An Asymptotically Stable Penalty Method for Multi-Domain Solution of the Unsteady, Compressible Navier-Stokes Equations

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Abstract

We develop a unified approach for dealing with open boundaries and patching of non-overlapping subdomain boundaries when performing simulations of the unsteady, three-dimensional, compressible Navier-Stokes equations given in conservation form. The appropriate boundary operators are derived by utilizing linearization and localization at the boundaries, and enforced through a penalty approach.

We apply a polynomial collocation method as a spatial approximation scheme and prove the semi-discrete initial-boundary value problem asymptotically stable through an energy method. The scheme converges uniformly to the limit of vanishing viscosity and, hence, remains valid also for the Euler equations.

The versatility of the scheme is demonstrated for multi-domain solutions of quasi-one-dimensional transonic nozzle flows and for flow around an infinitely long circular cylinder.

Key words: penalty method, compressible Navier-Stokes equations, open boundary conditions, domain decomposition.

AMS subject classifications: 65M12, 65M70, 65M55, 76N10.

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1 Introduction

The issue of the application of spectral methods to problems involving complex geometries has been a subject of active research in the last decade. Spectral methods require interpolation at the nodes of a Gauss type quadrature formula. Thus, the mesh points are predetermined and inflexible. In particular, the distribution of grid points is denser in the neighborhood of boundaries. This fact leads to considerable difficulties, even in one dimension, since for many problems the information is given in points different from those required by the spectral method. This fact manifests itself more severely when dealing with multi-dimensional problems and seems to limit the applicability of spectral methods to simple domains.

A powerful method used to overcome the limitations of spectral methods is the use of multi-domain techniques, in which a complex domain is decomposed into several geometrically simpler subdomains. An additional advantage of this approach is that multi-domain spectral methods are well suited for coarse grain parallel computing, with each subdomain being assigned to an individual processor.

The natural question posed by the multi-domain approach is how to specify appropriate patching conditions between the subdomains. For purely hyperbolic problems, it is well known that patching through the characteristic variables leads to a stable approximation. However, for dissipative wave problems the procedure is considerably more complicated.

Naturally, we must require the patching conditions to lead to a well-posed, continuous problem in each subdomain. For wave problems of dissipative type, the problem must, in order to be compatible with weak boundary layers, remain well-posed even in the limit where the dissipation vanishes and the problem becomes purely hyperbolic. In addition to this, we wish that the discrete approximation of the problem is asymptotically stable, and that the boundary conditions are easily implemented.

For general non-linear problems the issues of well-posedness and asymptotic stability are very complicated and for most problems only very little is known. However, as discussed by Kreiss and Lorenz [1], we may, for a large class of operators, simplify the problem significantly if the solutions are smooth. It was shown that in this case it is sufficient to consider the questions of well-posedness and asymptotic stability for the linearized and locally constant coefficient version of the full problem with homogeneous boundary conditions.

In this paper we present a unified approach for dealing with open boundaries and subdomain boundaries when performing simulations of the three-dimensional, compressible Navier-Stokes equation in conservation form. The presentation is partly based on results from [2, 3] where similar problems are treated in detail. The emphasis will be on, essentially, one dimensional patching schemes. However, by writing the Navier-Stokes equations in general curvilinear coordinates we obtain a scheme that is also applicable to multi-dimensional problems, provided patching is required along one coordinate axis only.

In the development of the scheme, we apply the energy method to the linearized, constant coefficient version of the continuous problem to obtain energy inequalities which bound the temporal growth of the solutions to the initial-boundary value problem. This approach allows us to derive a novel set of boundary conditions of Robin type, which ensure the complete problem to be well-posed. This result is obtained for the Navier-Stokes equations given in general curvilinear coordinates.

It has traditionally been found difficult to apply boundary conditions of Robin type when doing pseudospectral simulations of non-linear equations. Here, we show how to implement the boundary conditions as a penalty term, which allows for enforcing open boundary/patching conditions of a very general type. An attractive feature of this approach is that we may prove asymptotic stability of the semi-discrete scheme, thus gaining confidence in the computed results when addressing unsteady problems where long time integration is required.

A multi-domain scheme, where the patching of subdomains is based on a penalty method, is strictly local in space and time, thus making it well suited for implementation on contemporary parallel computer architectures with distributed memory.

Several schemes for calculating the multi-domain solution of viscous compressible flows have recently appeared [4, 5, 6]. However, the emphasis has been on methods for steady state problems. All previous methods for viscous flows are based on applying a separate treatment to the inviscid part of the equation; in most cases using methods known from the Euler equations, and applying a separate treatment to the viscous part of the equation. This second contribution is then applied as a correction to the result obtained from the inviscid patching.

The main difference between the previously proposed methods and the one developed here is that we develop a patching which accounts for the inviscid and viscous part of the equation simultaneously. Emphasis is directed towards developing methods that can handle general unsteady flows, and we apply high order explicit time integration schemes to verify that the proposed method, indeed, is well suited for simulating unsteady flows.

The remaining part of the paper is organized as follows. In Sec. 2. we introduce the penalty method and demonstrate the idea for the scalar wave equation. Section 3 contains the central parts of the paper, where well-posed boundary conditions to the three-dimensional, compressible Navier-Stokes equations in conservation form are derived and a semi-discrete asymptotically stable penalty method for enforcing these conditions is proposed. This leads to Sec. 4 where we present several examples of the use of the proposed scheme when addressing one- and two-dimensional compressible flow problems. Section 5 concludes with a brief summary.

2 The penalty method

Prior to developing the scheme for the compressible Navier-Stokes equations, we illustrate the idea behind the penalty method for the scalar problem

$$\begin{cases} U_t = \lambda U_x & |x| \leq 1 \ , \quad t > 0 \\ U(x,0) = h(x) \\ U(1,t) = g(t) \end{cases} ,$$

where $\lambda > 0$. We wish to solve this problem using a Legendre collocation method. The choice of the Legendre basis is made merely to simplify the example. The main results carry over to other polynomial collocation methods, e.g. Chebyshev methods. The Legendre interpolation operator, I_N, is defined as

$$I_N U(x) = \sum_{i=0}^{N} u(x_i)\, f_i(x) \ ,$$

where $f_i(x)$ are the Legendre-Lagrange interpolating polynomials [7], and x_i denotes the Legendre-Gauss-Lobatto collocation points. We seek a solution, $u(x,t)$, such that

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial u}{\partial x} \ \text{at} \ x = x_i \ , \ i \in [0 \ldots, N-1] \ .$$

The usual way to impose the boundary condition is to enforce $u(x_N) = g(t)$. However, this approach does not take into account the fact that the equation should hold arbitrarily close to the boundary. To circumvent this problem, Funaro and Gottlieb [8, 9] developed the penalty method which enforces the boundary conditions, as well as the equation at the boundary. They propose to use the scheme

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial u}{\partial x} - \tau Q^+(x)\left[u(1) - g(t)\right] \quad ,$$

where

$$Q^+(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)} \tag{1}$$

Here, P_N is the Legendre polynomial of order N. Note that $Q^+(x)$ equals zero at all the collocation points, except the endpoint, $x_N = 1$. By using the penalty method, the boundary condition becomes a part of the equation, although it also implies that the boundary condition is enforced only weakly.

The parameter τ is then to be determined such that the semi-discrete initial boundary value problem is asymptotically stable. Without loss of generality we may assume $g(t) = 0$ [1], and obtain a sufficient condition for stability through the use of an energy argument as

$$\begin{aligned} \sum_{i=0}^{N} u(x_i)\frac{\partial u(x_i)}{\partial t}\omega_i &= \lambda \sum_{i=0}^{N} u(x_i)\frac{\partial u(x_i)}{\partial x}\omega_i \\ &\quad -\tau \sum_{i=0}^{N} u(x_i)Q^+(x_i)\,u(1)\omega_i \le 0 \quad , \end{aligned} \tag{2}$$

where ω_i are the Legendre-Gauss-Lobatto weights [7]. Using the Legendre-Gauss-Lobatto quadrature formula, Eq.(2) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \frac{1}{2}\lambda\left[u^2(1) - u^2(-1)\right] - \tau\omega u^2(1) \le 0 \quad ,$$

where $\omega = 2/N(N+1)$ and $\|\cdot\|^2$ signifies the L_2-norm. Hence, asymptotic stability may be obtained if τ is chosen such that

$$\tau \ge \frac{\lambda}{2\omega} = \lambda\frac{N(N+1)}{4} \quad .$$

We note that for $N \to \infty$ the penalty method is equivalent to the traditional approach. Although not relevant for this simple problem, it is important to realize that since the boundary operator is applied as an independent part of the equation we may impose even very complicated types of boundary conditions in a straightforward and consistent manner when using the penalty method.

The only difference between the Legendre and the Chebyshev collocation scheme is that the expression for τ may not be found by analytical means in the latter case [8, 10].

In [9] the penalty method is extended to systems of hyperbolic equations, and it is shown that the results carry over when the boundary conditions are imposed through the characteristic variables.

3 The compressible Navier-Stokes equations

Consider, now, the non-dimensional, compressible Navier-Stokes equations in general curvilinear coordinates

$$\frac{\partial \overline{\mathbf{q}}}{\partial t} + \frac{\partial \overline{\mathbf{F}}_i}{\partial \xi_i} = \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\frac{\partial \overline{\mathbf{F}}_i^{\nu}}{\partial \xi_i} \quad . \tag{3}$$

Here, and in the remaining part of the paper, we will use the summation convention unless otherwise stated. The curvilinear coordinates, $(\xi_1, \xi_2, \xi_3) \in \Omega$, are defined as

$$\xi_i = \xi_i(x_1, x_2, x_3) \quad ,$$

and related to the Cartesian coordinates, (x_1, x_2, x_3), through the transformation Jacobian

$$J = \left|\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}\right| \quad .$$

The state vector, $\overline{\mathbf{q}}$, and the inviscid flux vectors are defined as

$$\overline{\mathbf{q}} = J\mathbf{q} \quad , \overline{\mathbf{F}}_i = J\mathbf{F}_j\xi_i^j \quad ,$$

where we have introduced the symbols

$$\xi_i^j = \frac{\partial \xi_i}{\partial x_j} \quad , \quad u_i^j = \frac{\partial u_i}{\partial \xi_j} \quad , \quad T^j = \frac{\partial T}{\partial \xi_j} \quad ,$$

and likewise for the Cartesian velocity components, (u_1, u_2, u_3), and the temperature field, T, which will be introduced shortly. The Cartesian components of the vectors are given as

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ E \end{bmatrix} \quad , \quad \mathbf{F}_i = \begin{bmatrix} \rho u_i \\ \rho u_1 u_i + \delta_{1i}\, p \\ \rho u_2 u_i + \delta_{2i}\, p \\ \rho u_3 u_i + \delta_{3i}\, p \\ (E+p)u_i \end{bmatrix} \quad .$$

Here, ρ is the density, E is the total energy, p is the static pressure and δ_{ij} represents the Kronecker delta-function.

The total energy

$$E = \rho\left(T + \frac{1}{2}u_i u_i\right) \quad ,$$

and the pressure are related through the ideal gas law

$$p = (\gamma - 1)\rho T \quad ,$$

where $\gamma = c_p/c_v$ is the ratio between the heat capacities at constant pressure (c_p) and volume (c_v), respectively, and is assumed constant.

The viscous flux vectors are in a similar manner defined as

$$\overline{\mathbf{F}}_i^\nu = J\mathbf{F}_j^\nu \xi_i^j \quad ,$$

with

$$\mathbf{F}_i^\nu = \begin{bmatrix} 0 \\ \tau_{x_1 x_i} \\ \tau_{x_2 x_i} \\ \tau_{x_3 x_i} \\ \tau_{x_j x_i} u_j + \frac{\gamma k}{\Pr}\frac{\partial T}{\partial x_i} \end{bmatrix} .$$

Considering only Newtonian fluids, the elements of the symmetric stress tensor are given as

$$\tau_{x_i x_j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\frac{\partial u_k}{\partial x_k} \quad .$$

Here, μ is the dynamic viscosity, λ is the bulk viscosity and k is the coefficient of thermal conductivity. The velocity flux and the temperature derivatives are obtained as

$$\frac{\partial u_i}{\partial x_j} = u_i^k \xi_k^j \quad , \quad \frac{\partial T}{\partial x_i} = T^k \xi_k^i \quad .$$

The equations are normalized using the reference values, $u_{\text{ref}} = \bar{u}$, $\rho_{\text{ref}} = \bar{\rho}$, $p_{\text{ref}} = \bar{\rho}\bar{u}^2$, $T_{\text{ref}} = \bar{u}^2/c_v$, where $(\bar{\rho}, \bar{u})$ is some characteristic state, and a reference length L. This yields a Reynolds number as $\text{Re} = \bar{\rho}\bar{u}L/\bar{\mu}$ and a Prandtl number as $\Pr = c_p\bar{\mu}/\bar{k}$. Note that the Reynolds number in Eq.(3), Re_{ref} (based on the reference values) is, in general, different from Re. In the remaining part of the paper we shall refer to the latter as the Reynolds number unless clarification is deemed necessary. With this normalization we need to specify the Mach number, M, the Reynolds number, Re, the length scale, L, and a dimensional temperature, T_0.

We consider only atmospheric air and take $\gamma = 1.4$ and $\Pr = 0.72$ in all problems. To model the temperature dependence of the dynamic viscosity we apply Sutherland's viscosity law [11]

$$\frac{\mu(T)}{\mu_s} = \left(\frac{T}{T_s}\right)^{3/2}\frac{T_s + S}{T + S} \quad ,$$

where we have $\mu_s = 1.716 \times 10^{-5}\ kg/msec^2$, $T_s = 273°K$ and $S = 111°K$ for atmospheric air. Assuming that the Prandtl number is constant allows for modelling the temperature dependency of the coefficient of thermal conductivity similarly, and in all simulations we adopt Stokes hypothesis (see e.g. [11]) to obtain $\lambda = -\frac{2}{3}\mu$; although the analytic results are given for the general case.

3.1 Well-posedness and boundary conditions for the continuous problem

In order to develop the appropriate boundary operator, we begin by splitting the viscous flux vectors as

$$\overline{\mathbf{F}}_i^\nu = \sum_{j=1}^{3} \overline{\Pi}_{\xi_i\xi_j} \quad ,$$

where we introduce the vector, $\overline{\Pi}_{\xi_i\xi_j} = J\Pi_{\xi_i\xi_j}$, defined as

$$\Pi_{\xi_i\xi_j} = \begin{bmatrix} 0 \\ \mu(\nabla\xi_i \cdot \nabla\xi_j)u_1^j + \lambda\xi_i^1\nabla\xi_j \cdot \mathbf{u}^j + \mu\xi_j^1\nabla\xi_i \cdot \mathbf{u}^j \\ \mu(\nabla\xi_i \cdot \nabla\xi_j)u_2^j + \lambda\xi_i^2\nabla\xi_j \cdot \mathbf{u}^j + \mu\xi_j^2\nabla\xi_i \cdot \mathbf{u}^j \\ \mu(\nabla\xi_i \cdot \nabla\xi_j)u_3^j + \lambda\xi_i^3\nabla\xi_j \cdot \mathbf{u}^j + \mu\xi_j^3\nabla\xi_i \cdot \mathbf{u}^j \\ \mu(\nabla\xi_j \cdot \mathbf{u})(\nabla\xi_i \cdot \mathbf{u}^j) + \lambda(\nabla\xi_i \cdot \mathbf{u})(\nabla\xi_j \cdot \mathbf{u}^j) \\ \ldots + (\nabla\xi_i \cdot \nabla\xi_j)(\mu\mathbf{u} \cdot \mathbf{u}^j + \frac{\gamma k}{\Pr}T^j) \end{bmatrix}$$

with $\mathbf{u}^j = (u_1^j, u_2^j, u_3^j)$. This splitting is equivalent to the one proposed in [12], although here it is given in general curvilinear coordinates. Borrowing the terminology from that work, we term the inviscid flux vectors as the hyperbolic part of the flux. For $i = j$ we obtain the parabolic part of the operator, while summing the parts with $i \neq j$ results in the mixed contribution.

We continue by introducing the transformation Jacobian

$$(4) \qquad \mathcal{A}_i = \frac{\partial\overline{\mathbf{F}}_i}{\partial\overline{\mathbf{q}}} \quad , \quad \mathcal{B}_{ij} = \frac{1}{2}\left(\frac{\partial\overline{\Pi}_{\xi_i\xi_j}}{\partial\overline{\mathbf{q}}_j} + \frac{\partial\overline{\Pi}_{\xi_j\xi_i}}{\partial\overline{\mathbf{q}}_i}\right) \quad ,$$

where we define

$$\overline{\mathbf{q}}_i = \frac{\partial\overline{\mathbf{q}}}{\partial\xi_i} \quad .$$

This allows for writing Eq.(3) as

$$\frac{\partial\overline{\mathbf{q}}}{\partial t} + \mathcal{A}_i\frac{\partial\overline{\mathbf{q}}}{\partial\xi_i} = \frac{1}{\text{Re}_{\text{ref}}}\mathcal{B}_{ij}\frac{\partial^2\overline{\mathbf{q}}}{\partial\xi_i\partial\xi_j} \quad ,$$

where summation over indices is assumed as usual. Note, that by construction we have $\mathcal{B}_{ij} = \mathcal{B}_{ji}$.

It is well known that Navier-Stokes equations, although being an incomplete parabolic system, support waves very similar to those encountered in the hyperbolic Euler equations. For hyperbolic systems, Gottlieb et al. [13] have shown that enforcing the boundary conditions through the characteristic variables of the system results in a stable approximation.

We will seek a boundary operator appropriate for enforcing boundary conditions in the ξ_1-direction. For simplicity we assume ξ_1 mapped such that $\xi_1 \in [-1,1]$.

For Navier-Stokes equations, we linearize around a uniform state, $\bar{\mathbf{q}}$, by fixing $\mathcal{A}_i$ and $\mathcal{B}_{ij}$, and transform into characteristic variables by diagonalizing $\mathcal{A}_1$ through a similarity transform, $\Lambda = \mathcal{S}^{-1}\mathcal{A}_1\mathcal{S}$, where Λ is the diagonal eigenvalue matrix with $\Lambda_{ii} = \lambda_i$ and $\mathcal{S}$ and $\mathcal{S}^{-1}$ are the matrix of right and left eigenvectors, respectively. We must require $|\mathcal{S}|$ and $|\mathcal{S}^{-1}|$ to be bounded. Applying this, the symmetrized, linearized set of equations transforms into

$$\mathcal{Q}^T\mathcal{Q}\frac{\partial\overline{\mathbf{R}}}{\partial t} + \mathcal{A}_i^s\frac{\partial\overline{\mathbf{R}}}{\partial\xi_i} = \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\mathcal{B}_{ij}^s\frac{\partial^2\overline{\mathbf{R}}}{\partial\xi_i\partial\xi_j} \quad , \tag{5}$$

where $\overline{\mathbf{R}} = \mathcal{S}^{-1}\overline{\mathbf{q}}$ are the characteristic variables. We have introduced a positive definite, symmetrizing diagonal matrix, $\mathcal{Q}^T\mathcal{Q} = \mathrm{diag}[1,2,2\bar{c}^2/(\gamma-1),2,1]$, where $\bar{c} = \sqrt{\gamma\bar{p}/\bar{\rho}}$ is the uniform state sound speed. Additionally, we define the symmetric matrices [2]

$$\mathcal{A}_i^s = \mathcal{Q}^T\mathcal{Q}\mathcal{S}^{-1}\mathcal{A}_i\mathcal{S} \quad , \quad \mathcal{B}_{ij}^s = \mathcal{Q}^T\mathcal{Q}\mathcal{S}^{-1}\mathcal{B}_{ij}\mathcal{S} \quad .$$

The expression for the explicit entries in $\mathcal{B}_{ij}^s$ may be obtained from [3], where the diagonalizing matrices, $\mathcal{S}$ and $\mathcal{S}^{-1}$, are also given.

The entries in the diagonal matrix, Λ, are found as

$$\lambda_1 = \bar{u}_i n_i + \bar{c} \;,\quad \lambda_2 = \lambda_3 = \lambda_4 = \bar{u}_i n_i \;,\quad \lambda_5 = \bar{u}_i n_i - \bar{c} \;,$$

and correspond to the velocities of the characteristic waves for the Euler equations. We have introduced the unit vector, $\hat{\mathbf{n}}$, pointing along ξ_1 as

$$\hat{\mathbf{n}} = (n_1, n_2, n_3) = \frac{\nabla\xi_1}{\sqrt{\nabla\xi_1\cdot\nabla\xi_1}} \quad .$$

Additionally, we obtain the characteristic functions, $\overline{\mathbf{R}}$, as

$$\overline{\mathbf{R}} = J\begin{bmatrix} (m_i - \rho\bar{u}_i)n_i + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \\ (m_2-\rho\bar{u}_2)n_1 - (m_1-\rho\bar{u}_1)n_2 \\ \rho - \frac{\gamma-1}{\bar{c}^2}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \\ (m_3-\rho\bar{u}_3)n_1 - (m_1-\rho\bar{u}_1)n_3 \\ -(m_i - \rho\bar{u}_i)n_i + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \end{bmatrix} ,$$

where $m_i = \rho u_i$ is the momentum. Here R_1 and R_5 correspond to co- and counter propagating sound waves, respectively; R_2 and R_4 represent vorticity waves whereas R_3 is an entropy wave.

By defining a viscous correction vector, $\overline{\mathbf{G}}$, as

$$\mathcal{Q}^T\mathcal{Q}\overline{\mathbf{G}} = \mathcal{B}_{11}^s\frac{\partial\overline{\mathbf{R}}}{\partial\xi_1} + (\mathcal{B}_{12}^s + \mathcal{B}_{21}^s)\frac{\partial\overline{\mathbf{R}}}{\partial\xi_2} + (\mathcal{B}_{13}^s + \mathcal{B}_{31}^s)\frac{\partial\overline{\mathbf{R}}}{\partial\xi_3} \quad ,$$

we are now ready to state the main result of this section. The Lemma will be given without proof, as a detailed proof for the Cartesian case may be found in [2].

Lemma 3.1 *Assume there exists a solution,* $\mathbf{q}$, *which is periodic or held at a constant value at the* ξ_2- *and* ξ_3-*boundary. If the boundary conditions in the* ξ_1-*direction are given such that*

$$\forall(\xi_2,\xi_3) \;:\; -\frac{1}{2}\left[\overline{\mathbf{R}}^T\mathcal{A}_1^s\overline{\mathbf{R}} - \frac{2}{\mathrm{Re}_{\mathrm{ref}}}\overline{\mathbf{R}}^T\mathcal{Q}^T\mathcal{Q}\overline{\mathbf{G}}\right]_{\xi_1=-1}^{1} \le 0 \;,$$

and the fluid properties are constrained by

$$\bar{\mu} \ge 0 \;,\quad \bar{\lambda} \le 0 \;,\quad \bar{\lambda} + \bar{\mu} \ge 0 \;,\quad \frac{\gamma\bar{k}}{\mathrm{Pr}} \ge 0 \;,\quad \gamma \ge 1 \;,$$

then Eq.(5) constitutes a well-posed problem and the solution is bounded as

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\overline{\mathbf{R}}\|^2 \le -\frac{1}{\mathrm{Re}_{\mathrm{ref}}}\int_\Omega\left(\frac{\partial\overline{\mathbf{R}}^T}{\partial\xi_i}\mathcal{B}_{ij}^s\frac{\partial\overline{\mathbf{R}}}{\partial\xi_j}\right)J\,d\Omega \le 0 \;.$$

The proof is based on the fact that $\mathcal{B}_{ij}^s = \mathcal{B}_{ji}^s$ all are symmetric matrices. Thus, well-posedness is ensured by proving the quadratic form under the integral semi-positive. This is obtained provided the fluid properties are constrained as given in the Lemma and the mapping is nonsingular. For details we refer to [2].

For most real fluids under non-extreme conditions, it is true that $\bar{\mu}$ is positive, $\bar{\lambda}$ is negative and the following relationship is obeyed [12]

$$\frac{\gamma\bar{\mu}}{\mathrm{Pr}} \ge \bar{\lambda} + 2\bar{\mu} \ge \bar{\mu} \;.$$

Thus, the conditions on the fluid properties as given in Lemma 3.1 are only natural. In fact, if this is not obeyed, Navier-Stokes equations may be shown to violate the second law of thermodynamics [14].

As stated in Lemma 3.1, the appropriate boundary operator must be determined by construction such that

$$-\frac{1}{2}\mathcal{Q}^T\left[\overline{\mathbf{R}}^T\Lambda\overline{\mathbf{R}} - \frac{2}{\mathrm{Re}_{\mathrm{ref}}}\overline{\mathbf{R}}^T\overline{\mathbf{G}}\right]_{-1}^{1}\mathcal{Q} \le 0 \;.$$

Since $\mathcal{Q}^T\mathcal{Q}$ is a diagonal positive definite, this may conveniently be reformulated as

$$-\frac{1}{2}\left[\lambda_i^{-1}\left(\left(|\lambda_i|\overline{R}_i - \varepsilon\frac{|\lambda_i|}{\lambda_i}\overline{G}_i\right)^2 - \left(\varepsilon\overline{G}_i\right)^2\right)\right]_{-1}^{1} \le 0 \;,$$

where λ_i are the wave speeds by which the characteristic variables are advected (as given by the diagonal elements of Λ), and we have introduced $\varepsilon = \mathrm{Re_{ref}}^{-1}$. This formulation makes it straightforward to devise inflow-outflow boundary conditions, which are maximally dissipative and ensure well-posedness of the complete problem.

We note, in particular, that the formulation takes into account the off-diagonal terms of the stress tensor, which may be of importance if the artificial boundary is introduced into a strongly vortical region of the flow.

Using this, the boundary operators on the characteristic variables may be expressed as [2]

$$\begin{array}{lcl} \text{Inflow Boundary} & : & \mathcal{R}^-\overline{\mathbf{R}} - \varepsilon\mathcal{G}^-\overline{\mathbf{G}} \\ \text{Outflow Boundary} & : & \mathcal{R}^+\overline{\mathbf{R}} + \varepsilon\mathcal{G}^+\overline{\mathbf{G}} \end{array} .$$

We have expressed the operators by introducing 4 diagonal matrices, $\mathcal{R}^-$, $\mathcal{R}^+$, $\mathcal{G}^-$ and $\mathcal{G}^+$, which ensures the correct construction of the operators. The four matrices are defined as $\mathcal{R}^- = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha\lambda_5]$ and $\mathcal{G}^- = \mathrm{diag}[1, 1, 1, 1, 1]$ at the inflow boundary with $\alpha = 0$ for subsonic conditions, and $\alpha = 1$ for supersonic conditions.

At the outflow we likewise define the operators $\mathcal{R}^+ = \mathrm{diag}[0, 0, 0, 0, \beta|\lambda_5|]$ and $\mathcal{G}^+ = \mathrm{diag}[0, 1, 1, 1, 1]$, where $\beta = 1$ for subsonic conditions, and $\beta = 0$ for supersonic outflow conditions. These result are obtained assuming $\bar{u}_i n_i > 0$ at the boundary. Similar results may be obtained in the case $\bar{u}_i n_i < 0$.

It was shown by Strikwerda [15] that the proper number of boundary conditions for an incomplete, parabolic system as the compressible Navier-Stokes equations is 5 in the inflow region and 4 in the outflow region. Our result clearly conforms with that.

We also note that in the limit of infinite Reynolds number, these boundary conditions converge uniformly toward the well known characteristic boundary conditions for the compressible Euler equations [16]. This property is important because it allows us to avoid weak boundary layers of the order $\exp(-\xi_1/\varepsilon)$ [17].

At first, it may seem as if all we have accomplished so far is to derive proper open boundary conditions to the compressible Navier-Stokes equations. When considering a multi-domain approach, the important observation to make is that we may equally well treat a subdomain boundary as an open boundary. However, it is a very special open boundary since we may at all times obtain boundary conditions from the neighboring subdomains.

The single remaining question is how to impose these complicated boundary operators in a spectral model of the compressible Navier-Stokes equations in a consistent and, preferably, simple way. This is the question we will address in the following section.

3.2 The semi-discrete penalty scheme

Following the line of thought leading to the asymptotically stable penalty scheme for the wave equation, we propose a collocation method for enforcing the boundary operator derived in the previous section for the compressible Navier-Stokes equations as

$$\begin{aligned} \frac{\partial \overline{\mathbf{q}}}{\partial t} + \frac{\partial \overline{\mathbf{F}}_i}{\partial \xi_i} &= \frac{1}{\mathrm{Re_{ref}}}\frac{\partial \overline{\mathbf{F}}_i^{\nu}}{\partial \xi_i} \\ &-\tau_1 Q^-(\xi_1)\mathcal{S}\left[\mathcal{R}^-(\overline{\mathbf{R}}_0 - \overline{\mathbf{R}}_{-1}) - \frac{1}{\mathrm{Re_{ref}}}\mathcal{G}^-(\overline{\mathbf{G}}_0 - \overline{\mathbf{G}}_{-1})\right] \\ &-\tau_2 Q^+(\xi_1)\mathcal{S}\left[\mathcal{R}^+(\overline{\mathbf{R}}_N - \overline{\mathbf{R}}_1) + \frac{1}{\mathrm{Re_{ref}}}\mathcal{G}^+(\overline{\mathbf{G}}_N - \overline{\mathbf{G}}_1)\right] . \end{aligned}$$

Here we have defined

$$Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N'(-1)} \quad , \quad Q^+(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)} \quad ,$$

and the symbols $\overline{\mathbf{R}}_0 = \overline{\mathbf{R}}(-1, \xi_2, \xi_3)$, $\overline{\mathbf{G}}_0 = \overline{\mathbf{G}}(-1, \xi_2, \xi_3)$, and, likewise; $\overline{\mathbf{R}}_N = \overline{\mathbf{R}}(1, \xi_2, \xi_3)$, $\overline{\mathbf{G}}_N = \overline{\mathbf{G}}(1, \xi_2, \xi_3)$. The values at the two boundaries are given through the remaining symbols depending on whether the boundary is an open boundary or a subdomain boundary. Hence, at $\xi = -1$ we obtain in the latter case that $\overline{\mathbf{R}}_{-1}$ represents the value of the characteristic vector in the neighboring subdomain and similarly for the viscous correction vector, $\overline{\mathbf{G}}_{-1}$. If, on the other hand, the inflow is an open boundary, the values of $\overline{\mathbf{R}}_{-1}$ and $\overline{\mathbf{G}}_{-1}$ must be specified, i.e.

$$\overline{\mathbf{R}}_{-1} = \mathcal{S}^{-1}(J\mathbf{g}(t)) \quad , \quad \overline{\mathbf{G}}_{-1} = \mathcal{S}^{-1}\overline{\mathbf{h}}(t) \quad ,$$

where $\mathbf{g}(t)$ gives the values of the state vector and $\overline{\mathbf{h}}(t)$ accounts for information of the gradients at the boundary. In most cases nothing is known about this and one may use $\overline{\mathbf{h}}(t) = 0$. At the outflow, $\xi_1 = 1$, the variables $\overline{\mathbf{R}}_1$ and $\overline{\mathbf{G}}_1$ are specified in a similar manner.

At an open boundary we may use the expected value of the state vector, $\mathbf{g}(t)$, to linearize around; whereas we use the value of the state vector at the previous time step as linearization vector at a subdomain boundary.

We now need to specify τ_1 and τ_2 such that the semi-discrete scheme is asymptotically stable. By noting that the proposed multi-domain scheme is algebraicly equivalent to a one domain scheme with open boundaries, we need only prove stability for one domain with homogeneous boundary conditions. We state the result without proof, but refer to [2] for a detailed proof in the Cartesian case.

Lemma 3.2 *Assume there exists a solution,* $\mathbf{q}$, *which is periodic or held at a constant value at the* ξ_2- *and* ξ_3-*boundary, and that the fluid properties of the uniform state,*

$\bar{\mathbf{q}}$, *are constrained by*

$$\bar{\mu} \geq 0 \ , \quad \bar{\lambda} \leq 0 \ , \quad \bar{\lambda} + \bar{\mu} \geq 0 \ , \quad \frac{\gamma \bar{k}}{\Pr} \geq 0 \ , \quad \gamma \geq 1 \ ,$$

and related as

$$\frac{\gamma \bar{\mu}}{\Pr} \geq \bar{\lambda} + 2\bar{\mu} \geq \bar{\mu} \ .$$

The linearized, constant coefficient version of the proposed scheme is asymptotically stable at the <u>*inflow*</u> *if*

$$\frac{1}{\omega\kappa}\left(1+\kappa+\sqrt{1+\kappa}\right) \geq \tau_1 \geq \frac{1}{\omega\kappa}\left(1+\kappa-\sqrt{1+\kappa}\right) \ .$$

Here

$$\kappa = \frac{1}{2\omega}\frac{1}{\mathrm{Re}_{\mathrm{ref}}}\frac{\gamma\bar{k}}{\Pr\bar{\rho}|\bar{u}_i n_i|} \ .$$

This result is dependent upon whether the inflow is subsonic or supersonic.

For <u>*supersonic outflow*</u>

$$\frac{1}{\omega}\left(1+\sqrt{\frac{1}{\kappa}}\right) \geq \tau_2 \geq \frac{1}{\omega}\left(1-\sqrt{\frac{1}{\kappa}}\right) \ .$$

For <u>*subsonic outflow*</u>

$$\frac{1}{\omega\kappa}\left(1+\kappa+\sqrt{1+\kappa}\right) \geq \tau_2 \geq \frac{1}{\omega\kappa}\left(1+\kappa+\sqrt{1+\kappa}\right) .$$

The value of ω depends on the choice of basis functions. For Legendre polynomials one should use

$$\omega = \frac{2}{N(N+1)} \ ,$$

where N is the number of modes in the expansion. For Chebyshev polynomials one gets

$$\omega = \frac{1}{N^2} \ .$$

This last result is based on extensive numerical experiments [2, 3, 10].

We wish to emphasize that the bounds on τ_1 and τ_2 given in Lemma 3.2 remain valid in the limit when the Reynolds number approaches infinity. This is realized by expanding the bounds for $\varepsilon \ll 1$ to obtain

$$\infty > \tau_1 \geq \frac{1}{2\omega} + \varepsilon\frac{1}{8\omega}\kappa \ ,$$

in the inflow region and

$$\infty > \tau_2 > -\infty \quad , \quad \infty > \tau_2 \geq \frac{1}{2\omega} + \varepsilon\frac{1}{8\omega}\kappa \ ,$$

for supersonic and subsonic outflow, respectively. The linearized, constant coefficient version of the Euler equations may be transformed into five independent hyperbolic equations for which we should expect the bounds on the penalty parameters to be given by the results in Sec. 2. We observe that the bounds given above converge uniformly to the expected values in the limit of vanishing viscosity and, thus, the scheme remains stable. The observation that no bounds are necessary on τ_2 for supersonic outflow simply reflects the fact that no boundary conditions are required for the Euler equations at such a boundary.

Note that the presented scheme assumes that the velocity, $(\bar{u}_i n_i)$, is positive, i.e. inflow at $\xi_1 = -1$ and outflow at $\xi_1 = 1$. However, in [2] stability was proven at inflow and outflow independently, and we may thus choose any combination of inflow-outflow patching consistent with the flow realization while maintaining the asymptotic stability.

We have used an adaptive 4th order Runge-Kutta scheme for temporal integration of the Navier-Stokes equations. The boundary conditions and the patching are performed at each intermediate time step followed by enforcing continuity across the subdomain boundaries.

The global time step, Δt, is found as [18]

$$\Delta t = CFL \times \min_{\Omega}\left[|v_i u_i| + c\sqrt{v_i v_i} + \frac{2\gamma}{\Pr\mathrm{Re}_{\mathrm{ref}}}\frac{\mu}{\rho}v_i v_i\right]^{-1}$$

where CFL is the CFL-number. We have defined the local curvilinear vector as

$$\mathbf{v} = (v_1, v_2, v_3) = \frac{|\nabla\xi_1|}{\Delta_i\xi_1} + \frac{|\nabla\xi_2|}{\Delta_j\xi_2} + \frac{|\nabla\xi_3|}{\Delta_k\xi_3} \ ,$$

where $\Delta_i\xi_1$, $\Delta_j\xi_2$ and $\Delta_k\xi_3$ is the local grid-size along the three coordinate axes with respect to the indices (i, j, k). Also $|\nabla\xi_i| = (|\xi_i^1|, |\xi_i^2|, |\xi_i^3|)$.

4 Numerical examples

In this section we will present schemes and results for the multi-domain solution of steady and unsteady compressible flows.

4.1 Quasi-one-dimensional nozzle flows

Consider the flow in a quasi-one-dimensional Laval nozzle. The dynamics of the fluid is then described by a simplified set of equations as

$$(6) \quad \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{H} = \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\frac{\partial \mathbf{F}^{\nu}}{\partial x} \ , \quad |x| \leq 1 \ , \ t > 0 \ .$$

Here we have

$$\mathbf{q} = \begin{bmatrix} \rho A \\ \rho u A \\ EA \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u A \\ (\rho u^2 + p)A \\ (E+p)uA \end{bmatrix},$$

$$\mathbf{F}^{\nu} = \begin{bmatrix} 0 \\ A\tau_{xx} \\ Au\tau_{xx} + A\frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial x} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ -p\frac{dA}{dx} \\ 0 \end{bmatrix}.$$

This set of equations is obtained from Eq.(3) by using $\xi_1 = x$ and $\xi_2 = \xi_3 = 0$, thus cancelling all u_2-and u_3-components and ξ_2-and ξ_3-derivatives. Additionally, we use the fact that for a slowly varying area variation, $A(x)$, the quasi-one-dimensional divergence of a vector function, $\mathbf{f} = (f, 0)$, may be approximated as

$$\nabla \cdot \mathbf{f} = \frac{\partial f A}{\partial x} .$$

As reference values for non-dimensionalizing the equations, we use the values at the throat.

For this problem the wave-speed of the characteristic waves becomes

$$\lambda_1 = \bar{u} + \bar{c} \ , \quad \lambda_2 = \bar{u} \ , \quad \lambda_3 = \bar{u} - \bar{c} \ ,$$

and the characteristic functions, $\mathbf{R}$, are defined as

$$\mathbf{R} = A(x) \begin{bmatrix} \rho u - \bar{u}\rho + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}^2 - \rho u\bar{u}\right) \\ \rho - \frac{\gamma-1}{\bar{c}^2}\left(E + \frac{1}{2}\rho\bar{u}^2 - \rho u\bar{u}\right) \\ -(\rho u - \bar{u}\rho) + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}^2 - \rho u\bar{u}\right) \end{bmatrix} .$$

For simulations of inviscid flows, specification of these characteristic functions whenever they enter the computational domain leads to a well-posed problem.

The viscous correction vector, $\mathbf{G}$, becomes

$$\mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} = \frac{1}{2\bar{\rho}} \begin{bmatrix} \frac{\bar{k}(\gamma-1)}{\mathrm{Pr}}\frac{\partial\zeta_1}{\partial x} + \frac{4}{3}\bar{\mu}\frac{\partial\zeta_2}{\partial x} \\ -\frac{\bar{k}(\gamma-1)}{\bar{c}\mathrm{Pr}}\frac{\partial\zeta_1}{\partial x} \\ \frac{\bar{k}(\gamma-1)}{\mathrm{Pr}}\frac{\partial\zeta_1}{\partial x} - \frac{4}{3}\bar{\mu}\frac{\partial\zeta_2}{\partial x} \end{bmatrix},$$

where we, for simplicity, introduce

$$\zeta_1 = R_1 + R_3 - \frac{2\bar{c}}{\gamma - 1}R_2 \ , \quad \zeta_2 = R_1 - R_3 \ .$$

Here $\partial\zeta_1/\partial x$ is a consequence of the normal heat flux, while $\partial\zeta_2/\partial x$ accounts for the normal stress at the boundary.

4.1.1 Numerical tests

As a test case for the proposed multi-domain scheme we have chosen a symmetric converging-diverging Laval nozzle with a cross-sectional area given as

$$A(x) = 1 - 0.8x(1-x) \quad , \quad 0 \le x \le 1 \ ,$$

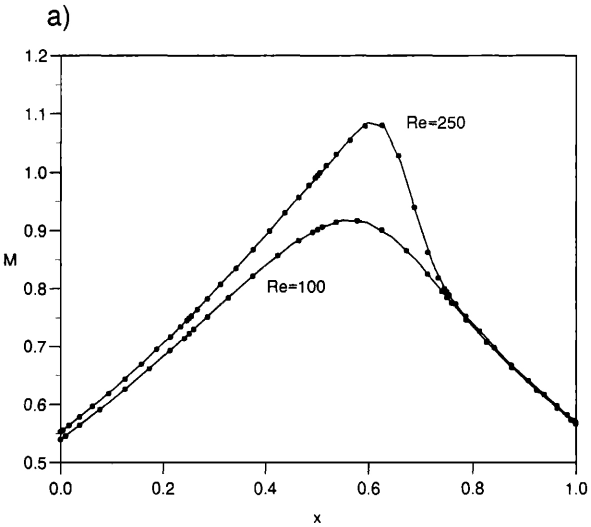

Figure 1: Steady state solution of the Mach profile for the viscous transonic nozzle flow. For the one-domain solutions (full line) at Re=100, N=32 collocation points were used, whereas N=48 were used at Re=250. The solutions symbolized by the dots represent the four-domain solution with the subdomains being equally sized and with N/4 modes in each. CFL=3.0 was used in all simulations.

and a ratio between the stagnation pressure and the back pressure of 0.78. This results in a choked flow through the nozzle with the supersonic flow being terminated by a stationary shock in the divergent part of the nozzle. In the inviscid limit, this problem has an analytic solution with a shock at $x \simeq 0.773$ with shock Mach number, $M_s = 1.32$. We have chosen the length of the nozzle, $L = 0.1m$, the stagnation temperature, $T_0 = 300°K$ and $M = 1.0$, as the flow is choked.

Although the transonic nozzle flow leads to a steady state solution, we have implemented the scheme as a fully unsteady problem using a 4'th order Runge-Kutta method. Additionally, we have in all simulations applied an high order exponential filter to the solution at each time step.

As initial condition we used, the inviscid solution, smoothed by a 4'th order exponential filter, i.e. it is far from the steady state solution. As boundary conditions at the open inflow and outflow boundaries is used the inviscid solution, which, at least at high Reynolds numbers, is a very good approximation.

In Fig. 1 we show the steady state solutions obtained for Re = 100 and Re = 250 and we observe good agreement between the one and the four domain solutions. As a way of testing the accuracy of the multi-domain scheme we calculate the residual in L_2 as

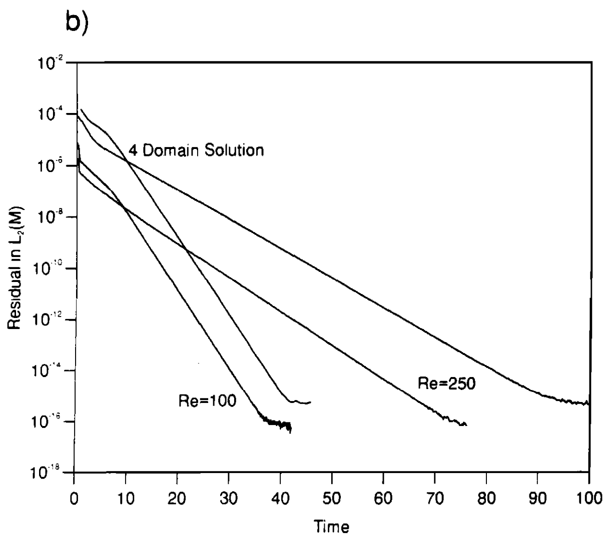

Figure 2: History of convergence of the Mach number for the one and four domain solutions.

$$\text{Residual}(f(t+\Delta t)) = \frac{\|f(t+\Delta t) - f(t)\|^2}{\|f(t)\|^2} \ .$$

In Fig. 2 we show the corresponding residuals of the Mach-number. The results for other variables are similar. We find that the one domain as well as the multi-domain solution converges to machine-precision with the same rate of convergence. The slight difference in the actual physical time of convergence is a consequence of the initial accuracy of the approximation. As we use the same total number of modes in the one domain and four domain solutions and both solutions converge at approximately the same physical time, we obtain a significant decrease in wall-clock time by employing the multi-domain approach. In this case the multi-domain scheme is more than 10 times faster than the one domain approach.

As final evidence of the performance of the scheme, we show in Fig. 3 the steady state Mach-profiles of the transonic nozzle flow at increasing Reynolds number, compared with the purely inviscid analytic solution. All viscous solutions are obtained using a five domain solution, with the domains clustered around the viscous shock. We observe that for a low Reynolds number, the flow becomes purely subsonic and consequently the Mach-profile changes upstream, as well as downstream, of the inviscid shock. For transonic flows, the steady state profiles are similar to the inviscid solution except in the highly viscous region in the neighborhood of the shock. For high Reynolds numbers (Re $\geq$ 500) we find that the solution converges to the inviscid solution as $\text{Re}^{-1/2}$, as expected. All viscous steady state profiles are computed with an L_2-residual less than

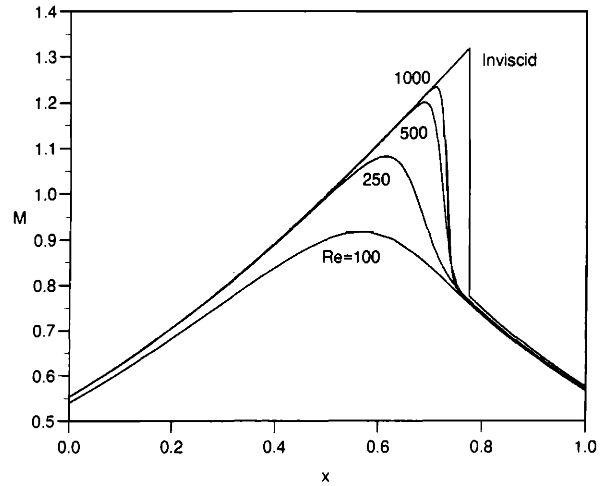

Figure 3: Steady state Mach profile for the transonic viscous nozzle flow at Re=100, 250, 500, 1000 and Re=∞. All solutions for finite Reynolds number are obtained as five domain solutions with CFL=3.0.

10^{-10}.

4.2 Flow around a circular cylinder

As a second test case for validating the proposed scheme, we have chosen unsteady compressible flow around an infinitely long circular cylinder. This flow is one of the most documented examples of simple exterior flows for which there exists an abundant amount of experimental results (see [19] and references therein).

We wish to simulate the unsteady subsonic flow in the von Karman shedding region for $60 < \text{Re} < 180$, where Re is based on the free stream values of the flow and the diameter of the cylinder. It is, therefore, sufficient to develop a two-dimensional model. The dynamics of the flow is described by the two-dimensional, compressible Navier-Stokes equations as given in Eq.(3), which we have normalized using the free stream values of the flow. We wish to simulate the dynamics of the flow by applying a multi-domain approach, where the full computational domain, Ω, is constructed by several non-overlapping concentric annular subdomains.

Each annular subdomain is mapped onto a rectangular computational domain, $(\xi_1, \xi_2) \in [0, 2\pi] \times [-1, 1]$. The branch cut, across which periodicity is enforced, is chosen at $\xi_1 = 0$, such that the physical grid relates to the computational grid as

$$x_1 = r(\xi_2)\cos\theta(\xi_1) \ , \ \ x_2 = r(\xi_2)\sin\theta(\xi_1) \ ,$$

where (x_1, x_2) are the Cartesian coordinates, (r, θ) are the corresponding polar coordinates and (ξ_1, ξ_2) are the general curvilinear coordinates. As a consequence of the geometry of the problem it is natural to choose a Fourier collocation method in ξ_1 and a Chebyshev collocation scheme in ξ_2.

By writing the problem in general curvilinear coordinates we obtain that although we treat a two-dimensional problem, we need only give boundary conditions and perform domain patching in the ξ_2-direction as boundary conditions in ξ_1 are given through periodicity. Thus, the one dimensional approach for patching may be applied for solving this problem.

We obtain the characteristic functions,

$$\overline{\mathbf{R}} = J \begin{bmatrix} (m_i - \rho\bar{u}_i)n_i + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \\ (m_2 - \rho\bar{u}_2)n_1 - (m_1 - \rho\bar{u}_1)n_2 \\ \rho - \frac{\gamma-1}{\bar{c}^2}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \\ -(m_i - \rho\bar{u}_i)n_i + \frac{\gamma-1}{\bar{c}}\left(E + \frac{1}{2}\rho\bar{u}_i\bar{u}_i - m_i\bar{u}_i\right) \end{bmatrix} ,$$

where

$$\hat{\mathbf{n}} = (n_1, n_2) = \frac{\nabla\xi_2}{\sqrt{\nabla\xi_2 \cdot \nabla\xi_2}} ,$$

is a unit vector pointing along ξ_2. The four corresponding eigenvalues are

$$\lambda_1 = \bar{u}_i n_i + \bar{c} , \quad \lambda_2 = \lambda_3 = \bar{u}_i n_i , \quad \lambda_4 = \bar{u}_i n_i - \bar{c} ,$$

yielding the wave speed of the co-propagating sound wave, the vorticity wave, the entropy wave and the counter-propagating sound wave, respectively.

Likewise, we obtain the viscous patching vectors, $\overline{\mathbf{G}} = J(G_1, G_2, G_3, G_4)^T$, as

$$G_1 = \frac{1}{\bar{\rho}r}\left[\frac{(\gamma-1)\bar{k}}{2\mathrm{Pr}}\frac{\partial(r\zeta_1)}{\partial r} + \frac{2}{3}\bar{\mu}\frac{\partial(r\zeta_2)}{\partial r} - \frac{1}{3}\bar{\mu}\frac{\partial R_2}{\partial\theta}\right],$$

$$G_2 = \frac{\bar{\mu}}{\bar{\rho}r}\left[\frac{\partial(rR_2)}{\partial r} - \frac{1}{6}\frac{\partial\zeta_2}{\partial\theta}\right],$$

$$G_3 = -\frac{(\gamma-1)\bar{k}}{\mathrm{Pr}}\frac{1}{2\bar{c}\bar{\rho}}\frac{1}{r}\frac{\partial(r\zeta_1)}{\partial r},$$

$$G_4 = \frac{1}{\bar{\rho}r}\left[\frac{(\gamma-1)\bar{k}}{2\mathrm{Pr}}\frac{\partial(r\zeta_1)}{\partial r} - \frac{2}{3}\bar{\mu}\frac{\partial(r\zeta_2)}{\partial r} + \frac{1}{3}\bar{\mu}\frac{\partial R_2}{\partial\theta}\right],$$

where we, for convenience, have introduced the symbols

$$\zeta_1 = R_1 + R_4 - \frac{2\bar{c}}{\gamma-1}R_3 , \quad \zeta_2 = R_1 - R_4 .$$

The terms depending on $\partial r\zeta_1/\partial r$ account for the effect of normal heat flux at the boundaries, while the remaining terms in $\overline{\mathbf{G}}$ represent the effects of normal and tangential stress at the boundaries.

Re	S_t computed	S_t experiment [19]
75	0.149	0.149
100	0.165	0.164
125	0.177	0.175

Table 1: Comparison of Strouhal number computed and reported from experiments.

At the subdomain boundaries we use the values of the state vector at the previous time-step as linearization parameters, and at the open boundary we use the free stream values. At the solid cylinder wall we assume no-slip, isothermal boundaries, i.e. $\mathbf{q}(r = L/2) = (\rho, 0, 0, \rho T_\infty)^T$, where ρ is determined numerically.

The scheme has been time-stepped using an adaptive, explicit 4'th order Runge-Kutta with the boundary conditions and the subdomain patching being enforced at intermediate time steps, where we also enforce continuity of the global solution and apply a high order exponential filter to the solution. All simulations to be presented were done with $CFL = 3.0$.

4.2.1 Numerical tests

We have performed tests with a cylinder of diameter $L = 0.1m$, a free stream Mach number, $M = 0.2$, and a stagnation temperature, $T_0 = 300°K$.

We have done simulations at various Reynolds numbers using a 5 domain non-conforming grid. We use high order interpolation between the different grids. In Fig. 4 we show contour plots of the normalized density, normalized pressure, vorticity and local Mach-number. This clearly shows the well known von Karman street rear of the cylinder. We observe that the contour lines are continuous across subdomain boundaries, and we note that the vortices propagate undisturbed across the subdomain boundaries without any reflections from the artificial boundaries. Also, the contour lines of the vorticity, $\nabla \times \mathbf{u}$, remain continuous across subdomain boundaries indicating that the gradient is continuous across subdomain boundaries as well. To evaluate the performance of the algorithm quantitatively, we have performed several computations at various Reynolds numbers. In Table 1 we compare the computed Strouhal number, i.e. the non-dimensional shedding frequency $S_t = \omega L/\bar{u}$, with that found in experiments [19]. The Strouhal number is calculated from time traces of the pressure in various positions in the computational domain. We observe very close agreement between computational and experimental results. These results lead us to conclude

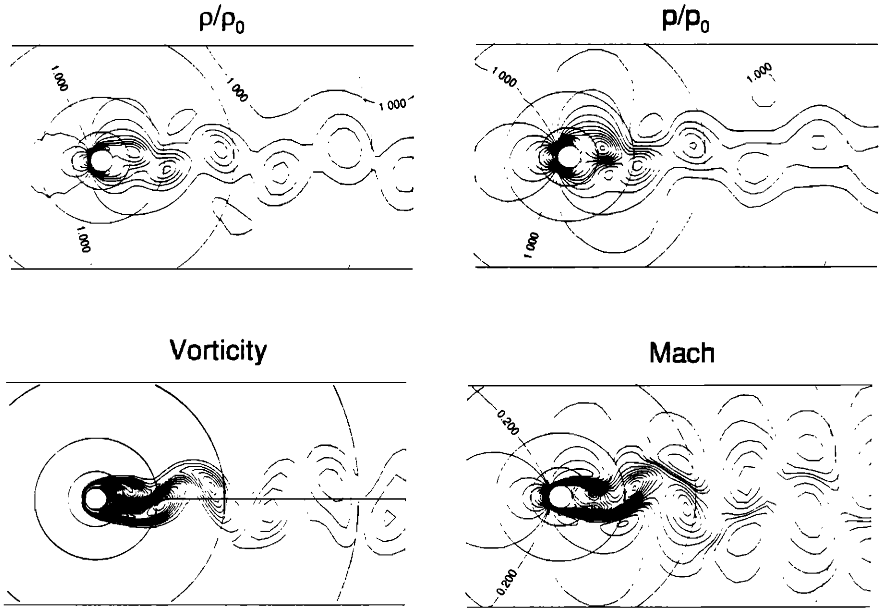

Figure 4: Contour plot of the normalized density, ρ/ρ_0, the normalized pressure, p/p_0, the vorticity, $\nabla \times \mathbf{u}$, and the local Mach-number at T=107, corresponding to 18 shedding cycles at Re = 100. The circles represent the subdomain boundaries.

that the scheme performs well, and is also applicable for fully unsteady, multi-dimensional flows.

5 Concluding remarks

In this paper we have developed a unified approach for dealing with open boundaries and subdomain boundaries when performing multi-domain simulations of the compressible Navier-Stokes equations given in conservation form.

The multi-domain scheme that we propose may be proven asymptotically stable and, as we have shown, is well suited for performing simulations of steady as well as unsteady compressible flows.

Although we have only considered one- and two-dimensional examples, we are confident that it is clear to the reader that the scheme may be applied for three-dimensional problems as well. However, at present we require that the boundaries may be described as curvilinear coordinate surfaces, i.e. treatment of boundaries is essentially one-dimensional. This may seem like a significant constraint. However, for a large group of flows (in particular interior flows), one may use the fact that the existence of solid walls allows for doing the patching, as all we require is that the state vector may be assumed constant or periodic in directions perpendicular to the one in which we are patching. Thus, patching in the streamwise direction in a channel flow is perfectly valid; as is the use of the open boundary conditions in such a situation. For exterior flows the assumption of a free stream at infinity allows for applying the conditions.

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