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# COORDINATIZATION FOR FELL BUNDLE ALGEBRAS

IGOR FULMAN

COMMUNICATED BY VERN I. PAULSEN

ABSTRACT. Let $C_r^*(E)$ be the reduced $C^*$-algebra generated by a Fell bundle $E$ over an $r$-discrete principal groupoid. We show that each element of $C_r^*(E)$ is represented by a continuous section of $E$. Also, the Coordinatization Theorem proved in this paper gives necessary and sufficient conditions for an abstract $C^*$-algebra $A$ to be isomorphic to $C_r^*(E)$ for some Fell bundle $E$.

## 1. INTRODUCTION

In this paper we study $C^*$-algebras arising from Fell bundles over groupoids, as defined in [5, 2, 10].

A good example of such bundle is the bundle over the set $\{1,\dots,m\}^2$, with the fiber above $(i,j)$ consisting of matrices of size $k_i \times k_j$. See the picture for $m = 3$, $k_1 = 2$, $k_2 = 1$, $k_3 = 3$:

$$\left(\begin{array}{cc|c|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline * & * & * & * & * & * \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right)$$

For $1 \le i,j,l \le m$, one can put: $(i,j)\cdot(j,l) = (i,l)$. At the same time, a matrix of the size $k_i \times k_j$ can be multiplied by a matrix of the size $k_j \times k_l$, and the product is a matrix of the size $k_i \times k_l$.

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Given such a construction, one can build the $C^*$-algebra generated by the Fell bundle. In our example this is the algebra of $n \times n$ matrices, where $n = \sum_{i=1}^{m} k_i$ (on the picture above $n = 6$).

The construction of the $C^*$-algebra generated by a Fell bundle has been studied on various levels of generality. The best studied (see [8, 4, 3, 6]) is the case where each fiber is a copy of the complex field, while the groupoid $R$ is an $r$-discrete principal groupoid (see the definitions in the articles cited). This case is referred to as the **commutative case**. In this case, there is an abstract description of $C^*$-algebras arising as $C^*$-algebras built by Fell bundles. Such algebras are described as $C^*$-algebras containing a so-called Cartan subalgebra (masa with some additional properties), or a **diagonal** in the terminology of Kumjian (see [4, 3]). Another case studied in [9] (and in the measure-theoretical setting in [1]) is the case of so-called crossed products where each fiber is a $C^*$-algebra. The most general setting (with no special assumptions about the fibers) was studied in [5, 10, 2]. The example considered above falls into this setting.

In Section 2 of this paper we show that, in the most general setting, each element of the generated $C^*$-algebra is actually a continuous section of the Fell bundle, i. e. each element has "coordinates". The algebraic operations can be naturally expressed in terms of the coordinates.

In Section 3 we give an abstract description of $C^*$-algebras arising as $C^*$-algebras generated by Fell bundles in the most general setting. This description involves the existence of a certain subalgebra that could be called a generalized Cartan subalgebra or a generalized diagonal. Importance of this generalization of Kumjian's theorem [4, §3, Theorem 1°] can be seen from the following two facts: (1) not every $C^*$-algebra possesses a diagonal (see the counter example by T. Natsume in [3, Appendix]), therefore not all $C^*$-algebras fit into the commutative setting; (2) even for the algebras that have diagonals and therefore can be represented in the commutative setting, this generalized representation can help to study their structure and properties. For example: block upper triangular matrix algebras and their inductive limits (see [7]) appear naturally as subalgebras of $C^*$-algebras arising in this setting.

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## 2. Coordinates of elements of the $C^*$-algebra generated by a Fell bundle

Throughout, $X$ will be a locally compact topological space, $R$ will be an equivalence relation on $X$ with countable equivalence classes such that the projections from $R \subset X \times X$ onto the first and the second component are local homeomorphisms (in the other terminology, $R$ is called an $r$-discrete principal groupoid), and $p : E \to R$ will be a Fell bundle over $R$ as in [2] and [5]. The space $R$ is assumed to be second countable. All $R$-sets (i. e. subsets of $R$ which are graphs of partial homeomorphisms of $X$) mentioned below in this paper are assumed to be open.

Let's consider the set $C_c(E)$ of all compactly supported continuous sections of $E$. This set is a $^*$-algebra with the operations defined as follows:

- $(fg)(x,y) = \sum_z f(x,z)g(z,y)$;
- $f^*(x,y) = f(y,x)^*$.

We let $\|\cdot\|_0$ denote the sup-norm on $C_0(E^0)$.
We let $\|\cdot\|_2$ denote the Hilbert $C^*$-norm on $C_c(E)$ as in [5]. So, for $f \in C_c(E)$:

$$\|f\|_2^2 = \sup_{x\in X} \|f^*f(x,x)\|_0 = \sup_{x\in X} \left\| \sum_z f(z,x)^* f(z,x) \right\|_0 .$$

We let $\|\cdot\|$ denote the operator norm on $C_c(E)$ as in [5]. So, for $f \in C_c(E)$:

$$\|f\| = \sup\{ \|fg\|_2 : g \in C_c(E), \|g\|_2 \le 1 \}$$

Finally, we let $C_r^*(E)$ be the $C^*$-algebra which is the completion of $C_c(E)$ in the operator norm $\|\cdot\|$, let $L^2(E)$ be the completion of $C_c(E)$ in the norm $\|\cdot\|_2$, and $\langle \cdot,\cdot \rangle$ be the inner product in $L^2(E)$, as in [5].

Our main goal in this section is to show that all elements of $C_r^*(E)$ are (represented by) continuous sections of the bundle $E$, and that the algebraic operations are given by the same formulas as for $C_c(E)$. More exactly this fact is expressed in the following theorem.

**Theorem 2.1.** (See [8, Proposition 4.2].) *The embedding of $C_c(E)$ into $C_0(E)$ extends to the one-to-one map $j : C_r^*(E) \to C_0(E)$ so that:*

1. $j(fg)(x,y) = \sum_z j(f)(x,z)j(g)(z,y)$;
2. $j(f^*)(x,y) = j(f)(y,x)^*$.

The element $j(f)$ will be then identified with $f$, and the values $j(f)(x, y)$ will be denoted by $\boldsymbol{f(x,y)}$ and called **the coordinates of $f$**.

To prove the theorem we need several lemmas. We begin with certain important inequalities for the norms on $C_c(E)$ that we have introduced. Let $\|\cdot\|_\infty$ denote the sup-norm on $C_c(E)$.

**Lemma 2.2.** (See [5, 3.13].) *For $f \in C_c(E)$:*

$$\|f\|_\infty \leq \|f\|_2 \leq \|f\|,$$

*so that the embedding $C_c(E) \to C_0(E)$ extends to a contractive map $j : C_r^*(E) \to C_0(E)$.*

PROOF. 1. If $(x_0, y_0) \in R$, then

$$\begin{aligned} & \|f(x_0,y_0)\|^2 = \|f(x_0,y_0)^* f(x_0,y_0)\| = \|f^*(y_0,x_0) f(x_0,y_0)\| \\ \leq \; & \|\sum_x f^*(y_0,x) f(x,y_0)\| \leq \sup_y \|\sum_x f^*(y,x) f(x,y)\| = \|f\|_2^2. \end{aligned}$$

This holds for every $(x_0, y_0) \in R$, so $\|f\|_\infty \leq \|f\|_2$.

2. Let $(x_0, y_0) \in R$. Let $h \in C_c(E^0) \subset C_c(E)$ be such that $\|h\|_0 = \|h(y_0)\| = 1$, and $h(y_0)^* f^* f(y_0, y_0) h(y_0)$ is arbitrarily (up to $\varepsilon > 0$) close to $f^* f(y_0, y_0)$. (This is possible because $E(y, y)$ is a $C^*$-algebra.) Then $\|h\|_2 = 1$, and furthermore:

$$\begin{aligned} \|fh\|_2^2 &= \sup_{y \in X} \|(h^* f^* f h)(y, y)\| \\ &\geq \|(h^* f^* f h)(y_0, y_0)\| \\ &= \|h(y_0)^* f^* f(y_0, y_0) h(y_0)\| \\ &\geq \|f^* f(y_0, y_0)\| - \varepsilon \\ &= \|\sum_x f^*(y_0, x) f(x, y_0)\| - \varepsilon \end{aligned}$$

So, $\|f\| \geq \|\sum_x f^*(y_0, x) f(x, y_0)\|$ for each $x_0 \in X$ and hence $\|f\| \geq \|f\|_\infty$.

□

**Lemma 2.3.** *The map $j : C_r^*(E) \to C_0(E)$ defined above is one-to-one.*

PROOF. Let $a \in C_r^*(E)$ with $j(a) = 0$, and let $a_n \in C_c(E)$ with $a_n \xrightarrow{\|\cdot\|} a$. Let $L$ be the representation of $C_r^*(E)$ by left multiplication on the Hilbert module

$L^2(E)$. For every $\varphi \in L^2(E)$ we have:

$$(L(a_n)\varphi)(x,y) = \sum_{z\sim x} a_n(x,z)\varphi(z,y).$$

Suppose $\operatorname{supp}\varphi$ is contained in some $R$-set which is the graph of a partial homeomorphism $\tau$ of $X$. Then:

$$(1) \qquad (L(a_n)\varphi)(x,y) = a_n(x,\tau^{-1}y)\varphi(\tau^{-1}y,y) = j(a_n)(x,\tau^{-1}y)\varphi(\tau^{-1}y,y).$$

We have: $a_n \to a$, so $L(a_n) \to L(a)$. This implies that $L(a_n)\varphi \to L(a)\varphi$. By Lemma 2.2, $(L(a_n)\varphi)(x,y) \to (L(a)\varphi)(x,y)$ almost everywhere on $R$.
On the other hand, $j$ is continuous, so $j(a_n) \to j(a)$, and so $j(a_n)(x,\tau^{-1}y) \to j(a)(x,\tau^{-1}y)$. Thus, we can pass to the limit in (1) and get:

$$(L(a)\varphi)(x,y) = j(a)(x,\tau^{-1}y)\varphi(\tau^{-1}y,y) = 0.$$

Such $\varphi$'s are total in $L^2(E)$, therefore $L(a) = 0$. □

So, every element $f \in C^*_r(E)$ is uniquely defined by the set of the values $f(x,y)$. Therefore, the name "coordinates" for these values is justified.

Our next goal is to "pass to the limit" in the formulas of algebraic operations for compactly supported functions. For this, first we need one particular case.

**Lemma 2.4.** *For $f \in C^*_r(E)$;*

$$f^*f(x,x) = \sum_z f^*(z,x)f(x,z),$$

*where the series converges in norm in the space $E(x,x)$.*

PROOF. Let $\{f_n\} \subset C_c(E)$, $f_n \xrightarrow{\|\cdot\|} f$. Then $(f_n)$ is a Cauchy sequence in $L^2(E)$, i. e. for every $\varepsilon > 0$ there exists $N$ such that for all $n,m > N$,

$$\sup_{x\in X}\left\|\sum_z (f_n - f_m)^*(x,z)\,(f_n - f_m)(z,x)\right\| < \varepsilon.$$

Let $\{F_i\}_{i=1}^{\infty}$ be an increasing sequence of subsets of $R$ such that $R = \bigcup_{i=1}^{\infty} F_i$ and for each $x$, each set $F_i \cap \{(x,y) | y \sim x\}$ is finite. (For example, $F_i = \bigcup_{j=1}^{i} R_j$, where $(R_j)$ is a sequence of (open) $R$-sets covering $R$.)

For $x \in X$ one has:

$$\left\|\sum_z (f_n - f_m)(z,x)^*(f_n - f_m)(z,x)\right\| < \varepsilon.$$

For every $i \in \mathbf{N}$:

$$\left\| \sum_{(x,z)\in F_i} (f_n - f_m)(z,x)^*(f_n - f_m)(z,x) \right\| < \varepsilon.$$

This sum is finite. One can pass to the limit as $m \to \infty$ and obtain:

$$\left\| \sum_{(x,z)\in F_i} (f_n - f)(z,x)^*(f_n - f)(z,x) \right\| \le \varepsilon.$$

Now one can pass to the limit as $i \to \infty$:

$$\left\| \sum_z (f_n - f)(z,x)^*(f_n - f)(z,x) \right\| \le \varepsilon.$$

We obtained that $f_n \to f$ in the "coordinate $l_2$-metric".

Using standard $l_2$-techniques one obtains that for every $f \in C_r^*(E)$ the series $\sum_z f^*(x,z)f(z,x)$ converges, and

$$\sup_{x\in X} \left\| \sum_z f^*(x,z)f(z,x) \right\| = \|f\|_2^2.$$

□

We have now developed all the techniques necessary to prove Theorem 2.1.

PROOF OF THEOREM 2.1.

We need to justify passing to the limit in the formulas for $C_c(E)$. Let $(f_n), (g_n) \subset C_c(E)$, and $f_n \to f$, $g_n \to g$. Then $f_n(x,y) \to f(x,y)$ for every $(x,y) \in R$, so passing to the limit in the second formula is justified. Analogously, $f_n g_n \to fg$, so $(f_n g_n)(x,y) \to (fg)(x,y)$, and the passing to the limit at the left hand side of the first formula is also justified.

For the right hand side of the first formula, let $\tau$ be a partial homeomorphism of $X$, with graph $\Gamma(\tau)$ in $R$, that takes $y$ to $x$. Let $h \in C_c(E)$ be such that $\operatorname{supp} h \subset \Gamma(\tau)$. Then:

$$\langle f_n^* h^*, g_n\rangle(y) = (h f_n g_n)(y,y) = \sum_z h(y,x) f_n(x,z) g_n(z,y).$$

Now, $f_n \to f$ and $g_n \to g$ in the $\|\cdot\|$-norm. So, $f_n \to f$ and $g_n \to g$ in the $\|\cdot\|_2$-norm. Consequently, $\langle f_n^* h^*, g_n\rangle \to \langle f^* h^*, g\rangle$ in $C_0(E)$. It follows from Lemma

2.4 that

$$\langle f^*h^*, g\rangle(y) = \sum_z h(y,x)f(x,z)g(z,y).$$

So, $\sum_z h(y,x)f_n(x,z)g_n(z,y) \to \sum_z h(y,x)f(x,z)g(z,y)$. Because this holds for every element $h$ of the above type, one obtains

$$\sum_z f_n(x,z)g_n(z,y) \to \sum_z f(x,z)g(z,y).$$

□

## 3. Coordinatization theorem

In this section, we give necessary and sufficient conditions for a $C^*$-algebra $A$ to be isomorphic to the algebra $C^*_r(E)$ generated by some Fell bundle $E$ over a $r$-discrete principal groupoid. These conditions are analogous to existence of so-called Cartan subalgebra in $A$ (see [8, Definition II.4.13]) and the definition of a diagonal pair in [4].

Before formulating the main result we need some notation. Let $A$ be a $C^*$-algebra, and let $D$ be a $C^*$-subalgebra of it. Assume that $D$ is the $C^*$-algebra generated by the continuous field $\{D(x)\}_{x\in X}$, where $X$ is a locally compact space. Suppose that for every $x \in X$, $D(x)$ is not trivial (i. e. $D(x) \neq \{0\}$).

We suppose that each $D(x)$ is unital, and that $D$ contains all scalar continuous sections vanishing at infinity. We denote the set of all such sections by $C$. Then $C$ is an Abelian subalgebra of $D$, and $C \cong C_0(X)$.

*Remark.* The condition of unitality is imposed only for convenience. We can drop it. But then we need the following condition: the algebra $A$ possesses an approximative unity $\{e_\lambda\}$ contained in $D$. Then, we can replace $D$ by $D \oplus C$ and $A$ by $A \oplus C$ where the multiplication $cf$ for $c \in C$, $f \in A$ is given by $\lim_\lambda (ce_\lambda)f$.

*Notation* 1. For each $x \in X$, let $I(x) = \{\, f \in D \mid f(x) = 0\,\}$. The set $I(x)$ is a closed ideal in $D$.

*Remark.* For $x \in X$: $D/I(x) \cong D(x)$. The isomorphism is given as follows: for $d \in D$, the class $[d] \in D/I(x)$ corresponds to $d(x)$.

**Definition 2.** We say that an element $s \in A$ is *normalizing* for $D$, or that $s$ *normalizes* $\boldsymbol{D}$, if $sDs^* \subseteq D$ and $s^*Ds \subseteq D$.

Suppose $s$ is normalizing for D and $(x,y) \in X \times X$. We say that *the element* $s$ *defines the pair* $(\boldsymbol{x}, \boldsymbol{y})$ (or that the pair $(x,y)$ is defined by $s$) if the following conditions hold:

1. $sDs^* \not\subseteq I(x)$;
2. $s^*Ds \not\subseteq I(y)$;
3. $sI(y)s^* \subseteq I(x)$;
4. $s^*I(x)s \subseteq I(y)$.

The set of all normalizing elements for $D$ will be denoted by $N$.

Note that $D \subset N$ and that for $s, t \in N$ one has: $s^* \in N$ and $st \in N$.

**Definition 3.** The subalgebra $D$ of $A$ will be called a *(generalized) Cartan subalgebra* if:

1. $A$ is spanned by $N$;
2. There exists a faithful conditional expectation $P : A \to D$.
3. $D$ is maximal in $A$ in the following sense: if $s \in N$ and if for every pair $(x, y)$ defined by $s$ we have $x = y$, then $s \in D$;
4. If $s \in N$ and for every pair $(x, y)$ defined by $s$ we have $x \neq y$, then $P(s) = 0$.

The main result of this section is the following theorem.

**Theorem 3.1.** *A $C^*$-algebra $A$ is isomorphic to the $C^*$-algebra $C_r^*(\mathcal{A})$ generated by some Fell bundle $\mathcal{A}$ over some $r$-discrete principal groupoid if and only if $A$ possesses a Cartan subalgebra.*

Proof. The sufficiency is evident. Indeed, suppose $A = C_r^*(\mathcal{A})$ for some Fell bundle $\mathcal{A}$ over a $r$-discrete principal groupoid $R$ as in [2] or [4]. Then, every normalizing element $s$ is supported on some (open) $R$-set. Elements of this $R$-set are exactly the pairs defined by $s$.

To prove the necessity, we need some additional notation. The proof is contained in several lemmas.

**Notation.** In what follows, normalizing elements will be denoted by lower case letters like $s$ or $t$, while (open) $R$-sets will be denoted by capital letters like $S$ or $T$. The set of all (open) $R$-sets will be denoted by $\mathcal{G}$. For $s \in N$ we denote:

$$R_s = \{ (x, y) \in R \mid s \text{ defines the pair } (x, y) \},$$

and for $S \in \mathcal{G}$ we denote

$$N_S = \{ s \in N \mid R_s \subseteq S \}.$$

Note that each $R_s$ is a $R$-set. Note also that for every $R$-set $S$, the set $N_S$ is a linear space. Indeed, if $s, t \in N_S$ then for $a \in D$ we have $sat^* \in D$ and $s^*at \in D$, by maximality of $D$ and so: $(s + t)a(s + t)^* = sas^* + sat^* + tas^* + tat^* \in D$.

Therefore, $s + t \in N$. Furthermore, if $s + t$ defines $(x, y)$ then one shows that $(x, y) \in R_S$.

For $s, t \in N_S$, we consider the map $s(\cdot)t^* : D \to D$. It actually maps the ideal $I(y)$ into the ideal $I(x)$. (This follows from the polarization identity.) So, the corresponding quotient map acts from $D(y) \cong D/I(y)$ into $D(x) \cong D/I(x)$.

**Notation.** The map from $D(y)$ into $D(x)$ defined above will be denoted by $\langle s, t\rangle^S_{(x,y)}$. Therefore, we have defined a sesquilinear $B(D(y), D(x))$-valued form on $N_S$.

For $s \in N_S$, let

$$\|s\|^S_{(x,y)} = \left\|\langle s, s\rangle^S_{(x,y)}\right\|^{\frac{1}{2}}.$$

This is a seminorm on $N_S$, because it is defined by a sesquilinear form.

Let

$$R = \{\, (x, y) \in X \times X \mid \exists s \in N : s \text{ defines the pair } (x, y) \,\}.$$

We equip $R$ with the topology generated by the collection of subsets $\{\, R_s \mid s \in N \,\}$.

The set $R$ is an equivalence relation on $X$. Indeed, $R$ is reflexive because for every $x \in X$ and for $d \in D$ such that $d(x) \neq 0$: the element $d$ belongs to $N$ and the pair $(x, x)$ is defined by $d$. Moreover, $R$ is symmetric because if $s$ defines the pair $(x, y)$ then $s^*$ defines $(y, x)$. And R is transitive because if $s$ defines the pair $(x, y)$ and $t$ defines $(y, z)$ then $st$ defines $(x, z)$.

The diagonal $\triangle = \{\, (x, x) \mid x \in X \,\}$ is open in $R$ in this topology, because $D \subset N$ and $\triangle = \bigcup_{s\in D} R_s$.

The induced topology on $\triangle$ coincides with the topology on $X$. Indeed, each open set on $X$ is the open support of some element $s \in D$, so it is $R_s$ for this $s$. Conversely, if $R_s \subset \triangle$ for some $s \in N$, then $s \in D$ and $R_s$ is the open support of $s$, so it is open in $X$.

The maps $r : (x, y) \mapsto x$ and $s : (x, y) \mapsto y$ are local homeomorphisms, because these maps must be one-to-one on $R_s$, and they take each $R_t$ onto $R_{tt^*}$ and $R_{t^*t}$ respectively.

**Lemma 3.2.** *Let $S, T \in \mathcal{G}$, $S \subset T$, $(x, y) \in S$ and $s \in N_S$. Then*

$$\|s\|^S_{(x,y)} = \|s\|^T_{(x,y)}.$$

PROOF. The proof is evident. □

**Lemma 3.3.** *Let $S \in \mathcal{G}$, $(x, y) \in S$, $s \in N_S$ and $f \in C$ be such that $f(x) = 1_{D(x)}$. Then $fs \in N_S$ and*

$$\|s - fs\|^S_{(x,y)} = 0.$$

PROOF. For every $t \in N_S$, the maps $s(\cdot)t^*$ and $fs(\cdot)t^*$ act the same on $D(x)$. □

**Lemma 3.4.** *Let $S, T \in \mathcal{G}$, $(x,y) \in S \cap T$ and $s \in N_S \cap N_T$. Then*

$$\|s\|^S_{(x,y)} = \|s\|^T_{(x,y)}.$$

PROOF. Let $U = S \cap T$. Let $f \in C_0(X) \cong C$ be such that $f(x) = 1$ and $\operatorname{supp} f \subseteq r(U)$. Then

$$\|s\|^S_{(x,y)} = \|fs\|^S_{(x,y)} = \|fs\|^U_{(x,y)} = \|fs\|^T_{(x,y)} = \|s\|^T_{(x,y)}.$$

□

**Notation.** Lemma 3.4 says that $\|s\|^S_{(x,y)}$ is actually independent of $S$. We will write $\|s\|_{(x,y)}$ for $\|s\|^S_{(x,y)}$.

**Lemma 3.5.** *For $s \in N$:*

$$\|s\| = \sup_{(x,y)\in R_s} \|s\|_{(x,y)}.$$

PROOF. $\|s\| = \|s^*s\|^{\frac{1}{2}}$ and $s^*s \in D$. Let $\|s\| = \|(s^*s)(x_0)\|^{\frac{1}{2}}$, $x_0 \in X$. Then

$$\|s\|_{(x_0,y_0)} = \sup_{a\in C} \|(s^*as)(x_0)\|^{\frac{1}{2}} = \|(s^*s)(x_0)\|^{\frac{1}{2}} = \|s\|.$$

Fix $(x,y) \in R$. Let

$$\overline{A(x,y)} = \{\, s \in N \mid s \text{ defines the pair } (x,y) \,\}.$$

The functional $\|\cdot\|_{(x,y)}$ is a seminorm on $\overline{A(x,y)}$. Let $\overline{A(x,y)}_0$ be the kernel of this seminorm and let

$$A(x,y) = \overline{A(x,y)} \Big/ \overline{A(x,y)}_0.$$

Then the functional $\|\cdot\|_{(x,y)}$, defined naturally on $A(x,y)$, becomes a norm. The space $A(x,y)$ is complete in this norm. Indeed, let $S \in \mathcal{G}$ be such that $(x,y) \in S$. It's easy to see that the space $N_S$ is complete with respect to the norm in $A$ and that $A(x,y)$ is a quotient of $N_S$.

Let $\mathcal{A}$ be the union of all $A(x,y)$ for $(x,y)$ running over $R$. We define the topology on $\mathcal{A}$ using a collection of sections as follows. For each $s \in N$ we define the section $\widehat{s}$ by the formula

$$\widehat{s}(x,y) = \begin{cases} [s] \in A(x,y) & \text{if } s \text{ defines the pair } (x,y), \\ 0 & \text{otherwise,} \end{cases}$$

where $[s]$ is the class of $s$ in $A(x,y) = \overline{A(x,y)} \Big/ \overline{A(x,y)}_0$. □

**Proposition 3.6.** *The set $\mathcal{A} = \cup_{(x,y)\in R} A(x,y)$ is a Fell bundle over $R$, with the operations given as follows: for $(x,y),(y,z) \in R$, $[s] \in A(x,y)$, $[t] \in A(y,z)$ we define:*

- $[s] \cdot [t] \stackrel{\text{def}}{=} [st] \in A(x,z)$;
- $[s]^* \stackrel{\text{def}}{=} [s^*] \in A(y,x)$.

PROOF. All the properties are evident. □

**Notation.** Let $C_r^*(\mathcal{A})$ be the $C^*$-algebra generated by $\mathcal{A}$. Let $\varphi : N \to C_r^*(\mathcal{A})$ be defined by

$$\varphi : s \mapsto \widehat{s}, \qquad s \in N.$$

It's easy to see that $\varphi$ is multiplicative. Let's extend the map $\varphi$ by linearity to the map denoted again by $\varphi$ from $\operatorname{lin} N$ into $C_r^*(\mathcal{A})$. The extended map is well defined. Indeed, let $s_1 + s_2 + \cdots + s_n = 0$. Let $(x,y) \in S$ where $S \in \mathcal{G}$. One can suppose that each $R_{s_i}$ either is contained in $S$ or doesn't intersect with $S$. Then, multiplying by appropriate functions from $C$ one can "exclude" those $s_i$ whose supports $R_{s_i}$ don't intersect with $S$. Finally, one deals with $N_S$ which is a linear space.

**Lemma 3.7.** (See [4, page 983].) *For $a \in A$:*

$$\|a\| = \sup\left\{ \|P(b^*a^*ab)\|^{1/2} : P(b^*b) \le 1 \right\}. \tag{2}$$

PROOF. The algebra $A$ equipped with the inner product $\langle a,b\rangle = P(a^*b)$ is a pre-Hilbert $D$-module. Let $\pi$ be the representation of the algebra $A$ on this pre-Hilbert module by left multiplications: $\pi(a)b = ab$. We have:

$$\begin{aligned} \|\pi(a)b\|_2 &= \|ab\|_2 = \langle ab, ab\rangle^{1/2} = \|P(b^*a^*ab)\|^{1/2} \\ &\le \|a^*a\|^{1/2}\|P(b^*b)\|^{1/2} = \|a\| \cdot \|b\|_2. \end{aligned}$$

so the representation $\pi$ is norm-decreasing, and it extends to the representation (denoted again by $\pi$) of $A$ on the completion $L^2(A)$ of this pre-Hilbert $D$-module.

The representation $\pi$ is one-to-one. Indeed, if $\pi(a) = 0$ then $P(b^*a^*ab) = 0$ for every $b \in A$, so $b^*a^*ab = 0$, so $ab = 0$ for every $b \in A$, thus $a = 0$.

Therefore, $\pi$ is isometric, and (2) follows. □

To finish the PROOF OF THEOREM 3.1 we need only show that the map $\varphi : \operatorname{lin} N \to C^*(\mathcal{A})$ is isometric. Indeed, for every $c \in \operatorname{lin} N$ one has: $P(c) = P'(\varphi(c))$, where $P'$ is the canonical conditional expectation on $C^*(\mathcal{A})$. Consequently, for $a \in \operatorname{lin} N$:

$$\begin{aligned} \|\varphi(a)\| &= \sup\{\, \|P'(\varphi(b)^*\varphi(a)^*\varphi(a)\varphi(b))\| : \|P'(\varphi(b^*b))\| \le 1\,\} \\ &= \sup\{\, \|P'(\varphi(b^*a^*ab))\| : \|P'(\varphi(b^*b))\| \le 1\,\} \\ &= \sup\{\, \|P(b^*a^*ab)\| : \|P(b^*b)\| \le 1\,\} = \|a\|. \end{aligned}$$

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MATHEMATICS INSTITUTE, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN N, DENMARK

*E-mail address*: ifulman@math.ku.dk, ifulman@member.ams.org, ifulman@math.uiowa.edu