A Global Algorithm in Spectral Methods for the Coupled Navier-Stokes/Euler Equations*

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Abstract

This paper deals with a viscous/inviscid coupled model. A new global variational formulation is introduced. The coupled equations are approximated by a spectral method using the discrete spaces $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_{N-2} \times \mathbb{P}_{N-2})$.

Key words: viscous/inviscid coupling, spectral method, Uzawa algorithm.

AMS subject classifications: 65M70, 65M12, 65N22.

1 Introduction

Coupling different partial differential equations, as a particular implementation of domain decomposition ideas, allows faster solution in many cases. Indeed, in the simulation of the fluid flow past an obstacle, for instance, often a complex and expensive model is only needed in a small fraction of domain. Outside this region we can use a simpler model. There has been much work on this research domain (see [3] for the theoretical justification, [1] for the numerical implementation on finite element methods and [6] for the first approximation by spectral element methods). An essential point in this type of approximation consists of finding correct conditions on the interface separating the viscous and inviscid subdomains. Secondly, appropriate algorithm is also important to the numerical implementation. The coupled model considered here has been first investigated in [6] where the discrete spaces $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_N \times \mathbb{P}_N)$ were used (where $\mathbb{P}_N \times \mathbb{P}_{N-2}$ are the discrete velocity and pressure spaces for the Navier-Stokes equations; $\mathbb{P}_N \times \mathbb{P}_N$ are the discrete velocity and pressure spaces for the Euler equations). In the existing literature, the numerical algorithm used to solve the resultant discrete equations was iteration-by-subdomain resolution. An effective iterative procedure requires exact convergence analysis and repeated resolutions to reach the convergence, which is often theoretically non trivial and numerically costly. Instead, the new variational formulation that we are going to introduce here allows us to globally solve the coupled problem. This global resolution method does not require the convergence analysis of the interface iterative procedure; it alleviates the need for repeat computations and offers potential advantages as regards to the overall computational cost. This paper follows the works of [6] and considers, furthermore, the discrete spaces $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_{N-2} \times \mathbb{P}_{N-2})$. We give the comparisons of the costs between the pure Navier-Stokes model and the coupled Navier-Stokes/Euler model.

2 Viscous/inviscid coupling

For the sake of simplification, consider domain $\Omega =]-2, 2[\times]-1, 1[$, which is broken into $\Omega^- =]-2, 0[\times]-1, 1[$ and $\Omega^+ =]0, 2[\times]-1, 1[$. Let $\Gamma^k = \partial\Omega \cap \partial\Omega^k, k = -, +; \Gamma = \partial\Omega^- \cap \partial\Omega^+$. $\vec{n}$ is the normal on $\partial\Omega$ to Ω, and $\vec{n}^-, \vec{n}^+$ are the normals on Γ to Ω^-, Ω^+, respectively.

For any integer m, let $H^m(\Omega)$ be the classical Hilbert Sobolev space, provided with the usual norm $\|\cdot\|_{m,\Omega}$, and also with the semi-norm $|\cdot|_{m,\Omega}$. $L_0^2(\Omega) = \{v; v \in L^2(\Omega), \int_\Omega v dx = 0\}$.

Throughout this paper, with any function φ defined in Ω, we identify by φ^k the restriction in Ω^k of φ, $k = -, +$. Reciprocally, for the functions φ^k defined in Ω^k, we denote by φ the pair (φ^-, φ^+).

Consider the viscous/inviscid coupled problem: Find two pairs $(\vec{u}^-, \vec{u}^+)$, (p^-, p^+) defined in (Ω^-, Ω^+) respec-

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tively, such that:

$$(1)\quad \begin{cases} \alpha\vec{u}^- - \nu\Delta\vec{u}^- + \nabla p^- = \vec{f} & \nabla\cdot\vec{u}^- = 0 \ \text{ in } \Omega^-, \\ \alpha\vec{u}^+ + \nabla p^+ = \vec{f} & \nabla\cdot\vec{u}^+ = 0 \ \text{ in } \Omega^+, \\ \vec{u}^- = 0 & \text{on } \Gamma^-, \\ \vec{u}^+\cdot\vec{n} = 0 & \text{on } \Gamma^+, \end{cases}$$

where $\vec{f}\in L^2(\Omega)^2$ and α,ν are two positive constants.

Define two real Hilbert spaces:

$$X = \{\vec{v}; \vec{v}|_{\Omega^-}\in H^1(\Omega^-)^2, \vec{v}|_{\Omega^+}\in L^2(\Omega^+)^2, \vec{v}|_{\Gamma^-} = 0\},$$
$$M = \{q; q|_{\Omega^-}\in L^2(\Omega^-), q|_{\Omega^+}\in H^1(\Omega^+), \textstyle\int_\Omega q = 0\}$$

with the norms

$$\|\vec{v}\|_X = \|\vec{v}^-\|_{1,\Omega^-} + \|\vec{v}^+\|_{0,\Omega^+},$$
$$\|q\|_M = \|q^-\|_{0,\Omega^-} + |q^+|_{1,\Omega^+}.$$

It has been proven [7] that the equations (1), with the following interface conditions:

$$\begin{cases} \nu\dfrac{\partial\vec{u}^-}{\partial\vec{n}^-} - p^-\vec{n}^- = p^+\vec{n}^+ & \text{on } \Gamma, \\ \vec{u}^-\cdot\vec{n}^- = -\vec{u}^+\cdot\vec{n}^+ & \text{on } \Gamma \end{cases}$$

are well-posed in $X\times M$, and the corresponding variational formulation writes: Find $\vec{u}\times p\in X\times M$, such that

$$(2)\quad \begin{array}{l} \alpha(\vec{u},\vec{v}) + \nu(\nabla\vec{u}^-,\nabla\vec{v}^-)_- - (p^-,\nabla\cdot\vec{v}^-)_- \\ +(\nabla p^+,\vec{v}^+)_+ - (p^+\cdot\vec{n}^+,\vec{v}^-)_\Gamma = (\vec{f},\vec{v}) \quad \forall\vec{v}\in X, \\ (\nabla\cdot\vec{u}^-,q^-)_- - (\vec{u}^+,\nabla q^+)_+ - (\vec{u}^-\cdot\vec{n}^-,q^+)_\Gamma = 0 \\ \hfill \forall q\in M, \end{array}$$

where $(\cdot,\cdot)_k$, $(\cdot,\cdot)_\Gamma$ are defined by

$$(\varphi,\psi)_k = \int_{\Omega^k}\varphi\psi\ , \quad (\varphi,\psi)_\Gamma = \int_\Gamma\varphi\psi\ , \quad k = -,+\ .$$

Theorem 2.1 *For all α and ν positive, problem (2) has one unique solution in $X\times M$.*

Proof The proof is standard by using the saddle-point theorem. We write problem (2) in the form: Find $\vec{u}\times p\in X\times M$, such that

$$\begin{cases} a(\vec{u},\vec{v}) + b(\vec{v},p) = (\vec{f},\vec{v}) & \forall\vec{v}\in X, \\ b(\vec{u},q) = 0 & \forall q\in M, \end{cases}$$

where forms a and b are defined as follow:

$$\begin{array}{ll} a(\vec{u},\vec{v}) = & \alpha(\vec{u},\vec{v}) + \nu(\nabla\vec{u}^-,\nabla\vec{v}^-)_- \quad \forall\vec{u},\vec{v}\in X, \\ b(\vec{v},q) = & -(q^-,\nabla\cdot\vec{v}^-)_- + (\nabla q^+,\vec{v}^+)_+ \\ & +(q^+,\vec{v}^-\cdot\vec{n}^-)_\Gamma \qquad \forall\vec{v}\in X,\ q\in M. \end{array}$$

The saddle-point theorem consists in verifying four properties: continuity and ellipticity of the form a; continuity and compatibility of the form b. The three firsts are proven in a classical way. The last one is proven if we show: there exists a positive constant β, such that

$$\inf_{q\in M}\sup_{\vec{v}\in X}\frac{b(\vec{v},q)}{\|\vec{v}\|_X\|q\|_M}\geq\beta$$

which can be found in [7]. □

3 Spectral discretizations and error estimations

Let $I\!P_N$ be the space of all polynomials of degree $\leq N$. $\xi^N_{ij,k}$ and $w^N_{ij,k}$ $(i,j = 0,\cdots,N)$ denote, respectively, the $(N+1)^2$ Gauss-Lobatto points and weights corresponding to the subdomain $\Omega^k (k = -,+)$. Let $\Xi^k_N = \{\xi^N_{ij,k}; i,j = 0,\cdots,N\}$.

A classical method of solving a coupled problem consists of exhibiting its solution as a limit of solutions of two subproblems within Ω^- and Ω^+. This is done by considering the following iterative procedure: first the one of two subproblems, in Ω^- for instance:

$$\begin{cases} \alpha\vec{u} - \nu\Delta\vec{u} + \nabla p = \vec{f} & \text{in } \Omega^-, \\ \nabla\cdot\vec{u} = 0 & \text{in } \Omega^-, \\ \vec{u} = 0 & \text{on } \Gamma^-, \\ \nu\dfrac{\partial\vec{u}}{\partial\vec{n}} - p\cdot\vec{n}^- = p^+\cdot\vec{n}^+ & \text{on } \Gamma \end{cases}$$

is solved with a Neumann-type condition $p^+\cdot\vec{n}^+$ arbitrary; then, knowing $\vec{u}^-$ on Γ, we solve the other subproblem:

$$\begin{cases} \alpha\vec{u} + \nabla p = \vec{f} & \text{in } \Omega^+, \\ \nabla\cdot\vec{u} = 0 & \text{in } \Omega^+, \\ \vec{u}\cdot\vec{n} = 0 & \text{on } \Gamma^+, \\ \vec{u}\cdot\vec{n}^+ = \vec{u}^-\cdot\vec{n}^+ & \text{on } \Gamma \end{cases}$$

which gives p^+; and so on until the convergence be reached. The procedure requires generally a certain number of repeat resolutions to reach the convergence.

But here, we choose the strategy called "global resolution", which has been first used in [6]. Precisely, we consider the discrete coupled problem: Find $\vec{u}_N\times p_N\in X_N\times M_N$, such that

$$(3)\begin{cases} a_N(\vec{u}_N,\vec{v}_N) + b_N(\vec{v}_N,p_N) = (\vec{f},\vec{v}_N)_N & \forall\vec{v}_N\in X_N, \\ b_N(\vec{u}_N,q_N) = 0 & \forall q_N\in M_N \end{cases}$$

where

$$X_N = X\cap(I\!P_N(\Omega^-)\times I\!P_{N-2}(\Omega^+)),$$
$$M_N = M\cap(I\!P_{N-2}(\Omega^-)\times I\!P_{N-2}(\Omega^+)),$$

and a_N, b_N are two bilinear forms, defined by

$$a_N(\vec{u}_N, \vec{v}_N) = \alpha(\vec{u}_N, \vec{v}_N)_N + \nu(\nabla\vec{u}_N^-, \nabla\vec{v}_N^-)_{-,N} \quad \forall \vec{u}_N, \vec{v}_N \in X_N,$$

$$b_N(\vec{v}_N, q_N) = -(q_N^-, \nabla\cdot\vec{v}_N^-)_{-,N} + (\nabla q_N^+, \vec{v}_N^+)_{+,N-2} + (q_N^+, \vec{v}_N^- \cdot \vec{n}^-)_{\Gamma,N} \quad \forall \vec{v}_N \in X_N,\ q_N \in M_N,$$

where

$$(\vec{u}, \vec{v})_N = (\vec{u}^-, \vec{v}^-)_{-,N} + (\vec{u}^+, \vec{v}^+)_{+,N-2}$$

with
$(\vec{u}^-, \vec{v}^-)_{-,N} = \sum_{i,j=0}^{N}(\vec{u}^-\cdot\vec{v}^-)(\xi_{ij,-}^N)w_{ij,-}^N,$
$(\vec{u}^+, \vec{v}^+)_{+,N-2} = \sum_{i,j=0}^{N-2}(\vec{u}^+\cdot\vec{v}^+)(\xi_{ij,+}^{N-2})w_{ij,+}^{N-2},$
$(\varphi, \psi)_{\Gamma,N} = \sum_{j=0}^{N}(\varphi\psi)(\xi_{Nj,-}^N)w_{Nj,-}^N$
(or equivalently, $= \sum_{j=0}^{N}(\varphi\psi)(\xi_{0j,+}^N)w_{0j,+}^N$).

We first state the following results [2].

Lemma 3.1 *There exist projection operators Π_N^k from $L^2(\Omega^k)$ in $\mathbb{P}_N(\Omega^k), k = -,+$; $\Pi_N^{-,1}$ from $\{\vec{v}; \vec{v} \in H^1(\Omega^-), \vec{v}|_\Gamma = 0\}$ in $\{\vec{v}_N; \vec{v}_N \in \mathbb{P}_N(\Omega^-), \vec{v}_N|_\Gamma = 0\}$ and $\Pi_N^{+,1}$ from $H^1(\Omega^+)$ in $\mathbb{P}_N(\Omega^+)$ such that*

$$\begin{aligned}
&\|\varphi - \Pi_N^k\varphi\|_{0,\Omega^k} \le CN^{-m}\|\varphi\|_{m,\Omega^k} \quad \forall\varphi \in H^m(\Omega^k),\ m\ge 0,\ k=-,+,\\
&\|\varphi - \Pi_N^{-,1}\varphi\|_{1,\Omega^-} \le CN^{1-m}\|\varphi\|_{m,\Omega^-} \quad \forall\varphi \in H^m(\Omega^-)\ \ m\ge 1,\\
&|\varphi - \Pi_N^{+,1}\varphi|_{1,\Omega^+} \le CN^{1-m}\|\varphi\|_{m,\Omega^+} \quad \forall\varphi \in H^m(\Omega^+)\ \ m\ge 1.
\end{aligned} \tag{4}$$

Theorem 3.1 *The discrete problem (3) is well posed in the space $X_N \times M_N$.*

Proof The proof is done as in the theorem 2.1. The verification of the "*Inf-Sup*" is given in Lemma 3.2 below. □

Lemma 3.2 *There exists a positive constant β_N, possibly depending on N, such that*

$$\inf_{q_N\in M_N}\sup_{\vec{v}_N\in X_N}\frac{b_N(\vec{v}_N, q_N)}{\|\vec{v}_N\|_X\|q_N\|_M} \ge \beta_N$$

Proof The proof follows the same lines as in the continuous case. We give only the estimation of the "*Inf-Sup*" constant β_N:

$$\beta_N = \frac{C\beta_N^-}{1+\beta_N^-}$$

where β_N^- is the local "*Inf-Sup*" constant corresponding to the Navier-Stokes part. □

Define the space

$$\mathbf{V}_N = \{\vec{v}_N; \vec{v}_N \in X_N, b_N(\vec{v}_N, q_N) = 0, \forall q_N \in M_N\}$$

The error estimations are given in the following theorem.

Theorem 3.2 *Assume that the solutions of the problem (1) satisfy $\vec{u} = \vec{u}^- \times \vec{u}^+ \in H^l(\Omega^-)^2 \times H^{m-1}(\Omega^+)^2$, $p = p^- \times p^+ \in H^{l-1}(\Omega^-)\times H^m(\Omega^+)$, where l and m are real numbers, $l \ge 2$, $m \ge 2$; furthermore, assume $\vec{f} \in H^\sigma(\Omega)^2$, where σ is a real number ≥ 2, then the approximate solutions of (3) $\vec{u}_N = \vec{u}_N^- \times \vec{u}_N^+, p_N = p_N^- \times p_N^+$ verify*

$$\begin{aligned}
&\|\vec{u}-\vec{u}_N\|_X + \beta_N\|p-p_N\|_M\\
&\le C[N^{1-l}(\frac{1}{\beta_N}\|\vec{u}^-\|_{l,\Omega^-} + \|p^-\|_{l-1,\Omega^-})\\
&+N^{1-m}(\frac{1}{\beta_N}\|\vec{u}^+\|_{m-1,\Omega^+} + \|p^+\|_{m,\Omega^+})\\
&+N^{-\sigma}\|\vec{f}\|_{\sigma,\Omega}\ ]
\end{aligned} \tag{5}$$

Proof Estimation (5) is a direct consequence of lemma 3.2 and following result [2]:

$$\begin{aligned}
&\|\vec{u}-\vec{u}_N\|_X + \beta_N\|p-p_N\|_M\\
\le\ & C[\inf_{\vec{v}_N\in\mathbf{V}_N}(\|\vec{u}-\vec{v}_N\|_X + \sup_{\vec{w}_N\in\mathbf{V}_N}\frac{(a-a_N)(\vec{v}_N,\vec{w}_N)}{\|\vec{w}_N\|_X})\\
&+\inf_{q_N\in M_N}(\|p-q_N\|_M + \sup_{\vec{w}_N\in\mathbf{V}_N}\frac{(b-b_N)(\vec{w}_N,q_N)}{\|\vec{w}_N\|_X})\\
&+\sup_{\vec{w}_N\in\mathbf{V}_N}\frac{(\vec{f},\vec{w}_N)-(\vec{f},\vec{w}_N)_N}{\|\vec{w}_N\|_X}]
\end{aligned}$$

□

Remark 3.1 *The estimate (5) is not optimal. It could be improved by looking for a better estimation for the term $\inf_{\vec{v}_N\in\mathbf{V}_N}\|\vec{u}-\vec{v}_N\|_X$. We refer this question to [2].*

4 Description of the algorithm

Let $\mathbf{U}^k, \mathbf{P}^k (k = -,+)$ be the values at the global collocation points of the velocity and the pressure. $\mathbf{D}^-$ and $(\mathbf{D}^+)^T$ denote the discrete divergence operators. $\mathbf{B}^-$ and $\mathbf{B}^+$ are the associated mass matrices. $\mathbb{\Gamma}$ denotes the discrete trace operator. Define

$$\mathbf{L}^k = \alpha\mathbf{B}^k + \nu\ (\mathbf{D}^-)(\mathbf{D}^-)^T\delta_{k-}.$$

with

$$\delta_{k-} = \begin{cases} 1 & k=- \\ 0 & k=+ \end{cases}$$

We write discrete problem (3) at the following matrix statement:

$$(6)\qquad \begin{aligned} \mathbb{L}\mathbf{U} + \mathbb{D}\mathbf{P} &= \mathbb{B}\mathbf{F} \\ \mathbb{D}^T\mathbf{U} &= \mathbf{0} \end{aligned}$$

where $\mathbf{0}$ is zero vector, and

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}^- \\ \mathbf{U}^+ \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}^- \\ \mathbf{P}^+ \end{pmatrix}, \quad \mathbb{L} = \begin{pmatrix} \mathbf{L}^- & 0 \\ 0 & \mathbf{L}^+ \end{pmatrix},$$

$$\mathbb{D} = \begin{pmatrix} -(\mathbf{D}^-)^T & \mathbb{F} \\ 0 & \mathbf{D}^+ \end{pmatrix} \quad \mathbb{B} = \begin{pmatrix} \mathbf{B}^- & 0 \\ 0 & \mathbf{B}^+ \end{pmatrix}.$$

It is assumed that the boundary conditions in the viscous part are already incorporated into the matrix operators.

We use the global iterative Uzawa procedure to solve (6). Formally, (6) can be equivalently replaced by the two separated systems:

$$(7)\qquad \mathbb{D}^T\mathbb{L}^{-1}\mathbb{D}\mathbf{P} = \mathbb{D}^T\mathbb{L}^{-1}\mathbb{B}\mathbf{F}$$

$$(8)\qquad \mathbb{L}\mathbf{U} = \mathbb{B}\mathbf{F} - \mathbb{D}\mathbf{P}$$

Noting that $\mathbb{D}^T\mathbb{L}^{-1}\mathbb{D}$ is a positive definite symmetric matrix, the pressure can be solved by an inner/outer conjugate gradient iterative procedure. An important point to note is that the matrix $\mathbb{L}$ in (7) and (8) is diagonal by bloc on the interface level, which means that the inner procedure is only needed in the viscous part.

5 Generalization to the coupled Navier-Stokes/Euler equations

We generalize the coupled model (1) to the coupled problem between the Navier-Stokes equations and the Euler equations:

$$(9)\qquad \begin{cases} \dfrac{\partial \vec{u}^-}{\partial t} + (\vec{u}^- \cdot \nabla)\vec{u}^- - \nu\Delta\vec{u}^- + \nabla p^- = f^- & \text{in } Q^- \\ \dfrac{\partial \vec{u}^+}{\partial t} + (\vec{u}^+ \cdot \nabla)\vec{u}^+ + \nabla p^+ = f^+ & \text{in } Q^+ \\ \vec{u}^-(0) = \vec{u}_0^- \text{ in } \Omega^- \qquad \vec{u}^+(0) = \vec{u}_0^+ \text{ in } \Omega^+ \\ \vec{u}^-|_{\Gamma^-} = 0 \qquad \vec{u}^+ \cdot \vec{n}^+|_{\Gamma^+} = 0 \end{cases}$$

with the incompressibility $\nabla \cdot \vec{u} = 0$, where $Q^k = \Omega^k \times (0,T), k = -,+$, and $\vec{u}_0^-, \vec{u}_0^+$ are two functions given. The non-linear term is treated by the method of characteristics. That is, we rewrite (9) under the form

$$(10)\qquad \begin{cases} \dfrac{D\vec{u}^-}{Dt} - \nu\Delta\vec{u}^- + \nabla p^- = f^- & \text{in } Q \\ \dfrac{D\vec{u}^+}{Dt} + \nabla p^+ = f^+ & \text{in } Q \\ \vec{u}^-(0) = \vec{u}_0^- \text{ in } \Omega^- \quad \vec{u}^+(0) = \vec{u}_0^+ \text{ in } \Omega^+ \\ \vec{u}^-|_{\Gamma^-} = 0 \quad \vec{u}^+ \cdot \vec{n}^+|_{\Gamma^+} = 0 \end{cases}$$

where D/Dt is the total derivative in the direction $\vec{u}$. We discretize (10) in time by an implicit scheme:

$$\begin{cases} \alpha\vec{u}^{-^{n+1}} - \nu\Delta\vec{u}^{-^{n+1}} + \nabla p^{-^{n+1}} \\ \qquad = f^{-^{n+1}} + \alpha\vec{u}^{-^{n}}(\chi^n(\cdot)) & \text{in } \Omega^- \\ \alpha\vec{u}^{+^{n+1}} + \nabla p^{+^{n+1}} = f^{+^{n+1}} + \alpha\vec{u}^{+^{n}}(\chi^n(\cdot)) & \text{in } \Omega^+ \\ \vec{u}^{-^{n+1}}|_{\Gamma^-} = 0 \quad \vec{u}^{+^{n+1}} \cdot \vec{n}^+|_{\Gamma^+} = 0 \end{cases}$$

where $\alpha = \frac{1}{\Delta t}$ with Δt the time step, and $\chi^n(x) = \chi(x,(n+1)\Delta t, n\Delta t)$ is the solution of

$$(11)\qquad \frac{d\chi}{d\tau} = \vec{u}^n(\chi), \quad \chi(x,(n+1)t;(n+1)t) = x$$

The time scheme is unconditionally stable, and each time iteration requires a coupled viscous/inviscid resolution plus a transport of the previous solution on the characteristics.

We note that, on the interface Γ, we have $\vec{u}^- \cdot \vec{n}^- = \vec{u}^+ \cdot \vec{n}^-$. Thus (11) is solved globally in all domain Ω without any additional interface conditions on Γ.

6 Numerical results

We give a numerical example obtained by using the algorithm presented in previous sections. We consider the equation (9) with an exact analytical solution: $u_1(x,y) = 1 - y^2$, $u_2(x,y) = 0$, $p(x,y) = \sin\pi x \sin\pi y$.

Table 1 lists the discrete L^2-error of $\vec{u} - \vec{u}_N$ using the pure Navier-Stokes (NS), coupled Navier-Stokes/Euler with $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_N \times \mathbb{P}_N)$ version (NS/EU(1)) and coupled Navier-Stokes/Euler with $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_{N-2} \times \mathbb{P}_{N-2})$ version (NS/EU(2)). The systems (7)-(8) are solved by the multigradient solver. The three methods give the same accuracy and converge exponentially.

N	NS	NS/Eu(1)	NS/Eu(2)
5	9.49E-04	1.04E-03	8.66E-03
7	2.56E-04	2.38E-04	6.23E-04
9	2.17E-05	2.08E-05	2.95E-05
11	9.67E-06	2.23E-06	5.03E-06
13	4.82E-07	2.56E-06	8.95E-07

Table 1: Discrete L^2-error of $\vec{u} - \vec{u}_N$

Table 2 lists the execution time in minutes on PC 486DX using the three models. The programs are written in Fortran and compiled with NDP486. The scheme in time is of 2-order. We take $N = 11, \Delta t = 0.01, \nu = 0.001$.

Time steps	NS	NS/Eu(1)	NS/Eu(2)
10	1.50m	1.20m	0.88m
20	2.10m	2.01m	1.47m
30	2.55m	2.38m	2.07m
40	3.35m	2.57m	2.15m
50	4.07m	3.20m	2.51m

Table 2: Execution time in minutes

7 Conclusions and discussions

1. In conclusion, we have presented a coupled model and its global resolution algorithm. The theoretical analysis and the numerical test show the effectiveness of the method. The comparisons of the execution time between the viscous/inviscid coupled resolution and the pure viscous (i.e. global Navier-Stokes equations) resolution have been done. The partial results show that the $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_N \times \mathbb{P}_N)$ approximative viscous/inviscid coupled model is more economical than the pure $\mathbb{P}_N \times \mathbb{P}_{N-2}$ viscous model and that $(\mathbb{P}_N \times \mathbb{P}_{N-2}) \times (\mathbb{P}_{N-2} \times \mathbb{P}_{N-2})$ version is even more so.

2. In our numerical test the gain is obtained for the domain splitted only into two same subdomains. The gain would be greatly increased if we used the Navier-Stokes equations only in a small fraction of domain.

3. The simulation of complex flows will produce a large and full matrix before the pressure **P**. The "simple" nested conjugate gradient algorithm, in this case, would no longer be efficient. One way to recover a rapid convergence of the Uzawa algorithm is to use a preconditioner.

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