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# ON THE EXPONENTIAL DECAY OF $C^1$ CUBIC LAGRANGE SPLINES ON NON-UNIFORM MESHES AND FOR NON-UNIFORM DATA POINTS

SANG DONG KIM AND BYEONG CHUNG SHIN

COMMUNICATED BY RIDGWAY SCOTT

ABSTRACT. The space of $C^1$ cubic splines on a given arbitrary mesh for $[0, 1]$ provides a unique interpolant to arbitrary data given at 0 and 1 and at two arbitrary points in the interior of each mesh interval. Sufficient conditions for the exponential decay of the corresponding Lagrange functions are given, in terms of the distribution of the data points.

## 1. INTRODUCTION

Consider the $C^1$ cubic Lagrange splines $\{\psi_i\}_{i=1}^{2N}$ for a partition $\Delta := \{t_i\}_{i=0}^{N}$ of $I := [0,1]$ satisfying

$$\psi_i(\xi_j) = \delta_{i,j} \ (i = 1, \dots, 2N,\ j = 0, 1, \cdots, 2N+1), \tag{1.1}$$

where

$$\Delta \ : \ 0 = t_0 < t_1 < \ \cdots \ < t_{N-1} < t_N = 1,$$

and $\{\xi_i\}_{i=0}^{2N+1}$ are given by $\xi_0 = 0$, $\xi_{2N+1} = 1$ and for $i = 1, \dots, N$,

$$\xi_{2i-1} = t_{i-1} + h_i p_i \in I_i \quad \text{and} \quad \xi_{2i} = t_{i-1} + h_i q_i \in I_i$$

with $h_i := t_i - t_{i-1}$, $I_i := (t_{i-1}, t_i)$ and $0 < p_i < q_i < 1$. Let

$$\Delta_l := \{(p_i, q_i)\,|0 < p_i < q_i < 1,\ i = 1, \dots, N\}$$

and $|\Delta| := \max_i \{h_i\}$.

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The existence and uniqueness of the Lagrange spline defined in (1.1) can be checked by using Schoenberg-Whitney conditions with respect to the space $\mathcal{S}$ of $C^1$ cubic splines for the given partition $\{t_1, t_1, t_2, t_2, \cdots, t_{N-1}, t_{N-1}\}$ (see [B1], [SW]). Denote by $i^* := \lfloor (i+1)/2 \rfloor$ the largest integer less than or equal to $(i+1)/2$. The exponential decay of such a Lagrange spline comes from de Boor and Birkhoff's observation (see [BB], [B2], [B3]) in the sense that the spline $\psi_i(t)$ on $I_j$ defined in (1.1) decays exponentially by $\left(\frac{1}{\omega}\right)^{|i^*-j|}$ where $\omega > 1$ as $j$ moves away from $i^*$. Of the exponential decay of the $C^1$ cubic Largange spline defined in (1.1), one may choose the $\omega \geq 5$ when $\xi_{2i-1}$ and $\xi_{2i}$ are chosen symmetrically with respect to the mid-point of $I_i$ (see [KK]). The goal of this paper is of the following. In the exponential decay of $C^1$ cubic Lagrange spline satisfying (1.1), the sufficient conditions are given when *any* two interior points $\xi_{2i-1}$ and $\xi_{2i}$ ($\xi_{2i-1} < \xi_{2i}$) are chosen on a non-uniform mesh of $I$. As a consequence of this result, following the idea in [B3], one may easily check a convergence of the $C^1$ cubic Lagrange spline interpolant $P_\Delta g$ defined by

$$P_\Delta g := \sum_{i=1}^{2N} g(\xi_i)\psi_i$$

for a given function $g$ on the mesh. For the fundamental cubic splines, such a question was studied by many authors (see, for example, [A], [K1], [K2] and [L]).

As a practical application of its exponential decay on $C^1$ cubic Lagrange spline, one may use it to develop a theory on the preconditioning cubic collocation method by other numerical method for an elliptic type partial differential equation(see [KP] for more detail).

The rest of this paper consists of the following way. In section 2 analyzing $\psi_i$ on $I_i$ in detail, we will give a condition on the general two interior points $\xi_{2i-1}$ and $\xi_{2i}$ in $I_i$, which leads to the exponential decay of $\psi_i$ on $I$. In section 3, using the results in section 2, we will extend the previous result of [KK].

## 2. A CONDITION ON THE INTERIOR POINTS

In this section we give a condition on two points $p$ and $q$ ($p < q$) in $(0, 1)$ which leads to the exponential decay of $C^1$ cubic Lagrange spline. With an affine transformation we have a condition on two local interpolatory points $\xi_{2i-1}$ and $\xi_{2i}$ on $I_i$. Therefore a condition on $p$ and $q$ also holds for the local interpolatory points.

Let $f$ be a cubic polynomial on $I$ satisfying $f(p) = f(q) = 0$. With a notation $\vec{f}(t) := (f(t), f'(t))^t$ for a function $f$ defined on $I$, an easy calculation leads to the following relation between $\vec{f}(1)$ and $\vec{f}(0)$ such that

$$\vec{f}(1) = D(p,q)\,\vec{f}(0)$$

where the elements of $2 \times 2$-matrix $D(p,q)$ are given by

$$\begin{aligned} d_{11}(p,q) &= 1 + \tfrac{(p+q)-(p+q)^2+pq}{p^2q^2} \\ d_{12}(p,q) &= 1 + \tfrac{1-(p+q)}{pq} \\ d_{21}(p,q) &= \tfrac{3(p+q)-2(p+q)^2+2pq}{p^2q^2} \\ d_{22}(p,q) &= 1 + \tfrac{3-2(p+q)}{pq}. \end{aligned} \tag{2.1}$$

Define two functions $g$ and $h$ on $[0,1]$ such that

$$g(p) = \frac{2-p-\sqrt{p(4-3p)}}{2} \quad \text{and} \quad h(p) = \frac{1-p+\sqrt{(1+3p)(1-p)}}{2}.$$

Let

$$\begin{aligned} E_L &= \{\, (p,q) \mid 0 < q < h(p), \quad 0 < p < q < 1 \,\}, \\ E_R &= \{\, (p,q) \mid g(p) < q < 1, \quad 0 < p < q < 1 \,\} \end{aligned}$$

and

$$E := E_L \cap E_R = \{\, (p,q) \mid g(p) < q < h(p), \quad 0 < p < q < 1 \,\}.$$

Then $E$ is a symmetric set with respect to the line $p+q=1$, *i.e.*,

$$(p,q) \in E \quad \text{if and only if} \quad (1-q, 1-p) \in E.$$

Contracting the set $E$ by $\delta$ for $p$ and $q-$directions, we define

$$E(\delta) := \{\, (p,q) \in E \,|\, g(p-\delta)+\delta \le q \le h(p+\delta)-\delta, \ q \ge p+2\delta \,\}.$$

Then $E(\delta)$ is also a symmetric set with respect to the line $p+q=1$. For the two curves $q = g(p-\delta)+\delta$ and $q = h(p+\delta)-\delta$, we can find the intersection point $(\kappa(\delta), 1-\kappa(\delta))$ which is located on the line $q = 1-p$, where

$$\kappa(\delta) := \frac{1-\sqrt{1-4\delta(1+3\delta)}}{2}.$$

Moreover, from the equation $p+2\delta = 1-p$, we have $p = 1/2-\delta$. Since $\kappa(\delta) \le \frac{1}{2}-\delta$, we have

$$0 < \delta \le \frac{\sqrt{5}-1}{8} \quad \text{and} \quad 0 < \kappa(\delta) \le \frac{5-\sqrt{5}}{8}.$$

The shapes of sets $E$ and $E(\delta)$ for $\delta = 0.05$ will be presented at the end of the last section. Furthermore, for any $(p,q) \in E(\delta)$ we have

$$\kappa(\delta) \le p \le \frac{2}{3} - 2\delta, \quad \frac{1}{3} + 2\delta \le q \le 1 - \kappa(\delta), \tag{2.2}$$

and

$$2\delta \le q - p \le 1 - 2\kappa(\delta). \tag{2.2$a$}$$

Now, by considering (2.1) we have the following results.

**Lemma 2.1.** *For any $(p,q) \in E(\delta)$ there is a positive constant $\omega$, only dependent on $\delta$, satisfying*

$$1 < \omega \le d_{11}(p,q) \le d_{22}(p,q) \tag{2.3}$$

*and*

$$1 < \omega \le d_{11}(1-q, 1-p) \le d_{22}(1-q,1-p). \tag{2.4}$$

*Moreover, we may choose $\omega \ge 5$ if $q = 1-p$, i.e., $p$ and $q$ are symmetric points. In particular, we may choose $\omega = 7$ if $p$ and $q$ are Legendre-Gauss points, i.e.,*

$$p = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad q = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right). \tag{2.5}$$

*Furthermore, $D(p,q)$ and $D(1-q,1-p)$ for $(p,q) \in E(\delta)$ are positive matrices.*

PROOF. An easy calculation yields the conclusions (2.3) and (2.4). The stronger results($\omega \ge 5$, $\omega = 7$) are found in [KK] and [KP]. □

**Corollary 2.2.** *Let $f$ be a cubic polynomial on $I$ vanishing at $p$ and $q$ where $(p,q) \in E(\delta)$. Then if $f(0)f'(0) \ge 0$, we have*

$$f(1)f'(1) > 0 \quad \text{and} \quad |f^{(r)}(1)| \ge \omega\, |f^{(r)}(0)|, \quad (r = 0,1) \tag{2.6}$$

*and if $f(1)f'(1) \le 0$, we have*

$$f(0)f'(0) < 0 \quad \text{and} \quad |f^{(r)}(0)| \ge \omega\, |f^{(r)}(1)|, \quad (r = 0,1). \tag{2.7}$$

PROOF. Using the positivity of $D(p,q)$ and $D(1-q,1-p)$, we have (2.6) from (2.3) and (2.7) from (2.4). □

From now on, we assume that $(p_i, q_i) \in \Delta_l \subset E(\delta)$ for all $i$. Following the arguments in [KK], we will show that the exponential decay of the function values of $\psi_i$ and $\psi_i'$ at knots $t_j$ as $j$ goes away from $i^*$ if the local interpolation points are chosen arbitrarily on a non-uniform mesh of $I$. By the linear change of variables from $[0,1]$ to $[t_{i-1}, t_i]$ for any $g \in S$ vanishing at $\xi_{2i-1}$ and $\xi_{2i}$, we have

$$\vec{g}(t_i) = D(h_i, p_i, q_i)\vec{g}(t_{i-1}), \tag{2.8}$$

where

$$D(h_i, p_i, q_i) := A(h_i)^{-1} D(p_i, q_i) A(h_i) \text{ and } A(h_i) := \begin{bmatrix} 1 & 0 \\ 0 & h_i \end{bmatrix}. \tag{2.9}$$

For the $C^1$ cubic Lagrange spline $\psi_i$ defined in (1.1), by (2.6), (2.8), (2.9) and mathematical induction, we have

$$\psi_i(t_{i^*-1})\psi_i'(t_{i^*-1}) > 0, \tag{2.10}$$

and

$$|\psi_i^{(r)}(t_j)| \le \left(\frac{1}{\omega}\right)^{(i^*-1-j)} |\psi_i^{(r)}(t_{i^*-1})|, \quad j < i^*, \quad (r = 0, 1). \tag{2.11}$$

Then (2.11) implies the decay of the function values of $\psi_i$ and $\psi_i'$ at $t_j$ as $j$ goes away from $i^*$ to 1.

Using (2.7), the change of variables $t \to (1-t)$ in (2.10) and (2.11) implies that

$$\psi_i(t_{i^*})\psi_i'(t_{i^*}) < 0$$

and

$$|\psi_i^{(r)}(t_j)| \le \left(\frac{1}{\omega}\right)^{(j-i^*)} |\psi_i^{(r)}(t_{i^*})|, \quad j \ge i^*, \quad (r = 0, 1). \tag{2.12}$$

Then (2.12) implies the decay of the function values of $\psi_i$ and $\psi_i'$ at $t_j$ as $j$ goes away from $i^*$ to $N$.

We will put the above observations as the following proposition.

**Proposition 2.3.** *Assume that $\Delta_l \subset E(\delta)$. For the $C^1$ cubic Lagrange splines $\{\psi_i\}_{i=1}^{2N}$ defined in (1.1), there is a positive constant $\omega > 1$, only dependent on $\delta$, satisfying the inequalities (2.11) and (2.12) for all $i = 1, \ldots, 2N$.*

## 3. Exponential decay

From now on we assume that $\Delta_l \subset E(\delta)$ for some $\delta$, $0 < \delta \le \frac{\sqrt{5}-1}{8}$. The main goal in this section is to show the exponential decay of a cubic Lagrange spline $\psi_i$ defined in (1.1). With (2.11) and (2.12), two estimates are required for the exponential decay of $\psi_i$. One is a bound for $|\psi_i|$ on $I_{i^*}$ and the other is a bound for $|\psi_i|$ on $I_{j^*}, (i \ne j)$ in terms of $\psi_i(t_{j^*-1})$ and $\psi_i(t_{j^*})$. The linear change of variables which converts $\psi_i$ on $I_{i^*}$ to a cubic polynomial $f$ on $[0,1]$ yields such bounds for $f$. Without loss of generality, we may assume that

$$f(p) = 1, \quad f(q) = 0, \quad (p,q) \in E(\delta) \tag{3.1}$$

and

$$f(0)f'(0) \geq 0, \quad f(1)f'(1) \leq 0. \tag{3.2}$$

**Lemma 3.1.** *Let $f$ be a cubic polynomial on $[0,1]$ satisfying (3.1) and (3.2) for $(p,q) \in E(\delta)$. Then there is a constant $C(\delta)$, only dependent on $\delta$, such that*

$$\max\{|f(0)|, |f(1)|\} < C(\delta) \tag{3.3}$$

*where*

$$C(\delta) := \frac{1}{\delta\, \kappa(\delta)}.$$

PROOF. By the Budan-Fourier theorem (see, for example, [BE]), $f$ has only one zero $q$ in (0,1), so that $f(0) > 0 > f(1)$. Then $f$ can be written as

$$f(t) = (t-q)(at^2 + bt - \frac{1}{q-p} - ap^2 - bp). \tag{3.4}$$

Let $F = F_1 \cap F_2 \cap F_3 \cap F_4$ where

$$\begin{aligned} F_1 &= \{(a,b) | f(0) \geq 0\}, \quad F_2 = \{(a,b) | f'(0) \geq 0\}, \\ F_3 &= \{(a,b) | f(1) \leq 0\}, \quad F_4 = \{(a,b) | f'(1) \geq 0\}. \end{aligned}$$

Define

$$T_0(a,b) := f(0) = p^2qa + pqb + \frac{q}{q-p} \tag{3.5}$$

and

$$T_1(a,b) := f(1) = (1-q)(a + b - bp - ap^2 - \frac{1}{q-p}). \tag{3.6}$$

Since $T_0$ and $T_1$ are bilinear on $F$, and since $F$ is the convex set bounded by straight lines, $T_0$ has the maximum value at the intersection point of $f'(0) = 0$ and $f(1) = 0$ and also $T_1$ has the maximum value at the intersection point of $f(0) = 0$ and $f'(1) = 0$ such that

$$\begin{aligned} \max_{(a,b)\in F} |T_0(a,b)| &= \left| \frac{q^2}{(q-p)(1-p)(p+q+pq)} \right| \\ &< \frac{3q^2}{q-p} < \frac{2}{p(q-p)} \end{aligned} \tag{3.7}$$

and

$$\max_{(a,b)\in F} |T_1(a,b)| = \left|\frac{-(1-q)^2}{p(q-p)(3-2p-2q+pq)}\right| < \frac{3(1-q)^2}{p(q-p)} < \frac{4}{3p(q-p)}. \tag{3.8}$$

By (2.2) the estimates (3.7) and (3.8) yield the conclusion. □

**Corollary 3.2.** *For $\psi_i$ on $I_{i^*}$ satisfying (1.1), there is the same constant $C(\delta)$ as in (3.3), only dependent on $\delta$, satisfying*

$$\max\{|\psi_i(t_{i^*-1})|, |\psi_i(t_{i^*})| < C(\delta).$$

PROOF. The change of variables yields the conclusion. □

**Lemma 3.3.** *Under the assumption of Corollary 3.2, there is the same constant $C(\delta)$ as in (3.3) satisfying*

$$|\psi_i(t)| < \frac{9}{2}C(\delta) \quad (t \in I_{i^*}). \tag{3.9}$$

PROOF. By the change of variables, it is enough to consider $f$ satisfying (3.1) and (3.2) for $(p,q) \in E(\delta)$. From (3.4), for $t \in [0,1]$, we have

$$\begin{aligned} |f(t)| &\le |(t+p)a+b| + \frac{1}{q-p} \\ &\le \max\{|pa+b|, |(1+p)a+b|\} + \frac{1}{q-p}. \end{aligned} \tag{3.10}$$

From (3.5) and (3.7), we have

$$\begin{aligned} |pa+b| = \frac{1}{pq}\left|f(0) - \frac{q}{q-p}\right| &< \frac{1}{pq}\left(\frac{3q^2}{q-p} + \frac{q}{q-p}\right) \\ &< \frac{4}{p(q-p)}. \end{aligned} \tag{3.11}$$

Since $0 < p < 2/3$ and $1/3 < q < 1$ by (3.6) and (3.8), we have

$$\begin{aligned} |(1+p)a+b| &= \frac{1}{(1-p)(1-q)}\left|f(1) + \frac{(1-q)}{q-p}\right| \\ &< \frac{1}{(1-p)(1-q)}\left(\frac{3(1-q)^2}{p(q-p)} + \frac{(1-q)}{q-p}\right) \\ &= \frac{1}{p(q-p)}\left(\frac{3+p-3q}{1-p}\right) \\ &< \frac{8}{p(q-p)}. \end{aligned} \tag{3.12}$$

Applying (3.11) and (3.12) to (3.10) and considering the ranges of $p$ and $q$ with respect to $\kappa(\delta)$ and $\delta$ in (2.2), we have

$$|f(t)| < \frac{8}{p(q-p)} + \frac{1}{q-p} < \frac{9}{p(q-p)} < \frac{9}{2\delta\,\kappa(\delta)}.$$

Therefore we have the conclusion. □

**Lemma 3.4.** *Let $f$ be a cubic polynomial on $[0,1]$ vanishing at $p$ and $q$ for $(p,q) \in E(\delta)$. Then there is a constant $K(\delta)$, only dependent on $\delta$, such that*

$$|f(t)| \;<\; K(\delta)\;\max\{|f(0)|,\;|f(1)|\} \tag{3.13}$$

*where*

$$K(\delta) \;:=\; \frac{4}{\kappa(\delta)^2}.$$

PROOF. Since $f(p) = f(q) = 0$, $f$ can be written as

$$f(t) = \frac{t(t-p)(t-q)}{(1-p)(1-q)} f(1) + \frac{(t-p)(t-q)(1-t)}{pq} f(0).$$

Since $0 < p < 2/3$ and $1/3 < q < 1$, we have

$$\left|\frac{t(t-p)(t-q)}{(1-p)(1-q)}\right| < \frac{3}{(1-q)}, \qquad \left|\frac{(t-p)(t-q)(1-t)}{pq}\right| < \frac{3}{p}.$$

Therefore we have

$$|f(t)| < \frac{3}{p(1-q)}\Big((1-q)|f(0)| + p|f(1)|\Big) < \frac{2}{p(1-q)}\Big(|f(0)| + |f(1)|\Big).$$

Observing $p \geq \kappa(\delta)$ and $1-q \geq \kappa(\delta)$ in (2.2), we have the conclusion. □

**Corollary 3.5.** *For $\psi_i$ on $I_j$ $(j \neq i^*)$ satisfying (1.1), there is the same constant $K(\delta)$ as in (3.13) such that*

$$|\psi_i(t)| \;<\; K(\delta)\max\{|\psi_i(t_{j-1})|, |\psi_i(t_j)|\}. \tag{3.14}$$

PROOF. It comes immediately by the change of variables applied to Lemma 3.4. □

**Theorem 3.6.** *Let $\{\psi_i\}_{i=1}^{2N}$ be the $C^1$ cubic Lagrange splines satisfying (1.1) with $\Delta_l \subset E(\delta)$. Then there is a constant $M(\delta)$, only dependent on $\delta$, such that*

$$|\psi_i(t)| \;<\; M(\delta)\,\left(\frac{1}{\omega}\right)^{|i^*-j|} \quad \textit{for} \quad t \in I_j \tag{3.15}$$

*where*

$$M(\delta) := \omega\, C(\delta)\, K(\delta) = \frac{4\omega}{\delta\, \kappa(\delta)^3}.$$

PROOF. For the case $I_j$ ($j \le i^* - 1$), using (3.14), (2.11) and (3.3) we have

$$|\psi_i(t)| < K(\delta)\, C(\delta) \left(\frac{1}{\omega}\right)^{i^*-j-1}. \tag{3.16}$$

For the case $I_j$ ($j \ge i^* + 1$), using (3.14),(2.12) and (3.3) we have

$$|\psi_i(t)| < K(\delta)\, C(\delta) \left(\frac{1}{\omega}\right)^{j-i^*-1}. \tag{3.17}$$

For the case $I_{i^*}$, using (3.9) and observing $K(\delta) > 5$ in (3.13) we have

$$|\psi_i(t)| \;<\; K(\delta)\, C(\delta). \tag{3.18}$$

Now, (3.16)-(3.18) with $M(\delta) := \omega\, C(\delta)\, K(\delta)$ yield the conclusion. Furthermore, using $C(\delta)$ and $K(\delta)$ defined in (3.3) and (3.13), respectively, we can express the constant $M(\delta)$ as $\frac{4\omega}{\delta\,\kappa(\delta)^3}$, which completes the conclusion (3.15). □

Now let us state stronger results on the exponential decay for the $C^1$ cubic Lagrange spline satisfying (1.1), where $\xi_i$ are chosen as Legendre Gauss points or the local symmetric points in $I_i$ (see [KK] and [KP]).

**Corollary 3.7.** *Let $\{\psi_i\}_{i=1}^{2N}$ be the $C^1$ cubic Lagrange splines satisfying* (1.1). *If $p$ and $q$ are chosen as Gauss-Legendre points as in (2.5), then there is a positive constant $Q_1$ such that*

$$|\psi_i(t)| \;\le\; Q_1 \left(\frac{1}{7}\right)^{|i^*-j|} \quad \textit{on} \quad I_j. \tag{3.19}$$

*If $p + q = 1$ and $0 < \underline{p} \le p \le \overline{p} < 1/2$ for all $(p,q) \in \Delta_l$, then there is a positive constant $Q_2$, dependent on $\underline{p}$ and $\overline{p}$, such that*

$$|\psi_i(t)| \;\le\; Q_2 \left(\frac{1}{5}\right)^{|i^*-j|} \quad \textit{on} \quad I_j. \tag{3.20}$$

PROOF. Suppose that $p$ and $q$ are Legendre-Gauss points, as in (2.5). Solving the equation $\kappa(\delta_1) = p$, we have

$$\{\ (p,q)\ \} \subset E(\delta_1), \quad \delta_1 = \frac{\sqrt{3}-1}{6}.$$

Then we have $\omega = 7$ from Lemma 2.1 and (3.19) from (3.15) with

$$Q_1 = M(\delta_1) = \frac{28}{\delta_1\,\kappa(\delta_1)^3}.$$

For the second case, $\delta_s$ can be chosen as a positive number satisfying $\kappa(\delta_s) = \min\{\underline{p},\ \frac{1}{3}\}$. Letting $\delta_2 = \min\{\delta_s,\ \frac{1}{2} - \overline{p}\}$, we have $\Delta_l \subset E(\delta_2)$. Then (3.20) comes from Theorem 3.6 with $Q_2 = M(\delta_2)$ and $\omega = 5$. $\square$

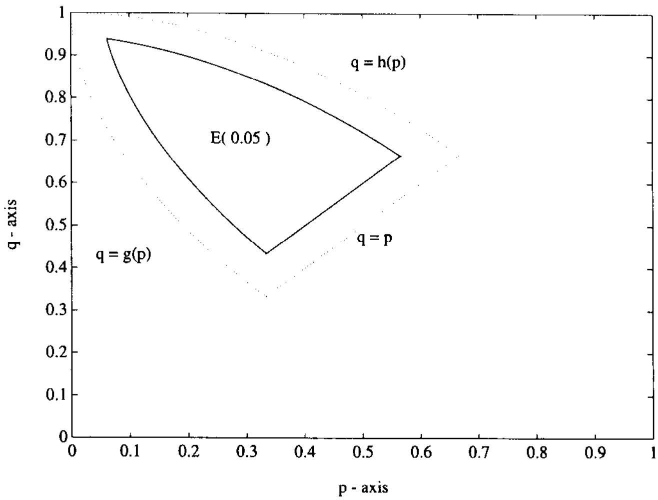


FIGURE 1. The shapes of sets $E$ and $E(\delta)$ when $\delta = 0.05$.

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(Kim) Dept. of Math., Teachers College, Kyungpook Nat'l Univ., Taegu, Korea.
*E-mail address*: `skim@sobolev.kyungpook.ac.kr`

(Shin) Basic Science Research Institution, Ajou University, Suwon, Korea.
*E-mail address*: `cshin@gauss.kyungpook.ac.kr`