A Numerical Study on the Accuracy of Fourier Spectral Methods Applied to the Nonlinear Burgers Equation

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Abstract

It is well known that spectral method can achieve exponential accuracy both on the approximation level and for solving partial differential equations if the solutions are analytic. However, the situation is very different when the function or solution is not smooth. For a linear partial differential equation with discontinuous solutions, Fourier spectral method will produce poor point-wise accuracy, but will still maintain exponential accuracy for all moments against analytic functions. In this presentation we address the issue of accuracy of Fourier spectral method applied to the nonlinear Burgers equation through a numerical study. Our numerical experiments show that, unlike in the linear case, the moments against analytic functions are no longer very accurate. However the numerical solution seems to still contain accurate information, since accurate point values can be extracted by a Gegenbauer polynomial based post-processing.

Key words: spectral method, accuracy, Gibbs phenomenon, Burgers equation.

AMS subject classifications: 65M15, 42A10.

1 Introduction

In this presentation we are concerned with the accuracy of Fourier spectral method when applied to a nonlinear conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \qquad -1 \le x < 1 \\ u(x,0) &= u^0(x) \end{aligned} \tag{1}$$

where the initial condition $u^0(x)$ is 2-periodic. As is well known, solutions to (1) typically contain discontinuities even if the initial condition $u^0(x)$ is analytic. Our purpose is to assess accuracy under such situation through a numerical study for the case of Burgers equation $f(u) = \frac{u^2}{2}$. See also [15].

First some notations. The Fourier approximation operator S_N to an L^2 function $u(x)$ is

$$S_N u(x) = \sum_{k=-N}^{N} \hat{u}_k e^{ik\pi x} \tag{2}$$

where the Fourier coefficients $\hat{u}_k$ are defined by

$$\hat{u}_k = \frac{1}{2}\int_{-1}^{1} u(x) e^{-ik\pi x} dx \tag{3}$$

for Fourier Galerkin, or by

$$\hat{u}_k = \frac{1}{2N+1} \sum_{j=-N}^{N} u(x_j) e^{-ik\pi x_j}, \tag{4}$$

where $x_j = \frac{2j}{2N+1}$, for Fourier collocation. To solve the partial differential equation (1), the standard Fourier spectral algorithm is

$$\begin{aligned} S_N(\partial_t v_N + \partial_x f(v_N)) &= 0, \qquad -1 \le x < 1 \\ v_N(x,0) &= S_N u^0(x) \end{aligned} \tag{5}$$

where $v_N(x,t) = \sum_{k=-N}^{N} \hat{v}_k(t) e^{ik\pi x}$ is supposed to approximate the exact solution $u(x,t)$ of (1), and S_N is either the Galerkin or the collocation Fourier approximation operator defined by (2)-(3) or by (2)-(4).

The approximation error

$$u(x) - S_N u(x) \tag{6}$$

is well known to be exponentially small (i.e., it is of the size $O(r^N)$ for some $0 < r < 1$) if $u(x)$ is analytic. In

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ICOSAHOM'95: Proceedings of the Third International Conference on Spectral and High Order Methods.

this presentation, we will only discuss analytic functions as representations of general smooth functions. Similar results can also be obtained for C^∞ or C^k functions. However, if $u(x)$ is only piecewise analytic but discontinuous, the approximation error (6) is $O(1)$ near the discontinuity and only first order $O(\frac{1}{N})$ elsewhere. This is known as the Gibbs phenomenon. See, e.g., [4] and [3]. Fortunately, even if the accuracy is poor in the point-wise sense, it is still excellent for the moments against any analytic functions. For any L^2 function $u(x)$ and any analytic function $w(x)$, we have [5]:

$$\left|\int_{-1}^{1}(u(x)-S_Nu(x))w(x)dx\right| \le Cr^N \tag{7}$$

for some constant C and $0 < r < 1$ independent of N. This property is the basis of all the "reconstruction" or "post-processing" techniques. These techniques try to recover exponential or at least high order accuracy for point values based on the Fourier approximation $S_Nu(x)$ of a piecewise analytic function. In other words, one tries to obtain a small post-processed approximation error

$$u(x) - P_NS_Nu(x) \tag{8}$$

where P_N is some post-processing operator. Examples of P_N include various high frequency filters [14], [11], [17], [2], which are of the form

$$P_NS_Nu(x) = \sum_{k=-N}^{N}\sigma\left(\frac{k}{N}\right)\hat{u}_ke^{ik\pi x} \tag{9}$$

with $S_Nu(x)$ given by (2). The function $\sigma(\xi)$ in (9) is even (or satisfies $\sigma(-\xi) = \overline{\sigma(\xi)}$ if it is complex valued as in [2]), smooth (the accuracy of the filter depends upon its smoothness), supported in $(-1,1)$ and satisfies $\sigma(0) = 1$ and $\sigma^{(k)}(0) = 0$ for $1 \le k \le K$ (with accuracy of the filter again depends upon K). These filters can recover high order or even exponential accuracy in the smooth regions *away from* the discontinuities (the filter in [2] can also recover high order accuracy up to the discontinuity from one side). A more recent example of P_N is the Gegenbauer polynomial based procedure discussed in [6], [7], [8], [9] and [10], which can give uniform exponential accuracy for all x right up to the discontinuity for piecewise analytic functions. In this sense spectral Fourier approximation is also exponentially accurate for piecewise analytic functions—one only has to extract the hidden information from the poor approximation $S_N(x)$ using the post-processor P_N.

When spectral method is used to solve the PDE (1), we can consider the following different types of errors. The strongest is the point-wise error from the exact solution $u(x,t)$:

$$u(x,t) - v_N(x,t), \tag{10}$$

which cannot be small even for $t = 0$ due to the Gibbs phenomenon, if the initial condition contains shocks. A more reasonable error is the point-wise error of the numerical solution $v_N(x,t)$ from the Fourier approximation of the exact solution $S_Nu(x,t)$:

$$S_Nu(x,t) - v_N(x,t). \tag{11}$$

If this error is exponentially small, we can claim the spectral method for (1) is exponentially accurate because of the post-processor (8) for the exact solution $u(x,t)$. An even weaker error is defined by the error in the first few Fourier coefficients, i.e.

$$\hat{u}_k(t) - \hat{v}_k(t) \tag{12}$$

for the first few k, where $\hat{u}_k(t)$ are the Fourier coefficients of the exact solution $u(x,t)$ of (1). This is actually an example of the more general definition of error in moments against any analytic function $w(x)$:

$$\int_{-1}^{1}(u(x,t) - v_N(x,t))w(x)dx \tag{13}$$

In fact, as long as this error in moments is exponentially small, we can claim that the spectral method is exponentially accurate in solving (1) by using property (7) for the exact solution $u(x,t)$ and the post-processing (8).

If the PDE (1) is linear (i.e. $f(u) = a(x,t)u$), it is proven in [5], [1] that spectral Fourier method is exponentially accurate in the sense that (13) is exponentially small. A post-processing (8) applied to $v_N(x,t)$ would then yield an exponentially accurate point-wise approximation to the exact solution $u(x,t)$. However, if (1) is nonlinear, it is still a theoretically open problem whether spectral Fourier method, equipped with either high frequency filtering or vanishing viscosity [16], [12], is exponentially (or high order) accurate in the sense of (13). Computational evidence in [13] seems to suggest that, even in this nonlinear case, highly accurate information is still implicitly contained in the numerical solution and can be extracted (at least *away from the discontinuity*) by a post-processing using high frequency filtering. However, because of the $O(1)$ Gibbs phenomenon, the global error of the post-processed solution is still only at most first order in L^1. Thus these numerical experiments do not establish global high order accuracy in the presence of shocks. In the next section we will perform a detailed numerical case study about this accuracy issue for the Burgers' equation ($f(u) = \frac{u^2}{2}$). We use a high frequency solution filter to stabilize the algorithm, and post

process the numerical result using the Gegenbauer polynomial based procedure [6], [10]. We observe that, unlike in the linear case, the spectral Fourier method is not very accurate in the sense of moments against analytic functions (13). However, numerical evidence does indicate the possibility of very high accuracy under some weaker definition, perhaps some average of Fourier coefficients, since the post-processed result $P_N v_N(x,t)$ is much more accurate than the Fourier coefficients themselves, and accurate Fourier coefficients can be "reconstructed" from this post-processed solution $P_N v_N(x,t)$.

Before finishing this introduction, we would like to describe briefly the Gegenbauer polynomial based postprocessing procedure in [6], [7], [8], [9] and [10]. Let's consider the simplest case of only one discontinuity, located at the interval boundary $x = \pm 1$. That is, $u(x)$ is an analytic function in $[-1,1]$ but not periodic. Let's also only consider the Galerkin case, i.e. assume that we are given the first $2N+1$ Fourier coefficients of $u(x)$ defined by (3), and hence $S_N u(x)$ in (2). The objective is to recover exponentially accurate point values of $u(x)$, *uniformly over* [-1,1]. To achieve this end, we consider the Gegenbauer expansion of $u(x)$ over [-1,1], i.e., expansions in the Gegenbauer polynomials $C_k^\lambda(x)$, which are orthogonal with the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$:

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k^\lambda C_k^\lambda(x) \tag{14}$$

Of course, we do not know the exact Gegenbauer coefficients $\hat{u}_k^\lambda$ of $u(x)$, but just the approximate Gegenbauer coefficients, $\hat{u}_{N,k}^\lambda$, obtained from $S_N u(x)$:

$$S_N u(x) = \sum_{k=0}^{\infty} \hat{u}_{N,k}^\lambda C_k^\lambda(x) \tag{15}$$

The two most important ingredients in the reconstruction procedure are: (1) do not fix the parameter λ, but choose it proportional to, but less than, N, and (2) do not sum (15) to infinity, but only to m:

$$P_N S_N u(x) = \sum_{k=0}^{m} \hat{u}_{N,k}^\lambda C_k^\lambda(x) \tag{16}$$

where m is again chosen proportional to, but less than, N. One could choose, for example, $m = \lambda = \frac{1}{6}N$, among many other possibilities. It can be proven [6] that, under such choices, the reconstruction error is *uniformly* exponentially small:

$$\max_{-1\le x\le 1} |u(x) - P_N S_N u(x)| \le C r^N \tag{17}$$

for some constants $C > 0$ and $0 < r < 1$ independent of N. This is in contrast with earlier filtering techniques which can obtain good accuracy *away from* the discontinuity but still leaves $O(1)$ errors near the discontinuity. The same conclusion also holds for more than one shocks, for Legendre, Chebyshev or general Gegenbauer expansions rather than Fourier expansions, and for collocation rather than Galerkin, [8], [9] and [10].

2 A numerical case study about accuracy

In our numerical solution reported in this section, time discretization is by a third order Runge-Kutta method, with a time step Δt sufficiently small such that the spatial error is dominant in all cases. We compute the exact solution of the Burgers equation by Newton iterations on the implicit characteristic equations, and compute the Fourier coefficients of a function (if not analytically given) by using a sufficiently accurate numerical quadrature.

We first solve a linear equation

$$\begin{aligned} \partial_t u + \frac{3}{5-4\cos(\pi x)}\,\partial_x u &= 0, \qquad -1 \le x < 1 \\ u(x,0) &= x \end{aligned} \tag{18}$$

with periodic boundary conditions, up to t=1, using the Fourier Galerkin method:

$$\begin{aligned} S_N\left(\partial_t v_N + \frac{3}{5-4\cos(\pi x)}\,\partial_x v_N\right) &= 0, \\ v_N(x,0) = S_N x = \sum_{\substack{k=-N \\ k\neq 0}}^{N} & \frac{(-1)^k\, i}{k\pi} e^{ik\pi x} \end{aligned} \tag{19}$$

Standard Galerkin method is stable for this linear problem but produces poor point value accuracy, see Figure 1. In all the figures, we have shown the solution in a period bordered by the single discontinuity. However, the accuracy in the first few Fourier coefficients, as representatives of moments against analytic functions, are computed more accurately, see Figure 2.

In order to compare with the nonlinear case reported later, we solve the same linear equation (18) using the filtered Fourier method, i.e., after each Runge-Kutta time step, the numerical solution is filtered by (9) with the exponential filter:

$$\sigma(\xi) = e^{-\alpha|\xi|^r} \tag{20}$$

where r is increasing with N and is related to the order of the filter, and α is chosen such that $e^{-\alpha}$ equals machine

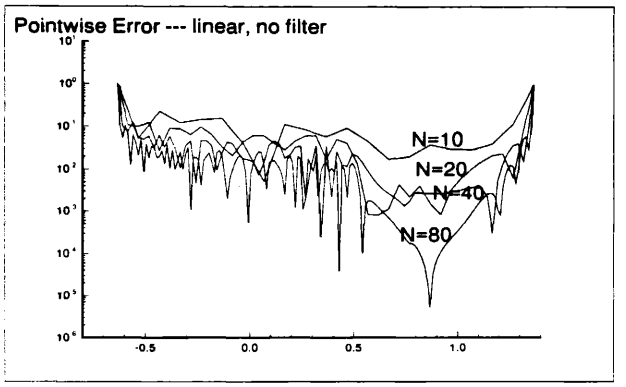

Figure 1: Point-wise errors in the logarithm scale, linear PDE (18). Fourier Galerkin using $2N+1$ modes, for $N = 10, 20, 40$ and 80.

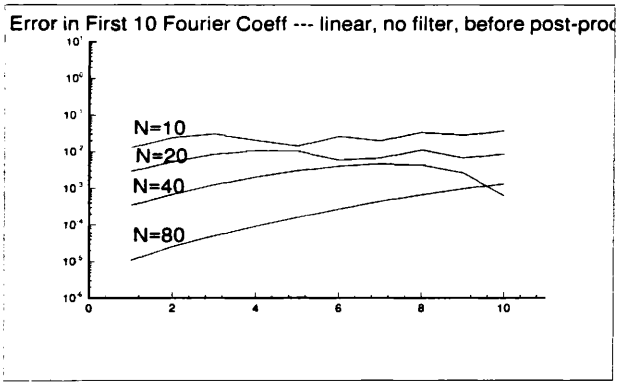

Figure 2: Errors in the first 10 Fourier coefficients, in the logarithm scale, linear PDE (18). Fourier Galerkin using $2N+1$ modes, for $N = 10, 20, 40$ and 80.

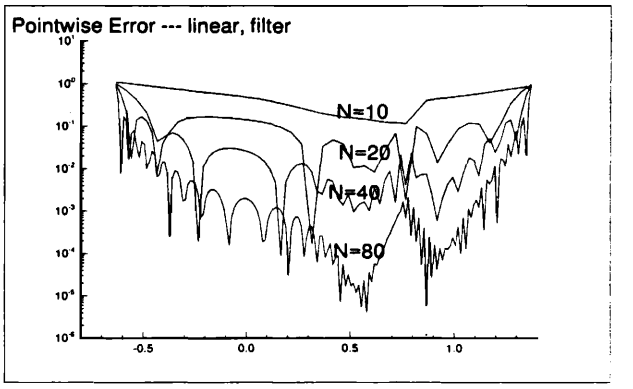

Figure 3: Point-wise errors in the logarithm scale, linear PDE (18). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r = 4$ for $N = 10$; $r = 6$ for $N = 20$; $r = 8$ for $N = 40$ and $r = 12$ for $N = 80$.

zero (10^{-16} for double precision). The exponential filter (20) has the advantage of simplicity, and usually it works equally well as more complicated filters [17]. For this linear problem, as well as for the nonlinear Burgers' equation later, we will use the Fourier method with the following choice of filter orders: $r = 4$ for $N = 10$; $r = 6$ for $N = 20$; $r = 8$ for $N = 40$ and $r = 12$ for $N = 80$. The result is shown in Figure 3 for the point-wise errors and in Figure 4 for the errors in the first few Fourier coefficients. Comparing with Figure 1 and Figure 2, we can see better point value accuracy in the smooth region because of the filters, and similar (good) accuracy for the first few Fourier coefficients.

The computational result for the linear equation is not surprising since it just verifies the proven fact [5], [1] that Fourier coefficients, as representatives of moments against analytic functions, are computed with exponential accuracy by the spectral Fourier method, and filtering will recover exponential point value accuracy in smooth regions away from the discontinuity. It should be noticed that, for the same N, the accuracy for the first few Fourier coefficients is at the same level at or better than the best point value accuracy in the smooth region after filtering. This is again not surprising since point value accuracy is obtained from the Fourier coefficients through filtering.

We now come to the nonlinear problem we are really interested in: we solve the nonlinear Burgers' equation

$$\begin{aligned} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) &= 0, \qquad -1 \le x < 1 \\ u(x,0) &= 0.3 + 0.7\sin(\pi x). \end{aligned} \tag{21}$$

The solution develops a shock at $t = \frac{1}{0.7\pi}$ and we compute

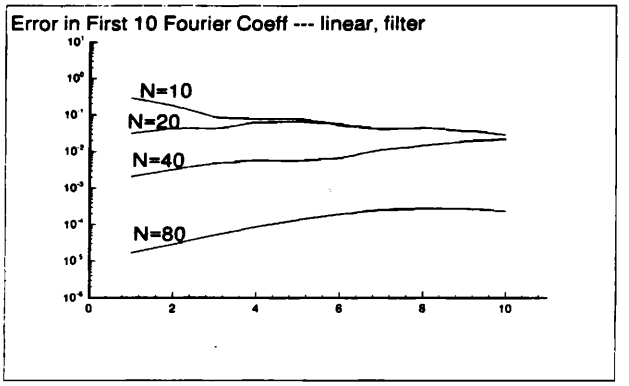

Figure 4: Errors in the first 10 Fourier coefficients, in the logarithm scale, linear PDE (18). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$.

the solution up to $t=1$. The initial condition is chosen such that the shock is moving with time. For this nonlinear PDE, the standard Galerkin method cannot converge to the entropy solution [16]. One would need to add dissipations either by the high frequency solution filtering (9) or by the spectral vanishing viscosity [16], [12], [13]. Numerical results for the Burgers' equation with the vanishing viscosity method can be found in, e.g., [13]. Here we will only report the results obtained by solution filtering, using the same r as in the previous linear case (18). We have also computed with the vanishing viscosity methods and have obtained similar results.

In Figure 5 we plot the point-wise error $u(x,t)-v_N(x,t)$, and in Figure 6 the error for the first 10 Fourier coefficients. While the pattern of the point-wise errors are similar to the linear case in Figure 3, the errors for the Fourier coefficients are clearly much worse in comparison. As a matter of fact, for the same N, the errors for the first few Fourier coefficients are a few magnitudes larger than the smallest point value error in the smooth region. This is clearly different from what we observe in the linear case in Figure 4, and suggests that the first few Fourier coefficients, again as representatives of moments against analytical functions, are no longer computed with exponential or high order accuracy. It is sort of puzzling that each difference in the Fourier coefficients $\hat{u}_k(t)-\hat{v}_k(t)$ is relatively large (Figure 6), but the point-wise error $u(x,t)-v_N(x,t)$, which is just an average (weighted sum) of $\hat{u}_k(t)-\hat{v}_k(t)$ (against $O(1)$ weight functions $e^{ik\pi x}$), is much smaller in the smooth region (Figure 5). Some cancelation must be present for this to happen.

Next, we apply the Gegenbauer post-processor [6], as

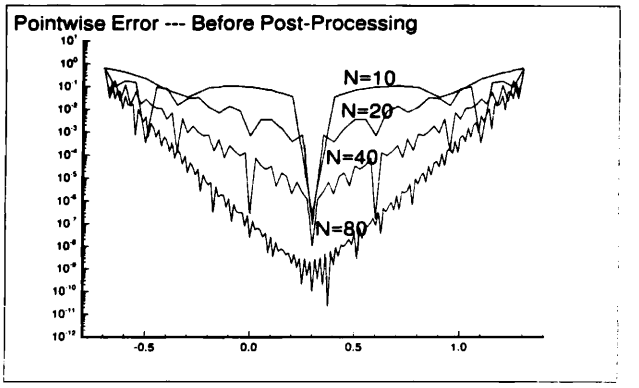

Figure 5: Point-wise errors in the logarithm scale, Burgers equation (21). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$.

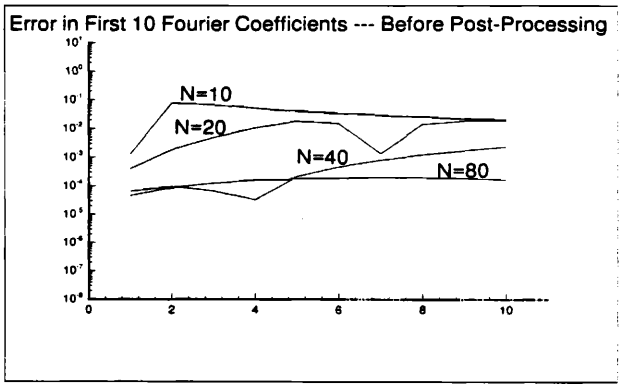

Figure 6: Errors in the first 10 Fourier coefficients, in the logarithm scale, Burgers equation (21). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$.

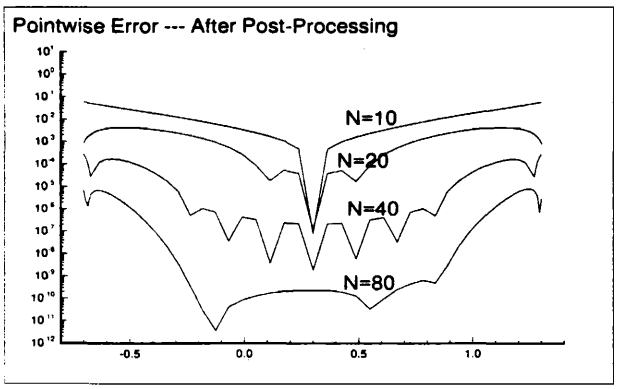

Figure 7: Point-wise errors in the logarithm scale, Burgers equation (21). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$. Gegenbauer post-processed, with parameters $\lambda=2, m=1$ for $N=10$; $\lambda=3, m=3$ for $N=20$; $\lambda=12, m=7$ for $N=40$ and $\lambda=62, m=15$, for $N=80$.

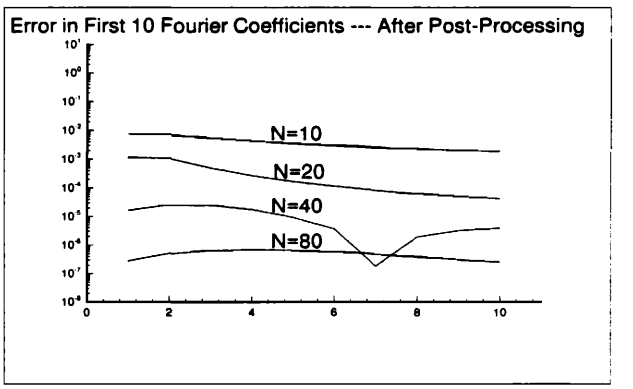

Figure 8: Errors in the first 10 Fourier coefficients, reconstructed from $P_N v_N(x,t)$, in the logarithm scale, Burgers equation (21). Fourier Galerkin using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$. Gegenbauer post-processed, with parameters $\lambda=2, m=1$ for $N=10$; $\lambda=3, m=3$ for $N=20$; $\lambda=12, m=7$ for $N=40$ and $\lambda=62, m=15$, for $N=80$.

briefly described in the previous section, to the numerical solution $v_N(x,t)$. We first use the exact shock location, and use the following values for the parameters m and λ: $\lambda=2, m=1$ for $N=10$; $\lambda=3, m=3$ for $N=20$; $\lambda=12, m=7$ for $N=40$ and $\lambda=62, m=15$, for $N=80$. We would like to point out that there is no theoretical justification in doing this post-processing for the current nonlinear case, since the post-processing procedure assumes that the Fourier coefficients are accurate, which is not true any more. However, the post-processed result is surprisingly good, see the point-wise errors in Figure 7. Just like in the approximation test cases [6], We can observe good accuracy everywhere including at the discontinuity $x=\pm 1+0.3$. From these very accurate point values we can reconstruct the Fourier coefficients, namely we can compute the Fourier coefficients of $P_N v_N(x,t)$, see Figure 8. These reconstructed Fourier coefficients are much more accurate than before the post-processing, comparing Figure 8 with Figure 6.

This suggests that, even if $v_N(x,t)$ or its Fourier coefficients $\hat{v}_k(t)$ are not very accurate, it must implicitly contain accurate information which is extracted in this case by the Gegenbauer polynomial based post-processor P_N. This numerical evidence suggests that in the nonlinear PDE case, Fourier coefficients $\hat{v}_k(t)$, just like point-wise values in the linear (or nonlinear) PDE case, are no longer good indicators of accuracy. They themselves are not very accurate, but they implicitly contain accurate information which can be extracted by adequate post-processors P_N. This accurate information might be contained in some averages of the Fourier coefficients (since the post-processing procedure based on Gegenbauer polynomials [6] uses certain averages of Fourier coefficients rather than the coefficients themselves).

We finally make two remarks:

Remark 2.1 *In the Gegenbauer reconstruction procedure above we have used the exact shock location. The procedure in [8] allows us to use an approximate shock location, determined from the Fourier coefficients themselves (e.g., [2]). Similarly good results can be obtained when the reconstruction is performed in a slightly smaller subinterval inside which the solution is guaranteed to be analytic. For example, we use the shock detector in [2], which in this case detects the shock location to within* 0.0000025 *for all the N values used, and a reconstruction inside the sub-interval* $[-0.999997, 0.999997]$, *which is just slightly smaller than* $[-1,1]$ *(when numerically detected shock is shifted to $x=-1$) and guarantees that the true shock is outside this region. The result is shown in Figure 9. It is clearly as good as the case where the exact shock location is used (comparing with Figure 7).*

Remark 2.2 *If we use collocation (4) instead of Galerkin, and the Gegenbauer reconstruction procedure in [10], the result is almost identically good: Compare Figure 10 with Figure 7.*

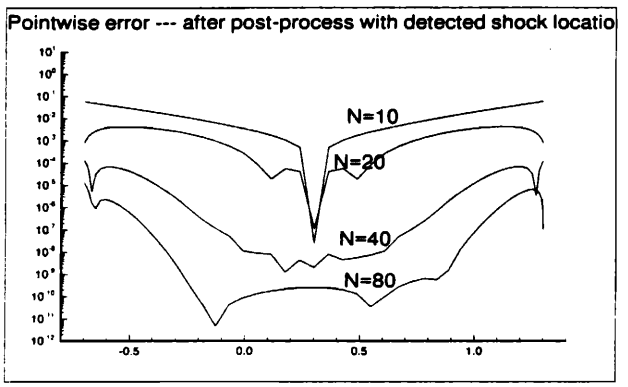

Figure 9: Point-wise errors in the logarithm scale, Burgers equation (21). Fourier Galerkin method using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$. Gegenbauer post-processed with a numerically determined shock location which for this problem produces shock locations to within 0.0000025 for all the N used. The reconstruction sub-interval is $[-0.999997, 0.999997]$ when the numerical shock is shifted to $x=-1$. Parameters: $\lambda=2, m=1$ for $N=10$; $\lambda=3, m=3$ for $N=20$; $\lambda=26, m=9$ for $N=40$ and $\lambda=52, m=17$, for $N=80$.

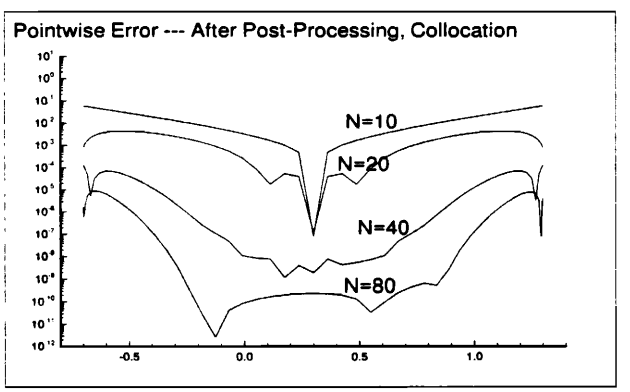

Figure 10: Point-wise errors in the logarithm scale, Burgers equation (21). Fourier collocation method using $2N+1$ modes with exponential solution filters of order r. $r=4$ for $N=10$; $r=6$ for $N=20$; $r=8$ for $N=40$ and $r=12$ for $N=80$. Gegenbauer post-processed, with parameters $\lambda=2, m=1$ for $N=10$; $\lambda=3, m=3$ for $N=20$; $\lambda=26, m=9$ for $N=40$ and $\lambda=60, m=15$, for $N=80$.

3 Acknowledgments

This research is supported by ARO grant DAAL03-91-G-0123 and DAAH04-94-G-0205, NSF grant DMS-9211820, NASA grant NAG1-1145 and AFOSR grant 93-0090.

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