Houston Journal of Mathematics
© 1998 University of Houston
Volume 24, No. 1, 1998

ON THE EXISTENCE AND GROWTH OF MILD SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM FOR OPERATORS WITH POLYNOMIALLY BOUNDED RESOLVENT

J.M.A.M. VAN NEERVEN[1] AND B. STRAUB[2]

COMMUNICATED BY GILES AUCHMUTY

Abstract. In this paper we study the growth of mild solutions of abstract Cauchy problems governed by a densely defined generator A of an α-times integrated semigroup $\{S^\alpha(t)\}_{t\geq 0}$. We prove the following results:

(i) If $\|S^\alpha(t)\| \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$, and all $t \geq 0$, then for all $\varepsilon > 0$, $\sigma > 0$ and $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ a unique mild solution exists. Moreover, this solution is exponentially bounded, and its exponential type is at most ω. If $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$, the solution is classical.

(ii) If $\|S^\alpha(t)\| \leq M(1+t^\gamma)$ for some constants $M \geq 1$, $\gamma \geq 0$, and all $t \geq 0$, then for all $\varepsilon > 0$, $\sigma > 0$ and all $x_0 \in D((-A_\sigma)^{\alpha+\varepsilon})$ a unique mild solution exists. Moreover, this solution is polynomially bounded, and its polynomial type is at most $\max\{\alpha - 1 + \varepsilon, \gamma + \varepsilon, 2\gamma - \alpha + \varepsilon\}$. If $x_0 \in D((-A_\sigma)^{1+\alpha+\varepsilon})$, the solution is classical.

These results are applied to study the growth of mild solutions of the Cauchy problem governed by a densely defined operator whose resolvent is polynomially bounded in the open right half plane.

1. Introduction

In this paper we study the asymptotic behaviour of solutions of the abstract Cauchy problem

$$\text{(ACP)} \qquad u'(t) = Au(t) \quad (t \geq 0), \qquad u(0) = x_0,$$

[1] Research supported by the Netherlands Organization for Scientific Research (NWO) and an Individual Fellowship in the Human Capital and Mobility Programme of the European Communities. Current address: Department of Mathematics, Technical University Delft, PO Box 5031, 2600 GA Delft, The Netherlands.

[2] Research partially supported by the Louisiana Education Quality Support Fund (LEQSF). Current address: School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia.

where A is a closed linear operator with domain $D(A)$ in a complex Banach space X. We will investigate this problem for densely defined operators A satisfying certain resolvent estimates in a right half plane. We prove existence and uniqueness of mild and classical solutions for initial values in optimal domains, and give optimal estimates for the growth of these solutions.

In order to motivate our approach, let A be a linear operator whose resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ exists and is polynomially bounded of order $O(|\lambda|^{\gamma-1})$ for some $\gamma \geq 0$ and all $\operatorname{Re}\lambda > \omega \geq 0$. By the general theory of Laplace transforms, the operator A generates an exponentially bounded, α-times integrated semigroup $\mathbf{S}^\alpha = (S^\alpha(t))_{t\geq 0}$ for every $\alpha > \gamma$ given by

$$S^\alpha(t)x := \frac{1}{2\pi i}\int_{\omega+\sigma+i\mathbb{R}} e^{\lambda t}\lambda^{-\alpha}R(\lambda, A)x\, d\lambda \tag{1.1}$$

for all $t \geq 0$ and $x \in X$ (see, e.g., [3, Theorem 3.1]). If one takes $\alpha = n \in \mathbb{N}$ then it is well-known that (ACP) admits a unique mild solution $u(\cdot, x_0)$ for every $x_0 \in D(A^n)$, which is given explicitly by

$$u(t, x_0) = \frac{d^n}{dt^n}S^n(t)x_0 = \sum_{k=0}^{n-1}\frac{t^k}{k!}A^k x_0 + S^n(t)A^n x_0, \quad t \geq 0. \tag{1.2}$$

In case $x_0 \in D(A^{n+1})$, then $u(\cdot, x_0)$ is a classical solution. Formula (1.2) shows that the exponential type of $u(\cdot, x_0)$ does not exceed that of the integrated semigroup $\mathbf{S}^n$. Also, if $\mathbf{S}^n$ is polynomially bounded of order β, then $u(\cdot, x_0)$ is polynomially bounded of order $\max\{n-1, \beta\}$.

Now, if the resolvent grows like $O(|\lambda|^{\gamma-1})$, integer exponents n in (1.1) are not optimal and should be replaced by exponents $\gamma + \varepsilon$. This leads us to study fractionally integrated semigroups $\mathbf{S}^\alpha$. Formally, one expects that, in analogy to the integer case, (ACP) has a mild solution u given by $u(t) = \frac{d^\alpha}{dt^\alpha}S^\alpha(t)x$ for every $x \in D((-A)^\alpha)$. Formal, but simple Laplace transform manipulations show that this fractional derivative of $S^\alpha(\cdot)x$ should be given by the singular integral

$$u(t) = \Gamma_\alpha \int_0^\infty \frac{1}{s-1}\Big(s^{[\alpha]-\alpha}\frac{d^{[\alpha]}}{dt^{[\alpha]}}S^\alpha(t) - s^{-1}\frac{d^{[\alpha]}}{dt^{[\alpha]}}S^\alpha(t/s)\Big)(-A)^{\alpha-[\alpha]}x_0\, ds. \tag{1.3}$$

Here, $\Gamma_\alpha = \pi^{-1}\sin((\alpha-[\alpha])\pi)$ and $[\alpha]$ denotes the integer part of α. This integral, however, has singularities in 0, 1, and ∞. Moreover, it is not clear whether the fractional powers of $-A$ exist. We overcome both problems by considering, for $\beta > \alpha$, the β-times integrated semigroup $\mathbf{S}^\beta_{\omega+\sigma}$ generated by $A_{\omega+\sigma} := A - \omega - \sigma$; here ω is the exponential type of $\mathbf{S}^\alpha$ and $\sigma > 0$ is arbitrary. The main results show that, by doing so, we obtain mild solutions for more initial values x_0 than by considering only the integer case, and that their behaviour is controlled by that of the integrated semigroup as in the integer case.

Theorem 1.1. *Let $\alpha \geq 0$ and A be the densely defined generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq Me^{\omega t}$ for some constants $M \geq 1$, $\omega \geq 0$, and all $t \geq 0$. Then, for all $\varepsilon > 0$, $\sigma > 0$ and $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP) has a unique mild solution. Moreover, this solution is exponentially bounded, and its exponential type is at most ω. If $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$, the solution is classical.*

Theorem 1.2. *Let $\alpha \geq 0$, and let A be the densely defined generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq M(1+t^\gamma)$ for some constants $M \geq 1$, $\gamma \geq 0$, and all $t \geq 0$. Then, for all $\varepsilon > 0$, $\sigma > 0$ and all $x_0 \in D((-A_\sigma)^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP) has a unique mild solution. Moreover, this solution is polynomially bounded, and its polynomial type is at most* $\max\{\alpha - 1 + \varepsilon, \gamma + \varepsilon, 2\gamma - \alpha + \varepsilon\}$. *If $x_0 \in D((-A_\sigma)^{1+\alpha+\varepsilon})$, the solution is classical.*

In these results, we use fractional powers for operators whose resolvent exists and is polynomially bounded in a sector $\{|\arg\lambda| \leq \varphi\}$; cf. Section 5. However, we would like to point out that this does not increase the difficulty of the argument. Its complexity is caused by the generality of the setting and our efforts to obtain the correct orders for the growth of the solutions. The fractional powers of $-A$ in Theorem 1.1 (resp. Theorem 1.2) exist in the classical sense if $\|S^\alpha(t)\| \leq Mt^\alpha e^{\omega t}$ for some $\omega \geq 0$ (resp. $\omega = 0$). In that case, we obtain in Theorem 1.2 polynomially bounded solutions of order α.

The results are optimal in the following sense. Suppose that A is a closed, densely defined linear operator such that fractional powers of $-A$ can be defined and let $\alpha \geq 0$. Then A is the generator of an exponentially bounded β-times integrated semigroup for every $\beta > \alpha$ if and only if the abstract Cauchy problem (ACP) has a classical solution $u(\cdot, x)$ for every $\beta > \alpha$ and $x \in D((-A)^{\beta+1})$ such that $u(\cdot, x)$ and $u'(\cdot, x)$ are exponentially bounded (cf. [23]). If $\alpha \in \mathbb{N}$, then the equivalence holds also for $\beta = \alpha$ (see, e.g., [5, Theorem 2.5], [16, Theorem 1] or [19, Theorem 4.2]).

Let us briefly sketch the history of the Theorems 1.1 and 1.2. If $0 < \alpha < 1$ and $\|S^\alpha(t)\| \leq Mt^\alpha e^{\omega t}$, it was shown by M. Hieber [10] that the Cauchy problem admits a unique classical solution for each $x_0 \in D((-A_{\omega+1})^{1+\alpha+\varepsilon})$, and that this solution is exponentially bounded. He states without a proof that this result is valid for all $\alpha \geq 0$. Although no bound for the exponential type is given, his proof shows that it is at most 3ω. Hieber's result was extended in [21] to α-times integrated semigroups, $0 < \alpha < 1$, satisfying $\|S^\alpha(t)\| \le Mt^\beta e^{\omega t}$ for some $0 \le \beta \le \alpha$. The proof there also covers some cases where α and/or β are greater than or equal to 1.

The paper is organized as follows. In Section 4, we show that for $\beta > \alpha$ and $x \in D((-A_{\omega+\sigma})^\beta)$ the integral

$$u_{\omega+\sigma}(t) = \Gamma_\beta \int_0^\infty \frac{1}{s-1}\Big(s^{[\beta]-\beta} \frac{d^{[\beta]}}{dt^{[\beta]}} S^\beta_{\omega+\sigma}(t) - \frac{1}{s}\frac{d^{[\beta]}}{dt^{[\beta]}} S^\beta_{\omega+\sigma}(\frac{t}{s})\Big)(-A_{\omega+\sigma})^{\beta-[\beta]}x_0\,ds$$

converges absolutely and represents a continuous, polynomially bounded function. Note that this is the equivalent to (1.3) for the β-times integrated semigroup $\mathbf{S}^\beta_{\omega+\sigma}$ generated by $A_{\omega+\sigma}$. In Section 6, we show that $u_{\omega+\sigma}(\cdot)$ is a mild solution of the problem

$$(\mathrm{ACP}_{\omega+\sigma}) \qquad u'(t) = A_{\omega+\sigma}u(t) \quad (t \ge 0), \qquad u(0) = x_0.$$

For the proof, we need detailed estimates on the behaviour of $\mathbf{S}^\beta_{\omega+\sigma}$ and its $[\beta]$-th derivative. We approach this problem in Section 3 by explicitly representing $\mathbf{S}^\beta_{\omega+\sigma}$ as a Stieltjes integral,

$$S^\beta_{\omega+\sigma}(t)x \;=\; \int_0^t e^{-(\omega+\sigma)(t-s)} S^\beta(t-s)x\, dg_{\omega+\sigma,\beta}(s),$$

where $g_{\omega+\sigma,\beta}$ is the unique non-negative, non-decreasing, left-continuous function whose Laplace-Stieltjes transform is $\lambda^{-\beta}(\lambda+\omega+\sigma)^\beta$. This representation allows us to reduce the problem of estimating $\mathbf{S}^{\alpha+\varepsilon}_{\omega+\sigma}$ to that of obtaining estimates for certain Stieltjes integrals involving $g_{\omega+\sigma,\alpha+\varepsilon}$. These are collected in Section 2.

We return to operators A whose resolvent is polynomially bounded in the right half plane in Section 7. Starting with different resolvent estimates we estimate the (polynomial) growth of the integrated semigroups generated by A. This leads to the following result.

Theorem 1.3. *Let A be a densely defined linear operator on X whose resolvent exists in the right half plane. Suppose there are $\alpha \ge 0$ and $\beta \ge 0$, $\alpha - 1 \le \beta \le \alpha$, such that*

$$\|R(\lambda, A)\| \le M|\lambda|^{\alpha-1}(Re\,\lambda)^{-\beta}, \qquad Re\,\lambda > 0.$$

Then, for all $\varepsilon > 0$, $\sigma > 0$, and $x_0 \in D((-A_\sigma)^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP) has a unique mild solution $u(\cdot, x_0)$. Moreover, this solution is polynomially bounded of order $\beta + \varepsilon$. If $x_0 \in D((-A_\sigma)^{1+\alpha+\varepsilon})$, the solution is classical.

Note that the condition $\alpha - 1 \leq \beta \leq \alpha$ is a consistency requirement which is automatically satisfied whenever A is an operator satisfying the remaining conditions.

If $\alpha = \beta$, the fractional powers are the classical ones, and we obtain polynomially bounded solutions of order α. In the special case that $\alpha = \beta = 1$, this holds even if $D(A)$ is not dense in X (cf. [6, Theorem 4.10 and Remark 4.11]).

A similar result is obtained if $\|R(\lambda, A)\| \leq M(1 + |\lambda|)^{\alpha-1}$ for some $\alpha \geq 0$ and all $\operatorname{Re}\lambda > 0$, in which case we obtain solutions in $D((-A)^{\alpha+\varepsilon})$ of polynomial order $\max\{0, \alpha - 1\} + \varepsilon$.

In the final Section 8 we apply our results to certain differential operators on $L^p(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$.

2. Preliminary estimates

In this section, we will derive estimates for certain Laplace-Stieltjes transforms that will be useful in the sequel.

We start by recalling some basic facts concerning completely monotonic functions. A C^∞-function $G : (0, \infty) \to \mathbb{R}$ is said to be *completely monotonic* if $(-1)^n G^{(n)}(\lambda) \geq 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. Here $G^{(n)}$ denotes the nth derivative of G. By Bernstein's theorem [26, Theorem IV.12.b], G is completely monotonic if and only if it is the Laplace-Stieltjes transform of a non-decreasing, non-negative, left-continuous function $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$. The function g is uniquely determined by G. Recall that the Laplace-Stieltjes transform G of g is defined as

$$G(\lambda) = \int_0^\infty e^{-\lambda t}\, dg(t), \quad \lambda > 0,$$

where the integral is an improper Stieltjes integral. In the above situation

$$\int_0^\infty e^{-\lambda t}\, dg(t) = \lim_{T\to\infty} \int_0^T e^{-\lambda t}\, dg(t) = g(0+) + \lim_{\substack{\varepsilon \downarrow 0 \\ T\to\infty}} \int_\varepsilon^T e^{-\lambda t}\, dg(t).$$

For $\mu \in \mathbb{C}$ and $\beta \geq 0$, we define the functions $g_{\mu,\beta} : [0, \infty) \to \mathbb{R}$ by

$$g_{\mu,\beta}(s) = \chi_{(0,\infty)}(s) + \sum_{k=1}^\infty \frac{<\beta>_k\ (\mu s)^k}{(k!)^2}, \quad s \geq 0.$$

Here $<\beta>_k := \beta(\beta - 1)\dots(\beta - k + 1)$, $k = 1, 2, \dots$, and $\chi_{(0,\infty)}$ denotes the characteristic function of $(0, \infty)$. For $\mu > 0$, the following lemma identifies the Laplace-Stieltjes transform of $g_{\mu,\beta}$.

Lemma 2.1. *For each $\mu > 0$ and $\beta \geq 0$, the function $G_{\mu,\beta}(\lambda) = \lambda^{-\beta}(\lambda+\mu)^{\beta}$, $\lambda > 0$, is completely monotonic. Moreover, it is the Laplace-Stieltjes transform of $g_{\mu,\beta}$. In particular, $g_{\mu,\beta}$ is non-negative and non-decreasing.*

PROOF. Fix $\mu > 0$ and $\beta \geq 0$. Observe that $G_{\mu,\beta}(\lambda) = F(H(\lambda))$, where $F(\lambda) = \lambda^{-\beta}$ and $H(\lambda) = \frac{\lambda}{\lambda+\mu}$. It is easy to verify that F is completely monotonic. The function H satisfies $H(\lambda) > 0$ for all $\lambda > 0$ and its derivative H' is completely monotonic. By [8, Criterion XIII.4.2], it follows that $G_{\mu,\beta}$ is completely monotonic.

By Bernstein's theorem, there is a unique non-decreasing, non-negative, left-continuous function $g : [0,\infty) \to \mathbb{R}$ with $g(0) = 0$, the Laplace-Stieltjes transform of which is $G_{\mu,\beta}$. On the other hand, we note that for all $\lambda > \mu$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda s}\, d\left(\chi_{(0,\infty)}(s) + \sum_{k=1}^{\infty} \frac{<\beta>_k\,(\mu s)^k}{(k!)^2}\right) &= 1 + \sum_{k=1}^{\infty} \frac{<\beta>_k}{k!}\,\frac{\mu^k}{\lambda^k} \\
&= \left(1+\frac{\mu}{\lambda}\right)^{\beta} \\
&= G_{\mu,\beta}(\lambda).
\end{aligned}$$

The interchange of Stieltjes integral and summation is justified by [26, Theorem I.16.4] applied to the partial sums. By the uniqueness theorem for the Laplace-Stieltjes transform, it follows that $g = g_{\mu,\beta}$. □

We will need a number of estimates concerning $g_{\mu,\beta}$. Throughout the paper, we adopt the convention that indices attached to a constant express on which parameters the constant depends.

Lemma 2.2. *For each $0 < \beta < 1$ there is a constant C_β such that for all $\omega > 0$ and $\sigma > 0$ we have*

$$\int_0^t e^{-\sigma(t-s)}\, dg_{\omega+\sigma,\beta}(s) \;\leq\; C_\beta \max\left\{1, \frac{1}{\ln(1+\frac{\sigma}{2\omega})}\right\} \sigma^{\beta-1}\, t^{\beta-1}, \qquad t > 0.$$

PROOF. The proof is a refinement of an argument in [21, Theorem 2.1]. Fix $0 < \beta < 1$, $\omega > 0$ and $\sigma > 0$. By [7, Formulas 4.1 (20), 4.3 (1), 5.4 (1) and 5.4 (8)], we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \int_0^t e^{-\sigma(t-s)}\, dg_{\omega+\sigma,\beta}(s)\, dt \\
&= \int_0^\infty e^{-(\lambda+\sigma)t}\, dt \int_0^\infty e^{-\lambda t}\, dg_{\omega+\sigma,\beta}(t) \\
&= \frac{1}{\lambda+\sigma} \left(\frac{\lambda+\omega+\sigma}{\lambda}\right)^\beta \\
&= \frac{1}{\Gamma(\beta)} \frac{\Gamma(\beta)}{\lambda^\beta} \frac{(\lambda+\omega+\sigma)^\beta}{\lambda+\sigma} \\
&= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^{\beta-1} dt\, \frac{1}{\Gamma(1-\beta)} \int_0^\infty e^{-\lambda t} t^{-\beta} e^{-(\omega+\sigma)t} \left(\sum_{k=0}^\infty \frac{k!}{(1-\beta)_k} \frac{(\omega t)^k}{k!}\right) dt \\
&= \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^\infty e^{-\lambda t} \int_0^t (t-s)^{\beta-1} s^{-\beta} e^{-(\omega+\sigma)s} \left(\sum_{k=0}^\infty \frac{(\omega s)^k}{(1-\beta)_k}\right) ds\, dt,
\end{aligned}$$

where $(u)_0 = 1$ and $(u)_k = u(u+1)\dots(u+k-1)$ if $k \geq 1$. The uniqueness theorem for Laplace transforms gives

$$\begin{aligned}
&\int_0^t e^{-\sigma(t-s)} dg_{\omega+\sigma,\beta}(s) \\
&\qquad = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-1} s^{-\beta} e^{-(\omega+\sigma)s} \left(\sum_{k=0}^\infty \frac{(\omega s)^k}{(1-\beta)_k}\right) ds.
\end{aligned} \tag{2.1}$$

We are going to estimate the integral in (2.1). Since $0 < \beta < 1$, we have $(1-\beta)_k \geq (1-\beta)(k-1)!$ for all $k \geq 1$. Hence,

$$\sum_{k=0}^\infty \frac{(\omega s)^k}{(1-\beta)_k} \leq \frac{1}{1-\beta}\left(1 + \sum_{k=1}^\infty \frac{(\omega s)^k}{(k-1)!}\right). \tag{2.2}$$

Put $\delta := \frac{\sigma}{2\omega}$. The function $\xi \longmapsto \xi(1+\delta)^{-\xi}$, $\xi > 0$, attains its maximum at the point $\xi = (\ln(1+\delta))^{-1}$, where it takes the value $(e\ln(1+\delta))^{-1}$. It follows that

$$k \leq \frac{1}{e\ln(1+\delta)}(1+\delta)^k, \qquad k = 1, 2, \dots.$$

Put $c_\delta = (\ln(1+\delta))^{-1}$. By this and the definition of δ,

$$
\begin{aligned}
1+\sum_{k=1}^{\infty}\frac{(\omega s)^k}{(k-1)!} &= 1+\sum_{k=1}^{\infty}\frac{k(\omega s)^k}{k!} \\
&\le \max\{1,c_\delta\}\sum_{k=0}^{\infty}\frac{1}{k!}((1+\delta)\omega s)^k \qquad (2.3)\\
&= \max\{1,c_\delta\}e^{(\omega+\frac{\sigma}{2})s}.
\end{aligned}
$$

Next, using that $e^{-\frac{\xi}{4}} \le C_\theta \xi^{-\theta}$ for all $\xi > 0$ and some constant C_θ, we have

$$
\begin{aligned}
\int_0^t \frac{(t-s)^{\beta-1}}{s^\beta}e^{-\frac{\sigma}{2}s}ds &= \int_0^{\frac{t}{2}} \frac{(t-s)^{\beta-1}}{s^\beta}e^{-\frac{\sigma}{2}s}ds + \int_{\frac{t}{2}}^t \frac{(t-s)^{\beta-1}}{s^\beta}e^{-\frac{\sigma}{2}s}ds \\
&\le \left(\frac{t}{2}\right)^{\beta-1}\int_0^\infty s^{-\beta}e^{-\frac{\sigma}{2}s}ds + \left(\frac{t}{2}\right)^{-\beta}e^{-\frac{\sigma}{4}t}\int_0^{\frac{t}{2}} s^{\beta-1}ds \qquad (2.4)\\
&= \left(\frac{t}{2}\right)^{\beta-1}\left(\frac{\sigma}{2}\right)^{\beta-1}\Gamma(1-\beta) + \frac{1}{\beta}e^{-\frac{\sigma}{4}t} \\
&\le k_\beta\,\sigma^{\beta-1}t^{\beta-1}
\end{aligned}
$$

for every $t > 0$. Combining (2.1), (2.2), (2.3), and (2.4), we obtain

$$
\begin{aligned}
\int_0^t e^{-\sigma(t-s)}dg_{\omega+\sigma,\beta}(s) &\le C_\beta\int_0^t (t-s)^{\beta-1}s^{-\beta}e^{-(\omega+\sigma)s}\left(1+\sum_{k=1}^{\infty}\frac{(\omega s)^k}{(k-1)!}\right)ds \\
&\le C'_\beta\int_0^t (t-s)^{\beta-1}s^{-\beta}e^{-(\omega+\sigma)s}\left(\max\{1,c_\delta\}e^{(\omega+\frac{\sigma}{2})s}\right)ds \\
&\le C''_\beta \max\{1,c_\delta\}\sigma^{\beta-1}t^{\beta-1}.
\end{aligned}
$$

□

Upon letting $\omega \downarrow 0$ we see that $g_{\omega+\sigma} \to g_\sigma$ pointwise. Since each of the functions $g_{\omega+\sigma}$, $\omega \ge 0$, is non-decreasing and non-negative, we can apply [26, Theorem I.16.4] and obtain:

Corollary 2.3. *For each* $0 < \beta < 1$ *there is a constant* C_β *such that for all* $\sigma > 0$ *and all* $t > 0$ *we have*

$$
\int_0^t e^{-\sigma(t-s)}dg_{\sigma,\beta}(s) \le C_\beta\sigma^{\beta-1}t^{\beta-1}.
$$

Next, we consider the case that there is also a polynomial term.

Corollary 2.4. *For each $0 < \beta < 1$ and $\gamma \geq 0$ there is a constant $C_{\beta,\gamma}$ such that for all $\omega > 0$, $\sigma > 0$, and $t > 0$ we have*

$$\int_0^t e^{-\sigma(t-s)}\,(t-s)^\gamma\,dg_{\omega+\sigma,\beta}(s) \;\leq\; C_{\beta,\gamma}\,\frac{\sigma^{\beta-\gamma-1}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\,t^{\beta-1}.$$

PROOF. Fix $0 < \beta < 1$, $\gamma \geq 0$, $\omega > 0$, and $\sigma > 0$. The function $\xi \mapsto \xi^\gamma e^{-\frac{\sigma}{2}\xi}$, $\xi > 0$, takes its maximum in the point $\xi = 2\gamma\sigma^{-1}$, and the maximum value is $(2\gamma\sigma^{-1})^\gamma e^{-\gamma}$. Therefore, there exists a constant C_γ such that

$$\xi^\gamma \leq C_\gamma \sigma^{-\gamma} e^{\frac{\sigma}{2}\xi}, \quad \xi > 0. \tag{2.5}$$

By Lemma 2.2 (applied to $\omega \mapsto \omega + \frac{1}{2}\sigma$ and $\sigma \mapsto \frac{1}{2}\sigma$) and (2.5),

$$\begin{aligned}
&\int_0^t e^{-\sigma(t-s)}\,(t-s)^\gamma\,dg_{\omega+\sigma,\beta}(s)\\
&\qquad\leq\; C_\gamma\,\sigma^{-\gamma}\int_0^t e^{-\frac{\sigma}{2}(t-s)}\;dg_{\omega+\sigma,\beta}(s)\\
&\qquad\leq\; C_\gamma\,\sigma^{-\gamma}\,C_\beta \max\left\{1,\left(\ln\left(1+\frac{\frac{1}{2}\sigma}{2(\omega+\frac{1}{2}\sigma)}\right)\right)^{-1}\right\}\sigma^{\beta-1}\,t^{\beta-1}.
\end{aligned}$$

Since the term containing the logarithm is always greater than 1, the proof of the corollary is complete. □

Upon letting $\omega \downarrow 0$, we obtain:

Corollary 2.5. *For each $0 < \beta < 1$ and $\gamma \geq 0$ there is a constant $C_{\beta,\gamma}$ such that for all $\sigma > 0$ and $t > 0$ we have*

$$\int_0^t e^{-\sigma(t-s)}\,(t-s)^\gamma\,dg_{\sigma,\beta}(s) \;\leq\; C_{\beta,\gamma}\,\sigma^{\beta-\gamma-1}\,t^{\beta-1}.$$

3. α-TIMES INTEGRATED SEMIGROUPS AND PERTURBATIONS WITH THE IDENTITY

The concept of n-times integrated semigroups, with $n \in \mathbb{N}$, was introduced by W. Arendt in 1987 ([1], see also [13] and [19]). A little later, M. Hieber [10] introduced α-times integrated semigroups for all $\alpha \in \mathbb{R}$, $\alpha \geq 0$.

Let $\alpha \geq 0$. A closed linear operator A is called the *generator of an exponentially bounded α-times integrated semigroup* if and only if $(\omega, \infty) \subset \varrho(A)$ for some $\omega \geq 0$, and there exists a strongly continuous mapping $\mathbf{S}^\alpha : [0,\infty) \to \mathcal{B}(X)$ satisfying $\|S^\alpha(t)\| \le Me^{\omega t}$ for some $M \ge 1$ and all $t \ge 0$, such that

$$R(\lambda, A)x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S^\alpha(t)x\,dt \tag{3.1}$$

for all $x \in X$ and $\lambda > \omega$. In this case, $\mathbf{S}^\alpha = (S^\alpha(t))_{t\ge 0}$ is called the exponentially bounded α-times integrated semigroup *generated* by A.

Note that an exponentially bounded 0-times integrated semigroup is a C_0-semigroup and vice versa.

If A is the generator of an exponentially bounded α-times integrated semigroup $\mathbf{S}^\alpha$, then A also generates an exponentially bounded β-times integrated semigroup for each $\beta > \alpha$, which is given by

$$S^\beta(t)x = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-s)^{\beta-\alpha-1} S^\alpha(s)x\,ds. \tag{3.2}$$

For every $x \in D(A)$ and $t \ge 0$, we have

$$\begin{aligned} S^\alpha(t)x \in D(A), \quad AS^\alpha(t)x &= S^\alpha(t)Ax, \quad \text{and} \\ S^\alpha(t)x &= \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t S^\alpha(r)Ax\,dr. \end{aligned} \tag{3.3}$$

For details, we refer to [10].

Throughout the following, we assume that A is the generator of an exponentially bounded α-times integrated semigroup $\mathbf{S}^\alpha$. By $[\alpha]$, we denote the integer part of α, i.e. the unique integer such that $[\alpha] \le \alpha < [\alpha]+1$.

Lemma 3.1. *Let $0 \le m \le [\alpha]$ and $x \in D(A^m)$. Then $S^\alpha(\cdot)x \in C^m([0,\infty), X)$ and*

$$\frac{d^m}{dt^m} S^\alpha(t)x = \sum_{k=1}^m \frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)} A^{m-k}x + S^\alpha(t)A^m x$$

for all $t \ge 0$.

PROOF. For $m = 0$, the assertion is trivial and for $m = 1$, it follows directly from (3.3). If $m \geq 2$, we obtain by (3.3) that

$$\begin{aligned} S^\alpha(t)x &= \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t \Big(\frac{r_1^\alpha}{\Gamma(\alpha+1)}Ax + \int_0^{r_1} \Big(\dots + \int_0^{r_{m-2}} \Big(\frac{r_{m-1}^\alpha}{\Gamma(\alpha+1)}A^{m-1}x \\ &\qquad + \int_0^{r_{m-1}} S^\alpha(r_m)A^m x\, dr_m\Big) dr_{m-1} \dots \Big) dr_2\Big)\, dr_1 \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)}x + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}Ax + \dots + \frac{t^{\alpha+m-1}}{\Gamma(\alpha+m)}A^{m-1}x \\ &\qquad + \int_0^t \int_0^{r_1} \dots \int_0^{r_{m-1}} S^\alpha(r_m)A^m x\, dr_m \dots dr_2\, dr_1 \end{aligned}$$

for all $t \geq 0$. This yields the claim. □

In particular, it follows that the Laplace transform of $(d^m/dt^m)S^\alpha(\cdot)x$ exists for all $\lambda > \omega$.

Lemma 3.2. *Let $0 \leq m \leq [\alpha]$ and $x \in D(A^m)$. For every $\lambda > \omega$ we have*

$$\int_0^\infty e^{-\lambda t}\frac{d^m}{dt^m}S^\alpha(t)x\, dt = \frac{R(\lambda, A)x}{\lambda^{\alpha-m}}.$$

PROOF. By Lemma 3.1, the Laplace transforms of $(d^k/dt^k)S^\alpha(\cdot)x$ $(0 \leq k \leq m)$ exist for all $\lambda > \omega$. Since $S^\alpha(0)x = 0$, it follows from Lemma 3.1 that $(d^k/dt^k)S^\alpha(t)x|_{t=0} = 0$ for every $0 \leq k \leq m-1$. Integrating by parts, we obtain

$$\int_0^\infty e^{-\lambda t}\frac{d^m}{dt^m}S^\alpha(t)x\, dt = \lambda^m \int_0^\infty e^{-\lambda t}S^\alpha(t)x\, dt = \lambda^m\,\frac{R(\lambda, A)x}{\lambda^\alpha} = \frac{R(\lambda, A)x}{\lambda^{\alpha-m}}$$

for every $\lambda > \omega$. □

Lemma 3.2 and formula (3.1) motivate us to introduce the notation

$$S^{\alpha-m}(t)x := \frac{d^m}{dt^m}S^\alpha(t)x, \quad 0 \leq m \leq [\alpha].$$

Whenever A generates an $(\alpha - m)$-times integrated semigroup $\mathbf{S}^{\alpha-m}$, it is given by $(d^m/dt^m)\ S^\alpha(\cdot)$, in agreement with our notation. Later, we will mainly work with $S^{\alpha-[\alpha]}(\cdot)$ rather than with $S^\alpha(\cdot)$ itself. In this way, what we essentially achieve is a reduction to the case $0 < \alpha < 1$.

The following perturbation result explains the importance of the functions $g_{\mu,\beta}$ introduced in Section 2.

Proposition 3.3. *Let $\alpha \geq 0$ and let A be the generator of an exponentially bounded α-times integrated semigroup $\mathbf{S}^\alpha$. Let $\mu \in \mathbb{C}$. Then $A_\mu := A - \mu$ generates an exponentially bounded α-times integrated semigroup $\mathbf{S}^\alpha_\mu$, and we have the following relation between $\mathbf{S}^\alpha$ and $\mathbf{S}^\alpha_\mu$. For all $t \geq 0$ and $x \in X$,*

$$S^\alpha_\mu(t)x = \int_0^t e^{-\mu(t-s)} S^\alpha(t-s)x \, dg_{\mu,\alpha}(s).$$

PROOF. By assumption, there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S^\alpha(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > |\mu| + \omega$, we obtain

$$\begin{aligned} \frac{R(\lambda, A_\mu)x}{\lambda^\alpha} &= \left(1 + \frac{\mu}{\lambda}\right)^\alpha \frac{R(\lambda+\mu, A)x}{(\lambda+\mu)^\alpha} \\ &= \left(1 + \sum_{k=1}^\infty \frac{<\alpha>_k \ \mu^k}{k!} \frac{1}{\lambda^k}\right) \frac{R(\lambda+\mu, A)x}{(\lambda+\mu)^\alpha} \qquad (3.4) \\ &= \frac{R(\lambda+\mu, A)x}{(\lambda+\mu)^\alpha} + \lim_{n\to\infty} R_n(\lambda), \end{aligned}$$

where

$$\begin{aligned} R_n(\lambda) &:= \sum_{k=1}^n \frac{<\alpha>_k \ \mu^k}{k!} \frac{1}{\lambda^k} \frac{R(\lambda+\mu, A)x}{(\lambda+\mu)^\alpha} \\ &= \sum_{k=1}^n \frac{<\alpha>_k \ \mu^k}{k!} \int_0^\infty e^{-\lambda t} f^{[k]}(t)\, dt \\ &= \int_0^\infty e^{-\lambda t} d\left(\sum_{k=1}^n \frac{<\alpha>_k \ \mu^k}{k!} f^{[k+1]}(t)\right), \end{aligned}$$

and

$$f^{[j]}(t) := \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} e^{-\mu s} S^\alpha(s)x \, ds \qquad t \geq 0, \ j \geq 1,$$

is the j-th antiderivative of $f(t) := e^{-\mu t}S^\alpha(t)x$. Choose $c > |\mu|$ such that $\|f(t)\| \leq M'e^{ct}$ for all $t \geq 0$. Then, for all $k \geq 1$ and $t, h \geq 0$,

$$\begin{aligned} \|f^{[k+1]}(t+h) - f^{[k+1]}(t)\| &\leq \int_t^{t+h} \|f^{[k]}(s)\| ds \\ &= \int_t^{t+h} \left\| \int_0^s \frac{(s-r)^{k-1}}{(k-1)!} f(r)\, dr \right\| ds \\ &\leq \int_t^{t+h} \int_0^s \frac{r^{k-1}}{(k-1)!} M'e^{c(s-r)} dr\, ds \\ &\leq M' \int_t^{t+h} e^{cs} \int_0^\infty \frac{r^{k-1}}{(k-1)!} e^{-cr} dr\, ds \\ &= \frac{M'}{c^k} \int_t^{t+h} e^{cs} ds. \end{aligned}$$

It follows that $s_n(t) := \sum_{k=1}^n (k!)^{-1} <\alpha>_k \mu^k f^{[k+1]}(t)$ $(n \geq 1)$ satisfies $s_n(0) = 0$ and

$$\begin{aligned} \|s_n(t+h) - s_n(t)\| &\leq M' \sum_{k=1}^n \frac{|<\alpha>_k|}{k!} \left(\frac{|\mu|}{c}\right)^k \int_t^{t+h} e^{cs} ds \\ &\leq M'[\alpha+1]! \left(\frac{1}{1-\frac{|\mu|}{c}}\right) \int_t^{t+h} e^{cs} ds \end{aligned}$$

for all $t, h \geq 0$. This estimate shows that the functions $s_n(\cdot)$ are locally of uniformly bounded variation.

Since for $\lambda > |\mu| + \omega$, $\lambda^{-1}R_n(\lambda) = \int_0^\infty e^{-\lambda t} s_n(t)\, dt$ converges as $n \to \infty$ (by (3.4)), the Trotter-Kato theorem of Laplace transforms (see, e.g., [9, Corollary 4.3]) yields that the functions $s_n(\cdot)$ converge (uniformly on compacta) to $s(\cdot)$ given by $s(t) = \sum_{k=1}^\infty \frac{<\alpha>_k}{k!} \mu^k f^{[k+1]}(t)$. Further, $s(\cdot)$ satisfies $s(0) = 0$, and $\lim_{n\to\infty} R_n(\lambda) = \int_0^\infty e^{-\lambda t}\, ds(t)$ for $\lambda > |\mu| + \omega$.

Similarly, we obtain that $s_n'(t) = \sum_{k=1}^n \frac{<\alpha>_k}{k!} \mu^k f^{[k]}(t)$ converges (uniformly on compacta). It follows that $s(\cdot)$ is differentiable and its derivative is given by $s'(t) = \sum_{k=1}^\infty \frac{<\alpha>_k}{k!} \mu^k f^{[k]}(t)$. Thus,

$$\frac{R(\lambda, A_\mu)x}{\lambda^\alpha} = \int_0^\infty e^{-\lambda t} \left(f(t) + \sum_{k=1}^\infty \frac{<\alpha>_k \mu^k}{k!} f^{[k]}(t) \right) dt, \quad \lambda > |\mu| + \omega$$

and

$$S_\mu^\alpha(t)x = e^{-\mu t}S^\alpha(t)x + \sum_{k=1}^\infty \frac{<\alpha>_k \mu^k}{k!} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{-\mu s} S^\alpha(s)x\, ds, \quad t \geq 0.$$

Next, we use that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{<\alpha>_k\ \mu^k}{k!}\ \frac{(t-s)^{k-1}}{(k-1)!}\ e^{-\mu s}S^{\alpha}(s)x$$
$$=\sum_{k=1}^{\infty}\frac{<\alpha>_k\ \mu^k}{k!}\ \frac{(t-s)^{k-1}}{(k-1)!}\ e^{-\mu s}S^{\alpha}(s)x$$

uniformly on compacta. Hence, we can interchange summation and integration. A change of variables yields

$$\begin{aligned} S^{\alpha}_{\mu}(t)x &= e^{-\mu t}S^{\alpha}(t)x + \int_0^t\sum_{k=1}^{\infty}\frac{<\alpha>_k\ \mu^k}{k!}\ \frac{s^{k-1}}{(k-1)!}\ e^{-\mu(t-s)}S^{\alpha}(t-s)x\,ds \\ &= \int_0^t e^{-\mu(t-s)}S^{\alpha}(t-s)x\,d\left(\chi_{(0,\infty)}(s)+\sum_{k=1}^{\infty}\frac{<\alpha>_k\ (\mu s)^k}{(k!)^2}\right) \\ &= \int_0^t e^{-\mu(t-s)}S^{\alpha}(t-s)x\,dg_{\mu,\alpha}(s). \end{aligned}$$

By the definition of $g_{\mu,\alpha}$, this completes the proof. □

Using the representation of $S^{\alpha}_{\mu}(t)$ given in Proposition 3.3, it follows that $\|S^{\alpha}_{\mu}(t)\|\le Ke^{ct}$, where $c=\max\{\omega-\operatorname{Re}\mu,|\mu|\}$, for some $K=K_{\alpha,\mu}\ge 0$ and all $t\ge 0$. By replacing the role of α by $\alpha-[\alpha]$ in Proposition 3.3, the same argument shows:

Proposition 3.4. *Let $x\in D(A^{[\alpha]})$ and $\mu\in\mathbb{C}$. Then for every $t\ge 0$,*

$$S^{\alpha-[\alpha]}_{\mu}(t)x = \int_0^t e^{-\mu(t-s)}S^{\alpha-[\alpha]}(t-s)x\,dg_{\mu,\alpha-[\alpha]}(s).$$

Note that if $\alpha-[\alpha]=0$, that is if α is an integer, the function $g_{\mu,0}$ in Proposition 3.4 is given by $g_{\mu,0}(t)=\chi_{(0,\infty)}(t)$, $t\ge 0$. Hence, in this case for all $x\in D(A^{[\alpha]})$ and $t\ge 0$ we have $S^0_{\mu}(t)x=e^{-\mu t}S^0(t)x$.

4. A SINGULAR INTEGRAL

In this section, we apply the results of Sections 2 and 3 to estimate the singular integrals

$$(4.1)\quad v_\rho(t,x):=\Gamma_{\alpha,\varepsilon}\int_0^{\infty}\frac{1}{s-1}\left(s^{[\alpha+\varepsilon]-\alpha-\varepsilon}S_\rho^{\alpha+\varepsilon-[\alpha+\varepsilon]}(t)-\frac{1}{s}S_\rho^{\alpha+\varepsilon-[\alpha+\varepsilon]}(\frac{t}{s})\right)x\,ds,$$

where $\Gamma_{\alpha,\varepsilon} := \pi^{-1}\sin((\alpha+\varepsilon-[\alpha+\varepsilon])\pi)$. As we will see later, for certain initial values the solutions of the abstract Cauchy problem can be represented by integrals of this form.

We start by proving estimates for $\|S_\rho^{\alpha+\varepsilon-[\alpha+\varepsilon]}(t)x\|$ $(x \in D(A^{[\alpha+\varepsilon]}))$. Later, it will be important to have such estimates in the cases that $\mathbf{S}^\alpha$ satisfies either $\|S^\alpha(t)\| \le M(1+t^\gamma)$ or $\|S^\alpha(t)\| \le Me^{\omega t}$. For this reason, we assume throughout that $\|S^\alpha(t)\| \le M(1+t^\gamma)e^{\omega t}$ and specialize later to the cases $\gamma = 0$ and $\omega = 0$. By $\|x\|_m := \sum_{k=0}^m \|A^k x\|$ we denote the graph norm on $D(A^m)$.

Lemma 4.1. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and A be the generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \le M(1+t^\gamma)e^{\omega t}$ for some $M \ge 1$, $\gamma \ge 0$, $\omega \ge 0$ and all $t \ge 0$. Then there is a constant $C_{\alpha,\gamma}$ such that for all $t > 0$, $0 < \sigma \le 1$ and $x \in D(A^{[\alpha]})$,*

$$\|S_{\omega+\sigma}^{\alpha-[\alpha]}(t)x\| \;\le\; MC_{\alpha,\gamma}\,\frac{\sigma^{\min\{-[\alpha],\alpha-[\alpha]-\gamma-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\,t^{\alpha-[\alpha]-1}\,\|x\|_{[\alpha]}.$$

PROOF. First, assume $\omega > 0$. Put $\beta := \alpha - [\alpha]$ and note that $0 < \beta < 1$. By Proposition 3.4, Lemma 3.1, Corollary 2.5, Lemma 2.2, and Corollary 2.4, and the facts that $g_{\omega+\sigma,\beta}$ is non-negative and non-decreasing and that $\frac{\sigma}{2\omega} \ge \frac{\sigma}{4\omega+2\sigma}$, we have

$$\begin{aligned}
&\|S_{\omega+\sigma}^{\beta}(t)x\| \\
&\le \int_0^t e^{-(\omega+\sigma)(t-s)}\|S^\beta(t-s)x\|\,dg_{\omega+\sigma,\beta}(s) \\
&\le \int_0^t e^{-(\omega+\sigma)(t-s)}\left(\sum_{k=1}^{[\alpha]}\frac{(t-s)^{\alpha-k}}{\Gamma(\alpha-k+1)} + M(1+(t-s)^\gamma)e^{\omega(t-s)}\right)\|x\|_{[\alpha]}\,dg_{\omega+\sigma,\beta}(s) \\
&\le \sum_{k=1}^{[\alpha]} C_{\alpha,k}(\omega+\sigma)^{\beta-\alpha+k-1}t^{\beta-1}\|x\|_{[\alpha]} + MC_{\alpha,\gamma}\,\frac{\sigma^{\beta-1}+\sigma^{\beta-\gamma-1}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\,t^{\beta-1}\|x\|_{[\alpha]}.
\end{aligned}$$

We now estimate $(\omega+\sigma)^{\beta-\alpha+k-1}$ by $\sigma^{\beta-\alpha+k-1}$ and use that $0 < \sigma \le 1$. By taking the most negative powers of σ in the resulting expression, the desired estimate is obtained.

Upon letting $\omega \downarrow 0$, and using that $S_\mu^\alpha(t)x$ depends continuously on $\mu \ge 0$ by Proposition 3.3, the result follows for $\omega = 0$. ☐

This lemma will be used to derive certain Hölder-type continuity properties of the map $t \longmapsto S_\mu^{\alpha-[\alpha]}(t)x$.

Lemma 4.2. *Let $\alpha \geq 0$ and let A be the generator of the α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq M(1+t^\gamma)e^{\omega t}$ for some $M \geq 1$, $\gamma \geq 0$, $\omega \geq 0$ and all $t \geq 0$. Let $\varepsilon > 0$ such that $\beta := \alpha + \varepsilon \notin \mathbb{N}$. Put $\delta := \frac{1}{2}\min\{\varepsilon, \beta - [\beta]\}$. Then there exists a constant $C_{\alpha,\varepsilon,\gamma}$ such that*

$$\left\| S^{\beta-[\beta]}_{\omega+\sigma}(t)x - S^{\beta-[\beta]}_{\omega+\sigma}(\tau)x \right\| \leq MC_{\alpha,\varepsilon,\gamma}\frac{e^{\omega+1}\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}(t-\tau)^\delta \|x\|_{[\beta]}$$

for all $0 \leq \tau \leq t$, $0 < \sigma \leq 1$ and $x \in D(A^{[\beta]})$.

Proof. Let $\rho := \varepsilon - \delta$ and $\eta := \alpha + \rho = \beta - \delta$. Then $\eta > \alpha$, $\eta \notin \mathbb{N}$ and $[\eta] = [\beta]$. By (3.2) we have

$$S^\eta(t)x = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}S^\alpha(s)x\,ds, \quad x \in X,\ t \geq 0.$$

Hence

$$\|S^\eta(t)\| \leq \frac{M}{\Gamma(\rho+1)}t^\rho(1+t^\gamma)e^{\omega t} \leq MC_\rho(1+t^{\gamma+\rho})e^{\omega t}, \quad t \geq 0. \tag{4.2}$$

Let $0 < \sigma \leq 1$ and $x \in D(A^{[\eta]})$. By (4.2) and Lemma 4.1, for $t > 0$ we have

$$\|S^{\eta-[\eta]}_{\omega+\sigma}(t)x\| \leq MC_{\eta,\gamma,\rho}\frac{\sigma^{\min\{-[\eta],\eta-[\eta]-\gamma-\rho-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\|x\|_{[\eta]}. \tag{4.3}$$

We need a better estimate for $0 \leq t \leq 1$, which we will produce next. By Proposition 3.4 and Lemma 3.1, (4.2), (2.5), and the fact that $g_{\omega+\sigma,\eta-[\eta]}$ is non-negative and non-decreasing, for all $0 \leq t \leq 1$ we obtain

$$\begin{aligned}
&\|S^{\eta-[\eta]}_{\omega+\sigma}(t)x\| \\
&\leq \int_0^t e^{-(\omega+\sigma)(t-s)}\|S^{\eta-[\eta]}(t-s)x\|\,dg_{\omega+\sigma,\eta-[\eta]}(s) \\
&\leq \int_0^t e^{-(\omega+\sigma)(t-s)}\left(\sum_{k=1}^{[\eta]}\frac{(t-s)^{\eta-k}}{\Gamma(\eta-k+1)} + MC_\rho(1+(t-s)^{\gamma+\rho})e^{\omega(t-s)}\right)\|x\|_{[\eta]} \\
&\qquad\qquad dg_{\omega+\sigma,\eta-[\eta]}(s) \qquad (4.4)\\
&\leq MC'_{\eta,\gamma,\rho}\left(\sum_{k=1}^{[\eta]}\sigma^{-\eta+k} + 1 + \sigma^{-\gamma-\rho}\right)\|x\|_{[\eta]}\,g_{\omega+\sigma,\eta-[\eta]}(t) \\
&\leq MC'_{\eta,\gamma,\rho}([\eta]\sigma^{-\eta+1} + 1 + \sigma^{-\gamma-\rho})\,\|x\|_{[\eta]}\,e^{\omega+1}.
\end{aligned}$$

In the last estimate, we used the assumptions $0 < \sigma \leq 1$, $0 \leq t \leq 1$, and the fact that by the definition of $g_{\omega+\sigma,\eta-[\eta]}$ we have $g_{\omega+\sigma,\eta-[\eta]}(t) \leq e^{(\omega+\sigma)t}$.

Combining (4.3) and (4.4), we see that there exists a constant $C''_{\eta,\gamma,\rho}$ such that

$$\|S^{\eta-[\eta]}_{\omega+\sigma}(t)x\| \leq MC''_{\eta,\gamma,\rho}\frac{e^{\omega+1}\sigma^{\min\{-[\eta],\eta-[\eta]-\gamma-\rho-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\|x\|_{[\eta]},\quad t>0. \tag{4.5}$$

Since $[\eta]=[\beta]$ we have

$$S^{\beta-[\beta]}_{\omega+\sigma}(t)x = S^{\eta-[\eta]+\delta}_{\omega+\sigma}(t)x = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}S^{\eta-[\eta]}_{\omega+\sigma}(s)x\,ds,$$

and, using (4.5), it follows that

$$\begin{aligned}
&\|S^{\beta-[\beta]}_{\omega+\sigma}(t)x - S^{\beta-[\beta]}_{\omega+\sigma}(\tau)x\|\\
&= \frac{1}{\Gamma(\delta)}\left\|\int_\tau^t (t-s)^{\delta-1}S^{\eta-[\eta]}_{\omega+\sigma}(s)x\,ds + \int_0^\tau \left((t-s)^{\delta-1}-(\tau-s)^{\delta-1}\right)S^{\eta-[\eta]}_{\omega+\sigma}(s)x\,ds\right\|\\
&\leq MC_{\eta,\gamma,\delta,\rho}\left(\int_\tau^t (t-s)^{\delta-1}ds + \int_0^\tau \left((\tau-s)^{\delta-1}-(t-s)^{\delta-1}\right)ds\right)\\
&\qquad\qquad\cdot\frac{e^{\omega+1}\sigma^{\min\{-[\eta],\eta-[\eta]-\gamma-\rho-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\|x\|_{[\eta]}\\
&= MC'_{\eta,\gamma,\delta,\rho}\left((t-\tau)^\delta+\tau^\delta-(t^\delta-(t-\tau)^\delta)\right)\frac{e^{\omega+1}\sigma^{\min\{-[\eta],\eta-[\eta]-\gamma-\rho-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\|x\|_{[\eta]}\\
&\leq 2MC'_{\eta,\gamma,\delta,\rho}(t-\tau)^\delta\frac{e^{\omega+1}\,\sigma^{\min\{-[\eta],\eta-[\eta]-\gamma-\rho-1\}}}{\ln(1+\frac{\sigma}{4\omega+2\sigma})}\|x\|_{[\eta]}
\end{aligned}$$

for all $0\leq\tau\leq t$. Since $[\eta]=[\beta]$, and since η, ρ and δ depend only on α and ε, the proof is complete. □

We are now in the position to give conditions under which the integral (4.1) converges.

Lemma 4.3. *Let $\alpha\geq 0$ and let $\mathbf{S}^\alpha$ be an α-times integrated semigroup satisfying $\|S^\alpha(t)\|\leq M(1+t^\gamma)e^{\omega t}$ for some $M\geq 1$, $\gamma\geq 0$, $\omega\geq 0$ and all $t\geq 0$. Let $\varepsilon>0$ such that $\alpha+\varepsilon\notin\mathbb{N}$. Then for all $x\in D(A^{[\alpha+\varepsilon]})$, $0<\sigma\leq 1$, and $t\geq 0$, the integral $v_{\omega+\sigma}(t,x)$ converges absolutely. Moreover, the map $t\longmapsto v_{\omega+\sigma}(t,x)$ $(t\geq 0)$ is continuous and polynomially bounded.*

PROOF. The integral has singularities in $0,1$, and ∞. We split it accordingly in three parts: $\int_0^\infty = \int_0^{\frac12}+\int_{\frac12}^2+\int_2^\infty = (I)+(II)+(III)$ and estimate these separately.

Put $\beta := \alpha + \varepsilon$ and $\delta := \frac{1}{2}\min\{\varepsilon, \beta - [\beta]\}$. By Lemma 4.2, there exists a constant $C = C_{\alpha,\gamma,\varepsilon,\omega,\sigma,M} > 0$ such that

$$\left\|S^{\beta-[\beta]}_{\omega+\sigma}(t)x - S^{\beta-[\beta]}_{\omega+\sigma}(\tau)x\right\| \;\le\; C\,(t-\tau)^{\delta}\,\|x\|_{[\beta]}, \quad 0 \le \tau \le t.$$

Since $S^{\beta-[\beta]}_{\omega+\sigma}(0)x = 0$, in particular we have

$$\left\|S^{\beta-[\beta]}_{\omega+\sigma}(t)x\right\| \;\le\; C\,t^{\delta}\,\|x\|_{[\beta]}, \quad t \ge 0.$$

The β-times integrated semigroup $\mathbf{S}^{\beta}$ satisfies $\|S^{\beta}(t)\| \le \frac{2M}{\Gamma(1+\varepsilon)}(1 + t^{\gamma+\varepsilon})e^{\omega t}$ $(t \ge 0)$, cf. the proof of (4.2). Therefore, by Lemma 4.1, there exists a constant $K = K_{\alpha,\gamma,\varepsilon,\omega,\sigma,M} > 0$ such that

$$\left\|S^{\beta-[\beta]}_{\omega+\sigma}(t)x\right\| \;\le\; K\,t^{\beta-[\beta]-1}\,\|x\|_{[\beta]}, \quad t > 0.$$

Combining these facts, it follows that

$$\begin{aligned}
\|(I)\| \;&\le\; \int_0^{\frac{1}{2}} 2s^{-\beta+[\beta]}Ct^{\delta}\|x\|_{[\beta]}\,ds \;+\; \int_{2t}^{\infty} \frac{2}{s}\left\|S^{\beta-[\beta]}_{\omega+\sigma}(s)x\right\|\,ds \\
&\le\; C_1 t^{\delta}\|x\|_{[\beta]} \;+\; \int_{\min\{2t,1\}}^{1} \frac{2}{s}Cs^{\delta}\|x\|_{[\beta]}\,ds \;+\; \int_1^{\infty} \frac{2}{s}Ks^{\beta-[\beta]-1}\|x\|_{[\beta]}\,ds \\
&=\; C_2\,(t^{\delta}+1)\,\|x\|_{[\beta]}; \\
\|(II)\| \;&\le\; \int_{\frac{1}{2}}^{2} \frac{1}{|s-1|}s^{-\beta+[\beta]}\left\|S^{\beta-[\beta]}_{\omega+\sigma}(t)x - S^{\beta-[\beta]}_{\omega+\sigma}(\frac{t}{s})x\right\|ds \\
&\qquad\qquad + \int_{\frac{1}{2}}^{2} \frac{1}{|s-1|}\left|s^{-\beta+[\beta]} - s^{-1}\right|\left\|S^{\beta-[\beta]}_{\omega+\sigma}(\frac{t}{s})x\right\|\,ds \\
&\le\; \int_{\frac{1}{2}}^{2} \frac{1}{|s-1|}s^{-\beta+[\beta]}C\left|t - \frac{t}{s}\right|^{\delta}\,\|x\|_{[\beta]}\,ds \\
&\qquad\qquad + \int_{\frac{1}{2}}^{2} \frac{1}{|s-1|}\left|s^{-\beta+[\beta]} - s^{-1}\right|C\big(\frac{t}{s}\big)^{\delta}\|x\|_{[\beta]}\,ds \\
&=\; C_3\,t^{\delta}\,\|x\|_{[\beta]}; \\
\|(III)\| \;&\le\; \int_2^{\infty} \frac{1}{s-1}\Big(s^{-\beta+[\beta]}Ct^{\delta}\|x\|_{[\beta]} + s^{-1}C(\frac{t}{s})^{\delta}\|x\|_{[\beta]}\Big)\,ds \;=\; C_4 t^{\delta}\|x\|_{[\beta]}.
\end{aligned}$$

These estimates yield that $v_{\omega+\sigma}(t,x)$ converges absolutely for all $x \in D(A^{[\beta]})$, $t \ge 0$ and $0 < \sigma \le 1$. Moreover, the convergence is uniform for t in compact subsets of $[0,\infty)$. Therefore, the map $t \longmapsto v_{\omega+\sigma}(t,x)$ $(t \ge 0)$ is continuous.

The polynomial boundedness of $v_{\omega+\sigma}(\cdot,x)$ $(t \geq 0)$ is again a consequence of the estimates above. $\square$

We now turn to the case $\omega = 0$ emphasizing the σ-dependence of the integral $v_\sigma(t,x)$. Such estimates are necessary to obtain polynomial bounds for our solutions of the abstract Cauchy problem.

Lemma 4.4. *Let $\alpha \geq 0$ and let $\mathbf{S}^\alpha$ be an α-times integrated semigroup satisfying $\|S^\alpha(t)\| \leq M(1+t^\gamma)$ for some $M \geq 1$, $\gamma \geq 0$ and all $t \geq 0$. Let $\varepsilon > 0$ such that $\alpha + \varepsilon \notin \mathbb{N}$. Then there exists a constant $C_{\alpha,\gamma,\varepsilon}$ such that*

$$\|v_\sigma(t,x)\| \;\leq\; MC_{\alpha,\gamma,\varepsilon}\,\sigma^{\min\{-[\alpha+\varepsilon],\alpha-[\alpha+\varepsilon]-\gamma-1\}}\,t^{\alpha+\varepsilon-[\alpha+\varepsilon]-1}\,\|x\|_{[\alpha+\varepsilon]}$$

for all $t \geq 2$, $0 < \sigma \leq 1$, and $x \in D(A^{[\alpha+\varepsilon]})$.

Proof. Fix $0 < \eta \leq \eta_0 < 1$ and split the integral in three parts as follows: $\int_0^\infty = \int_0^{1-\eta} + \int_{1-\eta}^{1+\eta} + \int_{1+\eta}^\infty = (I) + (II) + (III)$. Later, we will make a judicious choice for η_0 and η depending on t.

Put $\beta := \alpha + \varepsilon$ and $\delta := \frac{1}{2}\min\{\varepsilon, \beta - [\beta]\}$. By Lemma 4.2 we have

$$\left\|S_\sigma^{\beta-[\beta]}(t)x - S_\sigma^{\beta-[\beta]}(\tau)x\right\| \;\leq\; MC_{\alpha,\varepsilon,\gamma}\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}(t-\tau)^\delta\|x\|_{[\beta]}$$

for some constant $C_{\alpha,\gamma,\varepsilon} > 0$ and all $0 \leq \tau \leq t$, $0 < \sigma \leq 1$ and $x \in D(A^{[\beta]})$.

The β-times integrated semigroup $\mathbf{S}^\beta$ satisfies

$$\|S^\beta(t)\| \;\leq\; MC_\varepsilon(1+t^{\gamma+\varepsilon}), \qquad t \geq 0,$$

and hence by Lemma 4.1 we obtain

$$\|S_\sigma^{\beta-[\beta]}(t)x\| \;\leq\; MK_{\alpha,\gamma,\varepsilon}\,\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}t^{\beta-[\beta]-1}\|x\|_{[\beta]}.$$

for some constant $K_{\alpha,\gamma,\varepsilon} > 0$ and all $t > 0$, $0 < \sigma \leq 1$ and $x \in D(A^{[\beta]})$. We will now estimate (I), (II), and (III) separately.

First, for (I) we have

$$\begin{aligned}\|(I)\| &\leq MK_{\alpha,\gamma,\varepsilon}\|x\|_{[\beta]}\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}\\ &\qquad\cdot\int_0^{1-\eta}\frac{1}{|s-1|}\left(s^{-\beta+[\beta]}t^{\beta-[\beta]-1} + s^{-1}(t/s)^{\beta-[\beta]-1}\right)ds\\ &= 2MK_{\alpha,\gamma,\varepsilon}\|x\|_{[\beta]}\,\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}\,t^{\beta-[\beta]-1}\int_0^{1-\eta}\frac{1}{|s-1|}s^{-\beta+[\beta]}\,ds.\end{aligned}$$

Next, for (II) we have

$$\begin{aligned}
\|(II)\| \;\le\; & \int_{1-\eta}^{1+\eta} \frac{1}{|s-1|} s^{-\beta+[\beta]} \Big\| S_\sigma^{\beta-[\beta]}(t)x - S_\sigma^{\beta-[\beta]}(\frac{t}{s})x \Big\| \, ds \\
& \qquad + \int_{1-\eta}^{1+\eta} \frac{1}{|s-1|} \Big| s^{-\beta+[\beta]} - s^{-1} \Big| \Big\| S_\sigma^{\beta-[\beta]}(\frac{t}{s})x \Big\| \, ds \\
\le\; & MC_{\alpha,\gamma,\varepsilon} \|x\|_{[\beta]} \, \sigma^{\min\{-[\beta], \beta-[\beta]-\gamma-\varepsilon-1\}} \, t^\delta \int_{1-\eta}^{1+\eta} \frac{1}{|s-1|^{1-\delta}} \, s^{-\beta+[\beta]-\delta} \, ds \\
& + MK_{\alpha,\gamma,\varepsilon} \|x\|_{[\beta]} \, \sigma^{\min\{-[\beta], \beta-[\beta]-\gamma-\varepsilon-1\}} \, t^{\beta-[\beta]-1} \\
& \qquad \cdot \int_{1-\eta}^{1+\eta} \frac{\big| s^{-\beta+[\beta]} - s^{-1} \big|}{|s-1|} s^{-\beta+[\beta]+1} \, ds.
\end{aligned}$$

Finally, we estimate (III):

$$\|(III)\| \le 2MK_{\alpha,\gamma,\varepsilon} \|x\|_{[\beta]} \, \sigma^{\min\{-[\beta], \beta-[\beta]-\gamma-\varepsilon-1\}} \, t^{\beta-[\beta]-1} \int_{1+\eta}^{\infty} \frac{1}{s-1} s^{-\beta+[\beta]} \, ds.$$

Now,

$$\begin{aligned}
\int_0^{1-\eta} \frac{1}{|s-1|} s^{-\beta+[\beta]} \, ds \;&\le\; 2 \int_0^{\frac{1}{2}} s^{-\beta+[\beta]} \, ds + \left(\frac{1-\eta}{2}\right)^{-\beta+[\beta]} \int_{(1-\eta)/2}^{1-\eta} \frac{1}{1-s} \, ds \\
&\le\; \frac{2^{\beta-[\beta]}}{1-\beta+[\beta]} + \left(\frac{1-\eta_0}{2}\right)^{-\beta+[\beta]} \ln\frac{1}{\eta};
\end{aligned}$$

$$\begin{aligned}
\int_{1-\eta}^{1+\eta} \frac{1}{|s-1|^{1-\delta}} \, s^{-\beta+[\beta]-\delta} ds \;&\le\; (1-\eta)^{-\beta+[\beta]-\delta} \int_{1-\eta}^{1+\eta} \frac{1}{|s-1|^{1-\delta}} \, ds \\
&\le\; (1-\eta_0)^{-\beta+[\beta]-\delta} \, 2\delta^{-1} \eta^\delta;
\end{aligned}$$

$$\begin{aligned}
\int_{1-\eta}^{1+\eta} \frac{\big| s^{-\beta+[\beta]} - s^{-1} \big|}{|s-1|} s^{-\beta+[\beta]+1} ds &\le (1-\eta)^{-\beta+[\beta]} \int_{1-\eta}^{1+\eta} \frac{\big| s^{1-\beta+[\beta]} - 1 \big|}{|s-1|} \, ds \\
&\le (1-\eta_0)^{-\beta+[\beta]} 2\eta_0 (1-\eta_0)^{-\beta+[\beta]} (1-\beta+[\beta]) \\
&\le 2(1-\eta_0)^{2(-\beta+[\beta])};
\end{aligned}$$

$$\int_{1+\eta}^{\infty} \frac{1}{s-1} s^{-\beta+[\beta]} \, ds \;\le\; \int_{1+\eta}^{2} \frac{1}{s-1} \, ds + 2 \int_2^\infty s^{-\beta+[\beta]-1} \, ds \;=\; \ln\frac{1}{\eta} + \frac{2^{1-\beta+[\beta]}}{\beta}.$$

In the estimate of the third integral we applied the mean value theorem and in the estimate of the fourth integral we used that $(s-1)^{-1} \le 2s^{-1}$ for all $s \ge 2$.

Putting everything together, we obtain

$$\|v_\sigma(t,x)\| \le MC_{\alpha,\gamma,\varepsilon,\eta_0}\|x\|_{[\beta]}\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}\left(t^{\beta-[\beta]-1}(1+\ln\frac{1}{\eta})+t^\delta\eta^\delta\right).$$

So far, these estimates are valid for all $t > 0$ and $0 < \eta \le \eta_0 < 1$. We now fix $t \ge 2$, and take $\eta = t^{(\beta-[\beta]-1-\delta)/\delta}$ and $\eta_0 = 2^{(\beta-[\beta]-1-\delta)/\delta}$. This gives

$$\|v_\sigma(t,x)\| \le MC'_{\alpha,\gamma,\varepsilon,\eta_0}\|x\|_{[\beta]}\sigma^{\min\{-[\beta],\beta-[\beta]-\gamma-\varepsilon-1\}}\left(t^{\beta-[\beta]-1}(1+\ln t)+t^{\beta-[\beta]-1}\right).$$

Finally, since δ and η_0 depend only on α and ε, the proof of the lemma is complete. □

5. Fractional powers

In this section, we introduce the fractional powers of the generator of an exponentially bounded α-times integrated semigroup.

Let A be a closed, densely defined linear operator on X and assume that there are constants $0 < a < \frac{\pi}{2}$, $M \ge 1$, and $\gamma \ge -1$ such that the sector $\Sigma_a := \{\lambda \in \mathbb{C} : |\arg\lambda| \le a\} \cup \{0\}$ is contained in $\varrho(A)$ and

$$\|R(\lambda,A)\| \le M(1+|\lambda|)^\gamma, \qquad \lambda \in \Sigma_a. \tag{5.1}$$

Since $0 \in \varrho(A)$, it follows that in fact $R(\lambda,A)$ exists and satisfies (6.1) on $\Sigma_a \cup \mathrm{B}_\delta$ for some $\delta > 0$ and $\mathrm{B}_\delta := \{\lambda \in \mathbb{C} : |\lambda| \le \delta\}$ (with possibly a different constant M). Following [22], fractional powers of $-A$ can be defined as follows.

For every $b \in \mathbb{R}$, we let $\prec b \succ := \max\{0, [b]-[-\gamma]+2\}$. Let Γ be the (upwards oriented) boundary of $\Sigma_a \cup \mathrm{B}_\delta$, where $\delta > 0$ is as above. We define $(-A)^b$ as the closure of the operator J^b given on $D(J^b) = D(A^{\prec b \succ})$ by

$$J^b x = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (-\lambda)^b R(\lambda,A)x\,d\lambda & \text{if } b<0,\\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (-\lambda)^{b-[b]-1}R(\lambda,A)(-A)^{[b]+1}x\,d\lambda & \text{if } b \ge 0.\end{cases}$$

The space $D(A^0)$ is understood to be X.

In the case that $\gamma = -1$, the definition is consistent with the usual definition of fractional powers as given in [24].

The operators $(-A)^b$ satisfy the following properties. For the proofs of the statements, we refer to [22] and [23].

(P1) The operators $(-A)^b$ are closed and densely defined; if $b < -(\gamma+1)$, then $(-A)^b \in \mathcal{B}(X)$.

(P2) If b is an integer, then $(-A)^b$ is the usual power of $-A$. If b is not an integer, then for all $x \in D(A^{\prec b \succ})$,

$$(-A)^b x = \frac{\sin([b]+1-b)\pi}{\pi} \int_0^\infty t^{b-[b]-1} R(t,A)(-A)^{[b]+1} x \, dt.$$

If A is bounded, this holds for all $x \in X$.

(P3) The operators $(-A)^b$ are injective, and $(-A)^{-b}(-A)^b = I_{D((-A)^b)}$.

(P4) For all $b, c \in \mathbb{R}$, we have $(-A)^{b+c} \subseteq \overline{(-A)^b(-A)^c}$ with equality if $|b+c| > \gamma+1$ or $b+c$ is an integer.

(P5) Let $b \in \mathbb{R}$ such that $(-A)^{-b} \in \mathcal{B}(X)$ and let $c \in \mathbb{R}$. Then $D((-A)^b) \subseteq D((-A)^c)$ if and only if $(-A)^{c-b} \in \mathcal{B}(X)$.

For every $\sigma > 0$, the operator $A_\sigma = A - \sigma$ also satisfies condition (5.1) (with possibly a different constant M). Concerning the domains of the fractional powers of $-A$ and of $-A_\sigma = -(A_\sigma)$, we have the following results.

(P6) If $b > \gamma+1$, then $D((-A_\sigma)^b)$ does not depend on $\sigma \geq 0$, that is, $D((-A_\sigma)^b) = D((-A)^b)$ for all $\sigma > 0$.

(P7) For all $b \in \mathbb{R}$ and $0 < \sigma_0 < \sigma_1$, we have

$$\overline{(-A_{\sigma_0})^b(-A_{\sigma_1})^{-b}} = \left((-A_{\sigma_1})(-A_{\sigma_0})^{-1}\right)^{-b}.$$

If A is the densely defined generator of α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq Mt^\beta e^{\omega t}$ for some constants $M \geq 1$, $\omega \geq 0$, $\beta \geq 0$, and all $t \geq 0$, it follows from (3.1) that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ and that

$$(5.2) \qquad \|R(\lambda, A)\| \leq |\lambda|^\alpha \int_0^\infty e^{-\operatorname{Re}\lambda t} M t^\beta e^{\omega t} dt = M\Gamma(\beta+1) \frac{|\lambda|^\alpha}{(\operatorname{Re}\lambda - \omega)^{\beta+1}}$$

for all $\operatorname{Re}\lambda > \omega$. For $\sigma > 0$ consider the operator $A_{\omega+\sigma} = A - \omega - \sigma$. Let $0 < a < \frac{\pi}{2}$ and $0 < \delta < \sigma$. Then $\Sigma_a \cup \mathrm{B}_\delta \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\sigma\} \subseteq \varrho(A_{\omega+\sigma})$. Since $\operatorname{Re}\lambda \geq |\lambda| \cos a$ for all $\lambda \in \Sigma_a$, we obtain by (5.2) that

$$(5.3) \qquad \begin{aligned} \|R(\lambda, A_{\omega+\sigma})\| &= \|R(\lambda+\omega+\sigma, A)\| \\ &\leq \frac{M\Gamma(\beta+1)}{(\cos a)^{\beta+1}} \frac{(|\lambda|+\omega+\sigma)^\alpha}{(|\lambda|+\frac{\sigma}{\cos a})^{\beta+1}} \\ &\leq MC(1+|\lambda|)^{\alpha-\beta-1} \end{aligned}$$

for some $C = C_{\alpha,\beta,\omega,\sigma,a} > 0$ and all $\lambda \in \Sigma_a$. By applying this to $\beta = 0$ and $\beta = \gamma$ it follows that if $\|S^\alpha(t)\| \leq M(1+t^\gamma)e^{\omega t}$, then $\|R(\lambda, A_{\omega+\sigma})\| \leq MC\left((1+|\lambda|)^{\alpha-1} + (1+|\lambda|)^{\alpha-\gamma-1}\right) \leq 2MC(1+|\lambda|)^{\alpha-1}$. Together with (P6), this leads to:

Lemma 5.1. *Let $\alpha \geq 0$ and A be the generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq M(1+t^\gamma)e^{\omega t}$ for some $M \geq 1$, $\gamma \geq 0$, $\omega \geq 0$, and all $t \geq 0$. Then, for all $\sigma > 0$ and $b \in \mathbb{R}$ the fractional powers $(-A_{\omega+\sigma})^b$ are defined. Moreover, if $b > \alpha$, then the domain $D((-A_{\omega+\sigma})^b)$ is independent of $\sigma > 0$.*

In the next section, these fractional powers are used to show that for each $\sigma > 0$ the solutions of the abstract Cauchy problem (ACP) are given by

$$u(t, x_0) \;:=\; e^{(\omega+\sigma)t}\, v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0).$$

Here $v_{\omega+\sigma}(\cdot,\cdot)$ is the singular integral studied in the previous section, ω is the exponential type of the α-times integrated semigroup generated by A. Comparing this representation of $u(t,x_0)$ with the estimates in the previous section, we see that we need an estimate for the graph norm with respect to $A^{[\alpha+\varepsilon]}$ of $(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0$ that takes into account the σ-dependence. As it is a direct consequence of the properties of fractional powers, we give the estimate at this junction. The σ-dependence only plays a role in the case of polynomially bounded integrated semigroups, so we restrict ourselves to the case $\omega = 0$.

Lemma 5.2. *Let A be the densely defined generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \leq M(1+t^\gamma)$ for some $M \geq 1$, $\gamma \geq 0$, and all $t \geq 0$. Then for all $b < \alpha$ there is a constant $C_{\gamma,b,M} > 0$ such that for all $0 < \sigma \leq 1$ and $x \in D((-A)^b)$,*

$$\sum_{k=0}^{[b]} \|A^k(-A_\sigma)^{b-[b]}x\| \;\leq\; C_{\gamma,b,M}\, \sigma^{\min\{0,\alpha-\gamma\}} \sum_{k=0}^{[b]} \|A^k(-A_1)^{b-[b]}x\|.$$

Proof. The assertion follows immediately if $b \in \mathbb{N}$. Then $b - [b] = 0$ and $(-A_\sigma)^{b-[b]} = (-A_1)^{b-[b]} = I_X$. Thus assume $b \notin \mathbb{N}$. By (P3), (P7), Lemma 5.1 and the obvious fact that A^k commutes with the fractional powers of $-A_\sigma$, we have

$$\begin{aligned}
\sum_{k=0}^{[b]} \|A^k(-A_\sigma)^{b-[b]}x\| &= \sum_{k=0}^{[b]} \|A^k(-A_\sigma)^{b-[b]}(-A_1)^{-(b-[b])}(-A_1)^{b-[b]}x\| \\
&= \sum_{k=0}^{[b]} \|\big(-A_1(-A_\sigma)^{-1}\big)^{-(b-[b])}A^k(-A_1)^{b-[b]}x\| \qquad (5.4)\\
&\leq \big\|\big(-A_1(-A_\sigma)^{-1}\big)^{-(b-[b])}\big\| \sum_{k=0}^{[b]} \|A^k(-A_1)^{b-[b]}x\|.
\end{aligned}$$

We need to investigate the fractional powers of $-B := -A_1(-A_\sigma)^{-1} \in \mathcal{B}(X)$. The operator $\lambda - B$ is given by

$$\lambda - B = \Big(\lambda(\sigma - A) + (1-A)\Big)(\sigma - A)^{-1} = (\lambda+1)\Big(\frac{\lambda\sigma+1}{\lambda+1} - A\Big)(\sigma - A)^{-1}.$$

Hence, the inverse

$$(\lambda - B)^{-1} = \frac{1}{\lambda+1}(\sigma - A)\Big(\frac{\lambda\sigma+1}{\lambda+1} - A\Big)^{-1}$$

exists if $\frac{\lambda\sigma+1}{\lambda+1} \in \varrho(A)$, in particular for all $\lambda \geq 0$. By (5.3), the estimate $\|S^\alpha(t)\| \leq M(1+t^\gamma)$ implies $\|R(\mu, A)\| \leq MC_\gamma(\mu^{\alpha-1} + \mu^{\alpha-\gamma-1})$ for all $\mu > 0$. It follows that for $\lambda \geq 0$, $(\lambda - B)^{-1}$ satisfies

$$\begin{aligned}\|(\lambda - B)^{-1}\| &= \frac{1}{\lambda+1}\left\| I + \frac{\sigma-1}{\lambda+1}\Big(\frac{\lambda\sigma+1}{\lambda+1} - A\Big)^{-1}\right\| \\ &\leq \frac{1}{\lambda+1}\left(1 + \frac{1-\sigma}{\lambda+1}MC_\gamma\left\{\Big(\frac{\lambda\sigma+1}{\lambda+1}\Big)^{\alpha-1} + \Big(\frac{\lambda\sigma+1}{\lambda+1}\Big)^{\alpha-\gamma-1}\right\}\right).\end{aligned}$$

Since $\frac{1}{\lambda+1} \leq \frac{\lambda\sigma+1}{\lambda+1} \leq 1$ for all $\lambda \geq 0$ and $0 < \sigma \leq 1$, we majorize $(\frac{\lambda\sigma+1}{\lambda+1})^{\alpha-1}$ by $(\frac{1}{\lambda+1})^{\alpha-1}$ if the exponent $\alpha - 1$ is negative and by 1 otherwise. Thus the term $\frac{1-\sigma}{\lambda+1}(\frac{\lambda\sigma+1}{\lambda+1})^{\alpha-1}$ is bounded by 1. To estimate the term $\frac{1-\sigma}{\lambda+1}(\frac{\lambda\sigma+1}{\lambda+1})^{\alpha-\gamma-1}$ we proceed similarly unless $\alpha < \gamma$. Then, to avoid polynomial growth in λ, we have to use that $\frac{\lambda\sigma+1}{\lambda+1} \geq \sigma$ for all $\lambda \geq 0$ and $0 < \sigma \leq 1$. This yields that $\frac{1-\sigma}{\lambda+1}(\frac{\lambda\sigma+1}{\lambda+1})^{\alpha-\gamma-1}$ is bounded by $\sigma^{\alpha-\gamma}$. Summarizing, for all $0 < \sigma \leq 1$ and $\lambda \geq 0$ we have

$$\|(\lambda - B)^{-1}\| \ \leq \ \frac{1}{\lambda+1}(1 + 2\sigma^{\min\{0,\alpha-\gamma\}}MC_\gamma).$$

Now, by (P2) and the fact that B is bounded it follows that

$$(-B)^{-(b-[b])} \ = \ \frac{\sin(b-[b])\pi}{\pi}\int_0^\infty t^{[b]-b}(t-B)^{-1}dt.$$

Therefore, for all $0 < \sigma \leq 1$ we have

$$\|(-B)^{-(b-[b])}\| \ \leq \ \left|\frac{\sin(b-[b])\pi}{\pi}\right| \int_0^\infty t^{[b]-b}\frac{1}{t+1}(1 + 2\sigma^{\min\{0,\alpha-\gamma\}}MC_\gamma)dt.$$

Combining this with (5.4) gives the desired result. □

6. The abstract Cauchy problem

This section is devoted to the proofs of Theorems 1.1 and 1.2. As we already mentioned in Section 5, we are going to show that the mild solutions $u(\cdot, x)$ of (ACP) can be represented explicitly by the formula

$$u(t, x_0) := e^{(\omega+\sigma)t} v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]} x_0). \tag{6.1}$$

The proof consists of a series of lemmas, in which we have the following standing assumptions. The operator A is the densely defined generator of an α-times integrated semigroup satisfying $\|S^\alpha(t)\| \le M(1+t^\gamma)e^{\omega t}$ for some $M \ge 1$, $\gamma \ge 0$, $\omega \ge 0$ and all $t \ge 0$. Further $\varepsilon > 0$ is such that $\alpha + \varepsilon \notin \mathbb{N}$ and we fix $0 < \sigma \le 1$.

Lemma 6.1. *Let $x \in D(A^{[\alpha+\varepsilon]})$. Then the Laplace transform of the function $v_{\omega+\sigma}(\cdot, x)$ converges absolutely for all $\lambda > 0$ and*

$$\int_0^\infty e^{-\lambda t} v_{\omega+\sigma}(t, x)\, dt \;=\; (-A_{\omega+\sigma})^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda, A_{\omega+\sigma})x, \qquad \lambda > 0.$$

Proof. The convergence of the Laplace transform follows from Lemma 4.3. For $\lambda > 0$, we obtain, writing $\Gamma_{\alpha,\varepsilon} := \pi^{-1}\sin((\alpha+\varepsilon-[\alpha+\varepsilon])\pi)$ and using (P2),

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} v_{\omega+\sigma}(t, x)\, dt \\
&= \Gamma_{\alpha,\varepsilon} \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{1}{s-1}\Big(s^{[\alpha+\varepsilon]-\alpha-\varepsilon} S_{\omega+\sigma}^{\alpha+\varepsilon-[\alpha+\varepsilon]}(t)x - \frac{1}{s} S_{\omega+\sigma}^{\alpha+\varepsilon-[\alpha+\varepsilon]}(\frac{t}{s})x\Big)\, ds\, dt \\
&= \Gamma_{\alpha,\varepsilon} \int_0^\infty \frac{1}{s-1}\Big((\lambda s)^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda, A_{\omega+\sigma})x - \frac{1}{s} s(\lambda s)^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda s, A_{\omega+\sigma})x\Big)\, ds \\
&= \Gamma_{\alpha,\varepsilon} \int_0^\infty \frac{1}{s-1} (\lambda s)^{[\alpha+\varepsilon]-\alpha-\varepsilon} (\lambda s - \lambda) R(\lambda, A_{\omega+\sigma}) R(\lambda s, A_{\omega+\sigma})x\, ds \\
&= \Gamma_{\alpha,\varepsilon} \int_0^\infty t^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda, A_{\omega+\sigma}) R(t, A_{\omega+\sigma})x\, dt \\
&= (-A_{\omega+\sigma})^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda, A_{\omega+\sigma})x.
\end{aligned}$$

□

Lemma 6.2. *For every $x \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, $(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x \in D(A^{[\alpha+\varepsilon]})$ and*

$$\int_0^\infty e^{-\lambda t} v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x)\, dt \;=\; R(\lambda, A_{\omega+\sigma})x, \qquad \lambda > 0.$$

Proof. Since $x \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, by property (P3) we see that there exists a $y \in X$ such that $x = (-A_{\omega+\sigma})^{-\alpha-\varepsilon}y$. Then, by properties (P1), the first part of (P2), and (P4), we have $(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x = (-A_{\omega+\sigma})^{-[\alpha+\varepsilon]}y \in D(A^{[\alpha+\varepsilon]})$. Hence, $v_{\omega+\sigma}(\cdot, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x)$ is defined and by Lemma 6.1, it follows that

$$\begin{aligned}\int_0^\infty & e^{-\lambda t} v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x)\,dt \\ &= (-A_{\omega+\sigma})^{[\alpha+\varepsilon]-\alpha-\varepsilon} R(\lambda, A_{\omega+\sigma})(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x \\ &= R(\lambda, A_{\omega+\sigma})x\end{aligned}$$

for every $\lambda > 0$. □

Lemma 6.3. *Let $x \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$. Then $v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x) \in D(A_{\omega+\sigma})$ for all $t \geq 0$, and*

$$A_{\omega+\sigma} v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x) = v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]} A_{\omega+\sigma}x).$$

PROOF. First, note that the operators $(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}$ and $A_{\omega+\sigma}$ commute on $D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$. By assumption, there exists a $y \in X$ such that $x = (-A_{\omega+\sigma})^{-(1+\alpha+\varepsilon)}y$. Then

$$\begin{aligned} z := (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x &= (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}(-A_{\omega+\sigma})^{-(1+\alpha+\varepsilon)}y \\ &= (-A_{\omega+\sigma})^{-([\alpha+\varepsilon]+1)}y \in D(A_{\omega+\sigma}^{[\alpha+\varepsilon]+1}).\end{aligned}$$

By (3.3) and Lemma 3.1 we obtain $S_{\omega+\sigma}^{\alpha+\varepsilon-[\alpha+\varepsilon]}(t)A_{\omega+\sigma}z = A_{\omega+\sigma}S_{\omega+\sigma}^{\alpha+\varepsilon-[\alpha+\varepsilon]}(t)z$ for every $t \geq 0$. Since $A_{\omega+\sigma}$ is closed, this gives $v_{\omega+\sigma}(t, A_{\omega+\sigma}z) = A_{\omega+\sigma}v_{\omega+\sigma}(t, z)$ for all $t \geq 0$. □

Lemma 6.4. *If $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, then the map $u(\cdot, x_0)$ given by $u(t, x_0) := v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0)$ for $t \geq 0$, is the unique mild solution of the abstract Cauchy problem*

$$(\mathrm{ACP}_{\omega+\sigma}) \qquad u'(t) = A_{\omega+\sigma}u(t) \quad (t \geq 0), \qquad u(0) = x_0.$$

If $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$, then this solution is a classical solution of $(ACP_{\omega+\sigma})$.

PROOF. Assume that $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$. Then by Lemma 6.2 and [12, Theorem 2.1] the function $u(\cdot, x_0) = v_{\omega+\sigma}(\cdot, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0)$ is a mild solution of $(\mathrm{ACP}_{\omega+\sigma})$. Uniqueness is proved as follows. Suppose there is another mild solution $\tilde{u}(\cdot, x_0)$ of $(\mathrm{ACP}_{\omega+\sigma})$. Then $w(t) := \int_0^t u(s, x_0) - \tilde{u}(s, x_0)\,ds$ is a classical solution of $(\mathrm{ACP}_{\omega+\sigma})$ with initial value 0. By a theorem of Lyubič (see [20, Theorem 4.1.2]), $w(\cdot) = 0$.

If $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$, then also $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ and therefore the map $u(t) := v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0)$ solves the equation

$$u(t) \;=\; A_{\omega+\sigma}\int_0^t u(s)\,ds \;+\; x_0, \qquad t \geq 0.$$

By Lemma 6.3 and the closedness of $A_{\omega+\sigma}$, we obtain

$$v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0) \;=\; \int_0^t A_{\omega+\sigma}v_{\omega+\sigma}(s, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0)\,ds + x_0$$

for all $t \geq 0$. Hence, $v_{\omega+\sigma}(\cdot, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0)$ is continuously differentiable and

$$\begin{aligned} v'_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0) &= A_{\omega+\sigma}v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0), \\ v_{\omega+\sigma}(0, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0) &= x_0. \end{aligned}$$

The uniqueness follows by the above-mentioned theorem of Lyubič. □

Now we are in a position to prove Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1: If $\alpha + \varepsilon = n \in \mathbb{N}$, the theorem is an obvious consequence of (1.2). Therefore, we may assume that $\alpha + \varepsilon \notin \mathbb{N}$. By Lemma 6.4, for each $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ there exists a unique mild solution of (ACP), which is given by

$$u(t, x_0) \;=\; e^{(\omega+\sigma)t}v_{\omega+\sigma}(t, (-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0), \quad t \geq 0,$$

and it is a classical solution if $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$. By Lemma 4.3, there is a $\delta > 0$ such that

$$\|u(t, x_0)\| \;\leq\; MC_{\alpha,\varepsilon,\delta,\gamma,\omega,\sigma}\,\|(-A_{\omega+\sigma})^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0\|_{[\alpha+\varepsilon]}\,e^{(\omega+\sigma)t}(1 + t^\delta).$$

Since $\sigma > 0$ is arbitrary and $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ and $D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$ are independent of σ, this estimate shows that the solution has exponential type ω. □

PROOF OF THEOREM 1.2: First assume $\alpha + \varepsilon = n \in \mathbb{N}$. The n-times integrated semigroup generated by A is of polynomial type $\gamma + \varepsilon$. The desired result then follows from (1.2).

Therefore, we may assume $\alpha + \varepsilon \notin \mathbb{N}$. As in the proof of Theorem 1.1, for $0 < \sigma \leq 1$ the solution of (ACP) is given by

$$u(t, x_0) \;=\; e^{\sigma t}\,v_\sigma(t, (-A_\sigma)^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0), \quad t \geq 0.$$

Lemmas 4.4 and 5.2 yield that

$$\begin{aligned}\|u(t,x_0)\| &\le MC_{\alpha,\varepsilon,\delta,\gamma}\|(-A_\sigma)^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0\|_{[\alpha+\varepsilon]}\\ &\qquad\cdot t^{\alpha+\varepsilon-[\alpha+\varepsilon]-1}e^{\sigma t}\sigma^{\min\{-[\alpha+\varepsilon],\alpha-[\alpha+\varepsilon]-\gamma-1\}}\\ &\le C_{\alpha,\varepsilon,\delta,\gamma,M}\|(-A_1)^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0\|_{[\alpha+\varepsilon]}\\ &\qquad\cdot t^{\alpha+\varepsilon-[\alpha+\varepsilon]-1}e^{\sigma t}\sigma^{\min\{-[\alpha+\varepsilon],\alpha-[\alpha+\varepsilon]-\gamma-1\}+\min\{0,\alpha-\gamma\}}\end{aligned}$$

for all $t \ge 2$ and $0 < \sigma \le 1$. For $t \ge 2$ fixed, we now take $\sigma = t^{-1}$. This leads to

$$\|u(t,x_0)\| \le K'_{\alpha,\varepsilon,\delta,\gamma,M}\, e\, \|(-A_1)^{\alpha+\varepsilon-[\alpha+\varepsilon]}x_0\|_{[\alpha+\varepsilon]}\, t^{\max\{\alpha-1+\varepsilon,\gamma+\varepsilon,2\gamma-\alpha+\varepsilon\}}$$

for all $t \ge 2$. This concludes the proof of Theorem 1.2. □

Remark.

(i) Suppose that A is the densely defined generator of an α-times integrated semigroup for some $\alpha \ge 0$, $\alpha \notin \mathbb{N}$. Since A also generates an $([\alpha]+1)$-times integrated semigroup, it was known that (ACP) has a unique mild solution for every $x \in D(A^{[\alpha]+1})$. The Theorems 1.1 and 1.2 improve this result. In fact, take any $0 < \varepsilon < [\alpha] + 1 - \alpha$. If the fractional powers are the classical ones then clearly $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ is a larger set than $D(A^{[\alpha]+1})$. In the case that the fractional powers are not defined in the classical way, it follows by (P5) that $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ is not contained in $D(A^{[\alpha]+1})$. Hence, $\bigcup_{0<\varepsilon\le 1} D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, the maximal set of initial values for which we obtain mild solutions, is larger than $D(A^{[\alpha]+1})$.

(ii) If the fractional powers of $-A_\sigma$ exist in the classical sense, then the estimates for the polynomial growth bounds in Theorem 1.2 can be improved slightly. Indeed, since we have the inclusions $D((-A_\sigma)^\nu) \subset D((-A_\sigma)^\mu)$ whenever $\nu > \mu$, the solution $u(\cdot,x_0)$ satisfies an estimate

$$\|u(t,x_0)\| \le C_{\alpha,\varepsilon,\delta,\gamma,M,x_0}\, t^{\max\{\alpha-1+\varepsilon,\gamma+\varepsilon,2\gamma-\alpha+\varepsilon\}}, \quad t \ge 2,$$

for every $\varepsilon > 0$. Its polynomial growth bound is therefore given by $\max\{\alpha - 1, \gamma, 2\gamma - \alpha\}$, i.e. the ε's can be dropped. For example, if $\|S^\alpha(t)\| \le Mt^\alpha$ $(t \ge 0)$, the solutions are at most of polynomial type α.

7. Unbounded operators with polynomially bounded resolvent

If A is the generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying $\|S^\alpha(t)\| \le Mt^\beta e^{\omega t}$, then by (5.2) there is a constant $M \ge 1$ such that

$$\|R(\lambda,A_\omega)\| \le M\,|\lambda|^\alpha\,(\operatorname{Re}\lambda)^{-\beta-1}, \quad \operatorname{Re}\lambda > \omega. \tag{7.1}$$

In this section, we apply our results to arbitrary densely defined operators whose resolvent satisfies (7.1). We start with a general fact about Laplace transforms.

Lemma 7.1. *Let $\omega > 0$ and let $q : \{Re\,\lambda > \omega\} \longrightarrow X$ be a bounded analytic function. Let $b > 0$ and $\sigma > \omega$. Then, the function $f : [0,\infty) \longrightarrow X$ given by*

$$f(t) \;=\; \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{\lambda t}\,\lambda^{-b-1}\,q(\lambda)\,d\lambda, \quad t \geq 0,$$

is continuous, its Laplace transform converges absolutely on $\{Re\,\lambda > \omega\}$, and satisfies $q(\lambda) = \lambda^{b+1}\int_0^\infty e^{-\lambda t} f(t)\,dt$ there.

The straightforward proof is omitted. It can be found in [2, Theorem 2.7.1]. The following result is closely related to [3, Theorem 3.1].

Lemma 7.2. *Let A be a (not necessarily densely defined) linear operator on X whose resolvent exists in the right half plane. Suppose there are $\alpha \geq 0$ and $\beta \geq 0$ such that for each $\sigma > 0$ there is a constant M_σ such that*

$$\|R(\lambda, A)\| \;\leq\; M_\sigma\,|\lambda|^{\alpha-1}\,(Re\,\lambda)^{-\beta}, \qquad Re\,\lambda > \sigma.$$

Then, for every $\varepsilon > 0$, A generates an exponentially bounded $(\alpha+\varepsilon)$-times integrated semigroup. Moreover, there is a constant C_ε such that for each $\sigma > 0$,

$$\|S^{\alpha+\varepsilon}(t)\| \;\leq\; C_\varepsilon M_\sigma\,\sigma^{-\beta-\varepsilon}\,e^{\sigma t}, \qquad t \geq 0.$$

PROOF. Fix $\varepsilon > 0$ and $\sigma > 0$. For $x \in X$, define $q_x(\lambda) := \lambda^{1-\alpha}R(\lambda, A)x$. Then q_x is uniformly bounded in $\{\mathrm{Re}\,\lambda > \frac{1}{2}\sigma\}$. By Lemma 7.1, applied to $\omega \mapsto \frac{\sigma}{2}$ and $b \mapsto \varepsilon$, we obtain a continuous function f_x such that for every $t \geq 0$,

$$f_x(t) \;=\; \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{\lambda t}\frac{R(\lambda, A)x}{\lambda^{\alpha+\varepsilon}}\,d\lambda.$$

Define $S^{\alpha+\varepsilon}(t)x := f_x(t)$, $t \geq 0$. Then (7.1) holds and

$$\lambda^{\alpha+\varepsilon}\int_0^\infty e^{-\lambda t}S^{\alpha+\varepsilon}(t)x\,dt \;=\; \lambda^{\alpha+\varepsilon}\int_0^\infty e^{-\lambda t}f_x(t)\,dt \;=\; R(\lambda, A)x$$

for all $\lambda > 0$. Next we estimate $\|S^{\alpha+\varepsilon}(t)\|$:

$$\begin{aligned}
\|S^{\alpha+\varepsilon}(t)x\| &= \left\|\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{\lambda t}\frac{R(\lambda, A)x}{\lambda^{\alpha+\varepsilon}}\,d\lambda\right\| \\
&\leq \frac{1}{2\pi}M_\sigma\,\sigma^{-\beta}e^{\sigma t}\|x\|\int_{-\infty}^{\infty}|\sigma+is|^{-1-\varepsilon}\,ds \\
&= C_\varepsilon M_\sigma\,\sigma^{-\beta-\varepsilon}e^{\sigma t}\|x\|.
\end{aligned}$$

It follows that the family $\mathbf{S}^{\alpha+\varepsilon}$ is an exponentially bounded $(\alpha+\varepsilon)$-times integrated semigroup generated by A, and that the desired estimate is satisfied. $\square$

The proof actually shows that it is enough to assume that $|\lambda|^{1-\alpha}\|R(\lambda, A)\|$ is bounded in each right half plane $\{\mathrm{Re}\,\lambda > \sigma\}$ and $|\lambda|^{1-\alpha}\|R(\lambda, A)\| = O(\mathrm{Re}\,\lambda)^{-\beta}$ for $\mathrm{Re}\,\lambda \downarrow 0$.

PROOF OF THEOREM 1.3: We apply Lemma 7.2 with $M_\sigma \mapsto M$ for all $\sigma > 0$ and $\varepsilon \mapsto \frac{1}{2}\varepsilon$. Fix $t > 0$. Taking $\sigma = t^{-1}$, we see that A generates an $(\alpha + \frac{\varepsilon}{2})$-times integrated semigroup of polynomial order $\beta + \frac{\varepsilon}{2}$. Therefore, by Theorem 1.2, the unique mild solutions corresponding to initial values $x_0 \in D((-A_\sigma)^{\alpha+\varepsilon})$ are of polynomial order $(\beta + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2}$. $\square$

The restriction $\alpha - 1 \leq \beta \leq \alpha$ in the statement of Theorem 1.3 was necessary for reasons of consistency. Indeed, if $\beta < \alpha - 1$, then as $\lambda \to 0$ in a sector of angle less than $\pi/2$ we have $\|R(\lambda, A)\| \to 0$. This implies $0 \in \varrho(A)$ and $\|A^{-1}\| = 0$, which is impossible. If $\beta > \alpha$, then $\lim_{\lambda\to\infty} \lambda\|R(\lambda, A)\| = 0$. This is also impossible, as it yields $x = 0$ for all $x \in D(A)$ since for such x we have $x = \lambda R(\lambda, A)x - R(\lambda, A)Ax$ for all $\lambda > 0$.

Lemma 7.2 also allows us to deal with operators whose resolvent is of a slightly different type of polynomial growth.

Corollary 7.3. *Let A be a densely defined linear operator on X whose resolvent exists in the right half plane. Suppose there are $M \geq 1$ and $\alpha \geq 0$ such that*

$$\|R(\lambda, A)\| \leq M(1 + |\lambda|)^{\alpha-1}, \qquad \mathit{Re}\,\lambda > 0.$$

Then, for all $\varepsilon > 0$ and $x_0 \in D((-A)^{\alpha+\varepsilon})$, (ACP) has a unique mild solution $u(\cdot, x_0)$ which is polynomially bounded of order $\max\{0, \alpha - 1\} + \varepsilon$. *If $x_0 \in D((-A)^{1+\alpha+\varepsilon})$, the solution is classical.*

PROOF. If $0 \leq \alpha \leq 1$, then $(1 + |\lambda|)^{\alpha-1} \leq |\lambda|^{\alpha-1}$ for all $\mathrm{Re}\,\lambda > 0$ and the assumptions of Lemma 7.2 are satisfied with $\beta \mapsto 0$ and $M_\sigma \mapsto M$ for all $\sigma > 0$. Fix $t > 0$ and take $\sigma = t^{-1}$. It follows that A generates an $(\alpha+\frac{\varepsilon}{2})$-times integrated semigroup of polynomial order $\frac{\varepsilon}{2}$.

If $\alpha \geq 1$, for all $\sigma > 0$ and $\mathrm{Re}\,\lambda > \sigma$ we have $(1+|\lambda|)^{\alpha-1} \leq (1+\sigma^{-1})^{\alpha-1}|\lambda|^{\alpha-1}$. Therefore, with $M_\sigma \mapsto M(1 + \sigma^{-1})^{\alpha-1}$, $\beta \mapsto 0$, and $\sigma = t^{-1}$, we obtain that A generates an $(\alpha + \frac{\varepsilon}{2})$-times integrated semigroup of polynomial order $(\alpha - 1) + \frac{\varepsilon}{2}$.

As in Theorem 1.3, the desired conclusion now follows from Theorem 1.2, noting that since $0 \in \varrho(A)$ the fractional powers of $-A$ are defined. $\square$

Remark.

(i) Let A satisfy the condition of Theorem 1.3 with $\alpha = \beta = 0$. Then $A - \delta$ is invertible for all $\delta > 0$ and

$$\|R(\lambda, A_\delta)\| \leq M\delta^{-1}(1 + |\lambda|)^{-1}, \qquad \operatorname{Re}\lambda > 0.$$

Therefore, by [20, Theorem 2.5.2] A_δ generates a bounded holomorphic semigroup $\mathbf{T}_\delta$; the bound is proportional to δ^{-1}. Hence, mild solutions of (ACP) exist trivially for all $x_0 \in X$, and they are given by $u(t, x_0) = e^{\delta t}T_\delta(t)x_0$. Taking $\delta = t^{-1}$, it follows that $u(\cdot, x_0)$ is polynomially bounded of order 1. Note that by Theorem 1.3, the mild solutions in $D((-A)^\varepsilon)$ are polynomially bounded of order ε.

(ii) If A is a linear operator such that $\|R(\lambda, A)\| \leq M(1+|\lambda|)^{\alpha-1}$ for all $\operatorname{Re}\lambda > 0$ and some $\alpha \geq 0$, it is shown in [15] that (ACP) has a classical solution for each $x_0 \in D(A^{2+[\alpha]})$. The proof is based on a resolvent expansion formula

$$-R(\lambda, A)x_0 = -\frac{R(\lambda, A)y}{(\lambda - \lambda_0)^{1+[\alpha]}} - \sum_{k=1}^{1+[\alpha]} \frac{(-R(\lambda_0, A))^{2+[\alpha]-k}y}{(\lambda - \lambda_0)^k}$$

where $x_0 = (-R(\lambda_0, A))^{1+[\alpha]}y$, and Laplace inversion (this explains the occurrence of the term $[\alpha]$). Corollary 7.3 can be viewed as an improvement of this result for the densely defined case.

(iii) Whenever the fractional powers of $-A$ exist in the classical sense, then, as before (see Remark (ii) at the end of Section 6), the ε's occurring in the estimates for the polynomial growth bounds can be dropped. For example, if $\alpha = \beta$ in Theorem 1.3, the solutions are of polynomial type α.

Let us work out in more detail the case $\alpha = 1$.

Theorem 7.4. *Let A be a densely defined linear operator on X whose resolvent exists and is uniformly bounded in the right half plane. Then, for all $\varepsilon > 0$ and $x_0 \in D((-A)^{1+\varepsilon})$, (ACP) has a unique mild solution $u(\cdot, x_0)$. Moreover, this solution is exponentially stable. If $x_0 \in D((-A)^{2+\varepsilon})$, the solution is classical.*

Proof. By a standard argument, one sees that the resolvent is uniformly bounded in a half plane $\{\operatorname{Re}\lambda > -\delta\}$ for some $\delta > 0$. Therefore, we can apply Theorem 1.3 for $\alpha = 1$ and $\beta = 0$ or Corollary 7.3 for $\alpha = 1$ to the operator $A + \frac{\delta}{2}$. □

As another application of Lemma 7.2, we obtain an improvement of our results in Section 6 as far as the exponential growth of *classical* solutions is concerned. Noting that if A generates an α-times integrated semigroup $\mathbf{S}^\alpha$ satisfying

$\|S^\alpha(t)\| \le Mt^\beta e^{\omega t}$, then by (3.1) for each $\sigma > 0$ there is a constant $M_\sigma \ge 1$ such that

$$\|R(\lambda, A_\omega)\| \le M_\sigma |\lambda|^\alpha, \quad \operatorname{Re}\lambda > \sigma.$$

We will show that this inequality is in fact sufficient for the classical solutions to have exponential type of at most ω. In order to formulate the precise result, let $s^\alpha(A)$ denote the supremum of all $\omega \in \mathbb{R}$ such that $\|R(\lambda, A)\| = O(|\lambda|^\alpha)$ for $|\lambda| \to \infty$ in the right half plane $\{\operatorname{Re}\lambda > \omega\}$.

Theorem 7.5. *Let A be the densely defined generator of an exponentially bounded α-times integrated semigroup $\mathbf{S}^\alpha$. Then, for every $\omega > s^\alpha(A)$ and all $x_0 \in D((-A_\omega)^{1+\alpha+\varepsilon})$, the abstract Cauchy problem (ACP) admits a unique classical solution, and the exponential type of this solution does not exceed $s^\alpha(A)$.*

PROOF. Since for each $\sigma > 0$ the resolvent of $A_{\omega+\sigma}$ satisfies $\|R(\lambda, A_{\omega+\sigma})\| \le M_\sigma(1 + |\lambda|)^\alpha$ for some M_σ and all $\operatorname{Re}\lambda > 0$, Corollary 7.3 shows that for each $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\varepsilon})$, (ACP$_{\omega+\sigma}$) has a unique mild solution $u_{\omega+\sigma}(\cdot, x_0)$, which is of polynomial growth. But by the results of Section 6, there is also an exponentially bounded classical solution. Since classical solutions are mild solutions, the uniqueness of mild solutions implies that $u_{\omega+\sigma}(\cdot, x_0)$ is a classical solution. But then $u(t, x_0) := e^{(\omega+\sigma)t}u_{\omega+\sigma}(t, x_0)$ is the unique classical solution of (ACP), and this grows exponentially of type $\omega + \sigma$. Since $\omega > s^\alpha(A)$ and $\sigma > 0$ are arbitrary and $D((-A_\omega)^{1+\alpha+\varepsilon})$ is independent of ω, the theorem is proved. □

This result can be concisely formulated as saying that $\omega_{1+\alpha+\varepsilon}(A) \le s^\alpha(A)$. For generators A of C_0-semigroups (i.e. the case $\alpha = 0$), Weis and Wrobel [25] recently proved that $\omega_1(A) \le s^0(A)$ holds. Earlier, Wrobel [27] had shown that this is true in B-convex spaces, and in [18] it was shown that $\omega_{1+\varepsilon}(A) \le s^0(A)$ holds in arbitrary Banach spaces. An elementary proof of the Weis-Wrobel result was obtained in [17].

8. Application to elliptic differential operators

In this final section, we list some concrete examples of differential operators to which our results can be applied. For more details and proofs, we refer to the paper of Hieber [11].

Proposition 8.1. *Let A be a differential operator with constant coefficients on one of the spaces $X = L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, or $X = C_0(\mathbb{R}^n)$, and let ξ be its symbol. Assume that $\xi(\cdot) = iq(\cdot)$, with the polynomial $q(\cdot)$ either real homogeneous* *such that $q(x) = 0$ implies $x = 0$, or real elliptic. Then for all $\alpha > n|\frac{1}{2} - \frac{1}{p}|$, A is the generator of an α-times integrated semigroup $\mathbf{S}^\alpha$ on X which satisfies*

$$\|S^\alpha(t)\| \leq M(1 + t^\alpha), \quad t \geq 0.$$

If p is homogeneous, then $\quad \|S^\alpha(t)\| \leq Mt^\alpha, \quad t \geq 0.$

In the above result, if $X = C_0(\mathbb{R}^n)$ we take $p = \infty$.

Examples. The following operators generate α-times integrated semigroups.

(i) The Schrödinger operator $A = i\Delta$, where Δ is the Laplacian, generates an α-times integrated semigroup $\mathbf{S}^\alpha$ on the spaces $X = L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, or $X = C_0(\mathbb{R}^n)$, for all $\alpha > n|\frac{1}{2} - \frac{1}{p}|$. Moreover,

$$\|S^\alpha(t)\| \leq Mt^\alpha, \quad t \geq 0.$$

The space $D((-i\Delta)^\beta)$ can be calculated explicitly: it coincides with the Sobolev space $W^{2\beta,p}$.

(ii) The Korteweg-De Vries operator $A = \frac{\partial^3}{\partial x^3} + \frac{\partial}{\partial x}$ generates an α-times integrated semigroup $\mathbf{S}^\alpha$ on the spaces $X = L^p(\mathbb{R})$, $1 \leq p \leq \infty$, for all $\alpha > |\frac{1}{2} - \frac{1}{p}|$. Moreover,

$$\|S^\alpha(t)\| \leq M(1 + t^\alpha), \quad t \geq 0.$$

For $x_0 \in D((-A)^{\alpha+\varepsilon})$, in the above examples we obtain mild solutions of polynomial type α and $\alpha + \varepsilon$, respectively. By using more direct methods, the bound for the Schrödinger operator can be established more easily. Indeed, it follows from the results in [4] that in this case the mild solutions are of polynomial growth $O(t^{n|\frac{1}{2}-\frac{1}{p}|})$. More generally, Zheng and Lei [28] recently showed that if A is a constant coefficient differential operator of degree m whose symbol has range in the closed left half plane, then the abstract Cauchy problem for A has $O(t^{n|\frac{1}{2}-\frac{1}{p}|})$ mild solutions for all initial values in $D((-\Delta)^\beta)$, $\beta > \frac{mn}{2}|\frac{1}{2} - \frac{1}{p}|$. At least for integer α and $X = L^p(\mathbb{R}^n)$, the domain $D((-\Delta)^{\frac{m\alpha}{2}})$ equals $D((-A)^\alpha)$. Both [4] and [28] use the technique of regularized semigroups. Thus, up to an ε in the polynomial bounds, our results seem to be optimal. At present, we do not know whether, in our abstract setting, this extra ε can be removed.

Acknowledgement - We would like to thank Frank Neubrander for many discussions on this paper and for supplying the details of the proof of Proposition 3.3.

REFERENCES

[1] W. Arendt, Vector valued Laplace transforms and Cauchy problems, *Israel J. Math.* 59 (1987), 327-353.

[2] W. Arendt, M. Hieber and F. Neubrander, *Laplace Transforms and Evolution Equations*, monograph, in preparation.

[3] W. Arendt and H. Kellermann, Integrated solutions of Volterra integro-differential equations and applications, in: G. Da Prato and M. Iannelli (eds.), *Volterra integro-differential equations in Banach spaces and applications*, Proc. Conf. Trento (1987).

[4] K. Boyadzhiev and R. deLaubenfels, Boundary values of holomorphic semigroups, *Proc. Amer. Math. Soc.* 118 (1993), 113-118.

[5] R. deLaubenfels, Integrated semigroups, C-semigroups and the abstract Cauchy problem, *Semigroup Forum* 41 (1990), 83-95.

[6] R. deLaubenfels, Z. Huang, S. Wang and Y. Wang, Laplace transforms of polynomially bounded vector-valued functions and semigroups of operators, *Israel J. Math.*, 98 (1997), 189-207.

[7] A. Erdélyi (ed.), *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York - Toronto - London (1954).

[8] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, John Wiley & Sons, New York - London - Sydney (1966).

[9] B. Hennig and F. Neubrander, On representations, inversions, and approximations of Laplace transforms in Banach spaces, *Appl. Analysis* 49 (1993) 151-170.

[10] M. Hieber, Laplace transforms and α-times integrated semigroups, *Forum Math.* 3 (1991), 595-612.

[11] M. Hieber, Integrated semigroups and differential operators on L^p spaces, *Math. Ann.* 291 (1991), 1-16.

[12] M. Hieber, A. Holderrieth and F. Neubrander, Regularized semigroups and systems of linear partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, Ser. 4, 19 (1992), 363-379.

[13] H. Kellermann and M. Hieber, Integrated semigroups, *J. Funct. Anal.* 84 (1989), 160-180.

[14] H. Komatsu, Fractional powers of operators, *Pacific J. Math.* 19 (1966), 285-346.

[15] S.G. Krein, *Linear Differential Equations in Banach Spaces*, Translations Amer. Math. Soc. 29, Providence, R.I. (1971).

[16] I. Miyadera and N. Tanaka, Some remarks on C-semigroups and integrated semigroups, *Proc. Japan Soc.*, Ser. A, 63 (1987), 139-142.

[17] J.M.A.M. van Neerven, Individual stability of C_0-semigroups with uniformly bounded local resolvent, *Semigroup Forum* 53 (1996), 155-161.

[18] J.M.A.M. van Neerven, B. Straub and L. Weis, On the asymptotic behaviour of a semigroup of linear operators, *Indag. Math. (N.S.)* 6 (1995), 453-476.

[19] F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problem, *Pacific J. Math.* 135 (1988), 111-155.

[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[21] B. Straub, On α-times integrated semigroups and the abstract Cauchy problem, in: *LSU Seminar Notes in Functional Analysis and PDE's* (1991-92), Baton Rouge.

[22] B. Straub, Fractional powers of operators with polynomially bounded resolvent and the semigroups generated by them, *Hiroshima Math. J.* 24 (1994), 529-548.

[23] B. Straub, *On Fractional Powers of Closed Operators with Polynomially Bounded Resolvents and Applications to the Abstract Cauchy Problem*, Ph.D. Dissertation, Tübingen (1994).

[24] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Berlin VEB (1978).

[25] L. Weis and V. Wrobel, Asymptotic behavior of C_0-semigroups in Banach spaces, *Proc. Amer. Math. Soc.* 124 (1996), 3663-3671.

[26] D.V. Widder, *The Laplace Transform*, Princeton University Press, Princeton (1941).

[27] V. Wrobel, Asymptotic behavior of C_0-semigroups in B-convex spaces, *Indiana Univ. Math. J.* 38 (1989), 101-114.

[28] Q. Zheng and Y. Lei, The application of C-semigroups to differential operators in $L^p(\mathbb{R}^n)$, *J. Math. Anal. Appl.* 118 (1994), 809-818.

Received January 20, 1996

Alfred P. Sloan Laboratory of Mathematics, California Institute of Technology, CA 91125 Pasadena, U.S.A.

E-mail address: J.vanNeerven@twi.tudelft.nl

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

E-mail address: bernd@maths.unsw.edu.au