The h-p Version of Finite Element Method in $\mathbf{R}^3$: Theory and Algorithm

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Abstract

This paper gives a precise description of regularities of solutions and their derivatives of all orders for elliptic problems on polyhedral domains in the frame of the countably normed spaces with weighted $\mathbf{C}^k$-norms in neighborhoods of vertices, edges and vertex-edges. Under the guidance of the regularity theory, the geometric meshes and P-Q distribution (bilinear-linear or linear-uniform) of element degrees are designed accordingly in each of singular neighborhoods. The algorithms combining the geometric mesh and corresponding P-Q distribution of element degree achieve the exponential convergence and efficiency of computation. The performance of the h, p and h-p versions for a benchmark elasticity problem on a polyhedral domain is given and analysed.

Key words: the h-p version, the finite element method, geometric mesh, P-Q distribution, vertex singularity, edge singularity, vertex-edge singularity.

AMS subject classifications: 65N30, 65N15, 35B65, 35J25.

1 Introduction

The h-p version, its theory and algorithm, is a new development of the finite element method (FEM) in 1980's and 1990's. It was originated in and oriented to the structural mechanical problems on nonsmooth domains in $\mathbf{R}^2$ and $\mathbf{R}^3$, but now its use has been expanded to many other fields such as fluid mechanics, thermal analysis, electronic engineering, etc.. The methodology developed in the past decades has significantly influenced the theory and algorithm of FEM, the practices of engineering and scientific computation and the industry of commercial FEM codes, such as MSC/PROBE, PHLEX, MECHANICAL, STREE CHECK, and research code STRIPE (see [34, 35, 39]). The h-p version has been one of the most significant achievements of FEM's history since 1970's.

It is well known that the singularities of solutions for problems on nonsmooth domains may occur at the vertices and edges, which severely affect the effectiveness and efficiency of finite element solutions. The h-version, which reduces the element size h, and the p-version, which increase the element degree p, may not be able to achieve the desired accuracy in practical engineering range. Then the h-p version is the only reliable finite element approach which is able to provide effective and efficient algorithms. It reduces the element size h and increase the element degree p simultaneously and selectively in order to achieve the optimal rate of convergence and the efficiency of computations.

The h-p version of FEM in $\mathbf{R}^2$ was introduced in 1980's (see [2, 25, 26]) and has been well developed since then. It was originated in and oriented to the structure mechanical problems on nonsmooth domains. Under the guidance of the regularity theory in the frame of countably normed spaces for the problems on nonsmooth domains (see [3, 4, 27, 28]), the geometric mesh and the P-distribution of element degrees are properly designed, which leads to the exponential rate of convergence with respect to cubic root of the number of degree of freedom. The exponential convergence has been seen in numerous computations by using commercial and research codes mentioned above. The theory and algorithms have been generalized from elliptic boundary value problems to interface problems, eigenvalue problems, high-order problems, parabolic and hyperbolic problems (see [5, 7, 13, 16, 19, 29, 31, 38, 40]). Thus the h-p version in $\mathbf{R}^2$ has already been well established in 1980's. For survey of the h-p version in $\mathbf{R}^2$ we refer to [8, 11].

Although the computation and implementation of the h-p version in $\mathbf{R}^3$ was started in later 1980's (see [1, 12, 20]), there had been no progress on the approximation theory until the regularity theory in terms of countably normed

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spaces for elliptic problems on nonsmooth domains in $\mathbf{R}^3$ was established in early 1990's (see [20, 22, 23, 24]). There are several quite different features of the h-p version in three dimensional setting from those in two dimensional setting, due to the complexity of singularity at vertices and edges. First of all, the geometric meshes are designed differently in the neiborhoods of vertices, edges and vertex-edges. Secondly, in addition to the bilinear or linear P-distribution of element degrees, a linear or uniform Q-distribution of polynomials of one variable (in the direction of edges) have to be adopted in edge-neiborhoods and vertex-edge neighborhoods in order to achieve the efficiency of computations.

As a major development of the finite element method, in theory and practice, the h-p version of FEM in $\mathbf{R}^3$ involves the regularity theory of PDE on nonsmooth domain, the approximation theory of the h-p version, the parallel and iterative solvers for large-scale systems resulted from the h-p finite element discretization, implementation, applications to structural mechanics and engineering computation, etc.. This paper will focus on the regularity, approximation, and algorithm.

2 A model problem

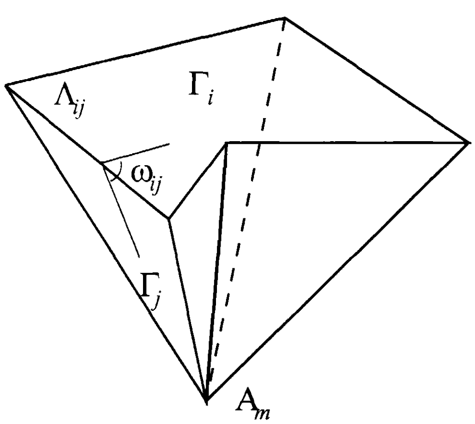

Fig. 1. Polyhedral Domain Ω

Let Ω be a polyhedral domain in $\mathbf{R}^3$ shown in Fig. 1, and let $\Gamma_i, i \in \mathcal{I} = \{1, 2, 3, \cdots, I\}$ be the faces (open), $A_m, m \in \mathcal{M} = \{1, 2, 3, \cdots, M\}$ be the vertices. By Λ_{ij} we denote the edge which is the intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$. Let $\mathcal{I}_m$ be a subset $\{j \in \mathcal{I} \mid A_m \in \bar{\Gamma}_j\}$ of $\mathcal{I}$ for $m \in \mathcal{M}$. Let $\mathcal{L} = \{ij | i, j \in \mathcal{I}, \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$, and let $\mathcal{L}_m$ denote a subset of $\mathcal{L}$ such that $\mathcal{L}_m = \{ij \in \mathcal{L} \mid A_m \in \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$. We denote by ω_{ij} the interior angle between Γ_i and Γ_j for $i, j \in \mathcal{L}$.

By $H^k(\Omega)$ we denote usual Sobolev spaces furnished with norms

$$\|u\|_{H^k(\Omega)} = \left\{ \sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_{L^2(\Omega)} \right\}^{1/2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $D^\alpha u = u_{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}$ and $H_0^1(\Omega) = \{u \in H^k(\Omega) \mid u|_{\Gamma^0} = 0\}$.

We now consider a problem on the polyhedral domain Ω

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u|_{\Gamma^0} = 0, & \\ \dfrac{\partial u}{\partial n}|_{\Gamma^1} = g, & \end{cases}$$

where $\Gamma^0 = \cup_{i \in \mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^1 = \cup_{i \in \mathcal{N}} \Gamma_i$ where $\mathcal{D}$ is a subset of $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \backslash \mathcal{D}$. We assume that f and g are analytic on $\bar{\Omega}$ and $\bar{\Gamma}^1$, respectively. Hence the singularity of the solution is caused solely by the unsmooth domain.

To illustrate effectively our main ideas, we consider here the Poisson equation with analytic data except the domain, but the regularity, the approximation results, and the algorithms presented in this paper are applicable to general elliptic problems with piecewise analytic data (see [9, 20, 21, 22, 23, 24]).

There exists a unique (weak) solution $u(x) \in H_0^1(\Omega)$ satisfying the variational equation

$$B(u, v) = F(v), \qquad \forall v \in H_0^1(\Omega) \tag{1}$$

where $B(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ and $F(v) = \int_\Omega f v dx + \int_{\Gamma^1} g v dS$ on $H_0^1(\Omega)$.

Let $S_N \subset H_0^1(\Omega)$ be a properly selected piecewise polynomial subspace, and u_N be the finite element solution in S_N satisfying

$$B(u_N, v) = F(v), \qquad \forall v \in S_N.$$

There holds the error estimates

$$\|u - u_N\|_{H^1(\Omega)} \le C \inf_{v \in S_N} \|u - v\|_{H^1(\Omega)}. \tag{2}$$

Therefore, the accuracy of the finite element solution depends solely on the preciseness of the description of the regularity of the solution, and the selection of the subspace S_N. To this end we shall decompose the domain Ω into various subregions, on which we introduce the countable normed spaces to precisely describe the regularity of the solution and its derivatives of all orders, and properly design the geometric mesh and P-Q distribution of the element degrees to achieve the exponential rate of convergence.

3 Regularity and approximation in neighborhoods of edges

We assume that the edge $\Lambda_{st} = \{x = (0,0,x_3) \mid a < x_3 < b\}$ lies on the x_3-axis, and introduce a neighborhood of the edge Λ_{st} shown in Fig. 2, $\mathbf{U} = \mathbf{U}_{\epsilon,\delta}(\Lambda_{st}) = \{x \in \Omega \mid 0 < r(x) = dist(x, \Lambda_{st}) < \epsilon, x_3 \in I_\delta = (a+\delta/2, b-\delta/2)\}$ where $\epsilon, \delta \in (0,1)$ are selected such that $\mathbf{U}_{\epsilon,\delta}(\Lambda_{st}) \cap \Gamma_\ell = \emptyset$ for $\ell \in \mathcal{I}$ and $\ell \neq s, t$.

By $\mathbf{C}^2_{\beta_{ij}}(\mathbf{U})$ we denote a countably normed space with weighted $\mathbf{C}^k$-norm which is a set of continuous functions $u(x)$ on $\bar{\mathbf{U}}$ such that for a real number $\beta_{ij} \in (0,1)$ and any α,

$$\|r(x)^{(\beta_{ij}+\alpha_1+\alpha_2-1)} D^\alpha(u(x) - u(0,0,x_3))\|_{\mathbf{C}^0(\bar{\mathbf{U}})} \leq C d^\alpha \alpha!$$

and

$$\|u_{x_3^{\alpha_3}}(0,0,x_3)\|_{\mathbf{C}^0(\bar{I}_\delta)} \leq C d_3^{\alpha_3} \alpha_3!$$

Hereafter $\alpha! = \Pi_{\ell=1}^3 \alpha_\ell!, d = (d_1, d_2, d_3)$ and $d^\alpha = \Pi_{\ell=1}^3 d_i^{\alpha_i}, C \geq 1$ and $d_\ell \geq 1$ are independent of α.

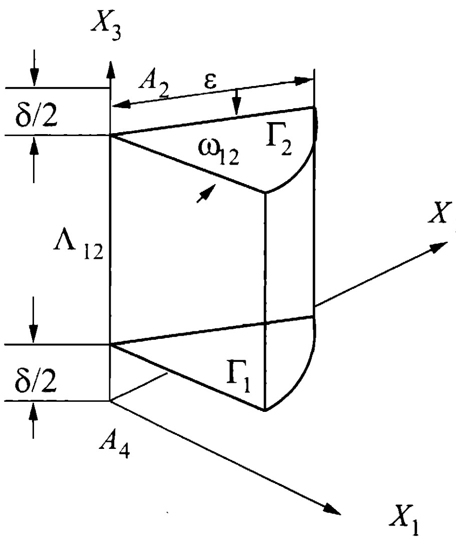

Fig. 2. Edge-Neighborhood $\mathbf{U}_{\epsilon,\delta}(\Lambda_{st})$

Theorem 3.1 *The (weak) solution $u(x)$ of (1) $\in \mathbf{C}^2_{\beta_{ij}}(\mathbf{U})$ with $\beta_{ij} \in (0,1)$ satisfying*

(3) $$\beta_{ij} \geq 1 - \kappa_{ij},$$

$$\kappa_{ij} = \begin{cases} \frac{\pi}{2\omega_{ij}} & \textit{if } \Gamma_i \subset \Gamma^0, \Gamma_j \subset \Gamma^1 \\ \frac{\pi}{\omega_{ij}} & \textit{otherwise.} \end{cases}$$

□

For the proof of the theorem we refer to [23].

Note that the space $\mathbf{C}^2_{\beta_{ij}}(\mathbf{U})$ is an anisotropic space which the solution belongs to. In the edge-neiborhood the solution behaves very differently in the direction parallel to the edge and the directions perpendicular to the edge. To achieve the best approximation by a piecewise polynomial we have to define mesh and element degrees accordingly.

First we divide the neighborhood $\mathbf{U} = \mathbf{U}_{\epsilon,\delta}(\Lambda_{st})$ into K levels along the edge with a uniform height H, which is not necessary to be small and will not be reduced when the mesh is refined. Then, according to the distance to the edge we divide $\mathbf{U}$ into n geometric layers. By $\Omega_{i,j,k}$ we denote an element in the i-th layer and the k-th level with $1 \leq j \leq J(i,k) \leq J$ (uniformly bounded with respect to i and k). The element $\Omega_{i,j,k}$ are hexahedral, or pentahedral, or tetrahedral, with h_l denoting the dimensions in the x_l directions, $l = 1,2,3$. Select a mesh factor $\sigma \in (0,1)$, the the geometric mesh $\mathbf{U}^n_\sigma = \{\Omega_{i,j,k},\ 1 \leq i \leq n, 1 \leq k \leq K, 1 \leq j \leq J(i,k)\}$ over the neighborhood $\mathbf{U}$ satisfying

(4) $$\begin{array}{l} h_1 = h_2 = c_1\sigma^{n-i}, \quad h_3 = H \approx 1; \\ dist(\Omega_{1,j,k}, \Lambda_{st}) = 0; \\ c_2\sigma^{n-i} \leq dist(\Omega_{i,j,k}, \Lambda_{st}) \leq c_3\sigma^{n-i} \\ \quad \text{for } 1 < i \leq n, \text{ and all } j, k. \end{array}$$

where $c_\ell, \ell = 1,2,3$ are some constants independent of i, j, k. A typical geometric mesh $\mathbf{U}^n_\sigma$ is shown in Fig. 3. For a precise description of the geometric mesh $\mathbf{U}^n_\sigma$ we refer to [6, 9, 20].

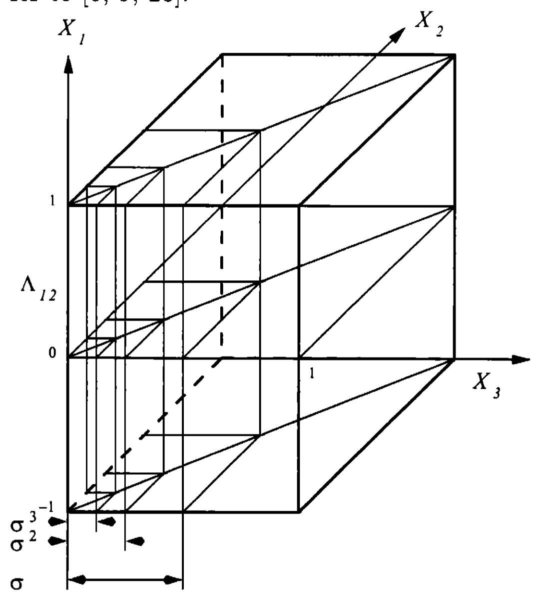

Fig. 3. Geometric Mesh $\mathbf{U}^n_\sigma$

We now define finite element space over the geometric mesh $\mathbf{U}_\sigma^n$

$$S^{P,Q}(\mathbf{U}_\sigma^n) = \{\phi(x) \mid \phi(x)|_{\Omega_{i,j,k}} = \phi_1(x_1) \oplus \phi_2(x_3), \\ \phi_1(x) \text{ is a polynomial of degree} p_{i,j,k}, \text{ and} \\ \phi_2(x_3) \text{ is a polynomial of degree } q_{i,j,k} \text{ in } x_3\}$$

and $S^{P,Q,1}(\mathbf{U}_\sigma^n) = S^{P,Q}(\mathbf{U}_\sigma^n) \cap H^1(\mathbf{U})$.

A linear P-distribution $\{p_{i,j,k},\ 1 \le i \le n, 1 \le k \le K, 1 \le j \le J(i,k)\}$ and a uniform Q-distribution $\{q_{i,j,k},\ 1 \le i \le n, 1 \le k \le K, 1 \le j \le J(i,k)\}$ should be associated with the geometric mesh $\mathbf{U}_\sigma^n$, with

$$p_{i,j,k} = [\mu i] \quad \text{for all } i,j,k \tag{5}$$

and

$$q_{i,j,k} = [\mu n] \quad \text{for all } i,j,k \tag{6}$$

where $\mu > 0$ is a degree factor. Hereafter [a] denotes the smallest positive integer $\geq$a. Then the combination of the geometric mesh and linear-uniform distribution of element degrees leads to the exponential convergence.

Theorem 3.2 *Let $u \in \mathbf{C}^2_{\beta_{st}}(\mathbf{U})$, and let $\mathbf{U}_\sigma^n$ be the geometric mesh defined by (4), and P-Q distribution be a linear-uniform distribution defined by (5,6) with the degree factor μ satisfying*

$$\mu > \frac{(1-\beta_{st})\ln(1/\sigma)}{\ln(1/F_H)} \tag{7}$$

where $F_H = \min\limits_{0\le\alpha\le1} \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}}(\alpha\bar{d}H)^\alpha$, $\bar{d} = \max\limits_i d_i$. The there exists $\phi(x) \in S^{P,Q,1}(\mathbf{U}_\sigma^n)$ such that

$$\|u - \phi(x)\|_{H^1(\mathbf{U})} \le ce^{-b_{st}N^{1/4}} \tag{8}$$

where $N = O(n^4)$ is the number of the degree of freedom of $S^{P,Q,1}(\mathbf{U}_\sigma^n)$, b_{st} depends on σ, μ and β_{st} but not on N.

For the proof of the theorem we refer to [9].

Remark 3.1 *Algorithms combining the geometric mesh and linear-uniform distribution of element degree achieve the optimal convergence and the efficiency of computations, because this combination reflect exactly the nature of singularity of the solution in the edge-neighborhood* $\mathbf{U}$. *Algorithms, associated with meshes which is not refined geometrically along the edge, is never able to reach the exponential convergence and computational efficiency. If a uniform P-distribution with $p_{i,j,k} = q_{i,j,k} = [\mu n]$ is associated with the geometric meshes, the exponential convergence may hold but with much smaller b_{st} in (8), which will severely affect the efficiency of the computation of finite element solution.*

Remark 3.2 *The design of long element $\Omega_{i,j,k}$ near the edge reflect the fact that the solution is analytic along the edge. The polynomial $\phi_2(x_3)$ of high degree $q_{i,j,k} = [\mu n]$ is used to approximate $u(0,0,x_3)$ effectively. Although the degree of $\phi_2(x_3)$ is higher than the degree of $\phi_1(x)$, the cost of computation for these polynomials $\phi_2(x)$ in x_3 direction is very minor, and it can be ignored comparing the cost of computations for those polynomials $\phi_1(x)$. An alternative approach to use of high-degree $\phi_2(x_3)$ is uniform refinement in the x_3-direction when the meshes are geometrically refined in the x_1-x_2 plane, but it will affect the efficiency of computation and significantly increase the cost of computation.*

4 Regularity and approximation in neighborhoods of vertex-edges

Let A_m be located in the origin. Then a neighborhood $\mathbf{O}_\delta(A_m)$ of the vertex A_m is defined by

$$\mathbf{O}_\delta(A_m) = \{x \in \Omega \mid 0 < \rho(x) = dist(x, A_m) < \delta\}$$

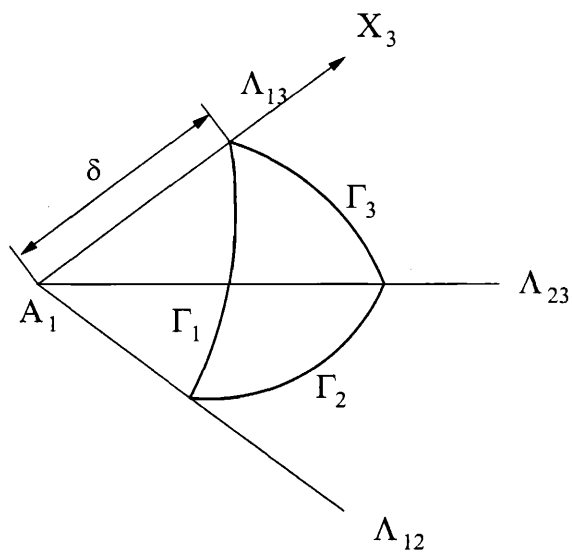

Fig. 4. Vertex-Neighborhood $\mathbf{O}_\delta(A_m)$

where $\delta \in (0,1)$ is selected such that $\bar{\mathbf{O}}_\delta(A_m) \cap \bar{\Gamma}_l = \emptyset$ for $l \notin \mathcal{I}_m$ (see Fig. 4). We further decompose $\mathbf{O}_\delta(A_m)$ into an inner neighborhood and several neighborhoods of vertex-edge. To this end we assume the edge $\Lambda_{st}, st \in \mathcal{L}_m$ lies in the positive x_3-direction, and introduce a vertex-edge neighborhood shown in Fig. 5

$$\mathbf{V} = \mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{ij}) = \{x \in \mathbf{O}_\delta(A_m),\ 0 < \phi < \sigma\}$$

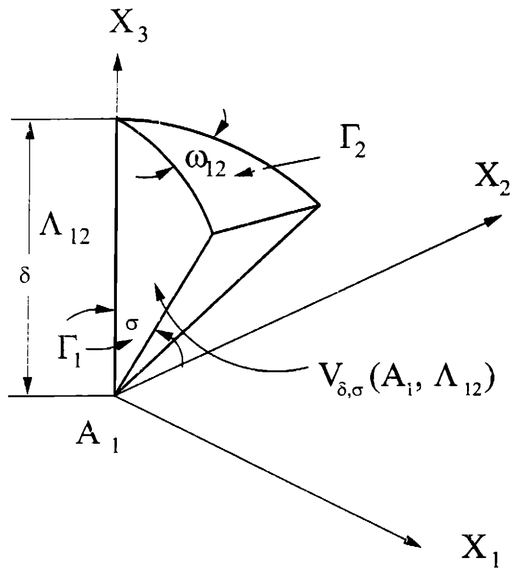

Fig. 5. Vertex-Edge Neighborhood $\mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{st})$

where ϕ is the angle between Λ_{st} and radical from A_m (the origin) to x, $0 < \delta, \sigma < 1$ such that $\bar{\mathbf{V}}_{\delta,\sigma}(A_m, \Lambda_{st}) \cap \bar{\mathbf{V}}_{\delta,\sigma}(A_m, \mathcal{L}_{k,l}) = A_m$ for any $st \in \mathcal{L}_m$ and $kl \in \mathcal{L}_m, st \neq kl$. (ρ, ϕ, θ) is the spherical coordinate with respect to A_m and Λ_{st}.

Similarly we introduce a countably normed space $\mathbf{C}^2_{\beta_{m,st}}(\mathbf{V})$ with weighted $\mathbf{C}^k$-norms, which is a set of continuous functions $u(x)$ on $\bar{\mathbf{V}}$ such that for a pair of real number $\beta_{m,st} = (\beta_m, \beta_{st}), 0 < \beta_m < 1/2, 0 < \beta_{st} < 1$ and for any α,

$$\|\Phi^{\alpha,2}_{\beta_{m,st}}(x) D^\alpha(u(x) - u(0,0,x_3))\|_{C^0(\bar{\mathbf{V}})} \leq c d^\alpha \alpha!$$

with $\Phi^{\alpha,2}_{\beta_{m,st}}(x) = \rho(x)^{\beta_m + |\alpha| - 1/2} (\sin\phi(x))^{\beta_{ij} + \alpha_1 + \alpha_2 - 1}$ and for $I_\delta = (0, \delta)$ and $\alpha_3 \geq 0$

$$\|\rho(x)^{\beta_m + \alpha_3 - 1/2}(u(0,0,x_3) - u(0,0,0))_{x_3^{\alpha_3}}\|_{C^0(\bar{I}_\delta)} \leq c d_3^{\alpha_3} \alpha_3!.$$

Theorem 4.1 *The (weak) solution $u(x)$ of (1) belongs to $\mathbf{C}^2_{\beta_{m,st}}(\mathbf{V})$ with $\beta_{st} \in (0,1)$ satisfying (4) and $\beta_m \in (0, 1/2)$ satisfying*

$$\beta_m > 1/2 - \lambda_m, \quad \lambda_m = \frac{1}{2}\sqrt{1 + 4\mu_1^{(m)}} - 1 \tag{9}$$

where $\mu_1^{(m)}$ is the smallest positive eigenvalue of the Laplace-Beltrami operator on the polygon S, which is a portion of the unit sphere subtended by an infinite cone which coincides with Ω in the neighborhood $\mathbf{O}_\delta(A_m)$. □

For the proof of the theorem we refer to [24].

The spaces $\mathbf{C}^2_{\beta_{m,st}}(\mathbf{V})$, like the spaces $\mathbf{C}^2_{\beta_{st}}(\mathbf{U})$, is an anisotropic space which the solution $u(x)$ belongs to. The solution behaves in this neighborhood very differently in the direction of the edge and in the direction perpendicular to the edge. Furthermore, note that $u(0,0,x_3)$ is no longer analytic, instead, it belongs to a countably normed space on an interval $I_\delta = (0, \delta)$. This features of the regularity of solution must be fully considered when we design the mesh and distribution of element degrees.

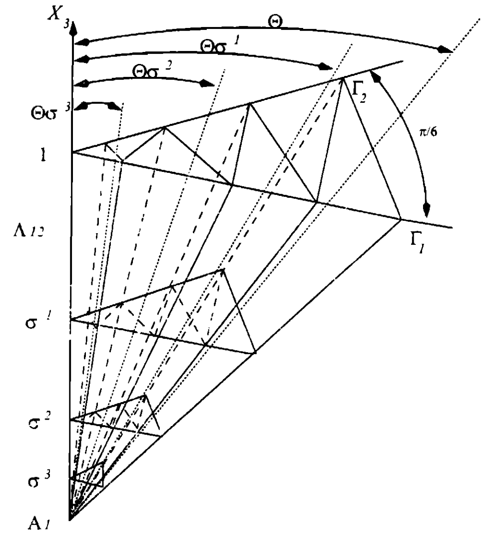

Fig. 6. Geometric Mesh $\mathbf{V}^n_\sigma$

The neighborhood $\mathbf{V}$ is divided into n geometric level according to the distance to the vertex A_m and n geometric layers according to angular distance to the edge Λ_{st}. The elements $\Omega_{i,j,k}$ located in i-th layer and k-th level with $1 \leq j \leq J(i,k) \leq J$ (uniformly bounded with respect to i, k) are hexahedral, or pentahedral or tetrahedral. Let $\sigma \in (0, 1)$ be a mesh factor, and let h_l denote the dimensions of element $\Omega_{i,j,k}$ in the x_l direction, $l = 1, 2, 3$. Then the geometric mesh $\mathbf{V}^n_\sigma = \{\Omega_{i,j,k},\ 1 \leq i \leq n, 1 \leq k \leq n, 1 \leq j \leq J(i,k)\}$ satisfies

$$\begin{array}{ll} h_1 \approx c_1\sigma^{2n-k-i}, \quad h_2 \approx c_1\sigma^{2n-k-i}, \quad h_3 \approx c_2\sigma^{n-k}; & \\ dist(\Omega_{i,j,1}, A_m) = 0, \quad dist^*(\Omega_{1,j,k}, \Lambda_{st}) = 0; & \\ dist(\Omega_{i,j,k}, A_m) = c_3\sigma^{n-k}, & \text{for } k > 1 \text{ and all } i, j; \\ dist^*(\Omega_{i,j,k}, \Lambda_{st}) = c_4\sigma^{n-i}, & \text{for } i > 1 \text{ and all } k, j; \end{array} \tag{10}$$

where $dist^*(\Omega_{i,j,k}, \Lambda_{st}) = \min_{x \in \bar{\Omega}_{i,j,k}} \phi(x) \approx \min_{x \in \bar{\Omega}_{i,j,k}} \sin\phi(x)$ is an angular distance between the element $\Omega_{i,j,k}$ and Λ_{st}. A geometric mesh $\mathbf{V}^n_\sigma$ is shown in Fig. 6. For the precise description of geometric mesh on $\mathbf{V}$ we refer to [6, 9, 20].

The corresponding finite element spaces over the geometric mesh $\mathbf{V}^n_\sigma$ is defined as

$$\begin{aligned} S^{P,Q}(\mathbf{V}^n_\sigma) = \{\phi(x) \mid\ & \phi(x)|_{\Omega_{i,j,k}} = \phi_1(x) + \phi_2(x_3), \\ & \phi_1(x) \text{ is a polynomial of degree } p_{i,j,k} \text{ and} \\ & \phi_2(x_3) \text{ is a polynomial of degree } q_{i,j,k} \text{ in } x_3\} \end{aligned}$$

and $S^{P,Q,1}(\mathbf{V}_\sigma^n) = S^{P,Q}(\mathbf{V}_\sigma^n) \cap H^1(\mathbf{V})$.

A bilinear P-distribution $\{p_{i,j,k}\ 1 \le i \le n, 1 \le k \le n, 1 \le j \le J(i,k)\}$ and a linear Q-distribution $\{q_{i,j,k}\ 1 \le i \le n, 1 \le k \le n, 1 \le j \le J(i,k)\}$ should be associated with the geometric mesh $\mathbf{V}_\delta^n$ with

$$p_{i,j,k} = [\mu i + \nu k - p_0] \quad \text{for all } i,j,k \tag{11}$$

and

$$q_{i,j,k} = [\nu k] \quad \text{for all } i,j,k \tag{12}$$

where $p_0 \ge 0$ is a properly selected integer, $\mu, \nu > 0$ are the degree factors.

Theorem 4.2 *Let $u \in \mathbf{C}^2_{\beta_{m,st}}(\mathbf{V})$, and let $\mathbf{V}_\sigma^n$ be the geometric mesh defined by (10) and P-Q distribution be bilinear-linear defined by (11) and (12) with μ and ν satisfying*

$$\mu > \frac{(1-\beta_{st})\ln(1/\sigma)}{\ln(1/F_1)}$$

and

$$\nu > \frac{(1-\beta_m)\ln(1/\sigma)}{\ln(1/F_1)} \tag{13}$$

where F_1 is the value of F_H at $H = 1$ given in (7). Then, there exists a $\phi(x) \in S^{P,Q,1}(\mathbf{V}_\sigma^n)$ such that

$$\|u - \phi(x)\|_{H^1(\mathbf{V})} \le ce^{-b_{m,st}N^{1/5}} \tag{14}$$

where $N = O(n^5)$ is the number of degree of freedom, $b_{m,st}$ depends on $\sigma, \beta_m, \beta_{st}, \mu$ and ν, but not on N. □

For the proof of the theorem, we refer to [9].

Remark 4.1 *The geometric meshes $\mathbf{V}_\sigma^n$ are refined both in ρ and in ϕ, and P-distribution is bilinear with respect to the layer number i and level number k, because the solution possesses two different types of singularities: edge-singularity and vertex-singularity in the neighborhood $\mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{st})$, which have been completely exposed in Theorem 4.1. The exponential convergence and the efficiency of computation are achieved only by those algorithms using a proper combination of geometric meshes and bilinear-linear distribution of element degrees.*

Remark 4.2 *Asymptotically the exponential rate with respect to $N^{1/5}$ is the best approximation result we can prove, due to the refinement in two directions. But for practical range of n e.g., $n < 10$, the exponential rate with respect to $N^{1/4}$ is possible by select suitable p_0, e.g., $p_0 = n$. For details, see [9].*

Remark 4.3 *The solution along the edge is not analytic, instead, $u(0,0,x_3)$ belongs to a countably normed space over an interval. Hence, in order to achieve the exponential convergence and efficiency of computation, the refinement in x_3-direction has to be carried out geometrically, and a linear distribution of element degrees has to be adopted for the polynomials $\phi_2(x_3)$ so that $u(0,0,x_3)$ can be approximated effectively. The increase of the number of degree of freedom and the computational cost for $\phi_2(x_3)$ is very minor and ignorable comparing the total number of degree of freedom and the total cost of computation of the finite element solution.*

5 Regularity and approximation in inner neighborhoods of vertices

We define an inner neighborhood of the vertex A_m by excluding all vertex-edge neighborhoods $\mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{st})$, $st \in \mathcal{L}_m$, $\tilde{\mathbf{O}} = \tilde{\mathbf{O}}_\delta(A_m) \setminus \cup_{st\in\mathcal{L}_m} \mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{st})$, which is shown in Fig. 7. We assume that A_m is located in the origin.

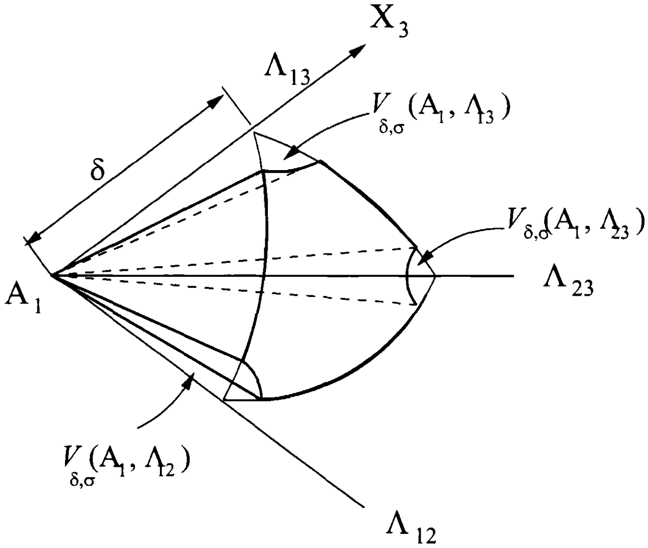

Fig. 7. Inner Vertex-Neighborhood $\tilde{\mathbf{O}}_\delta(A_m)$

The countably normed space $\mathbf{C}^2_{\beta_m}(\tilde{\bar{\mathbf{O}}})$ is defined as a set of continuous functions $u(x)$ on $\tilde{\bar{\mathbf{O}}}$ such that for a real number $\beta_m \in (0, 1/2)$ and any α

$$\|\rho(x)^{\beta_m+|\alpha|-1/2} D^\alpha(u(x) - u(0,0,0))\|_{\mathbf{C}^0(\tilde{\bar{\mathbf{O}}})} \le cd^\alpha \alpha!. \tag{15}$$

In this neighborhood the solution possesses only vertex-singularity.

Theorem 5.1 *The (weak) solution of (1) belongs to* $\mathbf{C}^2_{\beta_m}(\tilde{\mathbf{O}})$ *with* $\beta_m \in (0, 1/2)$ *satisfying (9).* □

For the proof of the theorem we refer to [24].

Unlike the space $\mathbf{C}^2_{\beta_{st}}(\mathbf{U})$ and $\mathbf{C}^2_{\beta_{m,st}}(\mathbf{V})$, the space $\mathbf{C}^2_{\beta_m}(\tilde{\mathbf{O}})$ is an isotropic space, the solution is singular in ρ, but behaves equally in the x_ℓ-direction, $\ell = 1, 2, 3$. Due to this character of the singularity of the solution, the mesh is refined in one direction, i.e., in ρ. According to the distance to the vertex A_m we divide $\tilde{\mathbf{O}}_\delta(A_m)$ into n levels. The elements in the k-th level with h_ℓ being the dimensions in the x_ℓ direction, $\ell = 1, 2, 3, 1 \le j \le J(k) \le J$ (uniformly bounded with respect to k), denoted by $\Omega_{j,k}$ are hexahedral, or pentahedral, or tetrahedral. A geometric mesh $\tilde{\mathbf{O}}^n_\sigma = \{\Omega_{j,k}, 1 \le k \le n, 1 \le j \le J(k)\}$ with a mesh factor $\sigma \in (0, 1)$ satisfying

$$(16)\quad \begin{aligned} & h_1 \approx h_2 \approx h_3 \approx c_1\sigma^{n-i}; \\ & dist(\Omega_{j,1}, A_m) = 0, \\ & dist(\Omega_{j,k}, A_m) = c_2\sigma^{n-i}, \quad \text{for } k > 1 \text{ and all } j. \end{aligned}$$

A geometric mesh $\tilde{\mathbf{O}}^n_\sigma$ is shown in Fig. 8. For the precise description on the geometric mesh over $\tilde{\mathbf{O}}_\delta(A_m)$ we refer to [6, 9, 20].

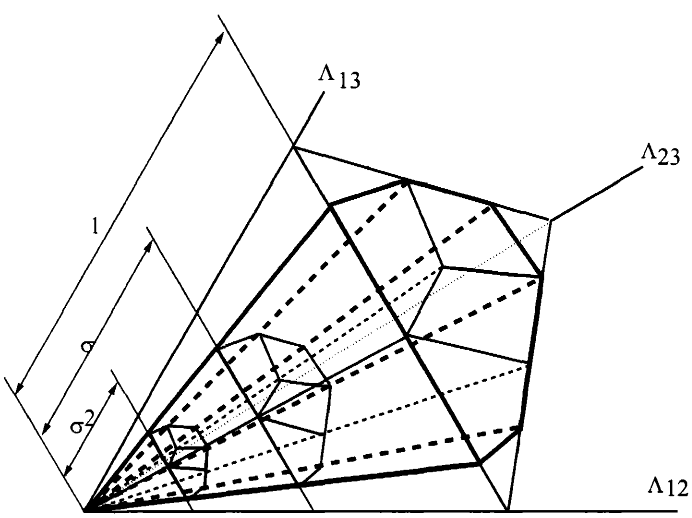

Fig. 8. Geometric Mesh $\tilde{\mathbf{O}}^n_\sigma$

We define a finite element space over $\tilde{\mathbf{O}}^n_\sigma$ by

$$S^P(\tilde{\mathbf{O}}^n_\sigma) = \{\phi(x) \mid \phi(x)|_{\Omega_{j,k}} \text{ is a polynomial of degree } p_{j,k}\}$$

and $S^{P,1}(\tilde{\mathbf{O}}^n_\sigma) = S^P(\tilde{\mathbf{O}}^n_\sigma) \cap H^1(\tilde{\mathbf{O}})$.

The character of singularity of the solution in the inner vertex-neighborhood is also reflected in the designing of element degrees. Only a linear P-distribution $\{p_{j,k}, 1 \le k \le n, 1 \le j \le J(k)\}$ is needed with

$$p_{j,k} = [\nu k], \quad \text{for all } j, k.$$

where $\nu > 0$ is the degree factor, because the solution behaves equally in all x_ℓ-direction and geometric mesh is refined in ρ.

The algorithms based on the geometric mesh and linear P-distribution achieve the exponential convergence.

Theorem 5.2 *Let* $u \in \mathbf{C}^2_{\beta_m}(\tilde{\mathbf{O}})$ *and let* $\tilde{\mathbf{O}}^n_\sigma$ *be the geometric mesh defined by (15) and the linear* P*-distribution be defined by (16) with* ν *satisfying (13). Then there exists a* $\phi(x) \in S^{P,1}(\tilde{\mathbf{O}}^n_\sigma)$ *such that*

$$(17)\qquad \|u - \phi(x)\|_{H^1(\tilde{\mathbf{O}})} \le ce^{-b_m N^{1/4}}$$

where $N = O(n^4)$ *is the number of degree of freedom,* b_m *depends on* β_m, ν *and* σ*, but not on* N. □

For the proof of the theorem we refer to [9].

6 Regularity and approximation of the *h-p* finite element solution on polyhedral domain

Let $\Omega_0 = \Omega \setminus \cup_{m\in\mathcal{M}}\mathbf{O}_\delta(A_m) \setminus \cup_{st\in\mathcal{L}} \cup_{\epsilon,\delta} (\Lambda_{st})$. Select δ and ϵ properly, Ω_0 will contain no edges and vertecies of the domain. Ω_0 is called the regular region in which the solution is analytic.

Theorem 6.1 *The (weak) solution of (1) is analytic on* $\bar{\Omega}_0$*, and for any* α

$$\|D^\alpha u(x)\|_{C^0(\bar{\Omega}_0)} \le cd^\alpha \alpha!$$

□

For the proof we refer to [24].

By β we denote a multi-index $(\beta_m, \beta_{st}, m \in \mathcal{M}, st \in \mathcal{L})$, with $\beta_m \in (0, 1/2)$ and $\beta_{st} \in (0, 1)$, and by $\mathbf{C}^2_\beta(\Omega)$ we define a countably normed space with weighted $\mathbf{C}^k$- norm, namely, for $u \in \mathbf{C}^2_\beta(\Omega)$, there hold

i) $u \in \mathbf{C}^0(\bar{\Omega})$;
ii) $u\, |_{\mathbf{U}_{\epsilon,\delta}(\Lambda_{st})} \in \mathbf{C}^2_{\beta_{st}}(\mathbf{U}_{\epsilon,\delta}(\Lambda_{st}))$;
ii) $u\, |_{\tilde{\mathbf{O}}_\delta(A_m)} \in C^2_{\beta_m}(\tilde{\mathbf{O}}_\delta(A_m))$;
ii) $u\, |_{\mathbf{V}_{\delta,\sigma}(A_m,\Lambda_{st})} \in C^2_{\beta_{m,st}}(\mathbf{V}_{\delta,\sigma}(A_m, \Lambda_{st}))$;
ii) $u\, |_{\Omega_0} \in \mathbf{C}^\infty(\bar{\Omega}_0)$, and $|D^\alpha u(x)| \le cd^\alpha \alpha!$.

Combining Theorem 3.1, 4.1, 5.1 and 6.1, we now have the regularity of the solution on whole polyhedral domain.

Theorem 6.2 *The (weak) solution $u(x)$ of (1) belongs to $\mathbf{C}^2_\beta(\Omega)$ with $\beta_m \in (0, 1/2), m \in \mathcal{M}$ and $\beta_{st} \in (0,1), st \in \mathcal{L}$ satisfying (4) and (9).* □

Remark 6.1 *The regularity results can be given in terms of countably normed spaces with other types of weighted norm and in various coordinate systems, for instance, the spaces $\mathbf{B}^2_\beta(\Omega)$ with weighted Sobolev norms. For the purpose of numerical approximation we prefer to the description in terms of the countably normed spaces with weighted $\mathbf{C}^k$-norms. The regularity theorems in these countably normed spaces and the relations of these spaces have been given in [22].*

Remark 6.2 *The regularity of solution on the polyhedral domain is completely described by the local asymptotic expansion of singular functions, which contains the power of $\rho(x), r(x)$ and $\sin(\phi(x))$ (see[14, 15, 17, 18, 30, 32, 33, 36, 37]). If the power and forms of leading singular functions are known, special elements can be constructed. The algebraic rate of convergence of the conventional finite element solutions may be improved because of the special elements, but the singular basis functions will destroy the nice band structure of the stiffness matric and deteriorate the condition number. Moreover, in practice the strength of singularity and form of singular functions are unknown, additional efforts and computations for these information are required (see [12]). Hence practically and theoretically the h-p finite element solution are the only effective and efficient numerical approach for the problems on nonsmooth domains, and the regularity theory in the frame of countably normed spaces is the only theory which is able to effectively guide the computational practice and to lead to the exponent rate of the convergence.*

We next design the *h-p* version finite element algorithms on whole polyhedral domain.

Since the solution $u(x)$ is analytic on $\bar{\Omega}_0$, we use a fixed and coarse mesh $\Omega^0_H = \{\Omega^0_\ell,\ 1 \le \ell \le L\}$. The elements Ω^0_ℓ's are hexahedral, pentahedral, or tetrahedral with $h_\ell \approx$ H ≈ 1 being their dimensions in the x_ℓ-directions, $1 \le \ell \le 3$. A uniform P-distribution $\{p_\ell = p,\ 1 \le \ell \le L\}$ with p coinciding with the highest degree used in elements in singular neighborhoods, is associated with the mesh Ω^0_H, and the finite element space is defined by

$$S^P(\Omega^0_H) = \{\phi(x) \mid \phi(x)|_{\Omega^0_\ell} \text{ is a polynomial of degree } p_\ell = p\}$$

and $S^{P,1}(\Omega^0_H) = S^P(\Omega^0_H) \cap H^1(\Omega_0)$.

Note that the accuracy of the finite element solution in this region is achieved by uniformly increasing the polynomial degree, but not by reducing the element sizes. It is well-known that for the analytic function u(x) on $\bar{\Omega}_0$, there is a polynomial $\phi(x) \in S^{P,1}(\Omega^0_H)$ such that

$$\|u - \phi(x)\|_{H^1(\Omega_0)} \le ce^{-b_0 N^{1/3}} \tag{18}$$

where N is the number of degree of freedom, b_0 is independent of N.

Let Ω^n_σ be the union of all geometric meshes $\mathbf{U}^n_\sigma, \tilde{\mathbf{O}}^n_\sigma, \mathbf{V}^n_\sigma$ and uniform mesh Ω^0_H, and let P-Q distribution of element degree on Ω^n_σ be the union of the linear-uniform P-Q distributions on $\mathbf{U}^n_\sigma$'s, the bilinear-linear P-Q distributions on $\mathbf{V}^n_\sigma$'s, the linear P-distribution on $\mathbf{O}^n_\sigma$'s and the uniform P-distribution on Ω^0_H. Further, by $S^{P,Q}(\Omega^n_0)$ we denote the finite element space over Ω^n_σ of piecewise polynomials whose restrictions on $\mathbf{U}^n_\sigma, \tilde{\mathbf{O}}^n_\sigma, \mathbf{V}^n_\sigma$ and Ω^0_H belong to the spaces $S^{P,Q}(\mathbf{U}^n_\sigma)$, $S^{P,Q}(\mathbf{V}^n_\sigma)$, $S^P(\tilde{\mathbf{O}}^n_\sigma)$ and $S^P(\Omega^0_H)$, respectively, and $S^{P,Q,1}(\Omega^n_\sigma) = S^{P,Q}(\Omega^n_\sigma) \cap H^1_0(\Omega)$.

We now come to the conclusion of the approximation of the *h-p* finite element solution.

Theorem 6.3 *Let the geometric mesh Ω^n_σ and the P-Q distribution associated with Ω^n_σ defined above. Then the h-p finite element solution $u_N \in S^{P,Q,1}(\Omega^n_\sigma)$ converges to the solution $u(x)$ of (1) exponentially,*

$$\|u - u_N\|_{H^1(\Omega)} \le ce^{-bN^{1/5}} \tag{19}$$

where N is the number of degree of freedom of $S^{P,Q,1}(\Omega^n_\sigma)$, b depends on β, σ, μ, ν, but not on N.

Proof Due to the definition of $S^{P,Q,1}(\Omega^n_\sigma)$ and the combination of (8), (14), (17) and (18), there exists $\phi(x) \in S^{P,Q,1}(\Omega^n_\sigma)$ such that

$$\|u - \phi(x)\|_{H^1(\Omega)} \le ce^{-bN^{1/5}}$$

where $b = \min\{b_{st}, b_m, b_{m,st}, b_0\}$ depending on β, σ, μ, ν but not on N, which together with (2) leads to (19) □

Remark 6.3 *Asymptotically the exponential rate with respect to $N^{1/5}$ is the best accuracy of finite element solution which we can prove. Although the rate is with respect to $N^{1/4}$ in neighborhoods of edges and inner neiborhoods of vertices, and $N^{1/3}$ in regular region Ω_0, the majority of the number of degree of freedom is concentrated in the neighborhoods of vertex-edges. Hence, as a total performance, the h-p finite element solution converges at the exponential rate with respect to $N^{1/5}$. In computational practices, if the integer p_0 in the bilinear P-distribution is properly selected and the number of layers and levels are not large, then the exponential rate with respect to $N^{1/4}$ is possibly achievable, as mentioned in Remark 4.3, and also as seen in practical computations.*

7 Numerical example

A benchmark elasticity problem on a polyhedral domain Ω with the modulus $E = 100$ and Poisson ratio $\nu = 0.3$, shown is Fig. 9, is computed by the h, p and h-p version of finite element method. Let $u = (u_1, u_2, u_3)$ be the displacement and $T = (T_1, T_2, T_3)$ be the traction on the boundary. The following boundary conditions are imposed to this problem:

i) On the faces ACNE, ABDC, AEFB, IJLM, the traction T=(T_1, T_2, T_3)=0;
ii) On the face DKLMNC, the displacement $u_1 = 0$, the traction $T_2 = T_3 = 0$;
iii) On the face NMJHFE, the displacement $u_2 = 1$, the traction $T_1 = T_3 = 0$;
iv) On the face KGIL, the displacement $u_2 = 0$, the traction $T_1 = T_3 = 0$;
v) On the face GHIJ, the traction $T_1 = 1, T_2 = T_3 = 0$;
vi) On the face KDBFHG, the displacement $u_3 = 0$, the traction $T_1 = T_2 = 0$.

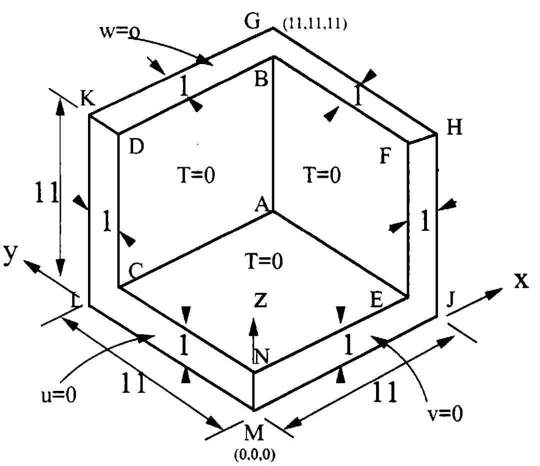

Fig. 9. Polyhedral Domain Ω

The h-version:

Uniform meshes of cubic elements with size $h = 1, 1/2, 1/3, 1/4$, shown in Fig. 10, are used with the uniform degree $p = 2$. Table 1 shows the relative error in energy norm $\approx 9\%$ when $h = 1/4$ and $N \approx 300,000$ and CPU time $t_1(s) = 29,800sec$ (single processor).

h	No. of Elm	N	$\|e\|_{E,R}\%$	$t_1(s)$
1	331	6708	20.53	120
1/2	2648	42679	13.32	1890
1/3	8937	131744	10.56	10130
1/4	21184	297735	8.99	29800

Table 1 Performance of the h-version ($p = 2$)

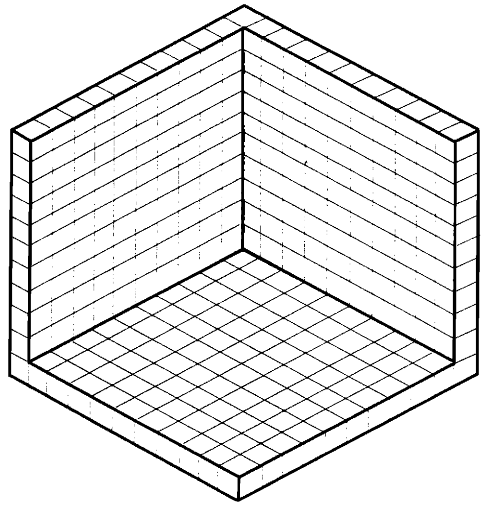

Fig. 10. Uniform Mesh with $h = 1$

The p-version:

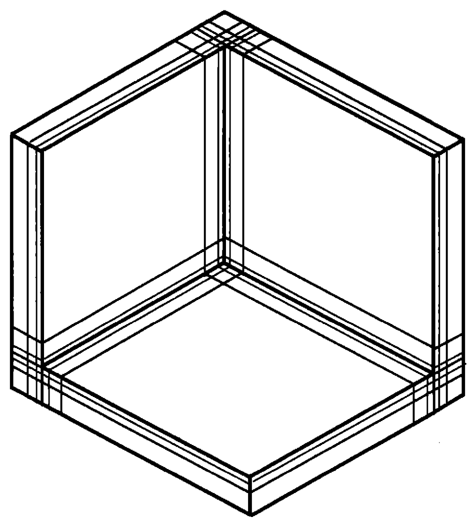

Fig. 11. Geometric Mesh $\Omega_\sigma^{n_1,n_2}$ with $n_1 = 2, n_2 = 1$

The finite element solutions of the p-version are computed on various geometric meshes. The geometric meshes of tensor product type are used, namely the meshes are heavily refined near the vertex A with layer number n_1 and slightly refined at other vertices with layer number n_2, because the singularity of solution near A is severe. The mesh factor $\sigma = 0.15$. A geometric mesh $\Omega_\sigma^{n_1,n_2}$ with $n_1 = 2, n_2 = 1$ is shown in Fig. 11. The degree p of element uniformly increases from 1 to 10. The performance of the p-version on Mesh 3 with $n_1 = 3$ and $n_2 = 2$ is given in Table 2. The relative error in energy norm reduces to 3% when $p = 7$ and CPU time $t_1 = 3305sec$ (single processor). The p-version perform on the geometric mesh much better than the h-version, but on quasi-uniform mesh, the p-version converges only as twice fast as the h-version. We will make more comments later.

p	N=DOF	$\|e\|_{E,R}\%$	$t_1(s)$
2	2925	43.84	33
3	5135	19.28	56
4	9328	8.21	169
5	15549	5.04	481
6	24404	3.78	1308
7	36496	3.01	3305
8	52428	2.49	7408
9	72803	2.11	14860
10	98224	1.84	31842

Table 2 Performance of the p-version (on Mesh 3, $n_1 = 3$, $n_2 = 2$)

The h-p version:

The tensor product meshes with the following combination of n_1 and n_2 are used:

Mesh 1: $n_1 = 1, \quad n_2 = 2;$
Mesh 3: $n_1 = 3, \quad n_2 = 2;$
Mesh 5: $n_1 = 5, \quad n_2 = 2;$
Mesh 7: $n_1 = 7, \quad n_2 = 2;$

Mesh(n_1)	p	N=DOF	$\|e\|_{E,R}\%$	$t_1(s)$
1	4	464	31.39	1
3	5	15549	5.04	481
3	6	24404	3.78	1308
5	7	67567	1.84	13337
5	8	97374	1.50	29392

Table 3 Performance of the h-p version (on Mesh 1,3,5, and uniform degree $p = \mu n_1$)

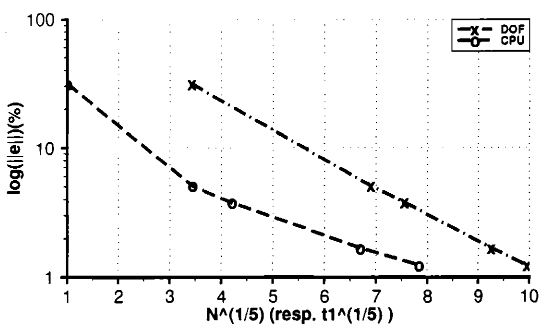

Fig. 12. Performance of the h-p Version

Associated with a uniform P-distribution with $p = [\mu n_1], \mu = 1.5 \sim 2.0$. The Table 3 shows that the 1.84% of relative error in energy norm is achieved at $n_1 = 5, n_2 = 2$ and $p = 7, N = 67,567$, and the CPU time $t_1 = 13,337 sec$. On the other hand, the p-version on Mesh 3 needs $p = 10, N = 98,224$ CPU time $t_1 = 31,842$.

The relative error of the h-p version in energy norm $\|e\|_{E,R}$ v.s. $N^{1/5}$ and t_1 are plotted in Fig. 12. The comparison between the h, p and h-p versions are shown in Fig. 13 and Fig. 14 where the relative error v.s. N and CPU time are plotted in log-log scales.

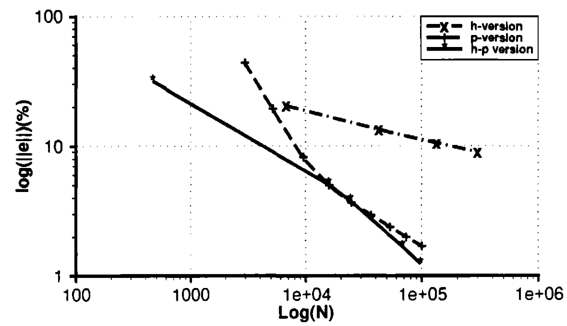

Fig. 13. Comparison between h, p and h-p versions: Error v.s. DOF

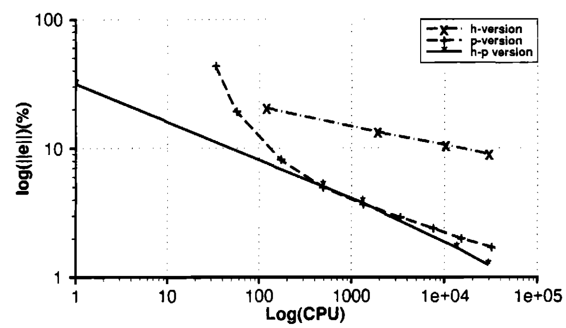

Fig. 14. Comparison between h, p and h-p versions: Error v.s. CPU

Remark 7.1 *The computations on this benchmark problem has shown that the h-p finite element solution can achieve the exponential rate in engineering practical range, e.g., 3%, as predicted by asymptotic analyses.*

Remark 7.2 *The computation of the h-p version in* $\mathbf{R}^3$ *have not been exhaustive enough, more experience in computations and implementations are needed, e.g., how to generate a geometric mesh, how to implement the P-Q distribution of element degrees (no existing 3-dimensional code has this feature yet), and what are the optimal mesh factors and degree factors, etc..*

Remark 7.3 *The computation has shown that the p-version on an over-refined geometric mesh converges to the desired accuracy exponentially before entering the asymptotic phase, it could be an alternative to the h-p version* *in R^3. The practical strategy to achieve the optimal convergence and efficiency for engineering applications in $\mathbf{R}^3$ needs to be developed further.*

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