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ON THE TOTAL CURVATURE OF PARALLEL FAMILIES OF CONVEX SETS IN 3-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH NONNEGATIVE SECTIONAL CURVATURE

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1. INTRODUCTION

Let M^m be a smooth complete oriented Riemannian manifold of nonnegative sectional curvature, and let $C \subset M$ be a closed convex subset. Put $\hat{\rho}(x) := \operatorname{dist}(x, M - C)$ for the distance function to the complement of C; thus each $C_t := \hat{\rho}^{-1}[t, \infty)$ is convex ([CE], Thm. 8.9). Denote by $k(t)$ the total curvature of C_t, i.e. the integral of the determinant of the second fundamental form of bdryC_t. G. Perelman has asked whether $k(t)$ is nondecreasing (for example, if $M = \mathbb{S}^m$ is a standard sphere and C is a hemisphere then the total curvature increases from $k(0) = 0$ to $k(\frac{\pi}{2}) = \operatorname{vol}(\mathbb{S}^{m-1})$). An affirmative answer would yield a proof of the Cheeger-Gromoll soul conjecture (although Perelman has proved the conjecture by other means [P]). In this note we prove that the answer is yes in case $m = 3$.

It is not difficult to prove by classical methods, and in general dimensions, that k is nondecreasing in the case that the sets C_t have C^2 (or even $C^{1,1}$) boundary (cf. the calculations of section 4 below). The difficulty arises when these sets develop singularities. Note that the question still makes sense in this case: if C is any convex set (with possibly nonsmooth boundary) put $\sigma(x) := \operatorname{dist}(x, C)$. Then for small s the sets $C^s := \sigma^{-1}[0, s]$ have $C^{1,1}$ boundaries, i.e. the outward unit normal field n_s on bdryC^s is Lipschitz. Thus by Rademacher's theorem the second fundamental form $II_s(v, w) := \langle \nabla_v n_s, w \rangle$ is well-defined at a.e. point $p \in$ bdryC^s. The total curvature of C is then

$$k(C) = \lim_{s \downarrow 0} k(C^s) = \lim_{s \downarrow 0} \int_{\mathrm{bdry} C^s} \det II_s. \tag{1.1}$$

As this definition is slightly cumbersome we give a more natural one which will serve as the framework for this note. The convex set C admits a *normal cycle* $N(C)$, which is an integral current of dimension $m-1$ in the tangent sphere bundle $\mathbb{S}M$ of M. If C has interior then $N(C)$ is the unique Legendrian cycle in $\mathbb{S}M$ such that $\pi_* N(C) = \partial[\![C]\!]$ and spt $N(C) \cap \pi^{-1}(p) \cap \mathrm{Tan(C, p)} = \emptyset$ for all $p \in \mathrm{bdry}C$. Given a vector $\xi_0 \in \mathbb{S}M$, consider a positively oriented orthonormal frame field $e_1(\xi), \ldots, e_m(\xi) = \xi \in \mathbb{S}_{\pi(\xi)}M$, defined for ξ close to ξ_0. Let ω_{ij} denote the connection forms for this frame. Then the $(m-1)$-form $\omega_{1m} \wedge \cdots \wedge \omega_{(m-1)m} =: \bar{\omega}$ is independent of the choice of the complementary vectors $e_1, \ldots, e_{m-1}$. Thus $\bar{\omega}$ is well-defined globally on $\mathbb{S}M$, and the total curvature of C is the evaluation of the normal cycle of C against this form: $k(C) = N(C)(\bar{\omega})$.

The equivalence of this definition with the one of the previous paragraph may be seen as follows. Let $\widetilde{\exp} : \mathbb{R} \times \mathbb{S}M \to \mathbb{S}M$ denote the natural lift of the exponential map. Then $N(C^s) = \widetilde{\exp}_{s*} N(C)$ for small $s \geq 0$, so

$$k(C^s) = N(C^s)(\bar{\omega}) = \widetilde{\exp}_{s*} N(C)(\bar{\omega}) \to N(C)(\bar{\omega}) = k(C) \tag{1.2}$$

as $s \downarrow 0$, since $\widetilde{\exp}_s \to$ the identity.

We are obliged to remark that in dimension 3 the Chern-Gauss-Bonnet formula [Ch] yields a simpler expression for $k(C)$ that does not involve the normal cycle. For sets $B \subset M^3$ with smooth boundary the formula of [Ch] reads

$$k(B) + \int_{\partial B} K_x \, dx = 4\pi\chi(B), \tag{1.3}$$

where χ is the Euler characteristic and K_x is the sectional curvature of M in the 2-plane $T_x \partial B$. Since the boundary of a convex set C is rectifiable, the second term on the left is well-defined in this case, and (1.3) remains valid with B replaced by C, so we may solve for $k(C)$. The Euler characteristics of the sets C_t satisfy $\chi(C_t) = \chi(C)$, so $k(C_t)$ is nondecreasing iff $\int_{\partial C_t} K_x$ is nonincreasing. It is tempting to think that this less technical definition might yield a correspondingly less technical proof of our theorem, but this seems an illusion: in the present framework $\int K_x$ may also be expressed as an integral over the normal cycle of a universal form $\tilde{\omega}$ on the sphere bundle. The idea of the proof of [Ch] is to apply Stokes' theorem on the graph of an appropriate section of the sphere bundle via the basic identity $d(\bar{\omega} + \tilde{\omega}) = d\bar{\omega} + d\tilde{\omega} = 0$. The idea of the proof presented here is to integrate $d\bar{\omega}$ over a graph associated to the distance function and apply Stokes' theorem; there is obviously nothing to be gained by integrating $d\tilde{\omega}$ instead.

2.

Our method is to work on the graph of the function $\rho := -\hat{\rho}$, or more precisely on the graph of its gradient. Although ρ is not everywhere differentiable, it is nevertheless *semiconvex* on intC — that is, about any given point there are smooth local coordinates $\phi : C \supset\supset U \to \mathbb{R}^m$, such that

$$\rho \circ \phi^{-1} = f + \alpha, \tag{2.1}$$

where f is smooth and α is the restriction to $\phi(U) \subset \mathbb{R}^m$ of a convex function. This fact is well known; for instance it is a consequence of the proof of Prop. 1.2 of [Fu1] (cf. also [Ba]). Therefore there is a closed Lagrangian current $[\![\nabla\rho]\!] \in \mathbb{I}_m(T(\text{intC}))$ that represents the graph of the gradient of ρ. Here T int C $\subset$ TM is endowed with the symplectic structure arising from the identification $TM \leftrightarrow T^*M$ induced by the Riemannian metric. In fact, in the semiconvex case the structure of this current is very simple: it is given by integration over a closed oriented Lagrangian Lipschitz submanifold $Q \subset T^*(\text{intC})$. This Q is precisely equal to the graph of the Clarke gradient $\nabla\rho$ of ρ, i.e.

$$\begin{aligned} \nabla\rho(x) \quad := \quad & \text{convex hull}\{v : v = \lim_{i\to\infty} \nabla\rho(x_i) \text{ for some sequence} \\ & x_i \to x,\ \rho \text{ differentiable at } x_i\}. \end{aligned}$$

The orientation of Q is determined by the condition that the projection π of the tangent bundle induce an orientation-preserving map $Q \to$ intC. Moreover, in view of the convexity of α, the orientation of Q is given explicitly as follows. For almost every $\xi \in Q$ the tangent space $T_\xi Q$ exists and is a Lagrangian subspace of the symplectic vector space $T_\xi TQ$. Put $V := \ker\pi_* \cap T_\xi Q \subset \ker\pi_* \simeq T_{\pi(\xi)}M$ and $W := \pi_*(T_\xi Q) \subset T_{\pi(\xi)}M$. The fact that $T_\xi Q$ is Lagrangian implies that V and W are orthogonal. Thus $T_\xi Q$ is canonically isomorphic to $T_{\pi(\xi)}M$. The orientation is now induced from that on the latter space. This corresponds to the fact that all curvature measures of a convex body are nonnegative.

We write

$$[\![Q]\!] = [\![\nabla\rho]\!],$$

with this orientation understood.

We will also make use of the following basic fact. Suppose $K \subset \mathbb{R}^m$ is a closed convex set. Put $\Sigma^k := \{x \mid \dim \text{Nor}(\text{K},\text{x}) = \text{m} - \text{k}\}$. Then Σ^k is a k-dimensional set which is C^2 rectifiable in the sense of [AS], i.e.

$$\Sigma^k \subset \bigcup_{i=1}^{\infty} V_i \cup N, \tag{2.2}$$

where each V_i is a k-dimensional submanifold of class C^2 and $\mathcal{H}^k(N) = 0$, where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure. For a proof of this fact see [A]. Applying this result to the epigraph of the function α appearing in (2.1), if we put

$$R^k := \{x \in \mathrm{intC} \mid \dim \nabla\rho(\mathrm{x}) = \mathrm{m} - \mathrm{k} - 1\},\ \mathrm{k} = 0, \ldots, \mathrm{m},$$

then each R^k is locally the image under ϕ^{-1} of a k-dimensional, C^2-rectifiable subset of $\mathbb{R}^m$, and is therefore itself a k-dimensional, C^2-rectifiable subset of M. Moreover, decomposing R^k as in (2.2), each $\rho|V_i \cap R^k$ is the restriction of a C^2 function ρ_i on V_i.

3.

The distance function ρ is differentiable at a point x_0 iff there is a unique minimizing geodesic γ from bdry$C$ to x_0, and in this case the gradient of ρ is precisely the tangent vector to γ at x_0. Therefore, at a general point x, the Clarke gradient $\nabla\rho(x)$ consists of the convex hull of the set of all tangent vectors to all minimizing geodesics from bdry$C$ to x. In particular, $0 \in \nabla\rho(x)$ iff x is a critical point of ρ in the usual generalized sense.

The convexity of C implies that ρ has at most one critical value $\rho_{\min}$. Therefore the normalizing map $\nu(v) := v/|v|$ is well-defined and locally Lipschitz on Q, at least away from this critical set. Thus the normal cycles of the sets C_t arise as

$$\begin{aligned} N(C_t) &= \langle \nu_*[\![Q]\!], \rho\circ\pi, t\rangle,\ t \in (\rho_{\min}, 0), \\ N(C) &= \lim_{t\uparrow 0} N(C_t). \end{aligned} \tag{3.1}$$

Therefore if $0 \le s < t < -\rho_{\min}$ then

$$\begin{aligned} k(t) - k(s) &= (N(C_t) - N(C_s))(\bar\omega) \\ &= \left(\langle \nu_*[\![Q]\!], \rho\circ\pi, -t\rangle - \langle \nu_*[\![Q]\!], \rho\circ\pi, -s\rangle\right)(\bar\omega) \\ &= -\partial(\nu_*[\![Q]\!] \llcorner \pi^{-1}\rho^{-1}(-t,-s))(\bar\omega) \\ &= -(\nu_*[\![Q]\!] \llcorner \pi^{-1}\rho^{-1}(-t,-s))(\bar\Omega) \end{aligned} \tag{3.2}$$

where

$$\bar\Omega := \sum_{j=1}^{m} (-1)^{j+1} \omega_{1m} \wedge \cdots \wedge \omega_{(j-1)m} \wedge \Omega_{jm} \wedge \omega_{(j+1)m} \wedge \cdots \wedge \omega_{(m-1)m}, \tag{3.3}$$

since $\bar\Omega = d\bar\omega$ for C^2 frames e_i. However (3.2) remains valid even if the frame is only C^1, as may be proved easily by approximation.

Our main result is the following.

Theorem 3.1. *If $m = 3$ then the 0-current (signed measure) $\pi_*(\nu_*[\![Q]\!] \llcorner \bar{\Omega})$ is non-positive.*

The proof is given in the next two sections.

4.

Put $Q_1 := \nu(Q)$ for the corresponding rectifiable set of unit vectors. By [F2,3.2.22], for $\mathcal{H}^3$ a.e. $\xi \in Q_1 \cap \pi^{-1}(R^i)$ the image under π_* of the approximate tangent 3-plane $T_\xi Q_1$ has dimension at most i. Since the curvature 2-forms Ω_{ij} are horizontal it follows that

$$\bar{\Omega}|Q_1 \cap \pi^{-1}(R^i) = 0, \; i = 0, 1.$$

It remains to compute the contributions due to R^2 and R^3. We shall see that both are nonpositive.

We deal first with R^3. From the first paragraph of §3 we see that $Q_1 \cap \pi^{-1}(R^3)$ consists of all values $\eta = -\widetilde{\exp}(t, -\xi)$ such that $\xi \in \mathrm{Nor(C)} \cap \mathbb{S}\mathrm{M}$, $t > 0$, and $\gamma(s) := \exp(-s\xi)$ is the unique minimizing geodesic from bdry$C$ to $\pi(\eta)$. Therefore, for $\mathcal{H}^3$ a.e $\eta \in Q_1 \cap \pi^{-1}(R^3)$, the tangent space $T_\eta Q$ is the direct sum of the tangent line to the lifted geodesic $\widetilde{\exp}(s\eta)$ and the tangent plane $T_\eta \mathrm{Nor}(\mathrm{C_t})$, where $t = \rho \circ \pi(\eta)$.

We claim first of all that for $\mathcal{H}^3$-a.e. such η the base point $\pi(\eta)$ is *not* a focal point of bdry$C$. In other words the restriction of the derivative $D(\widetilde{\exp}_{-t})$ to $T_\xi(\mathrm{Nor(C)})$ is nondegenerate. If not, then the usual second variation argument implies that t must be the smallest parameter for which this derivative degenerates. However if we put for tangent m-planes P to TM,

$$\delta(P) := \inf\{t : \text{ the restriction of } D(\widetilde{\exp}_{-t}) \text{ to } P \text{ is singular}\},$$

then δ is clearly lower semicontinuous, hence Borel measurable. So $\widetilde{\delta} : \xi \mapsto \delta(T_\xi(\mathrm{Nor(C)}))$ is a measurable function. In particular the graph of $\widetilde{\delta}$ has $\mathcal{H}^3$ measure zero in Nor(C) $\times \mathbb{R}$. So the image of this graph under the smooth map $\widetilde{\exp}$ also has $\mathcal{H}^3$ measure zero, as claimed.

Therefore the projection $D\pi$ is nondegenerate on $\Pi := T_\eta \mathrm{Nor}(\mathrm{C_t})$. Hence $D\pi : T_\eta Q \to T_{\pi(\eta)} M$ is nondegenerate and orientation preserving. Choose a local frame field $e_1(\eta'), e_2(\eta'), e_3(\eta') = \eta'$ for η' near η, in such a way that $e_1(\eta'), e_2(\eta')$ are principal directions for C_t at $\pi(\eta')$ and e_3 is the tangent vector field along geodesics normal to C_t. Then $\omega_{i3} \mid \Pi = \theta_i := \pi^*(e_i^*)$, $i = 1, 2$. Therefore

$$\begin{aligned} \bar{\Omega} \mid \Pi &= \Omega_{13} \wedge \omega_{23} - \omega_{13} \wedge \Omega_{23} \\ &= \Omega_{13} \wedge k_2\theta_2 - k_1\theta_1 \wedge \Omega_{23} \\ &= -(k_2 K_{13} + k_1 K_{23})\pi^*(vol_M), \end{aligned}$$

where the K_{ij} are sectional curvatures of M and the k_i principal curvatures of C_t at ξ_0. Since these quantities are all nonnegative, and $\pi|Q$ is orientation-preserving, we conclude that $\bar{\Omega}|Q \cap \pi^{-1}(R^3) \leq 0$.

5.

We write $R^2 = \bigcup_{i=1}^{\infty} V_i \cup N$, where the V_i are as in the last sentence of §2. Since $\bar{\Omega}$ is horizontal in two slots, the null set N may be neglected and we have the corresponding decomposition of 0-currents

$$(5.1) \qquad [\![Q_1 \cap \pi^{-1}R^2]\!] \llcorner \bar{\Omega} = \sum_{i=1}^{\infty} [\![Q_1 \cap \pi^{-1}V_i]\!] \llcorner \bar{\Omega}.$$

We will show that each of these terms is nonnegative.

Fix $V = V_i$. For simplicity we again denote by ρ the C^2 function on V extending $\rho|V \cap R^2$. By the first paragraph of §3, given $x \in R^2$ the set $Q \cap \pi^{-1}(x)$ is a line segment with endpoints $u_x, v_x \in \mathbb{S}_x M$. Since Q is Lagrangian, if $x \in V \cap R^2$ then σ_x is perpendicular to the tangent 2-plane $T_x V \subset T_x M$. Therefore

$$u_x - v_x = c(x)n(x),$$

where n is a unit normal to V and $c(x) > 0$. Furthermore the gradient $\nabla_V \rho(x) := \nabla(\rho|V)(x)$ is the common orthogonal projection to $T_x V$ of the elements of σ_x. Thus if we determine the positively oriented frame field $\epsilon_1, \epsilon_2, \epsilon_3$ by

$$\begin{aligned} \epsilon_2(x) &= n(x), \\ \epsilon_3(x) &= \nabla_V \rho(x)/|\nabla_V \rho(x)|, \end{aligned}$$

and put

$$\psi_x := \arcsin(c(x)/2) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

for the angle that u_x and v_x make with the tangent plane $T_x V$, then $Q_1 \cap \pi^{-1}(V \cap R^2)$ is the intersection with $\pi^{-1}(R^2)$ of the C^1 submanifold

$$\mathbb{S}M \supset \widetilde{V} := \{(x, \cos\phi\epsilon_3(x) + \sin\phi\epsilon_2(x)) : x \in V, -\psi_x \leq \phi \leq \psi_x\}.$$

Finally, note that if ι is the C^1 involution of $TM|\widetilde{V}$ defined by

$$\iota(x, a\epsilon_1 + b\epsilon_2 + c\epsilon_3) = (x, a\epsilon_1 - b\epsilon_2 + c\epsilon_3),$$

then

$$\iota_*[\![\widetilde{V}]\!] = -[\![\widetilde{V}]\!]. \tag{5.2}$$

Let $\phi := \arcsin\langle\xi, \epsilon_2(\pi(\xi))\rangle$ be the C^1 function on $\widetilde{V}$ determined by the relation

$$\xi = \cos\phi\epsilon_3 + \sin\phi\epsilon_2.$$

Now define the modified frame

$$\begin{aligned} e_1(\xi) &= \epsilon_1(\pi(\xi)), \\ e_2(\xi) &= -\sin\phi\epsilon_3(\pi(\xi)) + \cos\phi\epsilon_2(\pi(x)), \\ e_3(\xi) &= \xi. \end{aligned}$$

If ω_{ij}, Ω_{ij} denote the connection and curvature forms for this latter frame then obviously

$$\omega_{13}(\xi)\cdot v = 0$$

for vertical vectors $v \in T_\xi\widetilde{V}$. Therefore the summand $\omega_{13}\wedge\Omega_{32}$ of $\bar{\Omega}$ vanishes when restricted to $\widetilde{V}$. Denote by O_{ij} the curvature forms of M relative to the frame ϵ. Then

$$\Omega_{13}(\xi) = \cos\phi\, O_{13}(\pi(\xi)) + \sin\phi\, O_{12}(\pi(\xi)).$$

Therefore we may write

$$\Omega_{13} = (\cos\phi K_{13} + \sin\phi R_{1312})\theta_1\wedge\theta_3,$$

where the curvature tensor and sectional curvature are given in terms of the frame ϵ. Finally, on $\widetilde{V}$ we have $\omega_{32}\wedge\Omega_{13} = d\phi\wedge\Omega_{13}$ since the curvature forms are horizontal. To sum up, we may express the contribution of V by

$$\begin{aligned} [\![\widetilde{V}]\!] \llcorner \bar{\Omega} &= -[\![\widetilde{V}]\!] \llcorner \omega_{32}\wedge\Omega_{13} \\ &= -[\![\widetilde{V}]\!] \llcorner (\cos\phi K_{13} + \sin\phi R_{1312})\,\theta_1\wedge\theta_3\wedge d\phi \end{aligned}$$

Examining orientations and recalling the determination of the orientation of Q in the second paragraph of §2, we find that the contribution of the cosine term is nonpositive. As for the sine term, using (5.2) we may calculate

$$\begin{aligned} 2\pi_*([\![\widetilde{V}]\!] &\llcorner \sin\phi R_{1312} d\phi\wedge\theta_1\wedge\theta_3) \\ &= \pi_*\Big(([\![\widetilde{V}]\!] - \iota_*[\![\widetilde{V}]\!]) \llcorner \sin\phi R_{1312} d\phi\wedge\theta_1\wedge\theta_3\Big) \end{aligned}$$

$$= \pi_*\Big([\![\widetilde{V}]\!] \llcorner \sin\phi R_{1312} d\phi\wedge\theta_1\wedge\theta_3 - [\![\widetilde{V}]\!] \llcorner \iota^*(\sin\phi R_{1312} d\phi\wedge\theta_1\wedge\theta_3)\Big)$$

$$= \pi_* \left([\![\widetilde{V}]\!] \llcorner (\sin\phi R_{1312} d\phi \wedge \theta_1 \wedge \theta_3 - \sin(-\phi) R_{1312} d(-\phi) \wedge \theta_1 \wedge \theta_3) \right) = 0.$$

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