvectors of some second order differential operators on the interval Λ, degenerate at the ends ± 1 of Λ).

For the direction ξ in case ①
For both directions ξ and ζ in case ②

the Gauss–Lobatto formula is used: its nodes ξ_j and weights ρ_j are such that

$$\xi_0 = -1 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$$

and

$$\forall \Phi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^{1} \Phi(\xi)\, d\xi = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j.$$

For the direction ζ in case ①, we have constructed 3 formulas, indexed by $m \in \{1, 2, 3\}$:
• the formula $m = 1$ is a *Gauss-Radau* formula (in $\zeta = +1$) for the measure $(1 + \zeta)d\zeta$;
• the formula $m = 2$ is a *Gauss-Lobatto* formula for the measure $(1 + \zeta)d\zeta$;
• the formula $m = 3$ is derived from the *Gauss-Radau* formula for the measure $d\zeta$: it has the same nodes denoted $\zeta_j^{(3)}$ and its weights $\omega_j^{(3)}$ are obtained from the weights $\tilde{\omega}_j$ of the ordinary Gauss-Radau formula by $\omega_j^{(3)} = (1 + \zeta_j^{(3)})\, \tilde{\omega}_j$.

These quadrature formulas satisfy:
• $m = 1, 3$: the nodes $\zeta_j^{(m)}$ are such that

$$-1 < \zeta_1^{(m)} < \cdots < \zeta_N^{(m)} < \zeta_{N+1}^{(m)} = 1$$

and the exactness property reads

$$\forall \varphi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^{1} \varphi(\zeta)\,(1 + \zeta)d\zeta = \sum_{j=1}^{N+1} \varphi(\zeta_j^{(m)})\, \omega_j^{(m)}$$

— for $m = 1$, we even have the exactness for $\varphi \in \mathbb{P}_{2N}(\Lambda)$.

Type		φ_t^-	φ_t^0	φ_t^+
$m = 1, 3$ $L_1^2(\Lambda)$ norm	Experimental	$2t+2$	$t+\frac{1}{2}$	$2t+1$
	Theoretical	$2t+1$	$t+\frac{1}{2}$	$2t$
$m = 2$ $L_1^2(\Lambda)$ norm	Experimental	$2t+2$	$t+\frac{1}{2}$	$2t+1$
	Theoretical	$2t+2$	$t+\frac{1}{2}$	$2t$
$m = 2$ $L_{-1}^2(\Lambda)$ norm	Experimental	$2t$	$\times$	$\times$
	Theoretical	$2t$	$\times$	$\times$

Table 1: Table of convergence for the interpolation operators

- $m = 2$: the nodes $\zeta_j^{(2)}$ are such that

$$\zeta_1^{(2)} = -1 < \zeta_2^{(2)} < \cdots < \zeta_N^{(2)} < \zeta_{N+1}^{(2)} = 1$$

and the exactness property reads

$$\forall \varphi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^{1} \varphi(\zeta)\,(1+\zeta)d\zeta = \sum_{j=1}^{N+1} \varphi(\zeta_j^{(2)})\,\omega_j^{(2)}.$$

4.4 Projection operators on the interval

Let $L_1^2(\Lambda)$ and $L_{-1}^2(\Lambda)$ be the L^2 spaces on Λ associated with the measures $(1+\zeta)d\zeta$ and $(1+\zeta)^{-1}d\zeta$ respectively. The sharpest results for the orthogonal projection operators in $L^2(\Lambda)$, $L_1^2(\Lambda)$, $L_{-1}^2(\Lambda)$ use the Sobolev spaces $H_{\alpha\beta}^s(\Lambda)$ associated with the Jacobi weights $(1-\zeta)^\alpha(1+\zeta)^\beta$. We have

$$\|\varphi - \pi_N\varphi\|_{L^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s,s}^s(\Lambda)},$$

$$\|\varphi - \pi_N^+\varphi\|_{L_1^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s,s+1}^s(\Lambda)},$$

$$\|\varphi - \pi_N^-\varphi\|_{L_{-1}^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s,s-1}^s(\Lambda)}.$$

The Sobolev spaces in the right hand sides are the domains of the power $\frac{s}{2}$ of the Sturm-Liouville operators associated with $L^2(\Lambda)$, $L_1^2(\Lambda)$, $L_{-1}^2(\Lambda)$ respectively. Such an approach is given in [5, 6].

The interpolation operators associated with the Gauss–Lobatto formula on Λ are denoted i_N, whereas those associated to the formula (m) are denoted $i_N^{(m)}$. We have

$$\|\varphi - i_N\varphi\|_{L^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s,s}^s(\Lambda)}.$$

Let us give here the L^2 estimates for the $i_N^{(m)}$: if $s \ge 1$

$$\|\varphi - i_N^{(m)}\varphi\|_{L_1^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s-1,s}^s(\Lambda)}, \quad m = 1, 3,$$

$$\|\varphi - i_N^{(2)}\varphi\|_{L_1^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s-1,s+1}^s(\Lambda)},$$

and if, moreover $\varphi(-1) = 0$

$$\|\varphi - i_N^{(2)}\varphi\|_{L_{-1}^2(\Lambda)} \le c\,N^{-s}\,\|\varphi\|_{H_{s-1,s-1}^s(\Lambda)}.$$

The formula $m = 2$ is the only one which is convenient for the norm L_{-1}^2, and thus, for the variational space V_1^1 (when $k \ne 0$), since it is the only formula which preserves the zero trace in $\zeta = -1$, which corresponds to the rotation axis. The three formulas can be used for the variational space H_1^1 (when $k = 0$).

In the above Table 1, we show the results of numerical tests for the convergence rate of $\|\varphi - i_N^{(m)}\varphi\|$ in L^2 norms for the following functions φ:

$$\varphi_t^-(\zeta) = (1+\zeta)^t, \quad \varphi_t^0(\zeta) = |\zeta|^t, \quad \varphi_t^+(\zeta) = (1-\zeta)^t,$$

which have singularities in -1, 0 and $+1$ respectively. These numerical results show an order of convergence equal to those of the projection operators in $L_1^2(\Lambda)$ or $L_{-1}^2(\Lambda)$.

4.5 Discrete scalar product and forms

We return to the bidimensional domain Ω itself. Let $\mathcal{L}_1$ be the set of the indices ℓ of the domain decomposition in the situation ①, and $\mathcal{L}_2$ be the set of the ℓ in the situation ②.

By the help of the local maps $\mathcal{F}_\ell$, we define interpolation operators $\mathcal{I}_N^{(m)}$ on the domain Ω: for v in $\mathcal{C}^0(\overline{\Omega})$, $\mathcal{I}_N^{(m)}v$ is the only element of $\mathbb{X}_N(\Omega)$ such that

$$\begin{cases} \forall \ell \in \mathcal{L}_1, \quad \forall j, j' \quad \left(v - \mathcal{I}_N^{(m)} v\big|_{\Omega_\ell}\right) \circ \mathcal{F}_\ell(\zeta_j^{(m)}, \xi_{j'}) = 0 \\ \forall \ell \in \mathcal{L}_2, \quad \forall j, j' \quad \left(v - \mathcal{I}_N^{(m)} v\big|_{\Omega_\ell}\right) \circ \mathcal{F}_\ell(\xi_j, \xi_{j'}) = 0 \end{cases}$$

We have similar L^2 estimates as in the interval Λ. Here are the estimates that we can obtain in the norms $H^1_{(k)}(\Omega)$ of the variational spaces of problems (4) and (5): when $k = 0$, we have for $s > 1$ and $m = 1, 2, 3$

$$\|v - \mathcal{I}_N^{(m)} v\|_{H^1_1(\Omega)} \le c\, N^{-s}\, \|v\|_{H^{s+1}_1(\Omega)}, \tag{11}$$

and when $k \neq 0$, we have for $s > \frac{5}{2}$ and $m = 2$

$$\|v - \mathcal{I}_N^{(m)} v\|_{H^1_{(k)}(\Omega)} \le c\, N^{-s}\, \|v\|_{H^{s+1}_{(k)}(\Omega)}. \tag{12}$$

The definitions of the approximate scalar product and forms are obvious from the above quadrature formulas: for any function v which is integrable for the measure $r\,dr\,dz$, let us denote for each ℓ by $\mathcal{T}^\ell v$ the function on the square Σ such that

$$\int_\Omega v(r,z)\, r\, dr\, dz = \sum_{\ell \in \mathcal{L}_1} \int_\Sigma \mathcal{T}^\ell v(\zeta, \xi)\, (1+\zeta) d\zeta\, d\xi + \sum_{\ell \in \mathcal{L}_2} \int_\Sigma \mathcal{T}^\ell v(\zeta, \xi)\, d\zeta\, d\xi.$$

The approximate scalar product $\langle \cdot, \cdot \rangle_N$ is defined for v and w in $\mathcal{C}^0(\overline{\Omega})$ by

$$\langle v, w \rangle_N = \sum_{\ell \in \mathcal{L}_1} \sum_{j,j'} \mathcal{T}^\ell (v\overline{w})(\zeta_j^{(m)}, \xi_{j'})\, \omega_j^{(m)} \omega_{j'} + \sum_{\ell \in \mathcal{L}_2} \sum_{j,j'} \mathcal{T}^\ell (v\overline{w})(\xi_j, \xi_{j'})\, \omega_j\, \omega_{j'}.$$

For any $k \in \mathbb{Z}$ the sesquilinear forms $a_{k,N}$ which approach the forms a_k are defined thanks to the approximate scalar product $\langle \cdot, \cdot \rangle_N$: for u and v in $\mathcal{C}^1(\overline{\Omega})$ (with zero traces on Γ_0 if $k \neq 0$)

$$a_{k,N}(u,v) = \langle \partial_r u, \partial_r v \rangle_N + \langle \partial_z u, \partial_z v \rangle_N + k^2 \langle \frac{u}{r}, \frac{v}{r} \rangle_N.$$

For $k = 0$, any from the three quadrature formulas can be used, and if $k \neq 0$, only $m = 2$ is used. The discrete forms $a_{k,N}$ are coercive on $H^1_{(k)}(\Omega)$ for any k, uniformly in k.

5 Error estimates for the spectral element method

The error estimates for the discrete problems in Ω (7) and (8) obtained with the above definitions of the discrete spaces and quadrature formulas, are based on lemmas of classical type.

5.1 Lemmas of type Céa and Strang

Lemma 5.1 *For $k = 0$, we have the following estimate between the solution u^0 of the continuous problem (4) and the solution u_N^0 of the discrete problem (7): for any $v_N^0 \in \mathbb{X}_N(\Omega)$ such that $u_N^0 - v_N^0$ belongs to $\mathbb{X}_N^\circ(\Omega)$, for any $w_{N-1}^0 \in \mathbb{X}_{N-1}(\Omega)$, for any $f_{N-1}^0 \in \mathbb{Z}_{N-1}(\Omega)$,*

$$\|u^0 - u_N^0\|_{H^1_1(\Omega)} \le c \Big(\|u^0 - v_N^0\|_{H^1_1(\Omega)} + \|u^0 - w_{N-1}^0\|_{H^1_1(\Omega)} + \|f^0 - \mathcal{I}_N^{(m)} f^0\|_{L^2_1(\Omega)} + \|f^0 - f_{N-1}^0\|_{L^2_1(\Omega)} \Big)$$

Lemma 5.2 *For any $k \neq 0$, we have the following estimate between the solution u^k of the continuous problem (5) and the solution u_N^k of the discrete problem (8): for any $v_N^k \in \mathbb{X}_N^*(\Omega)$ such that $u_N^k - v_N^k$ belongs to $\mathbb{X}_N^0(\Omega)$, for any $w_{N-1}^k \in \mathbb{X}_{N-1}^*(\Omega)$, for any $f_{N-1}^k \in \mathbb{Z}_{N-1}^*(\Omega)$,*

$$\|u^k - u_N^k\|_{H^1_{(k)}(\Omega)} \le c \Big(\|u^k - v_N^k\|_{H^1_{(k)}} + \|u^k - w_{N-1}^k\|_{H^1_{(k)}} + \|f^k - \mathcal{I}_N^{(2)} f^k\|_{L^2_1(\Omega)} + \|f^k - f_{N-1}^k\|_{L^2_1(\Omega)} \Big)$$

with a constant c independent of k.

As v_N^0 and v_N^k we can take $\mathcal{I}_N^{(m)} u_N^0$ and $\mathcal{I}_N^{(2)} u_N^k$ respectively.

5.2 Evaluation of the constants $X(k,N)$

In order to estimate the constants $X(k,N)$ appearing in Theorem 3.2, we have to study the behavior of each of the four terms on the right hand side in each of the above estimates. Concerning the terms involving the data f^k, we can rely on the regularity of the data in problem (1). As for the terms involving the solutions themselves, we have to take into account the limitation of their regularity caused by the corners of Ω.

Let us denote by $\boldsymbol{e}$ the corners of Ω which do not belong to the rotation axis Γ_0. They correspond to the edges of $\breve{\Omega}$. Let $\omega(\boldsymbol{e})$ be the opening of the angle of Ω at the corner $\boldsymbol{e}$. We set:

$$\omega = \sup_{\boldsymbol{e}} \omega(\boldsymbol{e}).$$

Similarly, we denote by $\boldsymbol{c}$ the corners of Ω which belong to Γ_0. They correspond to the conical points of $\breve{\Omega}$. Let $\alpha(\boldsymbol{c})$ be the opening of the angle of Ω at the corner $\boldsymbol{c}$. We set:

$$\alpha = \sup_{\boldsymbol{c}} \alpha(\boldsymbol{c}).$$

The regularity of u^k is limited by the *first exponent of singularity* at each corner. For a corner $\boldsymbol{e}$, this first exponent does not depend on k and is equal to $\pi/\omega(\boldsymbol{e})$. For a corner $\boldsymbol{c}$, it depends on k and is equal to $\nu_k(\alpha(\boldsymbol{c}))$ with $\nu_k(\alpha)$ the smallest positive root of the equation

$$\nu \longmapsto P_\nu^k(\cos\alpha) = 0$$

where P_ν^k is the Legendre function of degree ν and order k. All of these functions decrease with respect to the opening; the smallest "edge exponent" is π/ω and, for each k, the smallest "corner exponent" is equal to $\nu_k(\alpha)$. Setting

$$s_k = \min\left\{\frac{\pi}{\omega}, \nu_k(\alpha) + \frac{1}{2}\right\}$$

we obtain the largest Sobolev exponent such that, if $s \geq s_k$

$$u^k \in H_1^{s_k+1-\varepsilon}(\Omega), \quad \forall \varepsilon > 0.$$

Then, when s is large enough, the constant $X(k,N)$ has a similar behavior as N^{-2s_k}: in the best polynomial approximation of singular functions, we also find the *doubling* of the convergence order well-known in the $p-$version of finite elements [5, 6, 1], and in spectral elements in the Cartesian case [2]. Indeed, setting

$$\tau_0 = \min\left\{\frac{\pi}{\omega}, \nu_0(\alpha) + \frac{1}{2}\right\}$$

and for $k \neq 0$

$$\tau_k = \min\left\{\frac{\pi}{\omega}, \nu_k(\alpha) - \frac{1}{2}\right\}$$

we obtain, combining the above lemmas with the estimates about the projection and interpolation operators

$$X(k,N) \leq c\Big(N^{-2\tau_k}(\log N)^{3/2} + N^{1-s}\Big). \tag{13}$$

6 Conclusions

6.1 Global estimates

We obtain a global estimate between the solution of the three dimensional problem (1) and the approximate solution

$$\breve{u}_{KN} = \sum_{k=-K}^{K} u_N^k \, e^{ik\theta}$$

with the help of the general estimate (10) and the above majoration (13). Taking into account

$$0 < \nu_0 \quad \text{and} \quad 1 < \nu_1 \leq \nu_2 \leq \cdots, \tag{14}$$

we get

$$\|\breve{u} - \breve{u}_{KN}\|_{H^1(\breve{\Omega})} \leq \Big(K^{-s} + N^{1-s} + (N^{-2\tau_0} + N^{-2\tau_1})(\log N)^{3/2}\Big) \times \Big(\|\breve{f}\|_{H^{s-1}(\breve{\Omega})} + \|\breve{g}\|_{H^{s+1}(\breve{\Omega})}\Big)$$

6.2 Examples

If the angles $\alpha(\boldsymbol{c})$ are equal to $\pi/2$, the domain $\breve{\Omega}$ has no conical point in $\boldsymbol{c}$. Thus, $\tau_0 = \tau_1 = \pi/\omega$. If Ω is a rectangle, then the convergence rate in the norm $H^1(\breve{\Omega})$ is

$$K^{-s} + N^{1-s} + N^{-4}(\log N)^{3/2},$$

and in the norm $L^2(\breve{\Omega})$, we prove by an Aubin-Nitsche argument

$$K^{-1-s} + N^{1-s} + N^{-5}(\log N)^{3/2}.$$

If Ω has the shape of a L, then the convergence rate in the norm $H^1(\breve{\Omega})$ is

$$K^{-s} + N^{1-s} + N^{-4/3}(\log N)^{1/2}.$$

Thanks to the minorations of (14), in the most general case, the exponents $2\tau_0$ and $2\tau_1$ are always > 1.

6.3 Hints about the numerical solution

For the discrete problems (7) and (8), there are several possibilities to derive matrices to be inverted for their numerical solution. Essential objects linking the discrete variational systems and the linear systems are the bases of Lagrange polynomials ℓ_j and $\ell_j^{(m)}$: the integer N being fixed, for $j \in \{0, \ldots, N\}$, ℓ_j is the unique polynomial belonging to $\mathbb{P}_N(\Lambda)$ which equals 1 in the node ξ_j and 0 in the other nodes. Similarly, concerning the "radial" variable, for $j \in \{1, \ldots, N+1\}$, $\ell_j^{(m)}$ is the unique polynomial belonging to $\mathbb{P}_N(\Lambda)$ which equals 1 in the node $\zeta_j^{(m)}$ and 0 in the other nodes.

To describe some peculiarities of these matrices, we focus on the radial variable $r = \zeta + 1$, for ζ in the interval Λ. Let us consider the discrete problems (7) and (8) without the variable z, in the only variable r. With the choice of basis functions $\ell_j^{(m)}$ for the test and trial functions, the *Galerkin method* reads as an algebraic problem $A_{\mathrm{GAL}\,k}^{(m)} U = F$ with a stiffness matrix $A^{(m)}_{\mathrm{GAL}\,k}$ equal to $\left(a_{k,N}(\ell^{(m)}_j,\ell^{(m)}_{j'})\right)_{j,j'}$. The matrix $A^{(m)}_{\mathrm{COL}\,k}$ of the collocation system at the nodes $\zeta^{(m)}_j$ reads

$$\left((-\partial_r^2-\frac{1}{r}\,\partial_r-\partial_z^2+\frac{k^2}{r^2})(\ell^{(m)}_j)(\zeta^{(m)}_{j'})\right)_{j,j'}.$$

It is known [3], that in the Cartesian case the Galerkin and the collocation systems are equivalent. In the radial case, when $k=0$, the Galerkin method is equivalent to the collocation method only for $m=3$; and when $k\neq 0$, the Galerkin method is equivalent to the collocation method for $m=2$.

Numerical evaluation of the condition numbers of the different rigidity matrices show a behavior in N^3 for all Galerkin matrices $A^{(m)}_{\mathrm{GAL}\,k}$, a behavior in N^4 for the collocation matrix $A^{(m)}_{\mathrm{COL}\,k}$ if $k=0$, $m=3$ or $k\neq 0$, $m=2$, and in N^5 for the "forbidden" collocation matrix with $k=0$, $m=1$.

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