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SHORT-TIME ASYMPTOTICS OF THE HEAT KERNEL OF THE LAPLACIAN OF A BOUNDED DOMAIN WITH ROBIN BOUNDARY CONDITIONS

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ABSTRACT. The basic problem in this paper is that of determining some geometrical properties of a general bounded domain in two or three dimensions with a smooth boundary where smooth functions are entering the boundary conditions which are not strictly positive, from complete knowledge of the eigenvalues for the negative Laplacian, using the asymptotic expansions of the trace of the heat kernel for short-time t. Further results are obtained.

1. INTRODUCTION

The underlying inverse problems are to deduce the precise shape of membranes from complete knowledge of the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq .... \leq \lambda_j \leq .... \to \infty \text{ as } j \to \infty, \tag{1.1}$$

for the negative Laplacian $-\triangle_n = -\sum_{k=1}^{n}(\frac{\partial}{\partial x^k})^2$ in R^n , n=2 or n=3

Let Ω be a given arbitrary simply connected bounded domain in R^n, (n = 2 or 3) with a smooth boundary $\partial\Omega$. Suppose that the eigenvalues (1.1) are given for the Helmholtz equation ($\triangle_n + \lambda$) u = 0 in Ω together with the following Robin boundary condition:

(B1): $(\frac{\partial}{\partial n} + \gamma)u = 0$ on $\partial\Omega$, where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$ and the impedance γ is assumed to be a smooth function which is not strictly positive. The object of this paper is to determine some geometrical quantities of the domain Ω, where the boundary condition (B1) is considered, from the asymptotic expansion of the trace of the heat kernel

$$\theta(t) = \sum_{j=1}^{\infty} exp(-t\lambda_j) \text{ as } t \to 0 \tag{1.2}$$

Note that the boundary condition (B1) has been investigated by Sleeman and Zayed [7] when $\Omega \subseteq R^2$ and by Zayed [9] when $\Omega \subseteq R^3$, in the case γ is a positive constant. Let us now mention some previously known results. If $\Omega \subseteq R^2$, then the Neumann problem (i.e.,$\gamma = 0$) has been investigated by many authors (see, Gottlieb [1], Pleijel [5], Hsu [2], McKean and Singer [4] and Zayed [11]) and have shown that

$$\theta_N(t) = \frac{|\,\Omega\,|}{4\pi t} + \frac{|\,\partial\Omega\,|}{8(\pi t)^1/2} + a_o + \frac{7}{256}(\frac{t}{\pi})^{1/2}\int_{\partial\Omega} K^2(z)dz + O(t) \tag{1.3}$$

as $t \to 0$. where $|\,\Omega\,|$ is the area of Ω , $|\,\partial\Omega\,|$ is the total length of the boundary $\partial\Omega$ and K(z) is the curvature $\partial\Omega$ at the point z. The constant term a_o has geometric significance, e.g., if the region Ω is smooth and convex, then $a_o = 1/6$ and if Ω is permitted to have a finite number "h" of smooth convex holes,then $a_o = (1-h)/6$.

Furthermore, if $\Omega \subseteq R^3$, then the Neumann problem has been discussed by many authors (see, Gottlieb [1], Pleijel[6], Hsu [2], McKean and Singer [4] and Zayed [13]) and have shown that

$$\theta_N(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|\,S\,|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\int_{\partial\Omega} H(z)dz + \frac{7}{128\pi}\int_{\partial\Omega}[H^2(z) - N(z)]dz$$

$$+O(t^{1/2}) \text{ as } t \to 0 \tag{1.4}$$

where V is the volume of Ω,$|\,S\,|$ is the surface area of the boundary $\partial\Omega$, while

$$H(z) = \frac{1}{2}[\frac{1}{R_1(z)} + \frac{1}{R_2(z)}]$$

and

$$N(z) = \frac{1}{R_1(z)R_2(z)}$$

are respectively the mean curvature and the Gaussian curvature of the boundary surface $\partial\Omega$ at the point z, in which R_1 and R_2 are the principal radii of curvature.

2. Statement of results

Theorem 2.1. *Suppose that the boundary $\partial\Omega$ of the region $\Omega \subseteq R^2$ is given locally by the equations $x^n = y^n(z)$ $(n = 1,2)$ in which z is the arc length of the counterclockwise-oriented boundary $\partial\Omega$ and $y^n(z) \in C^\infty(\partial\Omega)$. Then the asymptotic expansion of $\theta(t)$ with the Robin boundary condition (B1) has the form*

$$\theta(t) = \frac{|\,\Omega\,|}{4\pi t} + \frac{|\,\partial\Omega\,|}{8(\pi t)^{1/2}} + \frac{1}{6}[1 - \frac{3}{\pi}\int_{\partial\Omega}\gamma(z)dz] + \frac{7}{256}(\frac{t}{\pi})^{1/2}\int_{\partial\Omega}[K^2(z)-$$

$$\frac{32}{7}\left(\gamma(z)K(z)-2\gamma^2(z)\right)]dz+O(t) \tag{2.1}$$

as $t \to 0$.

Theorem 2.2. *Suppose that the boundary* $\partial\Omega$ *of the region* $\Omega \subseteq R^3$ *is given locally by infinitely differentiable functions*

$$x^n = y^n(z)\ ,(n=1,2,3)$$

of the parameters z^1, z^2. *If these parameters are chosen so that* $z^i = constant$, $i = 1, 2$ *are lines of curvature, the first and second fundamental forms of* $\partial\Omega$ *can be written in the form*

$$II_1(z,\triangle z) = \sum_{i=1}^{2} g_{ii}(z)(\triangle z^i)^2,$$

and

$$II_2(z,\triangle z) = \sum_{i=1}^{2} d_{ii}(z)(\triangle z^i)^2$$

In terms of the coefficients $g_{ii}, d_{ii}(i=1,2)$ *the principal radii of curvature are*

$$R_1(z) = g_{11}(z)/d_{11}(z), R_2(z) = g_{22}(z)/d_{22}(z)$$

Then the asymptotic expansion of $\theta(t)$ *with the Robin boundary condition (B1) has the form*

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\int_{\partial\Omega}[H(z)-3\gamma(z)]dz$$

(2.2)

$$+\frac{7}{128\pi}\int_{\partial\Omega}\left([H(z)-3\gamma(z)]^2-[N(z)-\frac{26}{7}\gamma(z)H(z)+\frac{47}{7}\gamma^2(z)]\right)dz+O(t^{1/2})$$

as $t \to 0$.

3. Construction of Results

Following the method of Kac [3] and Hsu [2], it is easily seen that $\theta(t)$ is given by the formula

$$\theta(t) = \int_{\Omega} G(t,x,x)dx \tag{3.1}$$

where the heat kernel G(t,x,y) is defined on$(0,\infty)X\overline{\Omega}X\overline{\Omega}$ which satisfies the following: For fixed $x \in \overline{\Omega}$,it satisfies the heat equation in t, y

$$\frac{\partial}{\partial t}G(t,x,y) = \triangle_{ny}G(t,x,y), \text{ (n=2 or 3)} \tag{3.2}$$

and the Robin boundary condition

$$[\frac{\partial}{\partial n_y} + \gamma(y)]G(t,x,y) = 0, \text{ on } \partial\Omega \tag{3.3}$$

and the initial condition

$$\lim_{t\to 0} G(t,x,y) = \delta(x-y), \tag{3.4}$$

where $\delta(x-y)$ is the Dirac delta function located at the source point x = y. Note that in (3.2) - (3.3) the subscript "y" means that the derivatives are taken in y-variables. Thus by the superposition principle of the heat equation, we write

$$G(t,x,y) = G_N(t,x,y) + \chi(t,x,y), \tag{3.5}$$

where $G_N(t,x,y)$ is the Neumann heat kernel on Ω which satisfies the heat equation

$$\frac{\partial}{\partial t}G_N(t,x,y) = \triangle_{ny}G_N(t,x,y), \text{(n = 2 or 3)} \tag{3.6}$$

and the Neumann boundary condition

$$\frac{\partial}{\partial n_y}G_N(t,x,y) = 0 \text{ on } \partial\Omega \tag{3.7}$$

and the initial condition

$$\lim_{t\to 0} G_N(t,x,y) = \delta(x-y), \tag{3.8}$$

while $\chi(t,x,y)$ satisfies the heat equation

$$\frac{\partial}{\partial t}\chi(t,x,y) = \triangle_{ny}\chi(t,x,y), (n = 2or3), \tag{3.9}$$

and the boundary condition

$$\frac{\partial}{\partial n_y}\chi(t,x,y) = -\gamma(y)G(t,x,y) \tag{3.10}$$

and the initial condition

$$\lim_{t\to 0}\chi(t,x,y) = 0 \tag{3.11}$$

Now, the solution of the problem (3.9)-(3.11) has the form

$$\chi(t,x,y) = -\int_0^t ds \int_{\partial\Omega} G_N(t-s,x,z)\gamma(z)G(s,z,y)dz \tag{3.12}$$

From (3.5) and (3.12) we have the following integral equation.

$$G(t,x,y) = G_N(t,x,y) - \int_0^t ds \int_{\partial\Omega} G_N(t-s,x,z)\gamma(z)G(s,z,y)dz \tag{3.13}$$

On applying the iteration method (see [10-12]) to the integral equation (3.13) we obtain the series

$$G(t,x,y) = \sum_{m=0}^{\infty}(-1)^m F_m(t,x,y) \tag{3.14}$$

where

$$F_o(t,x,y) = G_N(t,x,y), \tag{3.15}$$

and

(3.16)

$$F_m(t,x,y) = \int_0^t ds \int_{\partial\Omega} G_N(t-s,x,z)\gamma(z)F_{m-1}(s,z,y)dz \ , \ m=1,2,...$$

Consequently, we can get :

(3.17)

$$\theta(t) = \begin{cases} \theta_N(t) - \int_\Omega F_1(t,x,x)dx + \int_\Omega F_2(t,x,x)dx + O(t), & \text{if } \Omega \subseteq R^2, \\ \theta_N(t) - \int_\Omega F_1(t,x,x)dx + \int_\Omega F_2(t,x,x)dx + O(t^{1/2}), & \text{if } \Omega \subseteq R^3, \end{cases}$$

where $\theta_N(t) = \int_\Omega G_N(t,x,x)dx$, which has the same asymptotic expansions (1.3) if $\Omega \subseteq R^2$ and (1.4) if $\Omega \subseteq R^3$

The problem now is to study the integrals of $F_i(t,x,x)(i=1,2)$ over the region $\Omega \subseteq R^n(n = 2or3)$.

Lemma 3.1. *If $\Omega \subseteq R^2$, then in the case (B1) we deduce as $t \to 0$ that*

$$\int_\Omega F_1(t,x,x)dx = \frac{1}{2\pi}\int_{\partial\Omega}\gamma(z)dz + \frac{1}{8}(\frac{t}{\pi})^{1/2}\int_{\partial\Omega}\gamma(z)K(z)dz + O(t) \tag{3.18}$$

Further, if $\Omega \subseteq R^3$, then in the case (B1) we deduce as $t \to 0$ that

$$\int_\Omega F_1(t,x,x)dx = \frac{1}{4\pi^{3/2}t^{1/2}}\int_{\partial\Omega}\gamma(z)dz + \frac{1}{8\pi}\int_{\partial\Omega}\gamma(z)H(z)dz + O(t^{1/2}) \tag{3.19}$$

PROOF. The definition of $F_1(t,x,x)$ and the Chapman-Kolmogorov equation of the heat kernel imply that

$$\int_\Omega F_1(t,x,x)dx = t\int_{\partial\Omega} G_N(t,z,z)\gamma(z)dz. \tag{3.20}$$

Let us now introduce the following well known estimates (see [2]) of the Neumann heat kernel:

$$G_N(t,z,z) = \frac{2}{(4\pi t)^{n/2}}[1+\alpha(z)t^{1/2}] + O(t^{1-n/2}) \text{ as } t\to 0, \tag{3.21}$$

where

$$\alpha(z) = \begin{cases} \frac{1}{4}\pi^{1/2}K(z), & \text{if } \Omega\subseteq R^2, \\ \frac{1}{2}\pi^{1/2}H(z), & \text{if } \Omega\subseteq R^3 \end{cases} \tag{3.22}$$

On inserting (3.21) and (3.22) into (3.20), we arrive at the proof of lemma 3.1 □

Lemma 3.2. *If $\Omega\subseteq R^2$, then in the case (B1) we deduce as $t\to 0$ that*

$$\int_\Omega F_2(t,x,x)dx = \frac{1}{4}(\frac{t}{\pi})^{1/2}\int_{\partial\Omega}\gamma^2(z)dz + O(t) \tag{3.23}$$

Further, if $\Omega\subseteq R^3$, then in the case (B1) we deduce as $t\to 0$ that

$$\int_\Omega F_2(t,x,x)dx = \frac{1}{8\pi}\int_{\partial\Omega}\gamma^2(z)dz + O(t^{1/2}) \tag{3.24}$$

PROOF. From the definition of $F_2(t,x,x)$ and with the help of the expression of $F_1(t,x,x)$, we deduce that

(3.25)

$$\int_\Omega F_2(t,x,x)dx = \int_0^t (t-u)du\int_{\partial\Omega}\gamma(z)dz\int_{\partial\Omega} G_N(t-u,z,y)\gamma(y)G_N(u,y,z)dy$$

we replace $\gamma(y)$ in (3.25) as follows

$$\gamma(y) = \gamma(z) + O(\mid y-z\mid)$$

With the help of the following estimate for the Neumann heat kernel: There exist positive constants t_o, c_1 such that for all $t<t_o, (x,y)\in\overline{\Omega}X\overline{\Omega}$,

$$G_N(t,x,y) \le c_1 t^{-n/2} exp(-\frac{(\mid x-y\mid)^2}{c_1 t}), (n=2or3), \tag{3.26}$$

we deduce that

$$\begin{aligned}\int_{\partial\Omega} \mid z-y \mid G_N(t-u,y,z)G_N(u,z,y)dy \leq \\ c_1[u(t-u)]^{-n/2}\int_{R^{n-1}} \mid y \mid exp\left(-\frac{c_2|y|^2 t}{u(t-u)}\right)dy.\end{aligned} \tag{3.27}$$

Since the integral in the right-hand side of (3.27) is bounded by $c_3 t^{-n/2}$, where c_2 and c_3 are positive constants, we deduce as $t \to 0$ that

$$\int_\Omega F_2(t,x,x)dx = \int_{\partial\Omega} \gamma^2(z)g(t,z)dz + O(t^{(4-n)/2}), \tag{3.28}$$

where

$$g(t,z) = \int_0^t (t-u)du \int_{\partial\Omega} G_N(t-u,y,z)G_N(u,z,y)dy. \tag{3.29}$$

The right-hand side of (3.29) can be computed by taking the first term in the series expansion of the Neumann heat kernels $G_N(t-u,y,z) = 2q(t-u,y,z)$ and $G_N(u,z,y) = 2q(u,z,y)$ where

$$q(t,y,z) = \frac{1}{(4\pi t)^{n/2}} exp\left(-\frac{(\mid y-z \mid)^2}{4t}\right). \tag{3.30}$$

The explicit computation can be carried out with the help of a suitably chosen local coordinate system and the localization principle (see [2]). We leave the details of this computation to the interested reader and we content ourselves with the statement that the leading term of g(t,z) is equal to the same integral in the Euclidean space. Thus, we deduce that

$$\begin{aligned}g(t,z) = \frac{4}{(4\pi)^n}\int_0^t \frac{(t-u)du}{[u(t-u)]^{n/2}}\int_{R^{n-1}} exp\left(-\frac{\mid z-y \mid^2}{4(t-u)} - \frac{\mid z-y \mid^2}{4u}\right)dy \\ +O(t^{(4-n)/2}).\end{aligned} \tag{3.31}$$

After some reduction , we obtain

$$\begin{aligned}g(t,z) &= \frac{2}{(4\pi t)^{n/2}}(\frac{t}{\pi})^{1/2}\int_0^t (\frac{t-u}{u})^{1/2}du + O(t^{(4-n)/2}) \\ &= 2^{-n}\pi^{(1-n)/2}t^{(3-n)/2} + O(t^{(4-n)/2}).\end{aligned} \tag{3.32}$$

Consequently we write

$$g(t,z) = \begin{cases} (\frac{1}{4})(\frac{t}{\pi})^{1/2} + O(t), & \text{if } \Omega \subseteq R^2, \\ \frac{1}{8\pi} + O(t^{1/2}), & \text{if } \Omega \subseteq R^3 \end{cases} \tag{3.33}$$

On inserting (3.33) into (3.28) we arrive at the proof of lemma 3.2 □

Now, our results (2.1) - (2.2) follow immediately from the results of Lemmas 3.1 and 3.2 and the formulas (1.3), (1.4), (3.17) respectively.

4. Further results

If the boundary $\partial\Omega$ of the region $\Omega \subseteq R^n$ (n = 2 or 3) consists of a finite number of parts $\partial\Omega_i$ (i= 1, ..., m) such that $\partial\Omega = \cup_{i=1}^{m}\partial\Omega_i$ and if we consider the Helmholtz equation $(\Delta_n+\lambda)u = 0$ in Ω together with the impedance boundary conditions $(\frac{\partial}{\partial n_i} + \gamma_i)u = 0$ on $\partial\Omega_i$ (i=1,...,m) where the impedances γ_i (i = 1, ..., m) are assumed to be smooth functions defined on $\partial\Omega_i$ (i= 1, ..., m) which are not strictly positive, then on using arguments similar to those used in Sec. 3, we deduce if $\Omega \subseteq R^2$, that

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{\sum_{i=1}^{m} L_i}{8(\pi t)^{1/2}} + \frac{1}{6}\left[1 - \frac{3}{\pi}\sum_{i=1}^{m}\int_{\partial\Omega_i}\gamma_i(z)dz\right]$$
$$+\frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2}\sum_{i=1}^{m}\int_{\partial\Omega_i}\left[K^2(z) - \frac{32}{7}\left(\gamma_i(z)K(z) - 2\gamma_i^2(z)\right)\right]dz + O(t) \text{ as } t \to 0, \tag{4.1}$$

while, if $\Omega \subseteq R^3$, we deduce that

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{\sum_{i=1}^{m} | S_i |}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\sum_{i=1}^{m}\int_{\partial\Omega_i}[H(z) - 3\gamma_i(z)]dz$$
$$+\frac{7}{128\pi}\sum_{i=1}^{m}\int_{\partial\Omega_i}\left([H(z) - 3\gamma_i(z)]^2 - [N(z) - \frac{26}{7}\gamma_i(z)H(z) + \frac{47}{7}\gamma_i^2(z)]\right)dz$$
$$+O(t^{1/2}) \text{ as } t \to 0 \tag{4.2}$$

In the formula (4.1), L_i (i = 1,...,m) are the total lengths of the parts $\partial\Omega_i$ (i = 1, ..., m) respectively, while in the formula (4.2), $| S_i |$ (i = 1, ...,m) are the surface areas of the parts $\partial\Omega_i$. (i = 1, ..., m) respectively.

Note that the first three terms of the formula (4.1) have been constructed by Zayed [10] when γ_i (i=1,2) are positive constants and by Zayed and Younis [8] when γ_i (i=1,...,m) are positive constants, while the formula (4.2) has been constructed by Zayed [12] when γ_i (i=1,2) are positive constants and by Zayed [14] when γ_i (1= 1,...,m) are positive constants.

We close this section with the remark that the case n = 1 is easy and well known which has been discussed by many authors, and have shown that the length of a uniform vibrating string and the unknown boundary conditions can be found from complete knowledge of its eigenvalues, while the case $n > 3$ is more difficult and has been discussed in a little bite by few authors. Thus, the case $n > 3$ is still an open problem for the interested readers which can be discussed in a future work.

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