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CLOSED GEODESICS ON 2-DIMENSIONAL χ–GEOMETRIC POLYHEDRA

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ABSTRACT. For 2-dimensional finite χ-geometric polyhedra of curvature $K \leq \chi < 0$ it is shown that the polygonal flow, applied to a closed curve, converges to a geodesic. Moreover, it is shown that there exists a finite number of closed geodesics with length smaller than a given positive B. As an application of the polygonal flow, a way of constructing closed, in particular simple, curves is given as well as a condition which implies that a curve is non-homotopic to a point.

1. INTRODUCTION AND PRELIMINARIES

Classical questions about closed geodesics on Riemannian manifolds are about existence, uniqueness in each homotopy class, and about counting the number of closed geodesics with length bounded by a given real. See number [11] and [12] for results in these directions in the context of a Riemannian manifold. In this work we are concerned with similar questions in the context of 2-dimensional χ-geometric polyhedra (see definition 2 below) and with the application of a curve-shortening process on these spaces namely, the polygonal flow, originally introduced on surfaces by J.Hass and P.Scott in [10]. For such spaces with curvature K satisfying $K \leq \chi < 0$ we show that the unique closed geodesic contained in each non-zero homotopy class of closed curves can be obtained by applying the polygonal flow of Hass and Scott on any closed curve in the homotopy class. Then, using the notion of the developing surface associated to a closed curve we show (see theorem 3.2 below) that in each finite 2-dimensional χ-geometric polyhedron X with curvature K satisfying $K \leq \chi < 0$ there exists at most a finite number of closed geodesics of length less than a given constant $B > 0$. From the proof of this theorem, it is easily deduced that the number $\Pi(B)$ of closed geodesics of length less than B in a 2-dimensional χ-geometric polyhedron X, satisfies the inequality

$$\Pi(B) < e^{hB}$$

for some constant h depending on X, provided the curvature K of X satisfies $K \leq \chi < 0$. It remains to be examined whether the following asymptotic behavior of $\Pi(B)$, which is true for Riemannian manifolds, namely

$$\exists\ h > 0 \textit{ such that } \lim_{B\to\infty} \frac{\Pi(B)}{e^{hB}/hB} = 1$$

(see [13]), holds also for 2-dimensional χ-geometric polyhedra with curvature K satisfying $K \leq \chi < 0$.

On the other hand, for an important class of 2−dimensional geometric polyhedra constructed in [2, 3, 9], we give a method to construct closed, in particular simple, geodesics. From this construction and for 2−dimensional simplicial complexes which satisfy a purely topological condition, namely that each vertex is incident with at least ten 1− simplices, we obtain a way to identify closed, in particular, curves which are not homotopic to a point. We conclude this section with the necessary definitions.

Let (X, d) be a metric space. A *geodesic segment* in X is an isometry $c : I \to X$, where I is a closed interval in $\mathbb{R}$. A *geodesic* in X is a map $c : \mathbb{R} \to X$ such that for each closed interval $I \subset \mathbb{R}$, the map $c_{|I} : I \to X$ is a geodesic segment. A *local geodesic segment* (usually called *geodesic arc*) in X is a map $c : I \to X$ such that for each $t \in I$ there is an $\varepsilon > 0$ such that $c_{|[t-\varepsilon,t+\varepsilon]\cap I} : [t-\varepsilon, t+\varepsilon] \cap I \to X$ is a geodesic segment.

A *local closed geodesic* in X is a periodic map $c : \mathbb{R} \to X$ such that for each $t \in \mathbb{R}$ there is an $\varepsilon > 0$ such that c restricted to the subinterval $[t-\varepsilon, t+\varepsilon] \subset \mathbb{R}$ is a geodesic segment. A *closed geodesic* in X is a periodic map $c : \mathbb{R} \to X$ such that for each closed interval $I \subset \mathbb{R}$ the map $c_{|I} : I \to X$ is the shortest geodesic arc in its homotopy class with endpoints fixed. For the notion of $CAT(\chi)$ −inequality and curvature in a geodesic metric space see [1, ch.10], where the following lemma is also shown.

Lemma 1.1. *Let X be a geodesic space with curvature $K \leq \chi$. For each $p \in X$ there is $\varepsilon > 0$ such that the open ball $B(p, \varepsilon)$ is a convex neighborhood around p.*

Let X be a finite 2-dimensional polyhedron. A 1-dimensional simplex σ of X is said to have index $k_\sigma, k_\sigma \in \mathbb{N}$, if k_σ faces of 2-simplices (i.e. triangles) of X are glued together to form σ. A 1-dimensional simplex σ of X is called *singular* if

$k \geq 3$. We will denote by $X^{[1]}$ the singular 1-skeleton of X i.e., the union of all singular 1-simplices of X.

Definition 1. We will say that a closed curve c in X is *simple* if either c has no self-intersections or, in the case $c(t) = c(s)$ for some $t \neq s$ then $c(t) \in X^{[1]}$ and there exists $\varepsilon > 0$ sufficiently small such that the triangles containing $c(t-\varepsilon), c(t+\varepsilon)$ do not contain both $c(s-\varepsilon)$ and $c(s+\varepsilon)$.

For each point $x \in \sigma$ which is not a vertex, there exists a neighborhood of x in X, which is homeomorphic to k_σ half discs glued together along their boundary. If $v \in X$ is a vertex, then a closed neighborhood of v in X is piecewise linear homeomorphic to a cone on G, where G is a simplicial graph.

Definition 2. A polyhedron X is called χ-*geometric polyhedron* if each k-simplex of X, for all k, is isometric to a simplex in M^k_χ. The number $\chi \in \mathbb{R}$ is fixed for a given polyhedron and M^k_χ denotes the unique complete simply connected k-dimensional Riemannian manifold of constant sectional curvature χ. In particular, if X is 2-dimensional then every 2-simplex of X is isometric to a geodesic triangle in M^2_χ.

The following proposition is a useful property for simply connected polyhedra, which follows from [4, page 403].

Proposition 1.2. *Let X be a simply connected polyhedron of arbitrary dimension which satisfies $CAT(\chi)$ -inequality, $\chi < 0$, and which has finitely many isometry types of simplices. Then each local geodesic segment in X is a geodesic segment.*

2. The polygonal flow on χ-geometric polyhedra

In this section we apply the polygonal flow to 2-dimensional χ-geometric polyhedra. The importance of the polygonal flow lies on the fact that it allows control on the number of self-intersection points of the curve. For surfaces, this is explained in the original work [10] of Hass and Scott and for singular spaces in [6]. In order to define the polygonal flow on a space X one must cover the space with convex neighborhoods. Here we shall only describe these neighborhoods and refer the reader to [10] and [6] for the full definition of the polygonal flow. Throughout this section X will denote a 2-dimensional χ-geometric polyhedron. The union of all singular 1- simplices of X will be denoted by $X^{[1]}$. A vertex of X which belongs to at least one singular 1-simplex will be called *singular* vertex.
A closed neighborhood N of a point x in X is homeomorphic to either :

(a) a closed disc $D(x,\varepsilon)$ if x lies in $X \backslash X^{[1]}$, or

(b) a bouquet $b(x,\varepsilon)$ of closed half discs each of radius ε if x belongs to a singular 1- simplex without being a singular vertex , or

(c) a cone $C(x,\varepsilon)$ formed on $G=\{y\in X \mid d(x,y)=\varepsilon\}$ if x is a singular vertex of X.

Moreover, by choosing ε sufficiently small (cf. lemma 1.1), we may assume that the neighborhood of any point is convex.

Let $N(x,\varepsilon)$ be a neighborhood of X of type (a), (b) or (c). If $0<\delta<1$, denote by $\delta N = N(x,\delta\varepsilon)$ the shrinking, by the factor δ , of the neighborhood $N(x,\varepsilon)$. X can be covered by a finite collection of neighborhoods $N_i, i=1,\ldots,n$ such that

(1) $\bigcup_{i=1}^{n} \frac{1}{2}N_i = X$

(2) $\forall\ i, \forall\ x,y\in N_i$, there exists unique geodesic segment, which lies entirely in N_i, joining x and y.

(3) No neighborhood of type (a) intersects a singular 1-simplex of X and each singular vertex of X lies in exactly one neighborhood of type (c).

Achieving condition (1) is just a matter of increasing n and letting the neighborhoods N_i overlap. Condition (2) follows from the fact that an χ-geometric polyhedron satisfies locally the $CAT(\chi)$ −inequality (see lemma 1.1) and therefore, we can choose the neighborhoods N_i to be convex.

Let N be any such neighborhood of a point x in X and let γ denote a finite collection of piecewise linear arcs in N. Suppose no two arcs have a common boundary point. Since N is simply connected and convex, there exists a homotopy, with end points fixed, from γ to a union of geodesic arcs (see lemma 1.6 of [10] and 2.1 of [6]). This homotopy is called *neighborhood straightening process (NSP)* on the collection γ in N. Let now γ be a closed curve in X. Performing NSP repeatedly on $\gamma\cap N$ for all N and pasting the homotopies together we obtain the polygonal flow $\gamma_t, t\in[0,\infty)$.

A simple application of the Arzela-Ascoli theorem shows that γ_t converges uniformly to a rectifiable curve β and using the argument given in [10, thm. 1.8] β is shown to be a local geodesic. Hence we have the following

Proposition 2.1. *Let X be a 2-dimensional χ-geometric polyhedron and γ be a closed curve in X which is not homotopic to a point. Then the polygonal flow γ_t lies arbitrarily close to a local closed geodesic.*

We conclude this section with a corollary concerning χ-geometric polyhedra of negative curvature. Note that existence of closed geodesic mentioned in part (ii) of the corollary below is proved in [8].

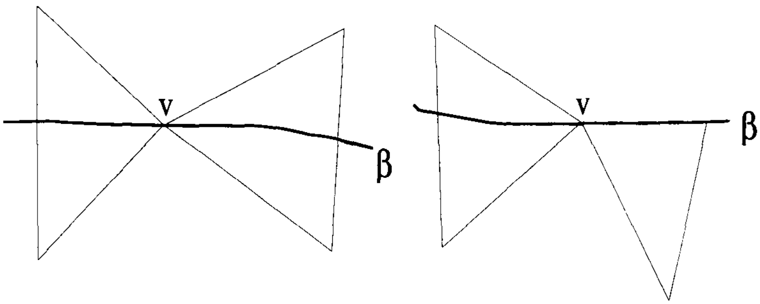

FIGURE 1

Corollary 2.2. *Let X be a finite χ-geometric polyhedron with curvature K satisfying $K \leq \chi < 0$. Then*
i) Each closed local geodesic of X is a closed geodesic
ii) The polygonal flow of Hass and Scott applied to closed curve γ non-homotopic to a point, converges to a closed geodesic which is unique in the homotopy class of γ.

PROOF. Let g be a local closed geodesic of X and let $\widetilde{g}$ be its lifting to the universal cover $\widetilde{X}$ of X. By proposition 1.2, $\widetilde{g}$ is a geodesic. Since $\widetilde{X}$ is simply connected, it satisfies the $CAT-(\chi)$ inequality globally. Hence, any two points in $\widetilde{X} \cup \partial\widetilde{X}$ can be joined by a unique geodesic (see for example [5, prop.2]). Combination of the above statements implies readily that g is a closed geodesic in X and it is unique in its homotopy class. $\square$

3. COUNTING GEODESICS

We begin with the definition of the developing surface associated to a closed curve β, provided that β is transverse to the 1-skeleton $X^{(1)}$ of X and does not have a back and forth. Recall that a curve $\beta : I \to X$ has a back and forth if $\exists\ t_1, t_2 \in I : \beta((t_1, t_2))$ lies in the interior of a single triangle T of X and $\beta(t_1), \beta(t_2)$ belong to the same side of T. X will always denote a 2-dimensional χ-geometric polyhedron with curvature $K \leq \chi < 0$.

Let $\beta : [0,1] \to X, \beta(0) = \beta(1)$, be a closed curve. Assume β is transverse to $X^{(1)}$, the 1-skeleton of X, with $\beta(0) \notin X^{(1)}$. Let T_0 be the triangle containing $p = \beta(t_0)$, $t_0 = 0$. Let t_1 be the smallest number in $(t_0, 1] = (0, 1]$ such that $\beta(t_1)$ belongs to $X^{(1)}$, and let T_1 be the unique triangle ($T_0 \neq T_1$) containing $\beta(t_1 + \varepsilon)$

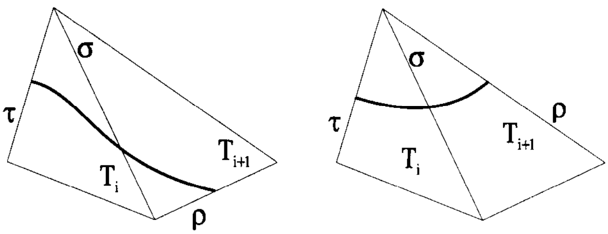

FIGURE 2. Non-contributing Contributing

for ε arbitrarily small. Glue T_0 with T_1 along their common 1-simplex (or vertex) which contains (resp. is) $\beta(t_1)$. Continue, in this way, attaching triangles T_i to $T_1 \cup T_2 \cup \cdots \cup T_{i-1}$ along the common face of T_{i-1} and T_i, $(T_{i-1} \neq T_i)$, which contains $\beta(t_i)$, until all triangles T_i intersected by the image of β are attached.

Definition 3. The process described above gives a singular surface S_β which we call the developing surface associated to β.

The singularities of S appear when $\beta(t_i)$ is a vertex and T_{i-1}, T_i have no common 1-simplex (see figure 1).

When β is a closed geodesic its developing surface is defined even in the case β is not transverse to $X^{(1)}$. For if β is a closed geodesic not transverse to $X^{(1)}$, its image of β must contain entire singular 1-simplices. We will be viewing β as a curve in S_β and use the same letter to denote it. The following lemma relates the length of a geodesic with the number of triangles in its developing surface.

Lemma 3.1. *Let X be a finite 2-dimensional χ-geometric polyhedron with curvature $K \leq \chi < 0$ and B a positive real number. There exists a natural number N_B such that any local geodesic β in X whose developing surface consists of at least N_B triangles has length $\ell(\beta) > B$.*

For the proof of this lemma we will need some terminology which we introduce next: A common 1-simplex σ of triangles T_i, T_{i+1} in the developing surface S_β associated to a local geodesic β is called *contributing* if the image of the curve β intersects 1-simplices τ and ρ of T_i and T_{i+1}, respectively, which have no vertex in common (apparently, $\tau \neq \sigma \neq \rho$). Otherwise, i.e. when τ and ρ do have a common vertex, it is called *non-contributing* (see figure 2).

Let a be the minimum angle formed among all pairs of 1-simplices of X and set m to be the integer $m = \left[\frac{\pi}{a}\right]$.

CLAIM. *In the developing surface S_β associated to a closed geodesic β, there can be at most m non-contributing 1-simplices which have one vertex in common.*

PROOF OF CLAIM. Suppose there exists exactly $m+1$ (any integer bigger than $m+1$ can be treated similarly) contributing 1-simplices with a vertex v in common. Then there exist triangles $T_0, T_1, \ldots, T_{m+1}$ glued together, consecutively, along non-contributing 1-simplices and triangles T_{-1} and T_{m+2} glued with T_0 and T_{m+1}, respectively, along contributing 1-simplices. Let β_{xy} be the part of the image of β which lies in $T_{-1} \cup T_0 \cup T_1 \cup \cdots \cup T_{m+1} \cup T_{m+2}$, see figure 3. Denote by β_{vx} and β_{vy} the geodesic segments joining v, x and v, y respectively. The sum of the angles at v is bigger than $\left(\left[\frac{\pi}{a}\right]+1\right)a$, and therefore, the angle formed by β_{vx} and β_{vy} at v is bigger than π. By [1, page 197], β_{xy} is not a geodesic segment, a contradiction, since β is a closed geodesic and, hence, β_{xy} is a geodesic segment. This completes the proof of the claim.

PROOF. (OF LEMMA 3.1) Let two triangles T_i and T_j of X be glued together along a common 1-simplex. By compactness of X there exists a positive lower bound, say l_{ij}, for the lengths of the curves with endpoints on 1-simplices of T_i and T_j which have no common vertex. Set $L = \min_{i,j}\{l_{ij}\}$ where T_i, T_j range among all 2-simplices of X. Note that L is smaller than or equal to the length of any 1-simplex of X. Define N_B to be the integer $N_B = \left(1+\left[\frac{B}{L}\right]\right)(m+1)$.

Let β be geodesic in X whose developing surface S_β consists of at least N_B triangles. By the claim above, any family of $m+1$ consecutive triangles in S_β contains a non-contributing 1-simplex. Hence, the length of the part of β which lies in the union of any $m+1$ consecutive triangles of S_β is at least L. Since S_β contains at least $\left(1+\left[\frac{B}{L}\right]\right)$ families of $m+1$ consecutive triangles, it follows that $\ell(\beta) > \left(1+\left[\frac{B}{L}\right]\right)L > B$. □

Theorem 3.2. *Let X be a finite 2-dimensional χ-geometric polyhedron. Assume the curvature K of X satisfies $K \le \chi < 0$. Then, given $B > 0$ there exists at most a finite number of closed geodesics with length smaller than B.*

PROOF. Let B be given and let N_B be the natural number provided by lemma 3.1. Each geodesic β in X determines a developing surface S_β which is the union of, say, k triangles $S_\beta = T_1 \cup \cdots \cup T_k$. If X is the union of n triangles $X = T_1 \cup \cdots \cup T_n$ for some $n \in \mathbb{N}$, then each geodesic determines a unique permutation with repetition (i.e. a triangle can appear more than once) of k triangles chosen out of n. Therefore, by lemma 3.1, the number of geodesics in X with length

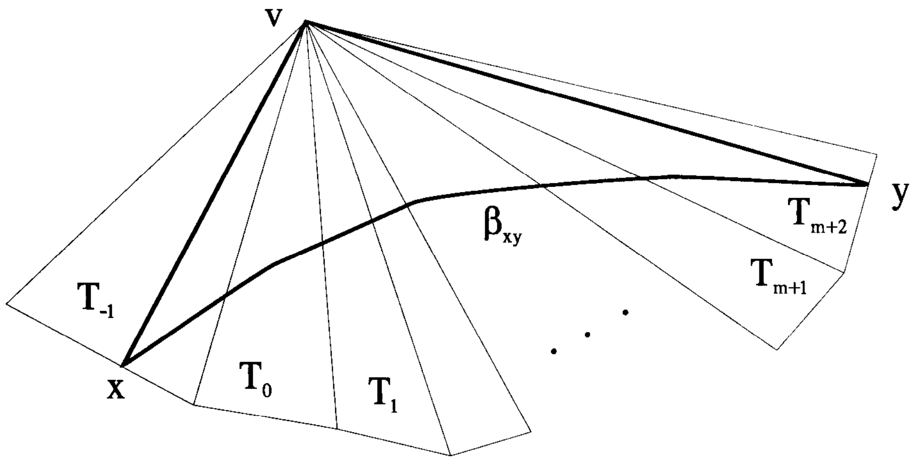

FIGURE 3

$< B$, is bounded by the number of permutations of k chosen out n with $k < N_B$. Obviously these permutations are finitely many. □

4. Application to simplicial complexes

In this section we present, as an application of the polygonal flow, a condition which implies that a closed curve in a 2−dimensional simplicial complex is non-homotopic to a point. Moreover, if at least ten 1-simplices are attached to each of the vertices of the complex, a way of constructing closed, in particular simple, curves non-homotopic to a point is given. We need the notion of convex developing surface which we introduce next.

A developing surface S_β, for some closed curve β, will be called *convex* if there exists a convex polygon P in the 2−dimensional model M^2_χ and vertices $v_i, v_{i+1}, v_j, v_{j+1}$ of P such that

(*i*) v_i, v_{i+1} and v_j, v_{j+1} determine non-consecutive sides of P

(*ii*) S_β can be obtained from P by identifying $v_{i+1}v_i$ with v_jv_{j+1}
or, by identifying v_iv_{i+1} with v_jv_{j+1}

(*iii*) $\widehat{v_i} + \widehat{v_{j+1}} \leq \pi$, $\widehat{v_j} + \widehat{v_{i+1}} \leq \pi$, if v_{i+1} is identified with v_j
or, $\widehat{v_i} + \widehat{v_j} \leq \pi$, $\widehat{v_{i+1}} + \widehat{v_{j+1}} \leq \pi$, if v_i is identified with v_j

where $\widehat{v}$ denotes the angle of P at the vertex v. Examples of curves whose developing surface is convex can be seen as follows.

CONSTRUCTION. Let X be a 2−dimensional simplicial complex with at least ten 1-simplices attached to each of its vertices. According to [7, page 121], X admits a hyperbolic structure such that

$$(1) \qquad \textit{any angle of any triangle in } X \textit{ is } \leq \pi/5.$$

Assume further that

$$(2) \qquad \textit{each } 1-\textit{simplex of } X \textit{ is incident with at least two } 2-\textit{simplices}$$

Consider a sequence of triangles as follows : let T_1 be arbitrary triangle in X with vertices x_1, y_1, z_1 (notation $T_1 = (x_1, y_1, z_1)$) and let $T_2 = (x_2, y_2, z_2)$ be a triangle attached (*in* X) to T_1 with the side $[x_1, y_1]$ of T_1 identified with the side $[x_2, y_2]$ of T_2 (by property 2, there is at least one such triangle). Next, let $T_3 = (x_3, y_3, z_3)$ be a triangle attached to T_2 with the side $[y_2, z_2]$ of T_2 identified with the side $[y_3, z_3]$ of T_3 (notation $[y_2, z_2] \sim [y_3, z_3]$). Observe that all three vertices y_1, y_2, y_3 coincide. The next triangle T_4 is a triangle (x_4, y_4, z_4) with $[z_3, x_3]$ identified with $[z_4, x_4]$. Continuing in this way we obtain a sequence of triangles $\{T_i\}$ such that

$$(3) \qquad T_i \textit{ is attached to } T_{i+1} \textit{ in } X \textit{ with } \begin{array}{l} [x_i, y_i] \sim [x_{i+1}, y_{i+1}] \Leftrightarrow i = 1 mod 3 \\ [y_i, z_i] \sim [y_{i+1}, z_{i+1}] \Leftrightarrow i = 2 mod 3 \\ [z_i, x_i] \sim [z_{i+1}, x_{i+1}] \Leftrightarrow i = 0 mod 3 \end{array}$$

We stop this procedure when we reach a triangle T_{k+1} for which $\exists\, k_0 < k : T_{k_0} = T_{k+1}$. Since X is a finite polyhedron, such triangle T_{k+1} will always exist making the sequence $\{T_i\}$ finite. By discarding all triangles T_i with $i < k_0$ we may assume that $k_0 = 1$ or 2.

If $k_0 = 1$ denote by S the surface obtained from $\cup_{i=1}^k T_i$ by gluing together T_i with T_{i+1} $\forall\, i$ according to (3) and then by identifying one (of the two) free sides of T_k (i.e. not identified with a side of T_{k-1}) with the side $[x_1, z_1]$ of T_1 or, $[z_1, y_1]$ of T_1. Similarly, if $k_0 = 2$, S is obtained from $\cup_{i=2}^k T_i$ by gluing together T_i with T_{i+1} $\forall\, i$ according to (3) and by identifying one (of the two) free sides of T_k (i.e. not identified with a side of T_{k-1}) with the side $[x_2, z_2]$ of T_2. It is readily seen that the angle at any vertex of S consists of at most five angles of triangles in X. By (1) it follows that the angle at any vertex of S is $\leq \pi$ and, hence, S is convex. Moreover, given any sequence of triangles as above with corresponding surface S, one can define a closed curve γ in X so that $S_\gamma = S$.

The following proposition gives a condition on a closed curve which implies that the closed curve is not homotopic to a point.

Proposition 4.1. *Let X be any 2−dimensional χ−geometric polyhedron and let γ be any closed curve in X with S_γ convex. Then*

(i) γ is not homotopic to a point
(ii) the unique closed geodesic γ_∞ in the homotopy class of γ has the property $S_\gamma = S_{\gamma_\infty}$. i.e. γ and γ_∞ intersect the same ordered sequence of triangles in X.

PROOF. Consider the subcomplex X_γ of X which is the image of the natural projection $p_\gamma : S_\gamma \to X$. Cover X_γ with X_γ−convex neighborhoods and apply the polygonal flow on γ within X_γ. For all times t during the flow, γ_t will intersect the same ordered sequence of triangles, i.e. $S_\gamma = S_{\gamma_t}$ $\forall\, t$. In particular, there is a lower bound for the lengths of the curves γ_t. Thus, proposition 2.1 can be applied to obtain a curve γ_∞ which is a local closed geodesic in X_γ. Moreover, $S_\gamma = S_{\gamma_\infty}$ and it remains to verify that γ_∞ is a closed geodesic in X. There is a unique closed curve $\widetilde{\gamma}_\infty$ in S_γ which is a local closed geodesic in S_γ and has the property $p_\gamma(\widetilde{\gamma}_\infty) = \gamma_\infty$. By convexity of S_γ, it follows that the projection $p_\gamma(\widetilde{\gamma}_\infty) = \gamma_\infty$ is also a local closed geodesic in X. Then by proposition 2.2, γ_∞ is a closed geodesic in X. □

Remark. The above proposition implies that if γ, γ' are closed curves with $S_\gamma, S_{\gamma'}$ convex then

$$\gamma, \gamma' \textit{are homotopic} \Longleftrightarrow S_\gamma = S_{\gamma'} \tag{4}$$

Moreover, given a 2−dimensional simplicial complex admitting a geometric structure with curvature $K \leq \chi < 0$ and a closed curve in it, one can use proposition 4.1 above in order to detect if the given closed, in particular simple, curve is not homotopic to a point.

The construction described at the beginning of this section can be performed with the same starting triangle in at least three distinct ways. Thus, we obtain three distinct convex (developing) surfaces $S_i, i = 1, 2, 3$. Moreover, we may choose closed curves $\gamma_i, i = 1, 2, 3$ such that $S_{\gamma_i} = S_i$, $\forall\, i = 1, 2, 3$. Property (4) implies that $\gamma_i, i = 1, 2, 3$ are pairwise non-homotopic and by proposition 4.1 each γ_i is not homotopic to a point. Hence, we have proved the following

Corollary 4.2. *Let X be a 2−dimensional simplicial complex having at least ten 1−simplices attached to each of its vertices. Then, there are at least 3 simple closed curves intersecting any given triangle of X which are pairwise non-homotopic and each not homotopic to a point.*

Remark. Let X be a (not necessarily finite) 2-dimensional χ-geometric polyhedron with curvature K satisfying $K \leq \chi < 0$. Let γ be a closed curve without back and forth and transverse to the 1-skeleton $X^{(1)}$ of X. Then for each singular 1-simplex σ of X, the singular number $\kappa_\sigma(\gamma)$ of γ at σ is defined by

$$\kappa_\sigma(\gamma) := card\{s \in \mathbb{R} | \gamma(s) \in \sigma\}$$

Let $\gamma_t, t \in [0, \infty)$ be the polygonal flow on γ. If the developing surface S_γ is convex, then it is easy to see that for all t, $S_\gamma = S_{\gamma_t}$ and $\kappa_\sigma(\gamma) = \kappa_\sigma(\gamma_t), \forall \sigma$. Moreover, by proposition 4.1, γ is not homotopic to a point. Using the argument of [6, thm.3.1] we obtain the the unique closed geodesic in the homotopy class of γ has at most B intersection points more than γ, where

$$B = \sum_{\sigma \in X^{[1]}} \frac{\kappa_\sigma(\kappa_\sigma - 1)}{2}$$

and $\kappa_\sigma = \kappa_\sigma(\gamma)$.

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