PRIME IDEALS OF FINITE HEIGHT IN POLYNOMIAL RINGS

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ABSTRACT. We investigate the structure of prime ideals of finite height in polynomial extension rings of a commutative unitary ring $R$. We consider the question of finite generation of such prime ideals. The valuative dimension of prime ideals of $R$ plays an important role in our considerations. If $X$ is an infinite set of indeterminates over $R$, we prove that every prime ideal of $R[X]$ of finite height is finitely generated if and only if each $P \in \text{Spec}(R)$ of finite valuative dimension is finitely generated and for each such $P$ every finitely generated extension domain of $R/P$ is finitely presented. We prove that an integrally closed domain $D$ with the property that every prime ideal of finite height of $D[X]$ is finitely generated is a Prüfer $v$-multiplication domain, and that if $D$ also satisfies d.c.c. on prime ideals, then $D$ is a Krull domain in which each height-one prime ideal is finitely generated.

1. INTRODUCTION

All rings considered in this paper are assumed to be commutative and to contain a unity element. Suppose $X = \{x_i\}_{i=1}^{\infty}$ is a countably infinite set of indeterminates over a Noetherian ring $R$ and $T$ is a localization of $R[X]$ with respect to a multiplicatively closed set of $R[X]$. (In particular, we are including the case where $T = R[X]$.) It is readily seen that a prime ideal of $T$ is finitely generated if and only if it is of finite height (cf. [8, Theorem 4, page 2]). In relation to this result, it is shown in [9, Theorem 3.3] that an ideal $c$ of $T$ is finitely generated if and only if $c$ has only finitely many associated prime ideals and each of the associated prime ideals of $c$ is finitely generated. Moreover, if this occurs, then $c$ has a finite primary decomposition.

Motivation for our work in the present paper comes from the following specific questions concerning a converse to the finite generation result.
Question 1.1. Suppose \( X = \{x_i\}_{i=1}^{\infty} \) is a countably infinite set of indeterminates over a ring \( R \).

1. If every prime ideal of \( R[X] \) of finite height is finitely generated, does it follow that every prime ideal of \( R \) of finite height is finitely generated?

2. Assume that each prime ideal of \( R \) has finite height. If each prime ideal of \( R[X] \) of finite height is finitely generated, does it follow that \( R \) is Noetherian?

We do not know the answer, in general, to either part of Question 1.1. For ease of reference in considering (1.1), we use the following terminology; here FH stands for finite height.

**Definition.** Suppose \( X = \{x_i\}_{i=1}^{\infty} \) is a countably infinite set of indeterminates over a ring \( R \). We say that \( R \) is an FH-ring if every prime ideal of \( R[X] \) of finite height is finitely generated.

The concept of valuative dimension is important in the consideration of Question 1.1. We recall that if \( D \) is an integral domain with quotient field \( K \), then the valuative dimension of \( D \), denoted \( \dim_v D \), is the positive integer \( h \) if there exists a valuation overring \(^1\) of \( D \) of rank \( h \) and no valuation overring of \( D \) of rank greater than \( h \). If there exist valuation overrings of \( D \) of rank greater than \( h \) for every positive integer \( h \), then \( D \) is said to have valuative dimension \( \infty \). The valuative dimension of a commutative ring \( R \) is defined to be the supremum of the valuative dimensions of domain homomorphic images of \( R \) [11, page 56]. For \( P \in \text{Spec}(R) \), the valuative dimension of \( P \) is \( \dim_v R_P \).

In general, for \( D \) an integral domain and \( P \in \text{Spec}(D) \), \( \dim_v D/P \) is at most \( \dim_v D - \text{dim} \ D_P \) [11, Prop. 2, page 57]. Since one also has \( \text{dim} \ D \leq \dim_v D \) [11, Théorème 1, page 56], \( \dim_v D/P \) is at most \( \dim_v D - \text{ht} \ P \). A summary of some basic properties of valuative dimension is given in [5, page 36]. An important property for us is:

**Observation 1.2.** If \( P \in \text{Spec}(R) \) has finite valuative dimension \( h \), where \( h \) is also the height of \( P \) (so \( \text{dim} R_P = \dim_v R_P \)), then for \( X \) a set of indeterminates over \( R \), the height of \( PR[X] \) in \( R[X] \) is also \( h \) (cf. [11, Théorème 3, page 62]).

**Discussion 1.3.** 1. In view of Cohen's theorem that a ring is Noetherian if every prime ideal of the ring is finitely generated [14, (3.4)], an affirmative

\(^1\)By an overring of an integral domain \( D \) with quotient field \( K \) we mean a subdomain of \( K \) that contains \( D \).
answer to part (1) of (1.1) implies that the answer to part (2) of (1.1) is also affirmative.

2. Suppose $P$ is a prime ideal of $R$ and $Y$ is a set of indeterminates over $R$. Then $Q = PR[Y]$ is a prime ideal of $S = R[Y]$. Since $S$ is a free $R$-module, it is readily seen that $Q$ is finitely generated in $S$ if and only if $P$ is finitely generated in $R$. Moreover, if $Y = \{y_1, \ldots, y_n\}$ is a finite set and $P$ has finite height, then $Q$ also has finite height. Indeed, if $P$ has height $h$, then the height of $PR[y_1]$ is at least $h$ and at most $2h$ (cf. [6, (30.2)]). Therefore if the set $Y$ is finite, then $Q = PR[Y]$ has finite height if $P$ has finite height and the question analogous to (1.1) for a finite set of indeterminates has an affirmative answer.

3. In the setting of (1.1), it is possible that there exists in $R$ a prime ideal $P$ having finite height such that $Q = PR[X]$ has infinite height in $R[X]$. Indeed, if $R$ is an integral domain, then $Q = PR[X]$ has infinite height precisely if the domain $R_P$ has infinite valuative dimension (cf. [6, page 360], [11, page 63]).

Suppose $R$ is an FH-ring and $Y$ is a set of indeterminates over $R$. Is every prime ideal of $R[Y]$ of finite height also finitely generated? We show in (1.4) below that this question has an affirmative answer if $Y$ is infinite. On the other hand, if $Y$ is finite, we show in (1.5) that an affirmative answer to this question is equivalent to an affirmative answer to Question 1.1.

**Proposition 1.4.** Suppose $R$ is an FH-ring and $Y$ is an arbitrary infinite set of indeterminates over $R$. Then each prime ideal of $R[Y]$ of finite height is finitely generated.

**Proof.** Let $P$ be a prime ideal of $R[Y]$ of finite height $h$ and let $P_0 < P_1 < \cdots < P_h = P$ be a chain of prime ideals of $R[Y]$ of length $h$ with terminal element $P$. Choose a polynomial $f_i \in P_i - P_{i-1}$ for $i = 1, 2, \ldots, h$. There exists a finite subset $\{y_i\}_{i=1}^n$ of $Y$ such that each $f_j \in R[y_1, \ldots, y_n]$. It follows that $P \cap R[y_1, \ldots, y_n]$ has height at least $h$. Extend $\{y_i\}_{i=1}^n$ to a countably infinite subset $Y'$ of $Y$. Then $P \cap R[Y']$ has height at least $h$, $P' = (P \cap R[Y'])R[Y] \subseteq P$ has height at least $h$, and hence $P = (P \cap R[Y'])R[Y]$. It follows that $P \cap R[Y']$ has height $h$. Since $R$ is an FH-ring, $P \cap R[Y']$ is finitely generated. Consequently, $P$ is finitely generated.

**Observation 1.5.** Suppose $x$ is an indeterminate over a ring $R$. As noted in part (2) of (1.3), a prime ideal $P$ of $R$ is finitely generated if and only if $Q = PR[x]$
is finitely generated in \( R[x] \), and \( Q \) has finite height if \( P \) has finite height. Thus if \( Y \) is a finite set of indeterminates over \( R \), and if every prime ideal of \( R[Y] \) of finite height is finitely generated, then \( R \) also has this property. The converse, however, is not true. There exists an integral domain \( R \) having the property that there exists in \( R \) no nonzero prime ideal of finite height and which also has the property that there exists in \( R[x] \) a prime ideal \( Q \) of height one that is not finitely generated. To obtain such a domain \( R \) one can begin with a valuation domain \( V \) of infinite rank having no nonzero prime ideal of finite height and having the form \( V = F(t) + M \), where \( M \) is the maximal ideal of \( V \), \( F \) is a field and \( F(t) \) is a simple transcendental extension field of \( F \). Let \( R = F + M \) and let \( Q \) be the kernel of the canonical \( R \)-algebra homomorphism \( R[x] \to R[t] \) of the polynomial ring \( R[x] \) mapping \( x \) to \( t \). Then \( Q \) is a prime ideal of \( R[x] \) of height one, for if \( K \) denotes the quotient field of \( R \), then \( R[x]_Q \) is a localization of the polynomial ring \( K[x] \) and hence is a DVR. Moreover, \( Q \) is not finitely generated, for the content ideal of \( Q \) in \( R \) is \( M \) and \( M \) as an ideal of \( R \) is not finitely generated.

In this example, the prime ideal \( Q \) of \( R \) has valuative dimension one. Hence if \( x = x_1 \), and \( X = \{x_i\}_{i=1}^{\infty} \), then \( QR[X] \) is a non-finitely generated prime ideal of \( R[X] \), and by (1.2), \( QR[X] \) has height one. Therefore the converse of part (1) of (1.1) is not true; that is, there exists a ring \( R \) in which each prime ideal of finite height is finitely generated such that \( R[X] \) fails to have this property.

**Question 1.6.** Suppose \( R \) is an \( FH \)-ring and \( c \) is an ideal of \( R[X] \) having finitely many associated primes, each of which is finitely generated.

1. Does it follow that \( c \) is finitely generated?
2. Does it follow that \( c \) has a finite primary decomposition?

**Observation 1.7.**
1. If \( R \) is an \( FH \)-ring, then every height-zero prime of \( R \) is finitely generated. For if \( P \) is a height-zero prime of \( R \), then \( PR[X] \) is a height-zero prime of \( R[X] \). Thus \( PR[X] \) is finitely generated and so \( P \) is finitely generated. It follows that \( R \) has only finitely many height-zero primes [9, Theorem 1.6].
2. In view of (1.4) and [8, Theorem 4], every Noetherian ring, or polynomial ring over a Noetherian ring, is an \( FH \)-ring. As we note in (2.1) below, it is also true in general that a localization of an \( FH \)-ring is again an \( FH \)-ring.
3. The case of (1.1) where \( R \) is an integral domain is already quite interesting. We consider this case in §3.
2. Stability properties of FH-rings and valuative dimension

Proposition 2.1. Suppose $R$ is an FH-ring.

1. If $U$ is a multiplicatively closed subset of $R$, then the localization $U^{-1}R = R_U$ is again an FH-ring.

2. If $Y$ is a set of indeterminates over $R$, then the polynomial ring $R[Y]$ is an FH-ring.

Proof. Since $R[X]_U$ is canonically isomorphic to $R_U[X]$ and since a prime ideal $Q$ of $R[X]_U$ has finite height if and only if $Q \cap R[X]$ has finite height in $R[X]$, the first assertion is clear. For (2), suppose $X$ is a countably infinite set of indeterminates over $R[Y]$. By (1.4), every prime ideal of $R[Y][X]$ of finite height is finitely generated. Therefore $R[Y]$ is an FH-ring. $\square$

Notation 2.2. We use $R^{(n)}$ to denote the polynomial ring in $n$ indeterminates over a ring $R$.

Proposition 2.3. Suppose $X$ is an infinite set of indeterminates over a ring $R$ and $P \in \text{Spec}(R)$. Then the following are equivalent.

1. $P[X]$ has finite height in $R[X]$.
2. $PR_P[X]$ has finite height in $R_P[X]$.
3. $R_P$ has finite valuative dimension.

Consequently, if $R$ is an FH-ring having finite valuative dimension, then $R$ is Noetherian.

Proof. The equivalence of (1) and (2) is clear. If $R_P$ has finite valuative dimension $h$, then for $n$ sufficiently large, the height of $P(R_P)^{(n)}$ is the height of $PR_P[X]$, which is $h$ (cf. [11, Théorème 3, page 62]). Thus (3) implies (2). On the other hand, if $R_P$ has infinite valuative dimension, then the sequence $\{\text{ht } P(R_P)^{(n)}\}_{n=1}^{\infty}$ is unbounded (cf. [11, Théorème 4, page 63]). Hence $PR_P[X]$ has infinite height and (2) implies (3). $\square$

Proposition 2.4. Suppose $R$ is a ring and $P \in \text{Spec}(R)$ contains only finitely many height-zero primes $P_1, \ldots, P_k$ of $R$. Let $X$ be an infinite set of indeterminates over $R$. The following are equivalent:

1. $PR[X]$ has finite height.
2. $PR[X]/P_iR[X]$ has finite height for each $i$, $1 \leq i \leq k$.
3. The domain $R_P/P_iR_P$ has finite valuative dimension for each $i$, $1 \leq i \leq k$.

Proof. The equivalence of (1) and (2) follows from the fact that $\{P_i[X]\}_i^k$ is the set of height-zero primes of $R[X]$ contained in $P[X]$. In view of the fact that
$P[X]/P_1[X] \cong (P/P_1)[X]$ and $(R/P_1)_{P/P_1} \cong R_{P}/P_{R}$, the equivalence of (2) and (3) follows from Proposition 2.3.

**Theorem 2.5.** A ring $R$ is an FH-ring if and only if for each positive integer $n$, each prime ideal of $R^{(n)}$ of finite valuative dimension is finitely generated.

**Proof.** Suppose $R$ is an FH-ring and $Q \in \text{Spec}(R^{(n)})$ is of finite valuative dimension. By (2.1), $R^{(n)}$ is an FH-ring and by (1.7), $R^{(n)}$ has only finitely many height-zero primes. Hence (2.4) implies that $QR^{(n)}[X]$ has finite height, where $X$ is an infinite set of indeterminates over $R^{(n)}$. Therefore $QR^{(n)}[X]$, and hence $Q$, is finitely generated.

Conversely, assume that each prime of $R^{(n)}$ of finite valuative dimension is finitely generated. It follows that every height-zero prime of $R$ is finitely generated. Hence by [9, Theorem 1.6], $R$ has only finitely many height-zero primes. Let $P$ be a prime ideal of $R[X]$ of finite height $h$. There is a finite subset $Y$ of $X$ such that $P \cap R[Y]$ has height at least $h$. We necessarily have $(P \cap R[Y])R[X] = P$, since the prime ideal $(P \cap R[Y])R[X]$ is contained in $P$ and has height at least $h$. By (2.4), it follows that $P \cap R[Y]$ has finite valuative dimension. By hypothesis, this means that $P \cap R[Y]$ is finitely generated, so that $P = (P \cap R[Y])R[X]$ is also finitely generated. Consequently, $R$ is an FH-ring. \qed

**Proposition 2.6.** Suppose $R$ is a ring, $n$ is a positive integer, $Q \in \text{Spec}(R^{(n)})$, and $P = Q \cap R$. Then $Q$ has finite valuative dimension if and only if $P$ has finite valuative dimension.

**Proof.** By passing from $R$ to $R_P$, we may assume that $R$ is quasilocal with maximal ideal $P$. If $P$ has finite valuative dimension $h$, then $R^{(n)}$ has valuative dimension $h + n$ [11, Théorème 2, page 60]. Since $Q \in \text{Spec}(R^{(n)})$, it follows that $Q$ has finite valuative dimension. On the other hand, if $P$ has infinite valuative dimension, then $PR^{(n)}$ has infinite valuative dimension. Since $R^{(n)}_{PR^{(n)}}$ is a localization of $R^{(n)}_Q$, it follows that $Q$ has infinite valuative dimension. \qed

**Observation 2.7.** Suppose $S = R[\zeta_1, \ldots, \zeta_n]$ is a finitely generated extension ring of $R$. If $Q' \in \text{Spec}(S)$ has infinite valuative dimension, then $P = Q' \cap R$ also has infinite valuative dimension. For $S$ is an $R$-algebra homomorphic image of $R^{(n)}$ and the preimage $Q$ of $Q'$ in $R^{(n)}$ has infinite valuative dimension and $Q \cap R = Q' \cap R = P$. Hence by (2.6), $P$ has infinite valuative dimension. However, as we observe in Observation 3.7 below, it can happen that there exists a prime ideal $Q' \in \text{Spec}(S)$ of finite valuative dimension such that $Q' \cap R = P$ has infinite valuative dimension.
Discussion 2.8. 1. Since every ring is a homomorphic image of a polynomial ring over \( \mathbb{Z} \) and since, as noted in part (2) of (1.7), a polynomial ring over a Noetherian ring is an FH-ring, the property of being an FH-ring is not in general preserved under homomorphic image.

2. It is unclear whether for \( P \) a height-zero prime of an FH-ring \( R \) it follows that \( R/P \) is again an FH-ring. A problem here is that for \( Q \in \text{Spec}(R) \) with \( P < Q \) it may happen that \( QR[X] \) has infinite height, but \( QR[X]/PR[X] \) has finite height.

3. It would be interesting to know if a finitely generated extension ring of an FH-ring is again an FH-ring.

3. FH-domains and condition \((\rho)\)

Discussion 3.1. Let \( D \) be an integral domain with quotient field \( K \) and let \( x_1, \ldots, x_n \) be indeterminates over \( K \). Then \( K[x_1, \ldots, x_n] = K^{(n)} \) is a localization of \( D[x_1, \ldots, x_n] = D^{(n)} \). Hence for \( P \in \text{Spec}(K^{(n)}) \) we have \( (K^{(n)})_P = (D^{(n)})_{P \cap D^{(n)}} \). Therefore \( P \cap D^{(n)} \) is of finite valuative dimension. In view of Theorem 2.5, for each positive integer \( n \), an FH-domain \( D \) satisfies the following condition which we denote by \( (\rho_n) \).

1. \((\rho_n)\) For each \( P \in \text{Spec}(K^{(n)}) \), the contraction \( P \cap D^{(n)} \) is finitely generated.

We say the integral domain \( D \) satisfies condition \((\rho)\) if \( D \) satisfies \((\rho_n)\) for each positive integer \( n \).

Observation 3.2. An equivalent form of condition \((\rho)\) on an integral domain \( D \) is that every finitely generated extension domain of \( D \) is finitely presented. It was proved by Nagata in [15] that a valuation domain has this property, and a result of Raynaud and Gruson in [16, (3.4.7), page 261] implies that a Prüfer domain also has this property.

Condition \((\rho)\) modulo prime ideals of finite valuative dimension of a ring \( R \) relates nicely to \( R \) being an FH-ring as we observe in Theorem 3.3.

Theorem 3.3. A ring \( R \) is an FH-ring if and only if each \( P \in \text{Spec}(R) \) of finite valuative dimension is finitely generated and for each such \( P \) the integral domain \( R/P \) satisfies condition \((\rho)\).

Proof. Assume that \( R \) is an FH-ring. By Theorem 2.5, each \( P \in \text{Spec}(R) \) of finite valuative dimension is finitely generated. To show \( R/P \) satisfies condition \((\rho)\), it suffices to show that if \( Q' \) is a prime ideal of the polynomial ring \((R/P)^{(n)}\)
such that \( Q' \cap (R/P) = (0) \), then \( Q' \) is finitely generated. Let \( Q \) denote the preimage of \( Q' \) in \( R^{(n)} \). Then \( Q \cap R = P \). By (2.6), \( Q \) has finite valuative dimension. Since \( R \) is an FH-ring, \( Q \) is finitely generated by (2.5). Therefore \( Q' \) is finitely generated.

Assume conversely that each \( P \in \text{Spec}(R) \) of finite valuative dimension is finitely generated and \( R/P \) satisfies condition (\( p \)). To show \( R \) is an FH-ring, by Theorem 2.5, it suffices to show for each positive integer \( n \) that each prime \( Q \) of \( R^{(n)} \) of finite valuative dimension is finitely generated. Proposition 2.6 implies that \( P = Q \cap R \) is of finite valuation dimension in \( R \). Therefore \( P \) is finitely generated. Since \( R/P \) satisfies condition (\( p \)), the image of \( Q \) in \( (R/P)^{(n)} \) is finitely generated. Therefore \( Q \) is finitely generated. \( \Box \)

A test case for part (2) of (1.1) asks whether a one-dimensional quasilocal FH-domain \( D \) is Noetherian. By (2.3), the answer is affirmative if \( \dim_{\nu} D \) is finite. On the other hand, Theorem 3.3 implies that a one-dimensional quasilocal domain having infinite valuative dimension and satisfying condition (\( p \)) is an FH-domain: hence the existence of such a domain would provide a negative answer to part (2) of (1.1).

Let \( D \) be an integral domain with quotient field \( K \). We recall that \( D \) is said to be quasi-coherent if \( I^{-1} = D :_K I = \{ a \in K : aI \subseteq D \} \) is finitely generated for each nonzero finitely generated ideal \( I \) of \( D \) [4].

**Proposition 3.4.** If \( D \) satisfies condition (\( p \)), then \( D \) is quasi-coherent.

**Proof.** Suppose \( I = (a_1, \ldots, a_n)D \) is a nonzero finitely generated ideal. Let \( x_1, \ldots, x_n \) be indeterminates over \( K \) and let \( f = a_1 x_1 + \cdots + a_n x_n \). Then \( fK[x_1, \ldots, x_n] \) is a height-one prime ideal of \( K[x_1, \ldots, x_n] = K^{(n)} \). Let \( P = fK^{(n)} \cap D^{(n)} \). Since \( D \) satisfies condition (\( p \)), \( P \) is a finitely generated homogeneous ideal, where \( D^{(n)} \) is regarded as a graded ring with \( D \) of degree zero and each \( x_i \) of degree one. The degree-one piece of \( P \) is \( I^{-1}f \), and \( P \) finitely generated as an ideal of \( D^{(n)} \) implies that \( I^{-1}f \) is finitely generated as a \( D \)-module. Therefore \( I^{-1} \) is finitely generated as a fractional ideal of \( D \). \( \Box \)

From (3.3) and (3.4), we have the following corollary.

**Corollary 3.5.** If \( R \) is an FH-ring, then for each \( P \in \text{Spec}(R) \) of finite valuative dimension, the domain \( R/P \) is quasi-coherent. In particular, since the ideal \( (0) \) of an integral domain is a prime ideal of finite valuative dimension, if \( D \) is an FH-domain, then \( D \) is quasi-coherent.
Question 3.6. Suppose $E = D[\zeta]$ is a simple integral extension of domains. Is there an implication in either (or both) directions between the condition that $D$ is an FH-domain and the condition that $E$ is an FH-domain?

Observation 3.7. In relation to Question 3.6, we remark that there can exist in $E$ a maximal ideal $M_2$ of finite valuative dimension such that $M_2 \cap D = M$ has infinite valuative dimension. This is illustrated by [7, Example 5.8, page 161], where $A$ is the field of algebraic numbers, $A((x))$ is the quotient field of the formal power series ring $A[[x]]$, $V_1$ is a valuation domain of infinite rank on $A((x))$ of the form $V_1 = A + M_1$, and $V_2 = A[[x]] = A + M_2$, where $M_2 = xA[[x]]$. Then with $M = M_1 \cap M_2$, and $\zeta \in M_1$ such that $\zeta$ is a unit in $V_2$, we define $D = A + M$ and $E = D[\zeta]$.

It is easy to see that condition $(\rho)$ lifts from $D$ to $E$. More generally we have:

Proposition 3.8. If $n \geq 2$, and if an integral domain $D$ satisfies condition $(\rho_n)$, then a simple extension domain $E = D[\zeta]$ of $D$ satisfies condition $(\rho_{n-1})$. Thus if $D$ satisfies condition $(\rho)$, then every finitely generated extension domain of $D$ also satisfies condition $(\rho)$.

Proof. Suppose $P' \in \text{Spec}(E^{(n-1)})$ is such that $P' \cap E = (0)$. Under the canonical $D$-algebra homomorphism of $D^{(n)}$ onto $D[\zeta]^{(n-1)}$ mapping $x_n \to \zeta$, the preimage of $P'$ is a prime ideal $P \in \text{Spec}(D^{(n)})$ such that $P \cap D = (0)$. Since $D$ satisfies condition $(\rho_n)$, $P$ is finitely generated. Therefore $P'$ is finitely generated and $E = D[\zeta]$ satisfies condition $(\rho_{n-1})$. The second statement of (3.8) follows from the first statement.

Corollary 3.9. Suppose $R$ is an FH-ring and $P \in \text{Spec}(R)$ is of finite valuative dimension. Then every finitely generated extension domain of $R/P$ is quasi-coherent. In particular, if $D$ is an FH-domain, then every finitely generated extension domain of $D$ is quasi-coherent.

Proof. Apply (3.5) and (3.8).

4. INTEGRALLY CLOSED FH-DOMAINS

We recall that an integral domain $D$ is a Prüfer $v$-multiplication ring, abbreviated PVMD, if the divisorial ideals of $D$ of finite type form a group [12, page 667], [6, page 427], [13]. It is well known that an integrally closed quasi-coherent

\footnote{The term $v$-multipliation ring is used in [10], while Bourbaki [3, page 96] calls such domains pseudo-Prüfer.}
domain is a PVMD. A simple direct proof for this is to observe that if \( I \) is a nonzero finitely generated ideal of a quasi-coherent domain \( D \), then \( J = II^{-1} \) is a finitely generated integral ideal of \( D \) with the property that \( J^{-1} = J : J \). Since \( J \) is finitely generated, the elements of \( J : J \) are integral over \( D \). If \( D \) is also integrally closed, then \( J^{-1} = J : J = D \), and it follows that \( D \) is a PVMD.

**Proposition 4.1.** Suppose \( R \) is an FH-ring and \( P \in \text{Spec}(R) \) is of finite valuative dimension. Then every finitely generated integrally closed extension domain of \( R/P \) is a PVMD. In particular, if \( D \) is an integrally closed FH-domain, then \( D \) is a PVMD.

**Proof.** This is immediate from (3.9) and the fact that an integrally closed quasi-coherent domain is a PVMD.

**Corollary 4.2.** Suppose \( D \) is a one-dimensional FH-domain such that the integral closure \( D' \) of \( D \) is a finitely generated \( D \)-module. Then \( D \) is Noetherian. In particular, a one-dimensional integrally closed FH-domain is a Dedekind domain.

**Proof.** By (4.1), \( D' \) is a PVMD. Since a one-dimensional PVMD is Prüfer, it follows that \( D' \), and hence \( D \), has valuative dimension one. Therefore, by (2.5), each prime ideal of \( D \) is finitely generated, and \( D \) is Noetherian.

In preparation for showing that certain integrally closed FH-domains are Krull domains, we note the following.

**Proposition 4.3.** A nontrivial valuation domain \( V \) is an FH-domain if and only if \( V \) is either a rank-one discrete valuation domain (DVR), or \( \text{Spec}(V) \) contains no prime ideal of finite positive height. ³

**Proof.** If \( V \) contains a prime ideal of finite positive height and \( V \) is not a DVR, then \( V \) contains a non-finitely generated prime ideal \( P \) of finite height. Then \( PV[X] \) is of finite height in \( V[X] \) and is not finitely generated. On the other hand, it is clear that if \( V \) is a DVR, then \( V \) is an FH-domain. If \( \text{Spec}(V) \) contains no prime ideal of finite positive height, then Theorem 3.3 implies that \( V \) is an FH-domain, for as noted in (3.2), \( V \) satisfies condition \((\rho)\).

**Theorem 4.4.** Suppose \( D \) is an integrally closed FH-domain that satisfies the descending chain condition (d.c.c.) on prime ideals. Then \( D \) is a Krull domain, and each prime ideal of \( D \) of height one is finitely generated.

³A nontrivial valuation domain \( V \) has no prime ideal of finite positive height if and only if the nonzero prime ideals of \( V \) intersect in \((0)\).
Proof. By Proposition 4.1, $D$ is a PVMD. Hence there exists a set $\{P_a\}_{a \in A}$ of prime ideals of $D$ such that $D = \cap_a D_{P_a}$, where each $D_{P_a}$ is a valuation domain. By (2.1), each $D_{P_a}$ is an FH-domain. Since $D$, and therefore $D_{P_a}$, satisfies d.c.c. on prime ideals, either $P_a = (0)$ or $D_{P_a}$ is a DVR. Therefore $P_a$ has finite valuative dimension, so by (2.5) each $P_a$ is finitely generated. Suppose $d \in D$ is a nonzero non-unit, and let $P$ be a minimal prime of $(d)$. Then $D_P$ is a PVMD whose maximal ideal $PD_P$ is the radical of a principal ideal. It follows that $D_P$ is a valuation domain, thus a DVR, and $P$ is finitely generated. Therefore each minimal prime of $(d)$ is finitely generated. Hence by [9, Theorem 1.6], $(d)$ has only finitely many minimal primes. It follows that the representation $D = \cap_a D_{P_a}$ is locally finite, and $D$ is a Krull domain in which each height-one prime ideal is finitely generated. 

Question 4.5. Suppose $(R, m)$ is a 2-dimensional quasilocal integrally closed FH-domain. Must $R$ be Noetherian?

With notation as in (4.5), we note that if $P$ is a height-one prime of $R$, then $P$ is finitely generated and has finite valuative dimension. Therefore $R/P$ is a one-dimensional quasilocal domain that satisfies condition $(\rho)$ and hence is quasi-coherent. If $R/P$ is Noetherian, then $m$ is finitely generated and $R$ is Noetherian.

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