HOUSTON JOURNAL OF MATHEMATICS
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Volume 24, No. 2, 1998

EXISTENCE AND NON EXISTENCE OF SOLUTIONS TO A VARIATIONAL PROBLEM ON A SQUARE

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COMMUNICATED BY HAÏM BREZIS

ABSTRACT. A non convex minimum problem on a square arising in shape optimization is studied. Conditions are discussed for the existence or the non existence of solutions.

1. INTRODUCTION

We consider the problem of minimizing the functional

$$\int_\Omega [h(\|\nabla v(x)\|) + v(x)]\, dx, \qquad v \in W_0^{1,1}(\Omega),$$

where the extended valued function $h\colon [0,\infty) \to [0,\infty]$ takes finite values only at $t = 1$ and $t = 2$, namely $h(1) = 0$ and $h(2) = 1$. In the case where Ω is a disk in $\mathbb{R}^2$, this problem admits a solution, no matter what the radius R of Ω might be (see [3]). In fact, in this case, a direct computation shows that a radial solution u_r exists having gradient ∇u_r of norm one on a disk, concentric with Ω, of radius $\min\{1, R\}$ and, when $R > 1$, having gradient of norm 2 on the remaining annulus. Here, we consider the very same problem when Ω is a square in the plane.

Minimum problems of this kind have a long history, having been considered in [7], [4], [6], [1], and [5] while investigating problems from optimal design. In these previous papers, the function h (the minimum between two parabolas having the same vertical axis or between one such parabola and a vertical half line) is not extended valued but is still not convex: numerical results in [5] concerning the convexified problem show that solutions to the original non convex problem do not exist. The function h considered here has been chosen in an attempt to simplify

1991 *Mathematics Subject Classification.* 49J45.

Key words and phrases. Non convex minimum problems.

the functions appearing in the quoted papers while retaining their essential feature of lacking convexity. For this function h, the construction provided in [2] applied to the case of a square shows that a solution exists whenever the length of the sides of the square does not exceed 2. The purpose of the present paper is to show that this existence result is sharp: a positive ε exists such that the given problem has no solutions whenever the length of the sides of the square is strictly between 2 and $2+\varepsilon$. This is achieved by building a solution to the corresponding convexified problem, i.e. the minimum problem for the convex integral

$$\int_{\Omega} [h^{**}(\|\nabla v(x)\|) + v(x)]\, dx, \qquad v \in W_0^{1,1}(\Omega),$$

and showing that, due to its properties, it cannot be a solution to the original problem and, moreover, that any other possible solution to the convexified problem would share the same properties.

A first result on non existence of solutions for problems similar to those considered here was obtained in [7], although in a rather different spirit: in the non existence result presented here, there is no a priori assumption on the properties of a possible solution.

Finally, we mention also that for problems of the kind here considered but with $h(t) = t^p$, $p > 1$, results in [8] imply that the sublevel sets of a solution are convex. Although the problem obtained from convexifying our functional has features similar to the problems considered in [8], the solution we build to the convexified problem is such that its sublevel sets are not convex, thus showing that this property is not to be expected to hold in general.

Finally, we wish to remark that the original problems of [7], [4] and [5] are more complex than the simplified one we consider and, so far, for them no precise conclusions concerning the existence versus the non existence of solutions exists.

2. Notations and preliminary results

Throughout this paper, we consider the non convex minimum problem

$$(\mathcal{P}) \qquad \min\left\{\int_{Q_r} [h(\|\nabla v(x)\|) + v(x)]\, dx\colon\ v \in W_0^{1,1}(Q_r)\right\}$$

where Q_r is the open square in $\mathbb{R}^2$ defined by $Q_r = (-r\,,r)\times(0\,,2r)$ with $r > 1$ and $h\colon [0\,,\infty) \to [0\,,\infty]$ is the extended valued, non convex function defined by

$$h(t) = \begin{cases} 0 & \text{if } t = 1, \\ 1 & \text{if } t = 2, \\ \infty & \text{elsewhere.} \end{cases}$$

As it was previously mentioned in the introduction, we wish to prove that problem $(\mathcal{P})$ fails to have solutions whenever $r > 1$ is sufficiently close to 1.

To this purpose, it is convenient to consider also the convexified minimum problem associated with $(\mathcal{P})$, i.e. the minimum problem

$$(\mathcal{P}^{**}) \qquad \min\left\{\int_{Q_r} [h^{**}(\|\nabla v(x)\|) + v(x)]\, dx\colon\ v \in W_0^{1,1}(Q_r)\right\}$$

where the function $h^{**}\colon [0\,,\infty) \to [0\,,\infty]$ is given by

$$h^{**}(t) = \begin{cases} \max\{0\,,t-1\} & \text{if } 0 \le t \le 2, \\ \infty & \text{if } t > 2. \end{cases}$$

The function h^{**} defined above, in spite of the notation used, is not the convex envelope of h but it is such that the function $\xi \in \mathbb{R}^2 \to h^{**}(\|\xi\|)$ is the convex envelope of the function $\xi \in \mathbb{R}^2 \to h(\|\xi\|)$.

Now, we introduce some notations that will be useful in the sequel. For $r > 1$, T_r is the open triangle in $\mathbb{R}^2$ whose vertices are $(0\,,0)$, $(r\,,0)$ and $(0\,,r)$. For $0 < a < r$, the open trapezoid whose vertices are $(0\,,0)$, $(a\,,0)$, $(a\,,r-a)$ and $(0\,,r)$ is denoted by $Q^*_{a,r}$ and the segment $\{(\xi_1,\xi_2)\colon\ \xi_1 = a,\, 0 \le \xi_2 \le r-a\}$ is denoted by $J_{a,r}$. Then, let $L^i\colon \mathbb{R}\times\mathbb{R}\times(-1\,,1) \to \mathbb{R}$, $i = 0,1$ be the smooth functions defined by

$$\begin{cases} L^0(x\,,y\,,z) = \dfrac{\sqrt{1+z^2}}{1-z}(r-x-y) \\ L^1(x\,,y\,,z) = 2\dfrac{\sqrt{1+z^2}}{2-\sqrt{1+z^2}}\left(1-\dfrac{1}{\sqrt{1+z^2}}-y\right) \end{cases}$$

for every $(x\,,y\,,z) \in \mathbb{R}\times\mathbb{R}\times(-1\,,1)$. The meaning of the function L^0 can be easily explained. Indeed, consider the line $\{(t\,,r-t)\colon\ t \in \mathbb{R}\}$, i.e. the line which contains the hypotenuse of T_r. In the sequel, we shall briefly refer to such line as the diagonal. Then, the value of L^0 computed at any point $(x\,,y\,,z)$ such that $y < r-x$ and $z < 1$ gives the distance in the plane of the point $(x\,,y)$ from the point

$$\left(\frac{x+yz-rz}{1-z}, \frac{rx-yz}{1-z}\right)$$

which lies on the intersection of the diagonal with the line through the point (x, y) orthogonal to the line $\{(x+t, y+zt)\colon\ t \in \mathbb{R}\}$. The meaning of the function L^1 will be clarified in the sequel.

For $\varphi \in \mathcal{C}^1([0, a])$, we denote the curve in the plane whose support is the graph Γ_φ of φ by $\Phi(x) = (x, \varphi(x))$, $0 \le x \le a$ and we let

$$n_\varphi(x) = \left(\frac{-\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}, \frac{1}{\sqrt{1+(\varphi'(x))^2}}\right), \qquad 0 \le x \le a,$$

be a unit normal to Γ_φ. Whenever $\varphi \in \mathcal{C}^1([0, a])$ satisfies $|\varphi'(x)| < 1$ for all $0 \le x \le a$, we set

$$l^i_\varphi(x) = L^i(x, \varphi(x), \varphi'(x)), \qquad 0 \le x \le a, \quad i = 0, 1$$

and we set also $\gamma^i_\varphi\colon [0, a] \to \mathbb{R}^2$, $i = 0, 1, 2$ to be the curves defined by

$$\begin{cases} \gamma^0_\varphi(x) = \Phi(x) + l^0_\varphi(x) n_\varphi(x) \\ \gamma^1_\varphi(x) = \Phi(x) + l^1_\varphi(x) n_\varphi(x) \\ \gamma^2_\varphi(x) = \Phi(x) + \left[l^1_\varphi(x) + 1\right] n_\varphi(x) - e_2 \end{cases} \qquad 0 \le x \le a$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Their supports will be denoted by Γ^i_φ. By the definition of L^0, it is clear that, whenever φ satisfies $\varphi(x) < r - x$ and $|\varphi'(x)| < 1$ for all $0 \le x \le a$, then the support Γ^0_φ of γ^0_φ is contained in the diagonal.

In the following, we shall consider a map $\varphi \in \mathcal{C}^3([0, a])$ with the following properties:

(P1) $\varphi(a) = 0$, the derivative of φ satisfies $0 < \varphi'(x) < 1$ for all $0 < x < 1$, vanishes for $x = 0$ and $x = a$ and the second derivative φ'' is strictly decreasing on the interval $[0, a]$;

(P2) the projection of minimal distance from the closure of $Q^*_{a,r}$ onto Γ_φ is single valued;

(P3) for $i = 0, 1, 2$, the first component of γ^i_φ, i.e. $(\gamma^i_\varphi)_1$, is an increasing diffeomorphism of the interval $[0, a]$ onto itself;

(P4) for $i = 1, 2$, the second component of γ^i_φ, i.e. $(\gamma^i_\varphi)_2$, is decreasing on the interval $[0, a]$.

Let us point out some consequences of the previous assumptions. First, notice that, for such φ, we have $\gamma_\varphi^1(0) = \gamma_\varphi^2(0)$ and $\gamma_\varphi^1(a) = \gamma_\varphi^2(a)$, i.e. the two curves Γ_φ^1 and Γ_φ^2 have the same initial and final points. On account of (P1), let $0 < x_0 < a$ be the unique point such that $\varphi''(x_0) = 0$ and set $I_1 = [0, x_0)$ and $I_2 = (x_0, a]$ so that the second derivative of φ is positive on the interval I_1. By (P2), the radius of curvature

$$R_\varphi(x) = \frac{\left[1 + (\varphi'(x))^2\right]^{3/2}}{\varphi''(x)}, \qquad x \in I_1 \cup I_2,$$

of Γ_φ at the point $\Phi(x)$ satisfies $l_\varphi^0(x) < R_\varphi(x)$ for all $x \in I_1$. As $R_\varphi(x) \to \infty$ when $x \to x_0$ from the left whereas l_φ^0 remains bounded on the interval I_1, we conclude that the function $1 - l_\varphi^0 / R_\varphi$ is uniformly bounded away from zero on the interval I_1. Then, consider the bounded sets

$$A_\varphi = \left\{(x, l) : \ 0 < x < a, \ 0 \le l \le l_\varphi^0(x)\right\},$$
$$B_\varphi = \{(\xi_1, \xi_2) : \ 0 < \xi_1 < a, \ \varphi(\xi_1) \le \xi_2 \le r - \xi_1\},$$

and let Ψ_φ be the function defined by

$$\Psi_\varphi(x, l) = \Phi(x) + l n_\varphi(x), \qquad (x, l) \in [0, a] \times \mathbb{R}.$$

Relying on (P1) again, it is easy to check that Ψ_φ is twice continuously differentiable on $[0, a] \times \mathbb{R}$ and that

$$\det \nabla \Psi_\varphi(x, l) = \sqrt{1 + (\varphi'(x))^2} \left\{1 - l \frac{\varphi''(x)}{[1 + (\varphi'(x))^2]^{3/2}}\right\},$$

for every $(x, l) \in [0, a] \times \mathbb{R}$. In particular, $\det \nabla \Psi_\varphi$ remains uniformly bounded away from zero on the closure of an open set $A'_\varphi \subset (0, a) \times \mathbb{R}$ which contains A_φ. Moreover, by (P2) again, it is clear that the open set A'_φ can be chosen in such a way that Ψ_φ is a homeomorphism of the closure of A'_φ onto the closure of an open set $B'_\varphi \subset (0, a) \times \mathbb{R}$ which contains B_φ. Thus, Ψ_φ is a twice continuously differentiable diffeomorphism of A'_φ onto B'_φ and the inverse function of Ψ_φ has bounded second derivatives on B'_φ.

3. Main result

The non existence of solutions to the minimum problem $(\mathcal{P})$ whenever the length $2r > 2$ of the sides of Q_r is sufficiently close to 2 follows immediately from the following two theorems.

Theorem 3.1. *Let $r > 1$ and $0 < a < r$ be such that $r - a \leq 1$. Assume that the differential equation*

$$\varphi''(x) = \frac{2\left[1 + (\varphi'(x))^2\right]^{3/2}\left[L^0(x,\varphi(x),\varphi'(x)) - L^1(x,\varphi(x),\varphi'(x)) - 1\right]}{\left[L^0(x,\varphi(x),\varphi'(x))\right]^2 - \left[L^1(x,\varphi(x),\varphi'(x))\right]^2 - 2L^1(x,\varphi(x),\varphi'(x))}$$

admits a solution $\varphi \in C^3([0,a])$ with the properties (P1), ... ,(P4). *Then, problem* $(\mathcal{P})$ *admits no solution in $W_0^{1,1}(Q_r)$.*

Theorem 3.2. *There exists $r_0 > 1$ with the property that, for all $1 < r \leq r_0$, there exists $0 < a(r) < r$ with $r - a(r) \leq 1$ such that the differential equation*

$$\varphi''(x) = \frac{2\left[1 + (\varphi'(x))^2\right]^{3/2}\left[L^0(x,\varphi(x),\varphi'(x)) - L^1(x,\varphi(x),\varphi'(x)) - 1\right]}{\left[L^0(x,\varphi(x),\varphi'(x))\right]^2 - \left[L^1(x,\varphi(x),\varphi'(x))\right]^2 - 2L^1(x,\varphi(x),\varphi'(x))}$$

admits a solution $\varphi \in C^3([0,a(r)])$ with the properties (P1), ... ,(P4).

The remaining part of this section is devoted to the proof of Theorem 3.1, our main result, while the proof of Theorem 3.2, a result of rather technical nature, is postponed to the subsequent Section 4.

PROOF OF THEOREM 3.1. The proof consists of the following steps. In (a), we investigate the geometric properties of the curves Γ_φ^1 and Γ_φ^2 associated to φ and in the following step (b) we use these properties to define a continuous function u on the closure of T_r, a candidate to being the restriction to T_r of a solution to the convexified problem $(\mathcal{P}^{**})$. In (c), we show that such function u is twice continuously differentiable off the curves Γ_φ^1, Γ_φ^2 and the segment $J_{a,r}$ while in the following step (d) we investigate the properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$ and, as a consequence, we obtain that u is actually continuously differentiable on the whole open set T_r. In (e), we prove that the norm of the gradient of u lies strictly between 1 and 2 in the open region bounded by the curves Γ_φ^1 and Γ_φ^2 and in (f) we compute the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$. In (g), we investigate the properties of the solutions to the differential equation

$$y'(t) = \frac{\nabla u(y(t))}{\|\nabla u(y(t))\|}$$

and in (h) we show that u is the restriction to T_r of a solution to problem $(\mathcal{P}^{**})$ by integrating along the trajectories of the differential equation considered in the previous step. In particular, such function u is not the restriction to T_r of a solution to the original problem $(\mathcal{P})$ as $\|\nabla u\|$ lies strictly between 1 and 2 on a set of positive measure. Finally, in (i) we conclude the proof of the theorem by showing that the properties of the gradient of u imply that problem $(\mathcal{P})$ has no solution.

In order not to overburden the notation, as the numbers a, r and the function φ are kept fixed throughout the proof of the theorem, we drop the indexes a, r and φ from now on.

(a) In order to define the function u on Q^*, we investigate the properties of the two curves Γ^1 and Γ^2. We begin by considering the unique point $0 < x_0 < a$ where $\varphi''(x_0) = 0$ and the two intervals $I_1 = [0\,, x_0)$ and $I_2 = (x_0\,, a]$. Since φ is a solution to the differential equation, we have $l^0(x_0) = l^1(x_0) + 1$ and the following identity

$$\frac{1}{R(x)} = 2\frac{l^0(x) - l^1(x) - 1}{[l^0(x)]^2 - [l^1(x)]^2 - 2l^1(x)}, \qquad x \in I_1 \cup I_2,$$

which yields

$$2[R(x) - l^1(x)] + [R(x) - l^0(x)]^2 - [R(x) - l^1(x)]^2 = 0, \qquad x \in I_1 \cup I_2.$$

As previously noticed, for all x in the interval I_1, we have $R(x) > l^0(x) > 0$ and hence $R(x) \neq l^1(x)$. Since $R(0) - l^1(0) > 0$, we conclude that $R(x) > l^1(x)$ for all $x \in I_1$. Now, fix $x \in I_1$ and consider the map

$$\theta(l) = -\frac{1}{2}\frac{[R(x) - l^0(x)]^2}{R(x) - l} + \frac{1}{2}[R(x) - l], \qquad l \neq R(x).$$

The map θ satisfies $\theta(l^1(x)) = 1$, $\theta(l^0(x)) = 0$ and its derivative is negative. Hence, $l^0(x) > l^1(x)$ for all $x \in I_1$. For $x \in I_2$, $R(x)$ is negative whereas $l^0(x)$ and $l^1(x)$ are both positive. Again, the derivative of θ shows that $l^0(x) > l^1(x)$. We claim that this implies that the curve Γ^1 lies below the diagonal. Such property is obvious for the initial and final points of Γ^1. Therefore, let $0 < x < a$ be fixed and set $\xi_1 = (\gamma^1)_1(x)$. We have to prove that $(\gamma^1)_2(x) < r - \xi_1$. If it were not so, the line $\{\Psi(x\,, l)\colon\ l \in \mathbb{R}\}$ would intersect the diagonal into two distinct points, i.e. it would coincide with the diagonal. This cannot be, as the slope of the line $\{\Psi(x\,, l)\colon\ l \in \mathbb{R}\}$ is $-1/\varphi'(x) < -1$. This proves the claim. Moreover, by (P1), we have for all $0 \leq x < a$

$$\begin{aligned}\frac{l^1(x)}{\sqrt{1 + (\varphi'(x))^2}} &= \frac{2}{2 - \sqrt{1 + (\varphi'(x))^2}}\left(1 - \frac{1}{\sqrt{1 + (\varphi'(x))^2}} - \varphi(x)\right) \geq \\ &\geq \frac{2}{2 - \sqrt{1 + (\varphi'(x))^2}}(-\varphi(x)) > -\varphi(x)\end{aligned}$$

so that the second component of γ^1 is positive on the interval $[0,a)$. As the first component of $\gamma^1(x)$ belongs to $(0,a)$ for $0 < x < a$, we see that all points of Γ^1, but the endpoints, are in Q^*.

As far as the curve Γ^2 is concerned, we notice that

$$\gamma^2(x) = \gamma^1(x) + n(x) - e_2, \qquad 0 \leq x \leq a, \tag{3.1}$$

so that, by (P1) and (P3), we have $(\gamma^2)_2(x) \leq (\gamma^1)_2(x)$ and $0 \leq (\gamma^2)_1(x) \leq (\gamma^1)_1(x)$ for all $0 \leq x \leq a$. Since $\gamma^1(x) \in Q^*$ for all $0 < x < a$, the point $\gamma^2(x)$ lies below the diagonal for the same values of x. Moreover,

$$\varphi(x) + \frac{l^1(x)+1}{\sqrt{1+(\varphi'(x))^2}} - 1 =$$
$$= \left(\frac{2}{2-\sqrt{1+(\varphi'(x))^2}} - 1\right)\left[\left(1 - \frac{1}{\sqrt{1+(\varphi'(x))^2}}\right) - \varphi(x)\right] > 0$$

for $0 \leq x < a$ so that the second component of $\gamma^2(x)$ is positive for these values of x. By (P3), the first component is in $(0,a)$ for $0 < x < a$ and hence all points of Γ^2, but the endpoints, are in Q^* as well.

Now, we claim that the curves Γ^1 and Γ^2 never touch in Q^* and that Γ^2 lies below Γ^1. Indeed, assume by contradiction that x' and x'' are two points in the interval $(0,a)$ such that $(\gamma^1)_1(x') = (\gamma^2)_1(x'')$. Since $(\gamma^2)_1(x) < (\gamma^1)_1(x)$ for every $0 < x < a$, it would follow that $x' < x''$ and hence, by (P3) and (3.1), that

$$(\gamma^2)_2(x'') < (\gamma^2)_2(x') = (\gamma^1)_2(x') + \left(\frac{1}{\sqrt{1+(\varphi'(x'))^2}} - 1\right) < (\gamma^1)_2(x'),$$

i.e. a contradiction. This proves the claim. Therefore, the open set Q^* is divided by the curves Γ^1 and Γ^2 into three non empty, open regions $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The closure of $\mathcal{O}_1$ is bounded above by the part of the boundary of Q^* which is contained in the hypotenuse of T and below by the curve Γ^1. It consists of the points $\Psi(x,l)$ where $0 \leq x \leq a$ and $l^1(x) \leq l \leq l^0(x)$. The closure of $\mathcal{O}_3$ consists of the points between Γ^2 and the x axis, i.e. the points

$$\left\{(\xi_1,\xi_2)\colon\ \xi_1 = (\gamma^2)_1(x) \text{ and } 0 \leq \xi_2 \leq (\gamma^2)_2(x) \text{ for } 0 \leq x \leq a\right\}.$$

The region $\mathcal{O}_2$ is the complement in Q^* of the union of the closures of $\mathcal{O}_1$ and $\mathcal{O}_3$. Finally, we let $\mathcal{O}_4$ be the open region defined as the complement in T of the closure of Q^*.

It is clear that $\overline{\mathcal{O}}_2 \cup \overline{\mathcal{O}}_3$ is given by the set

$$\left\{(\xi_1,\xi_2)\colon\ \xi_2 \geq 0 \text{ and } (\xi_1,\xi_2) = \Psi(x,l) \text{ with } 0 \leq x \leq a,\ 0 \leq l \leq l^1(x)\right\}. \tag{3.2}$$

Moreover, by (P3), each of the curves Γ^1 and Γ^2 has the property that no two distinct points on it have the same abscissa and hence, as Γ^2 lies above Γ^1, the intersections of the sets $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ with every vertical line are open intervals.

(b) On the closure of each region $\mathcal{O}_i$, we define a continuous function u^i in such a way that the definitions agree on the curves Γ^1, Γ^2 and on the segment J. We call u the continuous function defined on the closure of T whose restriction to the closure of $\mathcal{O}_i$ is u^i.

On the closure of $\mathcal{O}_1$, we set $u^1 = -\left(\Psi^{-1}\right)_2$ so that, for a point (ξ_1, ξ_2) in the closure of $\mathcal{O}_1$, we have $u^1(\xi_1,\xi_2) = -l$ where $(x\,,l)$ is the unique pair such that $(\xi_1,\xi_2) = \Psi(x\,,l)$. Hence, by this definition, the function $-u^1$ on $\mathcal{O}_1$ is the distance from Γ and, in particular, $u^1(\gamma^1(x)) = -l^1(x)$ for all $0 \leq x \leq a$. For a point (ξ_1,ξ_2) in the closure of $\mathcal{O}_3$, set $u^3(\xi_1,\xi_2) = -2\xi_2$. Hence, the function $-u^3/2$ is the distance from the base of T and, in particular, $u^3(\gamma^2(x)) = -2(\gamma^2)_2(x)$ for all $0 \leq x \leq a$. An easy computation shows that

$$u^1(\gamma^1(x)) = -l^1(x) = -2(\gamma^2)_2(x) = u^3(\gamma^2(x)), \qquad 0 \leq x \leq a.$$

For a point (ξ_1,ξ_2) in the closure of $\mathcal{O}_4$, set $u^4(\xi_1,\xi_2) = -\xi_2$ and notice that, as Ψ reduces to the identity map on the segment J, the functions u^1 and u^4 agree on it. Finally, we are left to define u^2 on the closure of $\mathcal{O}_2$ in a consistent way with the previous definitions. To this purpose, let $P(x)$ be the point defined by

$$P(x) = \gamma^2(x) + e_2 = \gamma^1(x) + n(x), \qquad 0 \leq x \leq a,$$

and notice that $\|P(x) - \gamma^1(x)\| = \|P(x) - \gamma^2(x)\| = 1$ for all $0 \leq x \leq a$. Now, we claim that, for every $\xi = (\xi_1\,,\xi_2)$ in the closure of $\mathcal{O}_2$, there exists a unique x in the interval $[0\,,a]$ such that

$$\left\{\begin{array}{l} \|P(x) - \xi\| = 1, \\ (P)_1(x) \leq \xi_1, \\ (P)_2(x) \geq \xi_2. \end{array}\right. \tag{3.3}$$

These conditions mean that the point ξ lies on the south-east arc of the circle of radius one centered at $P(x)$. Once this has been proved, we set $u^2(\xi) = u^1(\gamma^1(x)) = u^3(\gamma^2(x))$ for such x so that the arc of the circle of radius one connecting $\gamma^1(x)$ and $\gamma^2(x)$ on which ξ lies, turns out to be a level curve of u.

To prove the above mentioned claim, first notice that for a point ξ lying either on Γ^1 or on Γ^2, it is enough to choose such $x \in [0\,,a]$ that $\xi = \gamma^1(x)$ or $\xi = \gamma^2(x)$ respectively. Then, let $\xi = (\xi_1,\xi_2)$ be a point in $\mathcal{O}_2$ and, by (P2), let (x',l') be the unique pair such that $\xi = \Psi(x',l')$. As ξ does not belong to the closure of $\mathcal{O}_1$, it follows that $l' < l^1(x')$. Then, by (P3), there exists a unique point $x'' \in (0, a)$ such that $\xi_1 = (\gamma^2)_1(x'')$. By (P1) and $l' < l^1(x')$, we see that $(\gamma^1)_1(x') < (\gamma^2)_1(x'') = \xi_1$ and, as previously noticed, this implies that $x' < x''$. Moreover, since $(P)_1(x'') = (\gamma^2)_1(x'')$ and $(\gamma^2)_1(x') < (\gamma^1)_1(x')$, we see also that both points x' and x'' satisfy the second condition of (3.3). Now, consider the points $\gamma^2(x'')$, $P(x'')$ and ξ whose first components are all equal to ξ_1. The first of them belongs to the closure of $\mathcal{O}_3$ and, as ξ is not in the closure of $\mathcal{O}_3$, it follows that $(\gamma^2)_2(x'') < \xi_2$. The second point is neither in the closure of $\mathcal{O}_2$ nor in the closure of $\mathcal{O}_3$ by (3.2) and it has positive second component. As the set $\{\zeta\colon (\xi_1, \zeta) \in \mathcal{O}_2\}$ is an open interval in $(0, \infty)$, we conclude that $(\gamma^2)_2(x'') < \xi_2 < (P)_2(x'')$. Since $x' < x''$, we also have $(P)_2(x'') < (P)_2(x')$ by (P4). We have thus proved that there exist $0 < x' < x'' < a$ such that

$$\begin{cases} (P)_1(x') < (P)_1(x'') = \xi_1, \\ (P)_2(x') > (P)_2(x'') > \xi_2, \end{cases}$$

and it is now easy to check that, letting the function d be defined by $d(x) = \|P(x) - \xi\|$ for $0 \le x \le a$, we have $d(x') > 1$ and $d(x'') < 1$. Therefore, there exists at least one point $x \in (x', x'')$ such that (3.3) holds.

We have thus proved that, for every point $\xi = (\xi_1, \xi_2)$ in the closure of $\mathcal{O}_2$, there exists at least one point $x \in [0, a]$ such that (3.3) holds and we are left to prove the uniqueness of such x. To this purpose, let $x_1 < x_2$ be two points in the interval $[0, a]$ such that

$$\begin{cases} (P)_1(x_i) \le \xi_1 \\ (P)_2(x_i) \ge \xi_2 \end{cases} \qquad i = 1, 2.$$

By (P3), (P4) and the definition of P, we obtain that $(P)_1(x_1) < (P)_1(x_2)$ and $(P)_2(x_1) > (P)_2(x_2)$ so that $d(x_1) > d(x_2)$. This shows that (3.3) can hold true for at most one point. Thus, the function u^2 is well defined.

As far as the continuity of u^2 is concerned, we notice that, by elementary geometrical arguments, the function which maps a point ξ from the closure of $\mathcal{O}_2$ onto the unique $x \in [0, a]$ for which (3.3) holds, is continuous. Hence, by (P3) and (P4), u^2 is continuous as well.

(c) In this step, we show that each function u^i is twice continuously differentiable on the open set $\mathcal{O}_i$.

Indeed, the functions u^3 and u^4 are linear and the function u^1 agrees with $-\left(\Psi^{-1}\right)_2$ so that it is in $C^2(\mathcal{O}_1)$. Moreover, as it was noticed at the end of the previous Section 2, the inverse function of Ψ is twice continuously differentiable on the open set $B' \subset (0, a) \times \mathbb{R}$ and its second derivatives are bounded on such set.

Therefore, u^1 has a natural extension as a function in $\mathcal{C}^2(B')$ with bounded second derivatives. We still denote such extension by the same symbol and, for future purposes, we set $\mathcal{O}_2{}' = B'$. Now, we claim that the same regularity property is shared by u^2 as well. To see this, we show that the function u^2 is implicitly defined by an equation. To this purpose, recall that, by (P4), the map $(\gamma^2)_2$ is a homeomorphism of the interval $[0\,,a]$ onto a compact interval I of $\mathbb{R}$ and its inverse function is twice continuously differentiable on the interior of I. Set ψ to be

$$\psi(\xi_2) = (\gamma^2)_1 \circ (\gamma^2)_2^{-1}(\xi_2), \qquad \xi_2 \in I$$

so that $\xi_1 = \psi(\xi_2)$ for all $(\xi_1, \xi_2) \in \Gamma^2$ and notice also that ψ is decreasing by (P3) and (P4). Then, let $\xi = (\xi_1, \xi_2)$ be a point in $\overline{\mathcal{O}}_2 \setminus E$ where E is the set consisting of the initial and final points of the two curves Γ^1 and Γ^2 and let x be the unique point in the interval $(0\,,a)$ associated with ξ by (3.3). By the definition of u^2, we have $(\gamma^2)_2(x) = -(1/2)u^2(\xi)$ so that the components of $P(x)$ can be written as functions of the value of u^2 at the point ξ in the following way

$$(3.4) \qquad \begin{cases} (P)_1(x) = \psi\left((\gamma^2)_2(x)\right) = \psi\left(-\dfrac{1}{2}u^2(\xi)\right), \\ (P)_2(x) = (\gamma^2)_2(x) + 1 = -\dfrac{1}{2}u^2(\xi) + 1. \end{cases}$$

Hence, recalling the way $P(x)$ and $\xi = (\xi_1, \xi_2)$ are related by (3.3), we obtain the following identity

$$(3.5) \qquad \left[\xi_1 - \psi\left(-\frac{1}{2}u^2(\xi_1,\xi_2)\right)\right]^2 + \left[\xi_2 - \left(-\frac{1}{2}u^2(\xi_1,\xi_2) + 1\right)\right]^2 = 1$$

for all points $\xi \in \overline{\mathcal{O}}_2 \setminus E$ so that, letting

$$F(\xi_1,\xi_2,\zeta) = \left[\xi_1 - \psi\left(-\frac{1}{2}\zeta\right)\right]^2 + \left[\xi_2 - \left(-\frac{1}{2}\zeta + 1\right)\right]^2$$

for every $(\xi\,,\zeta) \in \mathbb{R}^2 \times \operatorname{int} I$, $\xi = (\xi_1, \xi_2)$, we see that the function u^2 verifies $F(\xi_1,\xi_2,u^2(\xi_1,\xi_2)) = 1$ for all points (ξ_1,ξ_2) in $\overline{\mathcal{O}}_2 \setminus E$. We notice also that F is twice continuously differentiable on $\mathbb{R}^2 \times \operatorname{int} I$ and, computing the derivative of F with respect to ζ at the point $(\xi_1,\xi_2,u^2(\xi_1,\xi_2))$, we obtain

$$F_\zeta(\xi_1,\xi_2,u^2(\xi_1,\xi_2)) =$$
$$= \left[\xi_1 - \psi\left(-\frac{1}{2}u^2(\xi_1,\xi_2)\right)\right]\psi'\left(-\frac{1}{2}u^2(\xi_1,\xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u^2(\xi_1,\xi_2) + 1\right)\right].$$

Now, recall that, due to (3.4) and (3.3), we have $(P)_1(x) = \psi(-u^2(\xi)/2)$ and $\xi_1 - (P)_1(x) \geq 0$. As ψ is decreasing, the first term in $F_\zeta(\xi_1,\xi_2,u^2(\xi_1,\xi_2))$ is at most zero. Then, since $(P)_2(x) = -u^2(\xi)/2 + 1$ and $\xi_2 \leq (\gamma^1)_2(x)$ by the geometric properties of the level curves of u^2, we have

$$\begin{aligned}\left[\left(-\frac{1}{2}u^2(\xi_1,\xi_2)+1\right)-\xi_2\right] &= (P)_2(x)-\xi_2 \geq (P)_2(x)-(\gamma^1)_2(x) = \\ &= (n)_2(x) \geq \frac{1}{\sqrt{2}}\end{aligned} \tag{3.6}$$

by (P1). Thus, $F_\zeta(\xi_1,\xi_2,u^2(\xi_1,\xi_2)) \leq -1/\sqrt{2}$ for all $\xi = (\xi_1,\xi_2)$ in $\overline{\mathcal{O}}_2 \setminus E$ and, as F is twice continuously differentiable on its domain of definition and u^2 is continuous, the implicit function theorem ensures that the equation (3.5) defines, on an open set $\mathcal{O}_2{}' \subset (0,a)\times\mathbb{R}$ containing $\overline{\mathcal{O}}_2 \setminus E$, a twice continuously differentiable function that agrees with u^2 on $\overline{\mathcal{O}}_2 \setminus E$. Again, we denote such extension by u^2. In particular, the same theorem and the properties of ψ ensure also that the second derivatives of u^2 are bounded on every subset of $\mathcal{O}_2$ having positive distance from the set E.

(d) In the previous steps, we have defined a continuous function u on the closure of T whose restriction to each of the open sets $\mathcal{O}_i$ is twice continuously differentiable. In this step, we investigate the properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$ and we show that it admits a unique continuous extension to the closure of T which turns out to be locally Lipschitz continuous on T in the sense that every point in T has a neighbourhood where the Lipschitz condition is satisfied, a property that will be useful in the following step (g). Moreover, as a consequence of the above mentioned properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$, we shall prove that u actually belongs to $C^1(T)$.

We begin by computing the explicit expression of the vector field $\frac{\nabla u}{\|\nabla u\|}$ in each of the open regions $\mathcal{O}_i$.

Recalling the definition of u in the region $\mathcal{O}_1$, we have

$$\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \nabla u(\xi) = -n(x), \qquad \xi \in \mathcal{O}_1$$

where $x = (\Psi^{-1})_1(\xi)$. In the region $\mathcal{O}_2$, the function u is implicitly defined by the equation (3.5). Hence, applying the implicit function theorem and recalling that $F_\zeta(\xi_1,\xi_2,u(\xi_1,\xi_2)) < 0$ for all $(\xi_1,\xi_2) \in \mathcal{O}_2$, we obtain

$$\frac{\nabla u(\xi_1,\xi_2)}{\|\nabla u(\xi_1,\xi_2)\|} = \frac{\nabla_\xi F(\xi_1,\xi_2,u(\xi_1,\xi_2))}{\|\nabla_\xi F(\xi_1,\xi_2,u(\xi_1,\xi_2))\|}, \qquad (\xi_1,\xi_2) \in \mathcal{O}_2$$

where $\nabla_\xi F = (F_{\xi_1}, F_{\xi_2})$. Computing the partial derivatives of F with respect to ξ_1 and ξ_2, we obtain

$$
\begin{cases} F_{\xi_1}(\xi_1, \xi_2, \zeta) = 2\left[\xi_1 - \psi\left(-\frac{1}{2}\zeta\right)\right] \\ F_{\xi_2}(\xi_1, \xi_2, \zeta) = 2\left[\xi_2 - \left(-\frac{1}{2}\zeta + 1\right)\right] \end{cases} \tag{3.7}
$$

for every $(\xi\,, \zeta) \in \mathbb{R}^2 \times \operatorname{int} I$, $\xi = (\xi_1, \xi_2)$, so that the following equation holds

$$
[F_{\xi_1}(\xi_1, \xi_2, u(\xi_1, \xi_2))]^2 + [F_{\xi_2}(\xi_1, \xi_2, u(\xi_1, \xi_2))]^2 = 4F(\xi_1, \xi_2, u(\xi_1, \xi_2)) = 4
$$

for all points (ξ_1, ξ_2) in $\mathcal{O}_2$. Therefore, we have

$$
\frac{\nabla u(\xi_1, \xi_2)}{\|\nabla u(\xi_1, \xi_2)\|} = \left(\xi_1 - \psi\left(\frac{1}{2}u(\xi_1, \xi_2\right), \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right)
$$

for every $(\xi_1, \xi_2) \in \mathcal{O}_2$. Since the vector field $\frac{\nabla u}{\|\nabla u\|}$ is constantly equal to $(0\,, -1)$ in the regions $\mathcal{O}_3$ and $\mathcal{O}_4$, we conclude that

$$
\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \begin{cases} -n\left(\left(\Psi^{-1}\right)_1(\xi)\right) & \xi \in \mathcal{O}_1 \\ \left(\xi_1 - \psi\left(\frac{1}{2}u(\xi_1, \xi_2\right) \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right) & \xi \in \mathcal{O}_2, \\ (0\,, -1) & \xi \in \mathcal{O}_3 \cup \mathcal{O}_4 \end{cases}
$$

where, as usual, $\xi = (\xi_1, \xi_2)$. Recalling the properties of φ, ψ and Ψ, it is clear that the vector field $\frac{\nabla u}{\|\nabla u\|}$ has a unique continuous extension to the closure of T. We denote such extension by G. Then, recall that, as shown in the previous step (c), all second derivatives of u are bounded on the open sets $\mathcal{O}_1$, $\mathcal{O}_3$, $\mathcal{O}_4$ and also on each subset of $\mathcal{O}_2$ having positive distance from the exceptional set E, the set of the initial and final points of the curves Γ^1 and Γ^2. Hence, the vector field $\frac{\nabla u}{\|\nabla u\|}$ has to be Lipschitz continuous on each of the sets $\mathcal{O}_1$, $\mathcal{O}_3$, $\mathcal{O}_4$ and also on each subset of $\mathcal{O}_2$ having positive distance from the set E. Since G is continuous on T and coincides with $\frac{\nabla u}{\|\nabla u\|}$ on a dense subset of T, it follows that it is actually locally Lipschitz continuous on T itself.

Finally, we are left to prove that u is actually continuously differentiable on T. We are going to prove this by showing that ∇u remains continuous on the curves Γ^1, Γ^2 and on the segment J. Indeed, let Γ^i be one such curve, fix a point $\xi \notin E$ on it and let $\tau(\xi)$ be the tangent vector to Γ^i at ξ. Set $(\nabla u)^{\pm}(\xi)$ and $\|\nabla u\|^{\pm}(\xi)$ to be the limits at ξ from the regions above and below Γ^i of ∇u and $\|\nabla u\|$ respectively. Such limits do exist due to the boundedness properties of the second derivatives of u in the regions $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$. From the continuity of u on Γ^i, we obtain

$$\langle (\nabla u)^+(\xi) - (\nabla u)^-(\xi), \tau(\xi)\rangle = \Big(\|\nabla u\|^+(\xi) - \|\nabla u\|^-(\xi)\Big) \langle G(\xi), \tau(\xi)\rangle = 0.$$

From the explicit expression of $\frac{\nabla u}{\|\nabla u\|}$ computed above and from the properties of the curves Γ^1 and Γ^2, we see that the scalar product appearing at the right hand side of the previous equality never vanishes. Hence, it follows that $\|\nabla u\|^+(\xi) = \|\nabla u\|^-(\xi)$ on Γ^i and this implies that ∇u is continuous at ξ. At last, it is clear that the limits of ∇u from the regions $\mathcal{O}_1$ and $\mathcal{O}_4$ at each point of the segment J exist and are equal. Therefore, u is continuously differentiable on T.

(e) In this step, we prove that the norm of the gradient of u lies strictly between 1 and 2 on the open set $\mathcal{O}_2$. This property shows that u cannot be the restriction to T of a solution to the original problem $(\mathcal{P})$.

To prove this, we begin by noticing that (3.5), (3.7) and the explicit expression of F_ζ computed in (c) yield that

$$\|\nabla u(\xi)\|^2 = \frac{4}{1+\beta(\xi)}, \qquad \xi \in \overline{\Omega}_2 \setminus E, \quad \xi = (\xi_1, \xi_2),$$

where we have set

$$\beta(\xi) = 2\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]\left[\xi_2 - \left(-\frac{1}{2}u(\xi_1,\xi_2)+1\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right) + \\ + \left\{\left[\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]^2 - 1\right\}\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right],$$

for all such ξ. We have to prove that β remains strictly between 0 and 3 on the open set $\mathcal{O}_2$. To prove this, fix $c \in u(\mathcal{O}_2)$ and consider the level set $\{u = c\} \cap \overline{\mathcal{O}}_2$. Since the restriction of u to the curve Γ^2 is easily seen to be injective, from the definition of u^2 described in (b) we obtain that such level set is an arc of a circle of radius one centered at $P(x)$ for a unique point $x \in (0, a)$. Moreover, its endpoints are the points $\gamma^1(x)$ and $\gamma^2(x)$ which lie on the curves Γ^1 and Γ^2 respectively. Since $\|\nabla u(\gamma^1(x))\| = 1$ and $\|\nabla u(\gamma^2(x))\| = 2$, it follows that $\beta(\gamma^1(x)) = 3$ and $\beta(\gamma^2(x)) = 0$.

We claim that $\beta(\xi)$ increases from 0 to 3 as the point ξ runs from $\gamma^2(x)$ to $\gamma^1(x)$ along the level curve $\{u = c\} \cap \overline{\Omega}_2$.

In order to prove the claim, we notice that the value of $\psi'(-u(\xi)/2)$ remains constant along such level curve. Therefore, for the sake of brevity, we set $\lambda =$

$\psi'(-u(\xi)/2)$ for all $\xi \in \{u = c\} \cap \overline{\Omega}_2$. The value of λ can be easily computed. Indeed, by (3.4) and the definition of P given in (b), we have

$$\begin{cases} (\gamma^1)_1(x) - \psi\left(-\frac{1}{2}u(\xi)\right) = (\gamma^1)_1(x) - (P)_1(x) = \frac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}, \\ (\gamma^1)_2(x) - \left(-\frac{1}{2}u(\xi)+1\right) = (\gamma^1)_2(x) - (P)_2(x) = -\frac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}. \end{cases}$$

Hence, as $\beta(\gamma^1(x)) = 3$, we obtain the following equation for λ

$$\frac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}(\lambda^2-1) - \frac{2\varphi'(x)}{1+(\varphi'(x))^2}\lambda = 3$$

which yields

$$\lambda = \frac{1}{\sqrt{1+(\varphi'(x))^2}} - \sqrt{\frac{1}{1+(\varphi'(x))^2} + 3\frac{\sqrt{1+(\varphi'(x))^2}}{\varphi'(x)} + 1}$$

as λ has to be negative. Then, notice that the level curve $\{u = c\} \cap \overline{\Omega}_2$ can be parametrized as $\delta(t) = ((P)_1(x) + \sin t\,, (P)_2(x) - \cos t)$, $0 \le t \le t_0$, where $t_0 = \arctan(\varphi'(x))$. In particular, $\delta(0) = \gamma^2(x)$ and $\delta(t_0) = \gamma^1(x)$ so that $(\beta \circ \delta)(0) = 0$ and $(\beta \circ \delta)(t_0) = 3$. Moreover, by (P1), we have $0 < t_0 < \pi/4$.

Now, consider the function $(\beta \circ \delta)(t) = (\lambda^2 - 1)\sin t - 2\lambda \sin t \cos t$ for $0 \le t \le t_0$. In order to prove the claim, it is enough to show that the derivative of $\beta \circ \delta$ is positive on the interval $[0\,, t_0]$. To prove this, consider the second derivative of $\beta \circ \delta$, i.e. $(\beta \circ \delta)''(t) = \sin t\left[8\lambda \cos t - (\lambda^2-1)\right]$, for $0 \le t \le t_0$. As λ is negative and $\lambda^2 - 1$ is easily seen to be positive, $\beta \circ \delta$ is concave on the interval $[0\,, t_0]$. Hence, we are left to prove that $(\beta \circ \delta)'(t_0) > 0$. To this purpose, a direct computation yields $(\beta \circ \delta)'(t_0) = -4\lambda \cos^2 t_0 + (\lambda^2-1)\cos t_0 + 2\lambda$. Now, $(\cos t_0, \sin t_0)$ is the unit tangent vector to the graph Γ of φ at the point x. Therefore, we have $\cos t_0 = \left[1 + (\varphi'(x))^2\right]^{-1/2}$ and this yields

$$(\beta \circ \delta)'(t_0) = \frac{3}{\varphi'(x)} - \frac{2(\varphi'(x))^2}{1+(\varphi'(x))^2}\sqrt{\frac{1}{1+(\varphi'(x))^2} + 3\frac{\sqrt{1+(\varphi'(x))^2}}{\varphi'(x)} + 1} + \frac{2(\varphi'(x))^2}{[1+(\varphi'(x))^2]^{3/2}}$$

where $0 < \varphi'(x) < 1$ by (P1). For $0 < z \le 1$, we have

$$\frac{3}{z} - \frac{2z^2}{1+z^2}\sqrt{\frac{1}{1+z^2} + 3\frac{\sqrt{1+z^2}}{z} + 1} + \frac{2z^2}{(1+z^2)^{3/2}} \ge \frac{3}{z} - \sqrt{2 + \frac{6}{z}}$$

and the function appearing at the right hand side of the above inequality is decreasing on the interval $(0,1]$ and is positive for $z=1$. Thus, $(\beta \circ \delta)'(t_0) > 0$ and this prove the claim.

(f) In this step, we wish to compute the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$ in each of the open regions $\mathcal{O}_i$.

As pointed out in the previous step (d), in the region $\mathcal{O}_1$ we have

$$\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \nabla u(\xi) = -n(x), \qquad \xi \in \mathcal{O}_1,$$

where $x = (\Psi^{-1})_1(\xi)$. To compute the divergence of $-n \circ \big((\Psi^{-1})_1\big)$ at a point $\xi \in \mathcal{O}_1$, we notice that, by the local inversion theorem, it depends only on the distance between the points ξ and $\Phi(x)$ and on the values of the first and second derivatives of φ computed at $x = (\Psi^{-1})_1(\xi)$. Hence, it coincides with the divergence of the normal to the graph of a function having the same value and the same first and second derivatives at x. By computing the divergence of the normal to the osculating circle, one obtains

$$\operatorname{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)(\xi) = \begin{cases} \dfrac{1}{R(x)-l} & \text{if } \varphi''(x) \neq 0 \\ 0 & \text{if } \varphi''(x) = 0 \end{cases} \tag{3.8}$$

where $\xi = \Psi(x, l)$.

As shown in (d), in the region $\mathcal{O}_2$, the function u is implicitly defined by the equation (3.5) so that we have

$$\frac{\nabla u(\xi_1,\xi_2)}{\|\nabla u(\xi_1,\xi_2)\|} = \frac{1}{2}\left(F_{\xi_1}(\xi_1,\xi_2,u(\xi_1,\xi_2)), F_{\xi_2}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right)$$

for every $(\xi_1,\xi_2) \in \mathcal{O}_2$ where the partial derivatives of F with respect to ξ_1 and ξ_2 are given by (3.7). Now, a direct computation yields

$$\begin{cases} \dfrac{\partial}{\partial \xi_1}\left[\dfrac{1}{2}F_{\xi_1}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right] = 1 + \dfrac{1}{2}\psi'\left(-\dfrac{1}{2}u(\xi_1,\xi_2)\right)u_{\xi_1}(\xi_1,\xi_2) \\ \dfrac{\partial}{\partial \xi_2}\left[\dfrac{1}{2}F_{\xi_2}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right] = 1 + \dfrac{1}{2}u_{\xi_2}(\xi_1,\xi_2) \end{cases}$$

for every $(\xi_1, \xi_2) \in \mathcal{O}_2$ so that, recalling the way the derivatives of u are related to the partial derivatives of F, we obtain

$$\operatorname{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)(\xi_1, \xi_2) =$$

$$= 2 - \frac{1}{2}\psi'\left(-\frac{1}{2}u(\xi_1, \xi_2)\right)\frac{F_{\xi_1}(\xi_1, \xi_2, u(\xi_1, \xi_2))}{F_\zeta(\xi_1, \xi_2, u(\xi_1, \xi_2))} - \frac{1}{2}\frac{F_{\xi_2}(\xi_1, \xi_2, u(\xi_1, \xi_2))}{F_\zeta(\xi_1, \xi_2, u(\xi_1, \xi_2))} =$$

$$= 2 - \frac{\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1, \xi_2)\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1, \xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right]}{\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1, \xi_2)\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1, \xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right]} = 1$$

for all $(\xi_1, \xi_2) \in \mathcal{O}_2$.

Finally, in the open regions $\mathcal{O}_3$ and $\mathcal{O}_4$, the vector field $\frac{\nabla u}{\|\nabla u\|}$ is obviously divergence free.

(g) In this step, we consider the Cauchy problem

$$\begin{cases} y'(t) = G(y(t)) \\ y(0) = (x, 0) \end{cases}$$

for $0 < x < r$ where G is the unique continuous extension of the vector field $\frac{\nabla u}{\|\nabla u\|}$ introduced in the previous step (d). By the properties of the vector field G, a unique local solution to this problem exists for $t \leq 0$ for every $0 < x < r$ and it can be extended to a left maximal interval of existence $(\vartheta_0(x), 0]$. For $0 < x < r$ and $\vartheta_0(x) < t \leq 0$, set $Y(x, t)$ to be this unique solution.

We aim at describing the geometrical behaviour of the integral lines $Y(x, t)$ as t ranges through the interval $(\vartheta_0(x), 0]$ for all $0 < x < r$. From now on, for reasons that will be apparent later, we set $\vartheta_3(x) = 0$ for all x.

First, consider $0 < x < a$. By integrating backwards in time, the solution $Y(x, t)$, as t decreases, rises vertically until it reaches the curve Γ^2 at $t = \vartheta_2(x)$. On the curve Γ^2, the vector field G is vertical and Γ^2 has no vertical tangents. Hence, the solution cannot touch Γ^2 at two different times. Moreover, as shown by (3.6), the second component of the vector field G satisfies

$$(G)_2(\xi_1, \xi_2) = \frac{u_{\xi_2}(\xi_1, \xi_2)}{\|\nabla u(\xi_1, \xi_2)\|} = \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right) \leq -\frac{1}{\sqrt{2}}$$

for all $(\xi_1, \xi_2) \in \overline{\Omega}_2 \setminus E$. This bound implies that, at a time $t = \vartheta_1(x)$, the solution Y reaches the curve Γ^1 at a point $\gamma^1(x')$ for some $x' \in (0, a)$. For $t \leq \vartheta_1(x)$, the solution Y remains on the line described by $\Psi(x', l)$ for $l^1(x') \leq l \leq l^0(x')$ and this line meets the curve Γ^0, the part of the boundary of Q^* contained in the hypotenuse of T, at the point $\gamma^0(x')$, the limit of the solution $Y(x, t)$ as $t \to \vartheta_0(x)$ from the right. Set $Y(x, \vartheta_0(x))$ to be such limit. Then, consider

$a \le x < r$. In this case, the solution Y remains in the region where the vector field G is constantly equal to $(0, -1)$ so that, integrating backwards, the solution rises vertically with constant speed -1 until it reaches the hypotenuse of T at time $\vartheta_0(x) = -(r - x)$. Again, set $Y(x, \vartheta_0(x))$ to be the point $(x, -\vartheta_0(x))$.

We have thus defined a continuous function Y on the set C defined by $C = \{(x, t)\colon\ 0 < x < r,\ \vartheta_0(x) \le t \le 0\}$. It is easy to check that it is injective and an argument similar to the previous one shows that it is also surjective onto the set $\{(\xi_1, \xi_2)\colon\ 0 < \xi_1 < r,\ 0 \le \xi_2 \le r - \xi_1\}$. Moreover, the uniqueness of solutions together with the fact that the vector field G is never tangent to any curve Γ^i for $i = 0, 1, 2$ implies that all functions ϑ_i are continuous on the interval $(0, a)$.

Now, we claim that Y is continuously differentiable on the set of all points $(x, t) \in C$ with $x \neq a$.

To this purpose, as pointed out in (c), we recall that we have $u^1 \in C^2(\mathcal{O}_1')$ and $u^2 \in C^2(\mathcal{O}_2')$ where the open sets $\mathcal{O}_1{}'$ and $\mathcal{O}_2'$ are contained in $(0, a) \times \mathbb{R}$ and in turn contain the closures (with respect to the strip $(0, a) \times \mathbb{R}$) of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Similarly, the functions u^3 and u^4 are linear and hence they can actually be regarded as twice continuously differentiable functions defined on open sets $\mathcal{O}_3'$ and $\mathcal{O}_4'$ containing the closures of $\mathcal{O}_3$ and $\mathcal{O}_4$ respectively. Therefore, each function u^i defines a continuously differentiable vector field G^i on the open set $\mathcal{O}_i'$ which agrees with G on the intersection of $\mathcal{O}_i'$ with the closure of $\mathcal{O}_i$. Let $g_i^t(\xi)$, $\xi \in \mathcal{O}_i'$ be the flows generated by the vector fields G^i and set

$$
(3.9) \qquad \begin{cases} Y^3(x, t) = g_3^t(x, 0) & (x, t) \in V_3 \\ Y^2(x, t) = g_2^{t-\vartheta_2(x)}\left(Y^3(x, \vartheta_2(x)\right) & (x, t) \in V_2 \\ Y^1(x, t) = g_1^{t-\vartheta_1(x)}\left(Y^2(x, \vartheta_1(x))\right) & (x, t) \in V_1 \end{cases}
$$

where $V_i = \{(x', t')\colon\ 0 < x' < a,\ t' \in U_i(x')\}$ and each set $U_i(x')$ is a suitable open neighbourhood of the interval $[\vartheta_i(x'), \vartheta_{i-1}(x')]$. It is clear that $Y(x, t) = Y^i(x, t)$ for $\vartheta_{i-1}(x) \le t \le \vartheta_i(x)$ and $0 < x < a$.

As a first step, we begin to prove that all functions ϑ_i are continuously differentiable on the interval $(0, a)$. To this purpose, assume for a while that we know that Y is continuously differentiable on an open set containing the graph of a continuously differentiable curve described as $\xi_2 = \sigma(\xi_1)$, $0 < \xi_1 < a$ and assume that the tangent to the curve and the vector field are never collinear. Then, if a continuous function ϑ satisfies

$$
(Y)_2(x, \vartheta(x)) = \sigma\left((Y)_1(x, \vartheta(x))\right), \qquad 0 < x < a,
$$

one can verify, by the implicit function theorem, that ϑ is continuously differentiable. Therefore, let $\xi_2 = \sigma^2(\xi_1)$, $0 \le \xi_1 \le a$ be a continuously differentiable representation of the curve Γ^2. Since the map Y^3 is continuously differentiable on the open set V_3, from the identity

$$\left(Y^3\right)_2(x\,,\vartheta_2(x)) = \sigma^2\left(\left(Y^3\right)_1(x\,,\vartheta_2(x))\right), \qquad 0 < x < a,$$

we infer that ϑ_2 is continuously differentiable on $(0\,,a)$. In turn, this implies that Y^2, as defined by (3.9), is continuously differentiable on the open set V_2 as well. The same argument applied to a continuously differentiable representation $\xi_2 = \sigma^1(\xi_1)$, $0 \le \xi_1 \le a$ of the curve Γ^1 yields that ϑ_1 is continuously differentiable on $(0\,,a)$ and hence the same is true for Y^1 on V_1. It is left to show that the map Y is also continuously differentiable at those points which are mapped by Y itself into the curves Γ^1 and Γ^2. It is certainly continuously differentiable with respect to t since its derivative is the continuous vector field G. Let us show that the derivative with respect to x exists also at those points which are mapped by Y into Γ^1.

In order to simplify the notations, set

$$\begin{cases} Y_-^i(x) = Y^i(x\,,\vartheta_{i-1}(x)), \\ Y_+^i(x) = Y^i(x\,,\vartheta_i(x)), \end{cases} \qquad 0 < x < a, \quad i = 1,2,3,$$

and let one such x be fixed. For $t \in U_1(x)$, we have

$$\frac{\partial Y^1}{\partial x}(x\,,t) = -\vartheta_1'(x)G^1\left(Y^1(x\,,t)\right) + \nabla_\xi g_1^{t-\vartheta_1(x)}\left(Y_-^2(x)\right)\left(Y_-^2\right)'(x)$$

and in particular, for $t = \vartheta_1(x)$, we obtain

$$\text{(3.10)} \qquad \frac{\partial Y^1}{\partial x}(x\,,\vartheta_1(x)) = -\vartheta_1'(x)G^1\left(Y_+^1(x)\right) + \left(Y_-^2\right)'(x)$$

since $\nabla_\xi g_i^0(\xi)$ is the 2×2 identity matrix for all $\xi \in \mathcal{O}_i'$. Analogously, for $t \in U_2(x)$, we have

$$\frac{\partial Y^2}{\partial x}(x\,,t) = -\vartheta_2'(x)G^2\left(Y^2(x\,,t)\right) + \nabla_\xi g_2^{t-\vartheta_2(x)}\left(Y_-^3(x)\right) + \left(Y_-^3\right)'(x)$$

and in particular, for $t = \vartheta_1(x)$, we obtain

$$\text{(3.11)} \quad \frac{\partial Y^2}{\partial x}(x\,,\vartheta_1(x)) = -\vartheta_2'(x)G^2\left(Y_-^2(x)\right) + \nabla_\xi g_2^{\vartheta_1(x)-\vartheta_2(x)}\left(Y_-^3(x)\right)\left(Y_-^3\right)'(x).$$

Now, the same kind of computation yields

$$\left(Y_-^2\right)'(x) = (\vartheta_1'(x) - \vartheta_2'(x))G^2\left(Y_-^2(x)\right) + \nabla_\xi g_2^{\vartheta_1(x)-\vartheta_2(x)}\left(Y_-^3(x)\right)\left(Y_-^3\right)'(x)$$

and hence, as $G^1\left(Y_+^1(x)\right) = G^2\left(Y_-^2(x)\right)$, we conclude that (3.10) and (3.11) are equal, i.e. the two derivatives coincide on the counterimage of Γ^1 with respect to Y. In a simpler way, the same argument shows that Y is differentiable on the counterimage of Γ^2 with respect to Y. Finally, it is obvious that Y is continuously differentiable at every point $(x\,,t) \in C$ with $a < x < r$. Thus, the claim is proved.

In particular, these results imply $\det \nabla Y$ exists and is continuous on the set of all points $(x\,,t) \in C$ with $x \neq a$. By Liouville theorem, it is given by

$$\tag{3.12} \begin{aligned} \det \nabla Y(x\,,t) &= \det \nabla Y(x\,,0) \exp\left\{\int_0^t \mathrm{tr}\,\left(\nabla_\xi G(Y(x\,,s))\right)\,ds\right\} = \\ &= \det \nabla Y(x\,,0) \exp\left\{\int_0^t \mathrm{div}\,\left(\frac{\nabla u(Y(x\,,s))}{\|\nabla u(Y(x\,,s))\|}\right)\,ds\right\} \end{aligned}$$

for $0 < x < r$, $x \neq a$ and $\vartheta_0(x) \le t \le 0$ where $\mathrm{tr}\,(A)$ denotes the trace of a square matrix A. As $\det \nabla Y(x\,,0) = -1$ for $0 < x < r$ with $x \neq a$, we conclude that the gradient matrix of Y has negative determinant for all values of $0 < x < r$, $x \neq a$ and $\vartheta_0(x) \le t \le 0$. Therefore, the inverse mapping of Y is continuously differentiable on the open set T outside the segment J.

(h) In this step, we let u be the Lipschitz continuous function defined on the closure of the square Q which is symmetric with respect to the axes of symmetry of Q and whose restriction to the closure of T has been described in (b). In particular, u vanishes on the boundary of Q. We wish to show that u is a solution to the convexified problem $(\mathcal{P}^{**})$. To this purpose, we prove the existence of a measurable function α with the properties that $\alpha(\xi) \in \partial h^{**}\left(\|\nabla u(\xi)\|\right)$ for almost every $\xi \in Q$ and

$$\int_Q \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}\,, \nabla\eta(\xi)\rangle + \eta(\xi)\right]\,d\xi = 0$$

for every $\eta \in \mathcal{D}(Q)$. As shown in [2], this implies that u is a solution to the convexified problem $(\mathcal{P}^{**})$.

Letting T_i be the eight open triangles in which Q is divided by its axes of symmetry, we have

$$\begin{aligned} &\int_Q \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}\,, \nabla\eta(\xi)\rangle + \eta(\xi)\right]\,d\xi = \\ &\qquad = \sum_{1\le i\le 8} \int_{T_i} \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}\,, \nabla\eta(\xi)\rangle + \eta(\xi)\right]\,d\xi \end{aligned}$$

and we claim that each of the eight terms above is null for every $\eta \in \mathcal{D}(Q)$. Of course, it is enough to prove the claim for the triangle T considered at the beginning of this section, the argument for all other triangles being similar up to translations and rotations. To see this, we consider the function Y defined in the previous step (g). As it was previously noticed, it is a diffeomorphism of the open set of all points $(x\,,t) \in C$ with $x \neq a$ onto the open set $T \setminus J$. Therefore, set

$$\Xi_\eta(\xi) = \langle \frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}\,, \nabla\eta(\xi)\rangle, \qquad \xi \in T,$$

to simplify the notations and apply the change of variable formula and Fubini's theorem to find

$$\int_T [\alpha(\xi)\Xi_\eta(\xi) + \eta(\xi)]\; d\xi =$$
$$= -\int_0^r \left(\int_{\vartheta_0(x)}^0 [\alpha\,(Y(x\,,t))\,\Xi_\eta\,(Y(x\,,t)) + \eta\,(Y(x\,,t))]\,[\det \nabla Y(x\,,t)]\,dt \right) dx.$$

Then, let $0 < x < r$ be fixed and consider the second summand of the inner integral appearing at the right hand side of the equality above. Integrating by parts and noticing that $\eta(Y(x\,,0)) = 0$ as $Y(x\,,0) = (x\,,0)$ belongs to one of the sides of Q for all $0 \leq x \leq r$, we obtain

$$\int_{\vartheta_0(x)}^0 \eta(Y(x\,,t)) \det \nabla Y(x\,,t)\,dt =$$
$$= -\int_{\vartheta_0(x)}^0 \Xi_\eta\,(Y(x\,,t)) \left(\int_{\vartheta_0(x)}^t \det \nabla Y(x\,,s)\,ds \right) dt.$$

Thus, setting

$$A(x\,,t) = \alpha(Y(x\,,t)) \det \nabla Y(x\,,t) - \int_{\vartheta_0(x)}^t \det \nabla Y(x\,,s)\,ds \tag{3.13}$$

for every $\vartheta_0(x) \leq t \leq 0$ and $0 < x < r$, we have

$$\int_T [\alpha(\xi)\Xi_\eta(\xi) + \eta(\xi)]\; d\xi = -\int_0^r \left(\int_{\vartheta_0(x)}^0 A(x\,,t)\Xi_\eta\,(Y(x\,,t))\; dt \right) dx.$$

It is our purpose to show that it is possible to find α with the properties listed at the beginning of this step and such that $A(x\,,t)$ defined by (3.13) is identically zero for every $\vartheta_0(x) \leq t \leq 0$ and $0 < x < r$.

We are going to define α on each integral curve $S_x = \{Y(x,t)\colon\ \vartheta_0(x) \leq t \leq 0\}$, $0 < x < r$. First, let $a \leq x < r$ be fixed and notice that, by (3.12), $\det \nabla Y$ is constantly equal to -1 along the curve S_x so that it is enough to set

$$\alpha(Y(x,t)) = t - \vartheta_0(x), \qquad \vartheta_0(x) \leq t \leq 0,$$

in order to have $A(x,t) = 0$ for the same values of t. Moreover, since $r - a \leq 1$ by assumption, we have $0 \leq \alpha(Y(x,t)) \leq r - a \leq 1$ along the curve S_x. As $\|\nabla u(Y(x,t))\| = 1$ for $\vartheta_0(x) < t < 0$ and $a \leq x < r$, we obtain that $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for the same values of t and x.

Then, let $0 < x < a$ be fixed and consider $\vartheta_0(x) \leq t \leq \vartheta_1(x)$. For such x and t, the function Y is given by

$$Y(x,t) = Y(x, \vartheta_0(x)) - (t - \vartheta_0(x))n(x')$$

where $x' \in (0,a)$ is the unique point such that $\gamma^0(x') = Y(x, \vartheta_0(x))$. Relying on (3.12) with zero replaced by $\vartheta_0(x)$ and on (3.8), we obtain that

$$\begin{aligned}\det \nabla Y(x,t) &= \det \nabla Y(x, \vartheta_0(x)) \exp\left\{ \int_{\vartheta_0(x)}^{t} \frac{1}{R(x') - [l^0(x') - s + \vartheta_0(x)]}\, ds \right\} = \\ &= \det \nabla Y(x, \vartheta_0(x)) \left\{ \frac{R(x') - [l^0(x') - t + \vartheta_0(x)]}{R(x') - l^0(x')} \right\}\end{aligned}$$

if $\varphi''(x') \neq 0$ while $\det \nabla Y(x,t)$ remains constantly equal to $\det \nabla Y(x, \vartheta_0(x))$ for all $\vartheta_0(x) \leq t \leq \vartheta_1(x)$ if $\varphi''(x') = 0$. Hence, set

$$\alpha(Y(x,t)) = -\frac{1}{2} \frac{[R(x') - l^0(x')]^2}{R(x') - [l^0(x') - t + \vartheta_0(x)]} + \frac{1}{2}\left\{R(x') - [l^0(x') - t + \vartheta_0(x)]\right\}$$

for every $\vartheta_0(x) \leq t \leq \vartheta_1(x)$ if $\varphi''(x') \neq 0$ and $\alpha(Y(x,t)) = t - \vartheta_0(x)$ for the same values of t if $\varphi''(x') = 0$. In the former case, we have

(i) $\alpha(Y(x, \vartheta_0(x))) = 0$;

(ii) $\displaystyle \frac{\partial}{\partial t}[\alpha(Y(x,t))] = \frac{1}{2} \frac{[R(x') - l^0(x')]^2}{\{R(x') - [l^0(x') - t + \vartheta_0(x)]\}^2} + \frac{1}{2} > 0$;

(iii) $\displaystyle \alpha(Y(x, \vartheta_1(x))) = -\frac{1}{2} \frac{[R(x') - l^0(x')]^2}{R(x') - l^1(x')} + \frac{1}{2}[R(x') - l^1(x')] = 1.$

In particular, this last equality follows by noticing that $\vartheta_1(x) = \vartheta_0(x) + [l^0(x') - l^1(x')]$ and recalling that φ is a solution to the differential equation. Since we have $\|\nabla u(Y(x,t))\| = 1$ for $\vartheta_0(x) < t \leq \vartheta_1(x)$, the properties (i), (ii) and (iii) imply that $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for $\vartheta_0(x) < t \leq \vartheta_1(x)$ and $0 < x < a$. Moreover, this choice of α shows that $A(x,t)$ is given by

$$\det \nabla Y(x,\vartheta_0(x)) \left\{ -\frac{1}{2}[R(x') - l^0(x')] + \frac{1}{2}\frac{\{R(x') - [l^0(x') - t + \vartheta_0(x)]\}^2}{R(x') - l^0(x')} + \right.$$
$$\left. -\frac{1}{R(x') - l^0(x')}\left[\frac{1}{2}\{R(x') - [l^0(x') - t + \vartheta_0(x)]\}^2 - \frac{1}{2}[R(x') - l^0(x')]^2\right]\right\} = 0$$

for all $\vartheta_0(x) \leq t \leq \vartheta_1(x)$. In particular, (iii) and the equality $A(x,\vartheta_1(x)) = 0$ yield

$$\det \nabla Y(x,\vartheta_1(x)) = \int_{\vartheta_0(x)}^{\vartheta_1(x)} \det \nabla Y(x,s)\,ds. \tag{3.14}$$

Next, consider those t in the interval $(\vartheta_1(x),\vartheta_2(x))$. For such t, the point $Y(x,t)$ is in $\mathcal{O}_2$, the region where the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$ is constantly equal to 1 as proved in (f). Therefore, taking the derivative with respect to t of both sides of (3.12), we see that $\det \nabla Y$ satisfies the differential equation

$$\frac{\partial \det \nabla Y}{\partial t}(x,t) = \det \nabla Y(x,t), \qquad \vartheta_1(x) \leq t \leq \vartheta_2(x).$$

Hence, on account of (3.14), we see that $A(x,t)$ vanishes for all $\vartheta_1(x) \leq t \leq \vartheta_2(x)$ provided we set $\alpha(Y(x,t)) = 1$ for the same values of t. In particular, we have

$$\det \nabla Y(x,\vartheta_2(x)) = \int_{\vartheta_0(x)}^{\vartheta_2(x)} \det \nabla Y(x,s)\,ds.$$

Moreover, by (e), the norm of the gradient of u lies strictly between 1 and 2 on $\mathcal{O}_2$. Hence, we have $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for all $\vartheta_1(x) < t < \vartheta_2(x)$ and $0 < x < a$.

Finally, consider those t in the interval $(\vartheta_2(x),0)$. The point $Y(x,t)$ is in $\mathcal{O}_3$ where the vector field is constant so that $\det \nabla Y(x,t)$ remains constantly equal to $\det \nabla Y(x,\vartheta_2(x))$. Setting $\alpha(Y(x,t)) = t - \vartheta_2(x) + 1$ for all $\vartheta_2(x) \leq t \leq 0$, it is easy to check that $A(x,t)$ vanishes for the same values of t. Moreover, it is clear that $\alpha \geq 1$ for $\vartheta_2(x) \leq t \leq 0$ and, as the norm of the gradient of u remains constantly equal to 2 on $\mathcal{O}_3$, we obtain that $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for all $\vartheta_2(x) \leq t < 0$.

We have thus proved that u is a solution to the convexified problem $(\mathcal{P}^{**})$. Moreover, as the open set $\mathcal{O}_2$ has positive measure and the norm of the gradient of u lies strictly between 1 and 2 on $\mathcal{O}_2$, we see that u cannot be a solution to the original problem $(\mathcal{P})$.

(i) In this step, we prove that the original problem $(\mathcal{P})$ has no solution. Assume that a solution v exists, a Lipschitz continuous function whose gradient has norm either 1 or 2 almost everywhere on Q. As v is also a solution to the problem $(\mathcal{P}^{**})$, we will reach a contradiction by showing that the norm of the gradient of the restriction of v to T has to lie strictly between 1 and 2 almost everywhere on the open set $\mathcal{O}_2$.
By the convexity of the functional minimized in $(\mathcal{P}^{**})$, the function $w = \frac{1}{2}(u+v)$ is a solution to $(\mathcal{P}^{**})$ as well. Therefore, we have

$$\int_Q \left[h^{**}\left(\left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\|\right) + \left(\frac{1}{2}u(\xi) + \frac{1}{2}v(\xi)\right)\right] d\xi =$$
$$= \frac{1}{2}\int_Q \left[h^{**}\left(\|\nabla u(\xi)\|\right) + u(\xi)\right] d\xi + \frac{1}{2}\int_Q \left[h^{**}\left(\|\nabla v(\xi)\|\right) + v(\xi)\right] d\xi$$

and hence,

$$\int_Q h^{**}\left(\left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\|\right) d\xi =$$
$$= \frac{1}{2}\int_Q h^{**}\left(\|\nabla u(\xi)\|\right) d\xi + \frac{1}{2}\int_Q h^{**}\left(\|\nabla v(\xi)\|\right) d\xi.$$

Since we have

$$h^{**}\left(\left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\|\right) \leq \frac{1}{2}h^{**}\left(\|\nabla u(\xi)\|\right) + \frac{1}{2}h^{**}\left(\|\nabla v(\xi)\|\right)$$

for a.e. $\xi \in Q$ by the convexity of h^{**}, we actually must have equality, i.e.

$$h^{**}\left(\left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\|\right) = \frac{1}{2}h^{**}\left(\|\nabla u(\xi)\|\right) + \frac{1}{2}h^{**}\left(\|\nabla v(\xi)\|\right)$$

for a.e. $\xi \in Q$. Now, consider the restrictions of u, v and w to the triangle T. We claim that, on the set where $\|\nabla u\| > 1$, in particular on the regions $\mathcal{O}_2$ and $\mathcal{O}_3$, the vectors ∇u and ∇v must be almost everywhere collinear. In fact, among those points such that the above equation holds, consider those $\xi \in T$ such that $\|\nabla u(\xi)\| > 1$ and v is differentiable at ξ with $\|\nabla v(\xi)\|$ equal to either 1 or 2. For such ξ, we must have from the equation above $\left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\| > 1$. If $\nabla u(\xi)$ and $\nabla v(\xi)$ were not collinear, we would have

$$1 < \left\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\right\| < \frac{1}{2}\|\nabla u(\xi)\| + \frac{1}{2}\|\nabla v(\xi)\| \leq 2$$

and hence, by the properties of h^{**}, we would also have

$$h^{**}\left(\left\|\frac{1}{2}\nabla u(\xi)+\frac{1}{2}\nabla v(\xi)\right\|\right)<h^{**}\left(\frac{1}{2}\|\nabla u(\xi)\|+\frac{1}{2}\|\nabla v(\xi)\|\right)\leq$$
$$\leq\frac{1}{2}h^{**}\left(\|\nabla u(\xi)\|\right)+\frac{1}{2}h^{**}\left(\|\nabla v(\xi)\|\right).$$

Such inequality can be true only on a null set. Thus, the claim is proved.

By Fubini's theorem, for almost every c in the range of u on $\mathcal{O}_2\cup\mathcal{O}_3$, on a set of full one dimensional Hausdorff measure along the level curve $\{u=c\}$, the gradient of v is orthogonal to the tangent to the level curve $\{u=c\}$ itself . Hence, v is constant along any such level curve. Actually, by the continuity of v and by the properties of u, the same is true for every level curve of u. Our assumption that v is a solution to the original problem $(\mathcal{P})$ in particular implies that, for almost every level curve $\{u=c\}$ in $\mathcal{O}_2\cup\mathcal{O}_3$, we have that ∇v exists, is collinear to ∇u and has norm either 1 or 2 on a set of full one dimensional Hausdorff measure along the curve. We will reach a contradiction by showing that this cannot be.

Let $\xi^0=(\xi_1^0,\xi_2^0)$ be a point in $\mathcal{O}_2$ where the gradient of v exists and is collinear to the gradient of u and let $\xi\in\mathcal{O}_3$, $\xi=(\xi_1,\xi_2)$ be a point along the level curve $\{u=u(\xi^0)\}$ where the gradient of v exists and equals $(0\,,\lambda)$ with $|\lambda|$ either 1 or 2. For all small enough t, let $\eta(t)$ be such that

$$u\left(\xi+\eta(t)e_2\right)=u\left(\xi^0+t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right)$$

where $\{e_1\,,e_2\}$ is the standard basis of $\mathbb{R}^2$. For all such t, the very definition of u yields $u\left(\xi+\eta(t)e_2\right)=-2\xi_2-2\eta(t)$ whereas the differentiability of u at ξ^0 yields

$$u\left(\xi^0+t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right)=u(\xi^0)+t\left\|\nabla u(\xi^0)\right\|+t\varepsilon_1(t)$$

where $\varepsilon_1(t)\to 0$ as $t\to 0$. As $u(\xi^0)=-2\xi_2$, we obtain

$$\eta(t)=-\frac{t}{2}\left\|\nabla u(\xi^0)\right\|-\frac{t}{2}\varepsilon_1(t)\tag{3.15}$$

and, in particular, we see that $\eta(t)\to 0$ as $t\to 0$. Now, recall that the set $\{u=u(\xi^0)\}$ is also a level curve of v. Hence, we have $v(\xi)=v(\xi^0)$ and

$$v\left(\xi+\eta(t)e_2\right)=v\left(\xi^0+t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right).$$

Therefore, the same kind of computation yields

$$v(\xi) + \lambda\eta(t) + \eta(t)\varepsilon_2(\eta(t)) = v\left(\xi + \eta(t)e_2\right) =$$
$$= v\left(\xi^0 + t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right) = v(\xi^0) + t\left\|\nabla v(\xi^0)\right\| + t\varepsilon_3(t)$$

where, again, both functions ε_2 and ε_3 approach zero as their arguments go to zero. Therefore, we see that the equation $\lambda\eta(t)+\eta(t)\varepsilon_2(\eta(t)) = t\left\|\nabla v(\xi^0)\right\|+t\varepsilon_3(t)$ holds and hence, by (3.15), we obtain

$$-\frac{\lambda t}{2}\left\|\nabla u(\xi^0)\right\| - \frac{\lambda t}{2}\varepsilon_1(t) + \eta(t)\varepsilon_2(\eta(t)) = t\left\|\nabla v(\xi^0)\right\| + t\varepsilon_3(t).$$

Finally, dividing by $t \neq 0$ and letting $t \to 0$, we obtain

$$-\frac{\lambda}{2}\left\|\nabla u(\xi^0)\right\| = \left\|\nabla v(\xi^0)\right\|.$$

Since $1 < \left\|\nabla u(\xi^0)\right\| < 2$ and $|\lambda|$ is either 1 or 2, we see that $\left\|\nabla v(\xi^0)\right\|$ is neither 1 nor 2.

This completes the proof of the theorem. $\square$

4. The differential equation

In this final section, we give the proof of Theorem 3.2, thus showing that all the hypotheses of Theorem 3.1 are fulfilled provided r is larger than 1 and is sufficiently close to 1.

Let us begin with some remarks. For $r > 1$, let $D\colon \mathbb{R}\times\mathbb{R}\times(-1,1) \to \mathbb{R}$ be the smooth function defined by

$$D(x,y,z) = [L^0(x,y,z)]^2 - [L^1(x,y,z)]^2 - 2L^1(x,y,z)$$

for every $(x,y,z) \in \mathbb{R}\times\mathbb{R}\times(-1,1)$ and let U be the open subset where D is positive. Both D and U depend on r by the function L^0 defined at the beginning of the previous Section 2. The function

$$F(x,y,z) = \frac{2\sqrt{1+z^2}}{D(x,y,z)}\left[L^0(x,y,z) - L^1(x,y,z) - 1\right], \qquad (x,y,z) \in U,$$

is the right hand side of the differential equation appearing in the statements of Theorems 3.1 and 3.2. We wish to estimate F and its first and second derivatives as functions of r. We have

$$D(x,0,0) = (r-x)^2 \geq \frac{1}{4}, \qquad 0 \leq x \leq \frac{1}{2},$$

and, by computing the derivatives of D, we obtain that

$$\|\nabla D(x,y,z)\| \leq C(r), \qquad 0 \leq x \leq 1/2, \quad 0 \leq |y|\,,|z| \leq 1/2, \tag{4.1}$$

where $C(r) \geq 1$ is a non decreasing function of r. Hence, the values of D are at least $1/8$ on the compact subset $K_r = [0,1/2] \times [-\delta_0(r),\delta_0(r)] \times [-\delta_0(r),\delta_0(r)]$ where we have set

$$\delta_0(r) = \frac{1}{8\sqrt{2}C(r)}.$$

Then, it is easy to check that there exists a non decreasing function $M(r) \geq 1$ such that

$$\begin{cases} |F(x,y,z)| \leq M(r) \\ \|\nabla F(x,y,z)\| \leq M(r) \\ \|HF(x,y,z)\| \leq M(r) \end{cases} \qquad (x,y,z) \in K_r, \tag{4.2}$$

where $\|HF\|$ denotes the euclidean norm of the 3×3 Hessian matrix of F.

PROOF OF THEOREM 3.2. For $r > 1$, set

$$\lambda_0(r) = \frac{1}{r-1} \min\left\{\frac{\delta_0(r)}{M(r)}, \sqrt{2\frac{\delta_0(r)}{M(r)}}\right\}$$

where $M(r)$ is the constant appearing in (4.2). Set also $a = a(\lambda,r) = \lambda(r-1)$ for $0 < \lambda \leq \lambda_0(r)$ and notice that, by the choice of $\lambda_0(r)$, we have $0 < a(\lambda,r) \leq 1/2$ for all $0 < \lambda \leq \lambda_0(r)$. For all such λ, the Cauchy problem

$$\begin{cases} \varphi''(x) = F(x,\varphi(x),\varphi'(x)) & 0 \leq x \leq a(\lambda,r) \\ \varphi(a(\lambda,r)) = 0, \\ \varphi'(a(\lambda,r)) = 0 \end{cases} \tag{4.3}$$

admits a unique solution defined on the whole interval $[0,a(\lambda,r)]$. Let $\varphi \in C^\infty([0,a(\lambda,r)])$ be such solution and notice that it satisfies

$$\|\varphi^{(k)}\|_\infty \leq \frac{1}{(2-k)!} M(r)\lambda^{2-k}(r-1)^{2-k}, \qquad k = 0,1,2.$$

These bounds can be improved. Indeed, by the definition of L^0 and L^1, one easily obtains that

$$\begin{cases} \left|L^0(x,y,z) - 1\right| \leq C\,[(r-1) + |x| + |y| + |z|] \\ \left|L^1(x,y,z)\right| \leq C\,[|x| + |z|] \end{cases} \tag{4.4}$$

whenever $(x,y,z) \in K_r$ for some positive C and hence

$$|F(x,y,z)| \leq 16C\,[(r-1)+|x|+|y|+|z|], \qquad (x,y,z) \in K_r.$$

Therefore, along the solution φ, we have

$$\sup\{|F(x,\varphi(x),\varphi'(x))| :\ 0 \leq x \leq \lambda(r-1)\} \leq M_0(\lambda,r)(r-1)$$

where $M_0(\lambda,r) \geq 1$ is a non decreasing function of its arguments. Thus, we also have

$$\|\varphi^{(k)}\|_\infty \leq \frac{1}{(2-k)!}M_0(\lambda,r)\lambda^{2-k}(r-1)^{3-k}, \qquad k=0,1,2. \tag{4.5}$$

We are left to prove that, for all sufficiently small $r-1>0$, we can choose $0<\lambda\leq\lambda_0(r)$ in such a way that the corresponding solution φ satisfies (P1), ... ,(P4). We split the remaining part of the proof into three steps.

Claim 1. For $r-1>0$ small enough, there exists $0<\lambda\leq\lambda_0(r)$ such that the corresponding solution φ of (4.3) satisfies $\varphi(0)<0$ and $\varphi'(0)=0$.

To prove this, we consider the asymptotic equation of $\varphi''(x) = F(x,\varphi(x),\varphi'(x))$ as $r \to 1_+$. Indeed, on account of (4.5), the Taylor's polynomial of F centered at $(0,0,0)$ and arrested at those terms which, computed along the solution φ of (4.3), approach zero not faster than $(r-1)$ as $r\to 1_+$, is given by

$$\widetilde{F}(x,y,z) = 2\frac{r-1}{r^2} - 2\frac{2-r}{r^3}x, \qquad (x,y,z)\in\mathbb{R}^3.$$

By (4.2), the associated remainder $R = F - \widetilde{F}$ satisfies

$$|R(x,y,z)| \leq M(r)\left(|x|^2+|y|+|z|\right), \qquad (x,y,z)\in K_r. \tag{4.6}$$

Now, for $r>1$ and $0<\lambda\leq\lambda_0(r)$, consider the asymptotic Cauchy problem

$$\begin{cases} \widetilde{\varphi}''(x) = \widetilde{F}(x,\widetilde{\varphi}(x),\widetilde{\varphi}'(x)) & 0\leq x\leq a(\lambda,r) \\ \widetilde{\varphi}(a(\lambda,r)) = 0, & \\ \widetilde{\varphi}'(a(\lambda,r)) = 0 & \end{cases}$$

whose unique solution is

$$\widetilde{\varphi}(x) = \frac{(r-1)}{r^2}(a-x)^2 + \frac{2-r}{3r^3}(a^3-x^3) - \frac{2-r}{r^3}a^2(a-x), \qquad 0\leq x\leq a,$$

where $a = a(\lambda,r)$. It satisfies

$$\begin{cases} \widetilde{\varphi}(0) = -\dfrac{\lambda^2(r-1)^3}{3r^3}\,[2\lambda(2-r)-3r], \\[2ex] \widetilde{\varphi}'(0) = \dfrac{\lambda(r-1)^2}{r^3}\,[\lambda(2-r)-2r], \end{cases} \tag{4.7}$$

and, for $1 < r < 2$, the expressions appearing at the right hand side of the above equalities change sign at the values $3r[2(2-r)]^{-1}$ and $2r(2-r)^{-1}$ respectively. Moreover, we have $3r[2(2-r)]^{-1} < 2r(2-r)^{-1}$ for $1 < r < 2$ and $3r[2(2-r)]^{-1} \to 3/2$ and $2r(2-r)^{-1} \to 2$ as $r \to 1_+$. Therefore, noticing that

$$\frac{\delta_0(r)}{r-1} \to \infty$$

as $r \to 1_+$ and recalling the definition of $\lambda_0(r)$, we see that $0 < 3r[2(2-r)]^{-1} < 2r(2-r)^{-1} < \lambda_0(r)$ for all $1 < r < 2$ sufficiently close to 1. For such r, we have

$$(4.8) \quad \begin{cases} \dfrac{3r}{2(2-r)} < \lambda < \dfrac{2r}{2-r} & \Longrightarrow \quad \widetilde{\varphi}(0) < 0, \quad \widetilde{\varphi}'(0) < 0; \\ \dfrac{2r}{2-r} < \lambda < \lambda_0(r) & \Longrightarrow \quad \widetilde{\varphi}(0) < 0, \quad \widetilde{\varphi}'(0) > 0. \end{cases}$$

Then, recalling (4.6), we obtain

$$|\varphi''(x) - \widetilde{\varphi}''(x)| \leq M(r)\left(|x|^2 + |\varphi(x)| + |\varphi'(x)|\right), \qquad 0 \leq x \leq a,$$

and hence, taking into account also (4.5), we conclude that

$$\|\varphi^{(k)} - \widetilde{\varphi}^{(k)}\|_\infty \leq \frac{1}{(2-k)!} M_1(\lambda, r)(r-1)^{4-k}, \qquad k = 0, 1.$$

where $M_1(\lambda, r) \geq 1$ is, as usual, a non decreasing function of λ and r. By comparing (4.7) and (4.8) with the previous estimate, we see that, for all $1 < r < 2$ sufficiently close to 1, there are values of λ in the interval $(0, \lambda_0(r)]$ such that the true solution φ and its derivative φ' computed at $x = 0$ have the same sign as the asymptotic solution $\widetilde{\varphi}(0)$ and its derivative $\widetilde{\varphi}'(0)$ respectively. As the values $\varphi(0)$ and $\varphi'(0)$ depend continuously on λ, the conclusion follows.

From now on, for all $1 < r < 2$ sufficiently close to 1, let $\lambda = \lambda(r)$ be the value whose existence was established in Claim 1 and let φ be the corresponding solution of (4.3). For such r, set also $a = a(r) = \lambda(r)(r-1)$ and notice that $0 < r - a(r) < 1$, since $\lambda(r) > 3r[2(2-r)]^{-1} > 1$, and that $\lambda(r) \to 2$ as $r \to 1_+$.

Claim 2. For $r - 1 > 0$ small enough, the solution φ of (4.3) satisfies (P1).

We prove this by showing that the third derivative of φ is negative on the interval $[0, a(r)]$ for all $1 < r < 2$ sufficiently close to 1. Indeed, the third derivative $\varphi^{(3)}$ of φ is given by

$$-\frac{2\sqrt{1+(\varphi'(x))^2}}{[1-\varphi'(x)]\, D(x, \varphi(x), \varphi'(x))} + \frac{\sqrt{1+(\varphi'(x))^2}}{2-\sqrt{1+(\varphi'(x))^2}} \frac{F(x, \varphi(x), \varphi'(x))}{D(x, \varphi(x), \varphi'(x))} + \\ + F_y(x, \varphi(x), \varphi'(x))\varphi'(x) + F_z(x, \varphi(x), \varphi'(x))\varphi''(x)$$

for all $0 \le x \le a(r)$ and hence, taking into account (4.2), (4.4) and (4.5), we see that $\varphi^{(3)}$ converges to -2 uniformly on the interval $[0, a(r)]$ as $r \to 1_+$. Therefore, the second derivative of φ is decreasing on the whole interval $[0, a(r)]$ provided $1 < r < 2$ is sufficiently close to 1 and

$$\varphi''(a(r)) = -\frac{2(r-1)}{[r-a(r)]^2}(\lambda - 1) < 0$$

since $\lambda = \lambda(r) > 3r[2(2-r)]^{-1} > 1$ for the same values of r. By Claim 1, it follows that φ' is positive on the open interval $(0, a)$ and, by (4.5), its maximum value is less than 1 provided $1 < r < 2$ is sufficiently close to 1. Thus, all the properties of (P1) are satisfied by φ.

Claim 3. For $r - 1 > 0$ small enough, the solution φ of (4.3) satisfies (P2), (P3) and (P4).

Consider the interval of those x where the radius of curvature

$$R_\varphi(x) = \frac{\left[1 + (\varphi'(x))^2\right]^{3/2}}{\varphi''(x)}, \qquad \varphi''(x) > 0,$$

of Γ_φ at $\Phi(x)$ is positive. By (4.5), for all such x, we have $R_\varphi(x) > l_\varphi^0(x)$ provided $r - 1 > 0$ is small enough. Thus, (P2) holds true for all such r.

Then, consider (P4). For $i = 1, 2$, the derivatives $(\gamma_\varphi^i)_2'$ of the second components of the functions γ_φ^i are given by

$$\varphi'(x)\left\{\left[1 - \frac{2}{2 - \sqrt{1 + (\varphi'(x))^2}}\right] - \frac{\varphi''(x)}{[1 + (\varphi'(x))^2]^{3/2}\left(2 - \sqrt{1 + (\varphi'(x))^2}\right)^2} \cdot \right.$$
$$\left. \cdot \left[2 + \left(2 - \sqrt{1 + (\varphi'(x))^2}\right)^2 [l_\varphi^1(x) + (i-1)] - 4\sqrt{1 + (\varphi'(x))^2}\,[1 - \varphi'(x)]\right]\right\}$$

for $0 \le x \le a(r)$ and $i = 1, 2$. Relying on (4.4) and (4.5) again, it is easy to check that the two terms appearing between curly brackets in the expression above converge uniformly on the interval $[0, a(r)]$ to -1 and 0 respectively when $r \to 1_+$. Hence, their sum is negative on $[0, a(r)]$ for $r - 1 > 0$ small enough whereas φ' is positive on $(0, a(r))$ by Claim 2. Thus, (P4) holds true provided $r - 1 > 0$ is small enough.

Finally, a completely analogous and even simpler argument shows that (P3) holds true as well for sufficiently small $r - 1 > 0$. This concludes the proof. $\square$

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Received July 3, 1997

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