AN ANALOGUE OF BERNSTEIN'S THEOREM

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ABSTRACT. We prove that for a function $f(x_1, x_2)$ defined on $\mathbb{R}^2$, the graph of $\nabla f$ is a minimal surface if and only if $f$ is harmonic or a quadratic polynomial. Using this result we prove the following classical result of Jögens: if $f$ satisfies the Monge-Ampère equation $f_{x_1}x_1 f_{x_2}x_2 - f_{x_1}x_2 = 1$, then $f$ must be a quadratic polynomial.

1. MAIN RESULTS

The classical Bernstein theorem says that if $f(x_1, x_2)$ is a function defined on $\mathbb{R}^2$ satisfying the minimal surface equation, then $f$ is a linear function. In this paper we prove the following analogue of Bernstein's theorem:

Theorem 1.1. Let $f(x_1, x_2)$ be a function defined on $\mathbb{R}^2$. Then the graph of $\nabla f$ is a minimal surface in $\mathbb{R}^2 \times \mathbb{R}^2$ if and only if $f$ is harmonic or a quadratic polynomial.

Recall the following result of Harvey and Lawson ([1]):

Proposition 1.2. Let $f(x_1, \ldots, x_n)$ be a real valued function defined in a connected open subset in $\mathbb{R}^n$. The graph of $\nabla f$ is a minimal submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ if and only if there exists a constant $\theta$ such that $f$ satisfies

$$\text{Im} \ det(e^{i\theta}(I + i\text{Hess}(f))) = 0,$$

where $\text{Im}$ denotes the imaginary part, $I$ is the identity matrix, and $\text{Hess}(f)$ is the hessian matrix of $f$. In particular if $f(x_1, x_2)$ is a function defined on $\mathbb{R}^2$, then the graph of $\nabla f$ is a minimal surface in $\mathbb{R}^2 \times \mathbb{R}^2$ if and only if

$$\sin \theta(1 - f_{x_1}x_1 f_{x_2}x_2 + f_{x_2}^2) + \cos \theta(f_{x_1}x_1 + f_{x_2}x_2) = 0.$$
This proposition follows from Lemma III 2.2, Theorem III 2.3 and Proposition III 2.17 in [1].

By Proposition 1.2, Theorem 1.1 is equivalent to the following theorem:

**Theorem 1.3.** Let \( f(x_1, x_2) \) be a function defined on \( \mathbb{R}^2 \). Assume \( f \) satisfies
\[
\sin \theta (1 - f_{x_1x_1} f_{x_2x_2} + f_{x_1x_2}^2) + \cos \theta (f_{x_1x_1} + f_{x_2x_2}) = 0
\]
for some constant \( \theta \). If \( \sin \theta \neq 0 \), then \( f \) must be a quadratic polynomial; if \( \sin \theta = 0 \), then \( f \) is harmonic.

As a corollary we obtain the following result of Jörgens ([2]):

**Corollary 1.4.** Let \( f(x_1, x_2) \) be a function defined on \( \mathbb{R}^2 \) satisfying the Monge-Ampère equation
\[
f_{x_1x_1} f_{x_2x_2} - f_{x_1x_2}^2 = 1.
\]
Then \( f \) must be a quadratic polynomial.

I don’t know whether we have a similar classification theorem in higher dimensional cases. Let’s ask the following:

**Question.** Let \( f(x_1, \ldots, x_n) \) be a convex function defined on \( \mathbb{R}^n \). Assume the graph of \( \nabla f \) is a minimal submanifold in \( \mathbb{R}^n \times \mathbb{R}^n \), that is, there exists a constant \( \theta \) such that
\[
\text{Im det}(e^{i\theta}(I + i\text{Hess}(f))) = 0.
\]
Is it true that \( f \) must be a quadratic polynomial.

We have the following results of Calabi and Flanders:

Let \( f(x_1, \ldots, x_n) \) be a function defined on \( \mathbb{R}^n \). Assume its hessian matrix \( \text{Hess}(f) \) is positive definite. Then \( f \) is a quadratic polynomial if one of the following conditions holds:

1. \( \det \text{Hess}(f) = 1 \) and \( 1 \leq n \leq 5 \) ([3]).
2. \( \text{tr} (I + \text{Hess}(f))^{-1} = \text{constant} \) ([4]).
3. \( \text{tr} (\text{Hess}(f))^{-1} = \text{constant} \) ([4]).

Using these results we can solve some very special cases of our question.

## 2. Proof of Theorem 1.1

We start with some lemmas:

**Lemma 2.1.** Let \( f(x_1, x_2) = (f_3(x_1, x_2), \ldots, f_n(x_1, x_2)) \) be a function defined on \( \mathbb{R}^2 \). If the graph of \( f \) is a minimal surface in \( \mathbb{R}^n \), then there exists a linear transformation
\[
x_1 = u_1, x_2 = au_1 + bu_2, \quad (b \neq 0)
\]
such that \((u_1, u_2)\) are global isothermal parameters for the graph of \(f\).

This lemma is Theorem 5.1 of [5].

We also need the following obvious fact:

**Lemma 2.2.** Two different conics in the \(x_1x_2\)-plane have at most four common points.

Now let’s prove Theorem 1.3, which is equivalent to Theorem 1.1. The second part of Theorem 1.3 is obvious. So let’s assume \(\sin \theta \neq 0\) and prove that \(f\) is a quadratic polynomial. It is enough to show that \(f_{x_1x_1}, f_{x_1x_2}, f_{x_2x_2}\) are constants.

By assumption the graph of \(\nabla f\) is minimal. By Lemma 2.1, there exists constants \(a\) and \(b \neq 0\) such that

\[
(u_1, u_2) \mapsto (u_1, a u_1 + bu_2, f_{x_1}, f_{x_2})
\]

is an isothermal parametrization.

We have

\[
\frac{\partial}{\partial u_1} (u_1, a u_1 + bu_2, f_{x_1}, f_{x_2}) = (1, a, f_{x_1x_1} + af_{x_1x_2}, f_{x_1x_2} + af_{x_2x_2}),
\]

\[
\frac{\partial}{\partial u_2} (u_1, a u_1 + bu_2, f_{x_1}, f_{x_2}) = (0, b, bf_{x_1x_2}, bf_{x_2x_2}).
\]

That \((u_1, u_2)\) are isothermal parameters is equivalent to that we have

\[
1 + a^2 + (f_{x_1x_1} + af_{x_1x_2})^2 + (f_{x_1x_2} + af_{x_2x_2})^2 = b^2 + b^2 f_{x_1x_2}^2 + b^2 f_{x_2x_2}^2,
\]

\[
ab + (f_{x_1x_1} + af_{x_1x_2}) bf_{x_1x_2} + (f_{x_1x_2} + af_{x_2x_2}) bf_{x_2x_2} = 0.
\]

For convenience we let \(X = f_{x_1x_1}, Y = f_{x_1x_2}\) and \(Z = f_{x_2x_2}\). The above equations can be rewritten as

\[
1 + a^2 + (X + aY)^2 + (Y + aZ)^2 = b^2 + b^2 Y^2 + b^2 Z^2,
\]

\[
ab + (X + aY) bY + (Y + aZ) bZ = 0.
\]

Simplifying these equations we get

\[
(1 + a^2 - b^2) + (1 + a^2 - b^2) Y^2 + (1 + a^2 - b^2) Z^2 + (X^2 - Z^2) + 2aY(X + Z) = 0,
\]

\[
ab(1 + Y^2 + Z^2) + bY(X + Z) = 0.
\]

So we have

\[
(1 + a^2 - b^2)(1 + Y^2 + Z^2) + (X - Z + 2aY)(X + Z) = 0
\]

\[
a(1 + Y^2 + Z^2) + Y(X + Z) = 0
\]
where in the second equation we omit the factor $b$ since $b \neq 0$. From these equations and the fact that $1 + Y^2 + Z^2 \neq 0$, we get

$$\det \begin{pmatrix} 1 + a^2 - b^2 & X & Z + 2aY \\ a & Y \\ \end{pmatrix} = 0,$$

that is

$$(3) \quad -aX + (1 - a^2 - b^2)Y + aZ = 0$$

By assumption there exists a constant $\theta$ such that

$$(4) \quad (1 - XZ + Y^2) \sin \theta + (X + Z) \cos \theta = 0$$

Note that $X + Z$ is everywhere nonzero. Indeed, if $X + Z = 0$ at some point, then $Z = -X$ at that point. Substituting this into (4) we get

$$(1 + X^2 + Y^2) \sin \theta = 0.$$ 

But this cannot happen since $1 + X^2 + Y^2 \neq 0$ and $\sin \theta \neq 0$.

We have the following two cases:

**Case A.** $a = 0$.

From equations (1) and (2), we get

$$(5) \quad (1 - b^2)(1 + Y^2 + Z^2) + (X - Z)(X + Z) = 0$$

$$(6) \quad Y(X + Z) = 0$$

Since $X + Z$ is everywhere nonzero, we get $Y = 0$ from (6). Substituting this into (5), we get

$$(7) \quad X^2 - b^2Z^2 + (1 - b^2) = 0$$

Substituting $Y = 0$ into (4) we get

$$(8) \quad (1 - XZ) \sin \theta + (X + Z) \cos \theta = 0$$

Equations (7) and (8) define two different conics. By Lemma 2.2 they have only finitely many common solutions. So there are only finitely many possible values for $X$, $Y$ and $Z$.

**Case B.** $a \neq 0$.

From (3) we get

$$X = \frac{1 - a^2 - b^2}{a} Y + Z.$$
Substituting this into (2) and (4) we get

\[ \frac{1 - b^2}{a} Y^2 + aZ^2 + 2YZ + a = 0 \]  

\[ (1 + Y^2 - Z^2 - \frac{1 - a^2 - b^2}{a} YZ) \sin \theta + (\frac{1 - a^2 - b^2}{a} Y + 2Z) \cos \theta = 0 \]

Equation (9) and (10) define two different conics. By Lemma 2.2 they have only finitely many common solutions. So there are only finitely many possible values for \( X, Y \) and \( Z \).

In any case, there are only finitely many possible values for \( X, Y \) and \( Z \), that is, there are only finitely many possible values for \( f_{x_1x_1}, f_{x_1x_2} \) and \( f_{x_2x_2} \). Since the domain of \( f \) is connected, \( f_{x_1x_1}, f_{x_1x_2} \) and \( f_{x_2x_2} \) must be constants. Therefore \( f \) is a quadratic polynomial.

REFERENCES


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