ON THE DYNAMICS OF SOME DIFFEOMORPHISMS OF $\mathbb{C}^2$
NEAR PARABOLIC FIXED POINTS

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ABSTRACT. In this paper we consider diffeomorphisms of $\mathbb{C}^2$ of the special
form $F(z, w) = (w, -z + 2G(w))$. For such maps the origin is a parabolic
fixed point. Under certain hypotheses on $G$ we prove the existence of a
domain $\Omega \subset \mathbb{C}$ with $0 \in \partial \Omega$ and of invariant complex curves $w = f(z)$
and $w = g(z)$, $z \in \Omega$, for $F^{-1}$ and $F$, such that $F^{-n}(z, f(z)) \to 0$ and
$F^n(z, g(z)) \to 0$ as $n \to \infty$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The field of complex dynamics in several variables has dramatically developed
in recent years. Global results in the theory, such as the study of the properties
of Julia and Fatou sets are obtained in [BS], [FS1], [FS2], etc. Besides the global
aspects, it is of interest to analyze the dynamics of holomorphic maps near fixed
points. Results in this direction are obtained in [HP], [U1], [U2], etc. We are
interested in the local behavior of the iterates of holomorphic maps near certain
fixed points.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a diffeomorphism which fixes 0 and is holomorphic
near 0, and let $\lambda_1$, $\lambda_2$ be the eigenvalues of $F'(0)$. If $|\lambda_1| < 1 < |\lambda_2|$ then it
is well known that for $r$ sufficiently small the sets $W^s_{loc}(0) = \{(z, w) \in \mathbb{C}^2 :$
$\|F^n(z, w)\| \leq r$ for all $n \geq 0$, $\lim_{n \to \infty} F^n(z, w) = 0\}$ and $W^u_{loc}(0) = \{(z, w) \in \mathbb{C}^2 :$
$\|F^n(z, w)\| \leq r$ for all $n \leq 0$, $\lim_{n \to -\infty} F^n(z, w) = 0\}$ are invariant com-
plex one dimensional manifolds called the local stable manifold and local unstable
manifold of $F$ at 0. In the case when $|\lambda_1| = 1$ or $|\lambda_2| = 1$ the above sets $W^s_{loc}(0)$
and $W^u_{loc}(0)$ are not necessarily manifolds anymore.

1991 Mathematics Subject Classification. Primary 32H50; Secondary 58F23.

85
In this paper we study diffeomorphisms $F$ of $\mathbb{C}^2$ of the following special form

\[(1.1) \quad F(z, w) = (w, -z + 2G(w)),\]

where $G \in C^1(\mathbb{C})$ is holomorphic near 0, $G(0) = 0$, $G'(0) = 1$. For such maps the origin is a fixed point, the eigenvalues of $F'(0)$ are both equal to 1 and $F'(0)$ is nondiagonalizable (thus equivalent to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). The real two dimensional case $(z, w) = (x, y) \in \mathbb{R}^2$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $G : \mathbb{R} \to \mathbb{R}$, is studied in [F]. Under the assumption that $G''(x) > 0$ for $x > 0$, it is shown in [F] that there are functions $f(x)$ and $g(x)$ defined for $x \geq 0$ such that the unique stable/unstable manifolds of $F$ in $\{(x, y) : x \geq 0\}$ are precisely the graphs of $g/f$.

As described in [F], Section 2, maps $F$ of form (1.1) have special symmetries. They come from the fact that $F = I \circ J$, where $I$ and $J$ are the involutions $I(z, w) = (w, z)$ and $J(z, w) = (z, -w + 2G(z))$, so $I \circ F \circ I = F^{-1}$ and $J \circ F \circ J = F^{-1}$. Assuming for the moment that there are unique stable and unstable manifolds for $F$ at 0, given by the graphs of the univalent (i.e. one-to-one holomorphic) functions $g$ and $f$ respectively, it follows that $I$ maps the stable manifold of $F$ to the stable manifold of $F^{-1}$, which is the unstable manifold of $F$; thus for every $z$ there is a $\zeta$ such that $(g(z), z) = I(z, g(z)) = (\zeta, f(\zeta))$, so $g = f^{-1}$. The same holds for $J$, so $(z, -g(z) + 2G(z)) = J(z, g(z)) = (\zeta, f(\zeta))$, and we conclude that $2G = f + f^{-1}$.

We now state the results of the paper; the proofs are given in section 2. One of our goals is to find suitable conditions on $G$ which ensure, in the complex case, the existence of invariant complex curves $w = f(z)$ and $w = g(z)$ for $F$ such that $F^n(z, w) \to 0$ as $n \to -\infty$ for $(z, w) \in \{w = f(z)\}$ and $F^n(z, w) \to 0$ as $n \to \infty$ for $(z, w) \in \{w = g(z)\}$.

We let $\Omega \subset \mathbb{C}$ be a convex domain (not necessarily bounded) such that $0 \in \partial \Omega$ and we assume that the function $G \in C^1(\mathbb{C})$ is holomorphic in a neighborhood of $\overline{\Omega}$ and satisfies the following conditions

- $(C1)$ $G(0) = 0$, $G'(0) = 1$, $\Omega \subseteq C(\Omega)$.
- $(C2)$ there exists $\alpha \in (-\pi/2, \pi/2)$ such that $\Re[e^{i\alpha}(G'(z) - 1)] > 0$ for $z \in \Omega$.
- $(C3)$ there is a ray $L = \{re^{i\theta} : 0 < r \leq r_0\} \subset \Omega$ such that $G'(re^{i\theta}) \in \mathbb{R}$ for all $0 \leq r < r_0$.

For future references, let us denote by $X(\Omega, \alpha)$ the class of functions $G \in C^1(\mathbb{C}) \cap O(\overline{\Omega})$ satisfying condition $(C1)$, $(C2)$, and $(C3)$ with $\theta = 0$ (i.e. the Taylor coefficients of $G$ at 0 are real).
For the function $F$ defined by (1.1) with $G$ satisfying conditions (C1), (C2) and (C3) we introduce the following sets:

$$W^u_\Omega(0) = \{(z, w) \in \mathbb{C}^2 : F^n(z, w) \in \Omega \times \Omega \text{ for all } n \geq 0, \lim_{n \to \infty} F^n(z, w) = 0\},$$

$$W^s_\Omega(0) = \{(z, w) \in \mathbb{C}^2 : F^n(z, w) \in \Omega \times \Omega \text{ for all } n \leq 0, \lim_{n \to -\infty} F^n(z, w) = 0\}.$$

We have the following:

**Theorem 1.1.** Let $F$, $G$ and $\Omega$ be as above. There exists a function $f$ univalent in $\Omega$ such that $\Omega \subseteq f(\Omega)$ and the functions $f$ and $g = f^{-1}$ have the following properties:

(i) the graphs $\{(z, g(z)) : z \in \Omega\}$ and $\{(z, f(z)) : z \in \Omega\}$ of $g$ and $f$ are invariant under of $F$ and $F^{-1}$ respectively, and $F^n(z, g(z)) \to 0$, $F^{-n}(z, f(z)) \to 0$, as $n \to \infty$, locally uniformly for $z \in \Omega$;

(ii) if condition (C2) on $G$ holds with $\alpha = 0$ then $W^u_\Omega(0) = \text{graph } g$ and $W^s_\Omega(0) = \text{graph } f$.

Following the proof of this theorem in section 2 we make some remarks regarding the analyticity of $f$ and $g$ at the origin.

We next apply Theorem 1.1 to the following general situation: $G \in C^1(\mathbb{C})$ is holomorphic near 0 and it has the expansion

$$G(z) = z + az^{j+1} + h(z),$$

where $a > 0$, $j \geq 1$ and $h(z) = \sum_{k \geq 1} \alpha_k z^k$, $\alpha_k \in \mathbb{R}$.

**Theorem 1.2.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be of form (1.1) with $G$ as above. There exist a domain $D \subset \mathbb{C}$ and functions $f$, $g$ holomorphic on $D$ with the following properties:

(i) $0 \in \partial D$, $D$ is starlike with respect to 0, $D \subseteq \{z : |\arg z| < \pi/j\}$ and the rays $\{z : |\arg z| = \pi/j\}$ are tangent to $\partial D$ at 0;

(ii) $g(D) \subseteq D \subseteq f(D)$ and $f \circ g = id$ on $D$ (hence $g$ is univalent on $D$);

(iii) the graphs $\{(z, g(z)) : z \in D\}$ and $\{(z, f(z)) : z \in D\}$ are invariant under $F$ and $F^{-1}$ respectively, and $F^n(z, g(z)) \to 0$, $F^{-n}(z, f(z)) \to 0$, as $n \to \infty$, locally uniformly for $z \in D$.

Finally, we consider some Henon maps $F_j$ of form (1.1), obtained for $G(z) = G_j(z) = z + az^{j+1}$, where $j \geq 1$ and $a > 0$. For such maps we can use Theorem 1.1 to obtain results of a global nature. We introduce the following domains $D_k \subset \mathbb{C}^k$:

$$D_k = \left\{re^{i\phi} : 0 < r < \infty, \frac{(2k - 1)\pi}{j} < \phi < \frac{(2k + 1)\pi}{j}\right\},$$
where \( k \in \{0, 1, \ldots, j-1\} \). Note that the union \( \bigcup_{k=0}^{j-1} D_k \) equals the complex plane minus the union of \( j \) rays joining 0 to \( \infty \). We have the following

**Theorem 1.3.** In the above setting, there exist functions \( f_k \) and \( g_k \) holomorphic on \( D_k, k \in \{0, 1, \ldots, j-1\} \), with the following properties

(i) \( g_k(D_k) \subset D_k \subset f_k(D_k) \) (properly) and \( f_k \circ g_k = \text{id} \) on \( D_k \);

(ii) the graphs \( \{(z, g_k(z)) : z \in D_k\} \) and \( \{(z, f_k(z)) : z \in D_k\} \) are invariant under \( F_j \) and \( F_j^{-1} \) respectively, and \( F_j^n(z, g_k(z)) \to 0, F_j^{-n}(z, f_k(z)) \to 0 \), as \( n \to \infty \), locally uniformly for \( z \in D_k \);

(iii) \( f_k \left( \exp \left( \frac{2k\pi i}{j} \right) z \right) = \exp \left( \frac{2k\pi i}{j} \right) f_0(z) \) for \( z \in D_0 \), and the same holds for \( g_k \) and \( g_0 \).

This theorem improves a result of [F] (see the remark in section 2 which follows the proof of the theorem).

2. PROOFS

We will use the following classic results in geometric function theory (see [D]):

**Lemma 2.1.** Let \( h \) be a holomorphic function in a neighborhood of the closed line segment \([z_0, z_1] \subset \mathbb{C}\). Then there exists a point \( z \) in the closed convex hull of the set \( h'([z_0, z_1]) \) such that \( h(z_1) - h(z_0) = Z(z_1 - z_0) \).

**Lemma 2.2.** (Noshiro-Warschawski Theorem) Let \( D \subset \mathbb{C} \) be a convex domain and let \( h \) be a holomorphic function in \( D \) satisfying \( \Re [e^{i\alpha} h'(z)] > 0 \) in \( D \), for some \( \alpha \in \mathbb{R} \). Then \( h \) is univalent in \( D \).

We also need the following lemma, which is essentially proved in [F] (note that we do not make any assumptions on the second derivative of \( H \)):

**Lemma 2.3.** Let \( H : [0, x_0] \to \mathbb{R} \) be a continuously differentiable function satisfying \( H(0) = 0 \) and \( H'(x) > 1 \) for \( 0 < x < x_0 \). Then the sequences of functions \( \{h_n\}_{n>0} \) and \( \{k_n\}_{n>0} \) given by \( h_1 = H, k_1 = H^{-1}, h_{n+1} = 2H - k_n, \) \( k_{n+1} = (h_{n+1})^{-1} \) are well defined and satisfy the following conditions for all \( 0 < x < x_0 \):

(i) \( 0 \leq h_n(x) \leq h_{n+1}(x) \leq 2H(x) \);

(ii) \( 0 \leq k_{n+1}(x) \leq k_n(x) \);

(iii) if \( k(x) = \lim_{n \to \infty} k_n(x) \) then the sequence of iterates \( \{k^j\}_{j>0} \) converges pointwise to 0 on \([0, x_0]\) as \( j \to \infty \).

**Proof.** An easy induction on \( n \) shows that \( \{h_n\} \) and \( \{k_n\} \) are well defined and satisfy (i) and (ii). Clearly \( k(x) < x \) and \( k_n^j(x) < 1 \) for all \( x \in (0, x_0) \), so
\[|k_n(x) - k_n(x')| \leq |x - x'| \] and hence \(k\) is Lipschitz. Then for any \(x \in (0, x_0)\), \(\{k^n(x)\}\) decreases to 0, which is the only fixed point of \(k\). \(\square\)

**Proof of Theorem 1.1.** Let \(P_{\alpha}\) be the half plane \(P_{\alpha} = \{z \in \mathbb{C} : \Re[e^{i\alpha}(z - 1)] > 0\}\) and let \(D_{\alpha}\) be the disc \(D_{\alpha} = j(P_{\alpha})\), where \(j(z) = 1/z\) (\(D_{\alpha}\) has 0 and 1 on its boundary and is tangent to \(\partial P_{\alpha}\) at 1). Set \(\lambda_{\alpha} = \sup\{|z| : z \in D_{\alpha}\} = 1/\cos \alpha\).

We first construct by induction sequences of holomorphic functions \(\{f_n\}_{n>0}\) and \(\{g_n\}_{n>0}\) with the following properties

\[
\begin{align*}
\text{(2.1)} & \quad f_n \in O(\Omega), \quad f_n(0) = 0, \quad f_n'(0) = 1 \\
& \quad \Re[e^{i\alpha}(f_n'(z) - 1)] > 0 \quad \text{for } z \in \Omega \\
& \quad f_n : \Omega \to f_n(\Omega) \quad \text{is univalent, } \Omega \subseteq f_n(\Omega), f_n(L) \supseteq L
\end{align*}
\]

\[
\begin{align*}
\text{(2.2)} & \quad g_n \in O(\Omega), \quad g_n(0) = 0, \quad g_n'(0) = 1 \\
& \quad g_n'(z) \in D_{\alpha} \quad \text{for } z \in f_n(\Omega) \\
& \quad g_n : f_n(\Omega) \to \Omega \quad \text{is the inverse of } f_n, g_n(L) \subseteq L
\end{align*}
\]

\[
\text{(2.3)} \quad 2G(z) = f_{n+1}(z) + g_n(z) \quad \text{for } n \geq 1 \text{ and } z \in \Omega.
\]

These sequences are constructed in analogy to the real two-dimensional case \([F]\).

Let \(f_1 = G\). By Lemma 2.2 \(f_1\) is univalent on \(\Omega\). Clearly conditions (C2) and (C3) imply that \(f_1(L) \supseteq L\), so \(f_1\) satisfies (2.1). We assume now by induction that \(f_n\) is defined so that it satisfies (2.1) and construct \(g_n\) and \(f_{n+1}\) such that (2.1), (2.2) and (2.3) hold.

Let \(g_n = (f_n)^{-1} : f_n(\Omega) \supseteq \Omega \to \Omega\). Then \(g_n'(z) = 1/f_n'(g_n(z))\) for \(z \in f_n(\Omega)\), so \(g_n'(z) \in D_{\alpha}\). In order to show that \(g_n\) extends holomorphically to a neighborhood of \(\overline{\Omega}\) it is enough to notice that for any \(\zeta \in \partial \Omega \cap \partial f_n(\Omega)\) there is a disc \(\Delta_{\zeta}\) centered at \(\zeta\) such that \(g_n\) extends holomorphically to \(\Delta_{\zeta}\). This follows since \(\zeta = f_n(\xi)\) for some \(\xi \in \partial \Omega\) and \(f_n\) is univalent in a neighborhood of \(\xi\), as \(f_n'(\xi) \neq 0\). Clearly \(g_n(0) = 0, g_n'(0) = 1\) and \(g_n(L) \subseteq L\). We also have

\[
\text{(2.4)} \quad |g_n(z)| = \left| \int_0^z g_n'(\zeta) d\zeta \right| \leq \lambda_{\alpha}|z|,
\]

for all \(z \in \Omega\). Set \(\tilde{f}_n(r) = e^{-i\theta}f_n(re^{i\theta})\) and \(\tilde{g}_n(r) = e^{-i\theta}g_n(re^{i\theta})\), \(0 \leq r \leq r_0\). Then \(\tilde{f}_n\) and \(\tilde{g}_n\) are real valued, \(\tilde{f}_n(0) = \tilde{g}_n(0) = 0\), \(\tilde{f}_n'(r) \geq 1\) and \(\tilde{g}_n = (\tilde{f}_n)^{-1}\).
We next define \( f_{n+1}(z) = 2G(z) - g_n(z) \). Then \( f_{n+1}(0) = 0 \), \( f'_{n+1}(0) = 1 \) and \( f_{n+1} \) is holomorphic on \( \Omega \), since \( G \) and \( g_n \) are holomorphic on \( \Omega \). Now \( g'_n(z) \in D_\alpha \) implies \( \Re [e^{i\alpha} (g'_n(z) - 1)] \leq 0 \), so
\[
\Re [e^{i\alpha} (f'_{n+1}(z) - 1)] = 2\Re [e^{i\alpha} (G'(z) - 1)] - \Re [e^{i\alpha} (g'_n(z) - 1)] > 0,
\]
for \( z \in \Omega \). Thus \( f_{n+1} \) is univalent in \( \Omega \) and \( f_{n+1}'(r) \geq 1 \), so \( f_{n+1}([0, r_0]) \supseteq [0, r_0] \) and \( f_{n+1}(L) \supseteq L \). Finally we let \( \zeta \in \partial \Omega \) and assume \( f_{n+1}(\zeta) \in \Omega \). As \( g_n(\zeta) \in \overline{\Omega} \) and \( \Omega \) is convex it follows that \( G(\zeta) = f_{n+1}(\zeta)/2 + g_n(\zeta)/2 \in \Omega \), which contradicts \( \Omega \subseteq G(\Omega) \). We conclude that \( \Omega \subseteq f_{n+1}(\zeta) \).

Relation (2.4) now shows that \( \{g_n\} \) is a normal family in \( \Omega \) and so is \( \{f_n\} \), by (2.3). If we let \( \tilde{G}(r) = e^{-i\theta}G(re^{i\theta}) \) then \( \tilde{f}_{n+1}(r) = 2\tilde{G}(r) - \tilde{g}_n(r) \), \( \tilde{g}_n = (\tilde{f}_n)^{-1} \) and, by (C2) and (C3), \( \tilde{G}'(r) > 1 \) for \( r \in (0, r_0) \). Thus Lemma 2.3 implies that \( \{\tilde{f}_n\} \) increases to a function \( \tilde{f} \) and \( \{\tilde{g}_n\} \) decreases to a function \( \tilde{g} \) which satisfies \( \tilde{g}'(r) \to 0 \) as \( n \to \infty \), for all \( 0 < r < r_0 \). It follows that any two subsequential limits of \( \{f_n\} \) and \( \{g_n\} \) respectively agree on \( L \), so there are functions \( f \) and \( g \) holomorphic on \( \Omega \) such that \( f_n \to f \) and \( g_n \to g \) locally uniformly in \( \Omega \). Since \( \Omega \subseteq f_n(\Omega) \), \( g_n = (f_n)^{-1} \) and \( f_{n+1} + g_n = 2G \) we have \( \Omega \subseteq f(\Omega) \), \( g = f^{-1} \) and \( f + g = 2G \) on \( \Omega \). By (2.4) \( |g(z)| \leq \lambda_\alpha |z| \), so \( g \), and hence \( f = 2G - g \), extend continuously at 0 by \( f(0) = g(0) = 0 \). As \( g(\Omega) \subseteq \Omega \) the iterates \( \{g^n\} \) form a normal family (in the larger sense that subsequences may diverge locally uniformly to infinity). But \( \tilde{g}^n \to 0 \) implies that \( g^n \to 0 \) locally uniformly on \( \Omega \).

To prove conclusion (i) of the theorem we use the facts that \( f \circ g = id \) and \( f + g = 2G \) on \( \Omega \) to see that
\[
F(z, g(z)) = (g(z), -z + 2G(g(z))) = (g(z), g^2(z))
\]
and
\[
F^{-1}(z, f(z)) = (2G(z) - f(z), z) = (g(z), z) = (g(z), f(g(z)))
\]
for all \( z \in \Omega \). We then get by induction that \( F^n(z, g(z)) = (g^n(z), g^{n+1}(z)) \) and \( F^{-n}(z, f(z)) = (g^n(z), f(g^n(z))) \), for \( z \in \Omega \) and for \( n \geq 0 \). Since \( g^n \to 0 \) as \( n \to \infty \), locally uniformly in \( \Omega \), and since \( f \) extends continuously at 0 by \( f(0) = 0 \), this shows that \( F^n(z, g(z)) \to 0 \) and \( F^{-n}(z, f(z)) \to 0 \) as \( n \to \infty \), locally uniformly for \( z \in \Omega \).

Finally, to prove (ii) we assume that \( \alpha = 0 \) and we notice by conclusion (i) that \( \text{graph } g \subseteq W_\alpha^1(0) \) and \( \text{graph } f \subseteq W_\alpha^n(0) \). Let \( F^n(z, w) = (z_n, w_n) \) for \( n \in \mathbb{Z} \). Assuming that \( (z, w) \in W_\alpha^1(0) \) and \( n > 0 \) we have
\[
F^n(z, w) = F(z_{n-1}, w_{n-1}) = (w_{n-1}, g(w_{n-1}) + f(w_{n-1}) - z_{n-1})
\]
so, since by (2.1) $\Re f'(z) > 1$ in $\Omega$, we see using Lemma 2.1 that there is $\beta \in \mathbb{C}$ with $\Re \beta > 1$ such that
\[
f(w_n) - z_n = f(g(w_{n-1}) + f(w_{n-1}) - z_{n-1}) - w_{n-1} = f(g(w_{n-1})) + \beta(f(w_{n-1}) - z_{n-1}) - w_{n-1} = \beta(f(w_{n-1}) - z_{n-1}).
\]
Thus by induction we get $|f(w_n) - z_n| \geq |f(w) - z|$ and since $(z_n, w_n) \to 0$ and $f$ is continuous at 0 we conclude that $f(w) = z$, so $(z, w) \in \text{graph } g$.

Similarly, if $(z, w) \in W^s(0)$ and $n < 0$ we have $|f(z_n) - w_n| \geq |f(z) - w|$, hence $(z, w) \in \text{graph } f$ and the theorem is completely proved.

**Remark.** We now make some remarks on the analyticity of $f$ and $g$ at 0. Without loss of generality we may assume that $\theta = 0$ in condition (C3), so $G$ has the following expansion at 0
\[
G(z) = z + \alpha_{j+1}z^{j+1} + \alpha_{j+2}z^{j+2} + \ldots, \quad \alpha_{j+1} \neq 0,
\]
where $j \geq 1$ and $\alpha_n \in \mathbb{R}$.

If $f$ and $g$ are analytic at 0 we write
\[
f(z) = a_1z + a_2z^2 + \ldots,
g(z) = 2G(z) - f(z) = (2 - a_1)z - a_2z^2 - \ldots + (2\alpha_{j+1} - a_{j+1})z^{j+1} + \ldots,
\]
expand $g \circ f$ around 0 and use $g \circ f(z) = z$ to find $a_n$ inductively. Clearly $a_1 = 1$.

We first notice by induction on $n$ that if $n \geq 2$ and $2n - 1 < j + 1$ then $a_n = 0$. Indeed, the coefficient of $z^3$ in the expansion of $g \circ f$ is $a_3 - 2a_2^2 - a_3 = 0$, so $a_2 = 0$; moreover, if $n$ is such that $2n - 1 < j + 1$ and $a_2 = \ldots = a_{n-1} = 0$ then the coefficient of $z^{2n-1}$ in $g \circ f$ is $a_{2n-1} - na_n^2 - a_{2n-1} = 0$, so $a_n = 0$.

There are two cases:

**Case 1.** $j + 1 = 2l$, $l \geq 1$. By above $f(z) = z + a_{l+1}z^{l+1} + \ldots$. The coefficient of $z^{2l}$ in $g \circ f$ is $a_{2l} + 2a_{2l} - a_{2l} = 0$, so $\alpha_{j+1} = 0$, a contradiction. Thus in the case when $j + 1$ is even there are no functions $f$, $g$ holomorphic around 0 such that $g = f^{-1}$ and $f + g = 2G$. Consequently the functions $f$, $g$ of Theorem 1.1 do not extend analytically at 0.

**Case 2.** $j + 1 = 2l - 1$, $l \geq 2$. By above $f(z) = z + a_lz^l + \ldots$, and computing the coefficients as indicated before it is not hard to see that $a_2^2 = 2\alpha_{j+1}/l \neq 0$ and the coefficient of $z^{2l+n-1}$ in the expansion of $g \circ f$ yields a formula for $a_{l+n}$, $n > 0$, of the type $a_l a_{l+n} = E(a_l, \ldots, a_{l+n-1}; \alpha_k, k \leq 2l + n - 1)$. So a formal
power series at 0 exists for $f$ (and hence for $g$) such that $g = f^{-1}$ and $f + g = 2G$. Moreover the coefficients $a_n$ are real, since $\alpha_n$ are real.

*Remark.* The proof of Theorem 1.1 shows that the operator $T : X(\Omega, \alpha) \to X(\Omega, \alpha), Tu = 2G - u^{-1}$ is well defined and that $f$ is a fixed point of $T$, obtained as the limit of $\{T^nG\}_{n \geq 0}$.

**Proof of Theorem 1.2.** For $r > 0$ small enough we let

$$
S(r) = \{ z \in \mathbb{C} : |z| < r, |\arg z| < \pi/j \},
$$

$$
Q(r) = S(r)^{-j} = \{ w \in \mathbb{C} : |w| > r^{-j}, |\arg w| = \pi \},
$$

and we consider the holomorphic function

$$
(2.5) \quad H(w) = \left[ G \left( w^{-1/j} \right) \right]^{-j} = w - aj + O(|w|^{-1/j}),
$$

defined for $w \in Q(r)$. Here $G$ has the form (1.2). We need the following

**Lemma 2.4.** There exist positive constants $r_0$ and $C_0$ such that

(i) $r_0C_0 > a(j + 1)$;

(ii) $|h(z)| < C_0|z|^{j+1}$, $|h'(z)| < C_0|z|^{j+1}$, for $z$ with $|z| \leq r_0$;

(iii) $|H(w) - w + aj| < C_0|w|^{-1/j}$, for $w \in Q(r_0)$;

(iv) $|H(w)|^{j+1} > |w|^{j+1} - C_0|w|^{-1/j}$, for $w \in Q(r_0)$.

**Proof.** By (1.2) and (2.5) one can clearly choose some constants $r_0$, $C_0$ so that (i), (ii) and (iii) hold. If $z = w^{-1/j}$ for $w \in Q(r_0)$ then

$$
|H(w)|^{j+1} = |G(z)|^{(j-1)} = \left| \frac{1}{z^{j-1}}(1 - (j - 1)az^j + O(|z|^{j+1})) \right|
$$

$$
> \frac{1}{|z|^{j-1}} - O(|z|) = |w|^{j+1} - O(|w|^{-1/j}),
$$

so (iv) also holds for $r_0$, $C_0$ suitably chosen.

For $\theta \in \left( -\frac{\pi}{2j}, \frac{\pi}{2j} \right)$ we define

$$
(2.6) \quad C(\theta) = \left[ \frac{C_0 \left( 1 + \frac{C_0}{aj(j+1)} \right)}{aj \cos j\theta} \right]^j,
$$

$$
(2.7) \quad \Omega(\theta) = \left\{ z = re^{i\phi} \neq 0 : \cos j(\phi - \theta) > \max \left( \frac{C_0}{a(j+1)}r, C(\theta)r^{j} \right) \right\}.
$$
Clearly $0 \in \partial D$. Using the fact that a domain $\{r < \beta(\phi)\}$ is convex if and only if $\beta^2 + 2(\beta')^2 - \beta\beta'' \geq 0$ it is easy to see that $\Omega(\theta)$ is the intersection of two convex domains, hence it is convex. It follows that $D$ is starlike with respect to 0, hence simply connected. Now $\partial \Omega$ has two tangents at 0, namely the rays $\{z : |\arg z - \theta| = \frac{\pi}{2}\}$. This implies that $D \subseteq \{z : |\arg z| < \pi/j\}$ and that the rays $\{z : |\arg z| = \pi/j\}$ are tangent to $\partial D$ at 0, so assertion (i) of the theorem holds.

**Lemma 2.5.** $G \in X(\Omega(\theta), -j\theta)$ and $G(\Omega(\theta)) \supset \overline{\Omega(\theta)} \setminus \{0\}$.

**Proof.** Recall the definition of $X(\Omega, \alpha)$ from section 1, using the conditions (C1), (C2), (C3). By (i) and (ii) of Lemma 2.4 and by the definition (2.7) of $\Omega(\theta)$ we have that if $z = re^{j\phi} \in \Omega(\theta)$ then $|z| < r_0$ and

$$\Re \left[ e^{-ij\theta}(G'(z) - 1) \right] = a(j + 1) \Re (z^j e^{-ij\theta}) + \Re (e^{-ij\theta} h'(z)) > a(j + 1) r^j \cos j(\phi - \theta) - C_0 r^{j+1} > 0,$$

so (C2) holds. We now show that $C(\Omega(\theta)) \supset \overline{\Omega(\theta)} \setminus \{0\}$. This is equivalent to showing (since $G$ is conjugated to $H$ by $w = z^{-j}$) that $H(\omega(\theta)) \supset \overline{\omega(\theta)}$, where

$$\omega(\theta) = \Omega(\theta)^{-1} = \left\{ w : \Re (we^{ij\theta}) > \max \left( \frac{C_0}{a(j + 1)} |w|^{\frac{j-1}{j}}, C(\theta) \right) \right\}.$$

It suffices to prove that $H(\partial \omega(\theta)) \cap \overline{\omega(\theta)} = \emptyset$. Let $w \in \partial \omega(\theta)$. Then $|w| \geq C(\theta)$. We have two cases.

Case 1. $\Re (we^{ij\theta}) = \frac{C_0}{a(j + 1)} |w|^{\frac{j-1}{j}}$. Then by (2.5) and by (iii) and (iv) of Lemma 2.4 we have

$$\Re \left( H(w)e^{ij\theta} \right) < \Re (we^{ij\theta}) - a j \cos j\theta + C_0 |w|^{-1/j}$$

$$= \frac{C_0}{a(j + 1)} |w|^{\frac{j-1}{j}} - a j \cos j\theta + C_0 |w|^{-1/j}$$

$$< \frac{C_0}{a(j + 1)} |H(w)|^{\frac{j-1}{j}} + \frac{C_0^2}{a(j + 1)} |w|^{-1/j} - a j \cos j\theta + C_0 |w|^{-1/j}$$

$$\leq \frac{C_0}{a(j + 1)} |H(w)|^{\frac{j-1}{j}} + C_0 \left( 1 + \frac{C_0}{a(j + 1)} \right) C(\theta)^{-1/j} - a j \cos j\theta$$

$$= \frac{C_0}{a(j + 1)} |H(w)|^{\frac{j-1}{j}},$$

the last equality following from the definition (2.6) of $C(\theta)$. So $H(w)$ is not in $\omega(\theta)$. 
Case 2. $\Re(we^{ij\theta}) = C(\theta)$. Then by (2.5) and (2.6) we get

$$\Re(H(w)e^{ij\theta}) < \Re(we^{ij\theta}) - aj\cos j\theta + C_0|w|^{1/j} \leq C(\theta) - aj\cos j\theta + C_0C(\theta)^{-1/j} < C(\theta),$$

hence $H(w)$ is not in $\overline{\omega(\theta)}$.

We already proved that $\Omega(\theta)$ is convex, so the proof of the lemma is complete once we notice that $(0, x(\theta)] \subset \Omega(\theta) \cap \mathbb{R}$, for some $x(\theta) > 0$. □

We now return to the proof of the theorem. By Theorem 1.1 there are univalent functions $f_\theta : \Omega(\theta) \rightarrow f_\theta(\Omega(\theta))$ and $g_\theta = (f_\theta)^{-1}$ satisfying conclusion (i) of Theorem 1.1. It follows from the proof of Theorem 1.1 and from Lemma 2.3 that for $\theta, \theta' \in (-\frac{\pi}{2j}, \frac{\pi}{2j})$ we have $f_\theta - f_{\theta'}$ and $g_\theta = g_{\theta'}$ on $\Omega(\theta) \cap \Omega(\theta') \cap \mathbb{R}$. Since $D$ is simply connected we conclude that there are holomorphic functions $f$ and $g$ defined on $D$ such that $f = f_\theta$ and $g = g_\theta$ on $\Omega(\theta)$ for all $\theta \in (-\frac{\pi}{2j}, \frac{\pi}{2j})$. Now $g(D) = \bigcup_\theta g(\Omega(\theta)) \subseteq \bigcup_\theta \Omega(\theta) = D$ and since $f_\theta \circ g_\theta = id$ and $f_\theta + g_\theta = 2G$ on $\Omega(\theta)$ it follows that $f \circ g = id$ and $f + g = 2G$ on $D$. Since $(g_\theta)^n \rightarrow 0$ on $\Omega(\theta)$ we see that $g^n \rightarrow 0$ pointwise on $D$, and hence locally uniformly on $D$, by Montel's theorem. Assertion (iii) of the theorem is now proved as in Theorem 1.1.

Remark. The domain $D$ constructed here has maximal aperture at 0, in the sense that $G$ doesn’t satisfy condition (C2) on domains with aperture larger than $2\pi/j$. Indeed, if $z = re^{i\phi}$ is such that $|\phi| = \pi/j$ then for any $\alpha \in (-\pi/2, \pi/2)$ we have

$$\Re(e^{i\alpha}(G'(z) - 1)) \leq -a(j + 1)r^j \cos \alpha + C_0r^{j+1} < 0,$$

provided that $C_0r < a(j + 1) \cos \alpha$.

Proof of Theorem 1.3. We fix $k \in \{0, \ldots, j - 1\}$ and consider the following rays $L_k$:

$$L_k = \left\{ r \exp \left( \frac{2k\pi i}{j} \right) : 0 < r < \infty \right\}.$$

We also define, for $\beta \in (0, \pi/j)$,

$$\Omega(k, \beta) = \left\{ r \exp(i\beta) : 0 < r < \infty, \frac{2k - 1}{j} \pi + \beta < \phi < \frac{2k\pi}{j} + \beta \right\}.$$

We have $L_k \subset \Omega(k, \beta)$ for every $\beta$, $\bigcup_{\beta \in (0, \pi/j)} \Omega(k, \beta) = D_k$, where $D_k$ is defined by (1.3). Also, if $\alpha = \pi/2 - j\beta$ then $\alpha \in (-\pi/2, \pi/2)$ and $G_j$ satisfies $\Re(e^{i\alpha}((G_j)'(z) - 1)) > 0$ for $z \in \Omega(k, \beta)$. We claim that $\Omega(k, \beta) \subseteq G_j(\Omega(k, \beta))$, for all $\beta$. Indeed, if we write $\beta \Omega(k, \beta) = R_- \cup R_+$, where $\phi_- = (2k - 1)/j + \beta$, $\phi_+ = 2k\pi/j + \beta$, $R_- = \{ r \exp(i\phi_-) \}$, $R_+ = \{ r \exp(i\phi_+) \}$, then $(R_-)^{j+1}$ is a ray lying in the half plane $H_- = \{ z : \phi_- - \pi < \arg z < \phi_- \}$, so $G_j(R_-) \subset$
Similarly, $G_j(R_+) \subset R_++(R_+)^{j+1} \subset H_+$, where $H_+ = \{ z : \phi_+ < \arg z < \phi_+ + \pi \}$, and the claim is proved.

By Theorem 1.1 there are univalent functions $f_{k,\beta} : \Omega(k,\beta) \to f_{k,\beta}(\Omega(k,\beta))$ and $g_{k,\beta} = (f_{k,\beta})^{-1}$ satisfying conclusion (i) of Theorem 1.1. As the rays $L_k$ are invariant for $G_j$, it follows from the proof of Theorem 1.1 that they will be invariant for $f_{k,\beta}$ and $g_{k,\beta}$ as well and that all the functions $f_{k,\beta}$ agree on $L_k$ and all the functions $g_{k,\beta}$ agree on $L_k$. Thus there are holomorphic functions $f_k$ and $g_k$ defined on $D_k$ so that $f_{k,\beta} = f_k$ and $g_{k,\beta} = g_k$ on $\Omega(k,\beta)$, for all $\beta$. As in the proof of Theorem 1.2 we have $g_k(D_k) \subseteq D_k$ and $f_k \circ g_k = id$ on $D_k$. Assuming for a contradiction that $g_k(D_k) = D_k$ (or $f_k(D_k) = D_k$) we get $f_k(D_k) = D_k$ (or $g_k(D_k) = D_k$) and $2G_j(D_k) \subseteq f_k(D_k) + g_k(D_k) = D_k$, so $G_j(D_k) \subseteq D_k$, which is false.

Finally, as $G_j(\exp(2k\pi i/j)z) = \exp(2k\pi i/j)G_j(z)$, $z \in D_0$, it follows easily from the uniqueness part (conclusion (ii)) of Theorem 1.1 that

$$f_k(\exp(2k\pi i/j)z) - \exp(2k\pi i/j)f_0(z),$$

$$g_k(\exp(2k\pi i/j)z) - \exp(2k\pi i/j)g_0(z),$$

for $z \in D_0$. The rest of the assertions of Theorem 1.3 follow directly from Theorem 1.1.

**Remark.** In [F], Theorem 4, the author considers the real Henon map $F(x, y) = (y, -x + 2G(y))$, obtained for $G(x) = x + x^2$. He shows, via real methods, that the functions $g(x)$ and $f(x)$, defined for $x \geq 0$ and whose graphs give the stable/unstable manifolds of $F$ in $\{(x, y) : x \geq 0\}$, are real analytic on $(0, \infty)$ and the radii of convergence of their Taylor series at $x_0 \in (0, \infty)$ are $x_0$; thus $f$ and $g$ are analytic on the right half plane $\{ \Re z > 0 \}$. Theorem 1.3 shows that these functions $f$ and $g$ are actually analytic on the slit plane $\mathbb{C} \setminus \{ z \leq 0 \}$.

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Received September 6, 1997

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