BSE BANACH MODULES AND BUNDLES OF BANACH SPACES

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Communicated by Vern I. Paulsen

Abstract. In a recent paper, S.-E. Takahasi defined the notion of a BSE Banach module over a commutative Banach algebra $A$ with bounded approximate identity. We show that the multiplier space $M(X)$ of $X$ can be represented as a space of sections in a bundle of Banach spaces, and we use bundle techniques to obtain shorter proofs of various of Takahasi’s results on $C^*$-algebra modules and to answer several questions which he raised.

1. Introduction

In this paper, $A$ will denote a commutative Banach algebra with bounded approximate identity $\{u_j\}$. Denote by $\Delta = \Delta_A$ the space of multiplicative functionals on $A$, and, for $h \in \Delta$, let $K_h = \ker h \subset A$ be the corresponding maximal ideal. We give $\Delta$ its weak-* topology. Let $X$ be a Banach $A$-module, and, for $h \in \Delta$, let $K_h X$ be the closure in $X$ of $\operatorname{span}\{ax : a \in K_h, x \in X\}$. As usual, $C_0(\Delta)$ is the space of continuous complex-valued functions on $\Delta$ which vanish at infinity, and $\hat{\cdot} : A \to C_0(\Delta)$ is the Gelfand representation of $A$. If $A$ is a $C^*$-algebra, we will identify $A$ and $\hat{A}$. Following the notation of Takahasi [7], if $h \in \Delta$, we choose $e_h \in A$ such that $\hat{e}_h(h) = h(e_h) = 1$, and we let $X^h$ be the closure in $X$ of $K_h X + (1 - e_h)X$. Then $X^h$ is independent of the choice of $e_h$. We set $X_h = X/X^h$. If $A$ is a $C^*$-algebra, for each $h \in \Delta$ we will choose $\|e_h\| = 1$.

Let $\mathcal{E} = \bigcup \{X_h : h \in \Delta\}$ be the disjoint union of the $X_h$. (We can, if we like, identify $\mathcal{E}$ with $\bigcup \{\{h\} \times X_h : h \in \Delta\}$; the $h$'s are useful for bookkeeping purposes.) We give an element $x + X^h \in \mathcal{E}$ ($x \in X, h \in \Delta$) its coset.
norm $\|x + X^h\|$. Let $\pi: \mathcal{E} \to \Delta$ be the obvious projection map. Given this data, let $\mathcal{C}(\mathcal{E})$ denote the linear space of selections (= choice functions) $\sigma: \Delta \to \mathcal{E}$, and let $\mathcal{C}^b(\mathcal{E})$ denote the subspace of bounded selections. Since each $X^h$ is a Banach $A$-module, so is each $X_h$. In particular, for each $a \in A$ and $x \in X$ we have $ax - \hat{a}(h)x \in X^h$. (Proof: $ax - \hat{a}(h)x = (ax - ae_hx) + (ae_hx - a(h)e_hx) + (\hat{a}(h)e_hx - \hat{a}(h)x)$; the first two terms of the sum are in $KhX$, and the last term is in $(1 - e_h)X$.) Hence, $a(x + X^h) = \hat{a}(h)x + X^h$. It follows that $\mathcal{C}(\mathcal{E})$ and $\mathcal{C}^b(\mathcal{E})$ are both $A$- and $\hat{A}$-modules under the operation $(a \cdot \sigma)(h) = \hat{a}(h)\sigma(h) = (\hat{a} \cdot \sigma)(h)$.

Consider the space $M(X) = Hom_A(A, X)$ of multipliers of $X$. Again, following [7], for $T \in M(X)$, we define a selection $\widehat{T}: \Delta \to \mathcal{E}$ by $\widehat{T}(h) = T(e_h) + X^h \in X_h$, where $e_h(h) = 1$. We see that this definition is also independent of the choice of $e_h$. The map $\widehat{\cdot}: M(X) \to \mathcal{C}^b(\mathcal{E})$ is an $A$-module homomorphism of $M(X)$ into $\mathcal{C}(\mathcal{E})$, as noted in [7]. (In fact, $M(X)$ is an $A$- and $\hat{A}$-submodule of $\mathcal{C}^b(\mathcal{E})$.) We also note that $T(A) \subset X_e$, the essential part of $X$. Further, if $x \in X$, then the map $T_x: A \to X, a \mapsto ax$, is an element of $M(X)$. If $X$ is an $A$-module, then so is $X^*$, with the multiplication given by $\langle x, f \cdot a \rangle = \langle ax, f \rangle$ ($a \in A, x \in X, f \in X^*$).

We associate two topologies with the fibered space $\mathcal{E}$, and study the properties of the spaces of continuous and bounded selections from $\Delta$ to $\mathcal{E}$ under these topologies. In [7], Takahasi explores some of the consequences of endowing $\mathcal{E}$ with the quotient topology induced by the product topology on $\Delta \times X$ and the projection map $p : \Delta \times X \to \mathcal{E}, (h, x) \mapsto x + X^h = \overline{T_x}(h) = \overline{x}(h)$. We denote $\mathcal{E}$ with this topology by $\mathcal{E}_1$.

In this paper, we show that $\mathcal{E}$ can be endowed with a topology which makes $\pi: \mathcal{E} \to \Delta$ a bundle of Banach spaces; we denote $\mathcal{E}$ with this topology by $\mathcal{E}_2$, and call it the multiplier bundle of $X$. In particular, we show that if $A$ has a bounded approximate identity, then the bundle and quotient topologies coincide; i.e. $\mathcal{E}_1 = \mathcal{E}_2$. We then show that the multiplier bundles for $X$ and for $X_e$ are homeomorphic, and we use this result and the machinery of section spaces of Banach bundles to subsume several of the examples concerning $C^*$-algebra modules adduced in [7]. We also answer several questions posed in [7]. The reader may also wish to consult [8] for more recent developments and applications of this construction of a field of quotient modules.

2. The bundle topology

We refer the reader to [1], [2], or [4] for fundamental notions regarding bundles of Banach spaces and Banach modules, and we especially draw upon the following results, which are key to our investigations.
Proposition 2.1. ([2, Corollary 3.7]) Suppose that $U$ is a topological space, and that $\{Y_p : p \in U\}$ is a collection of Banach spaces. Let $E$ be the disjoint union of the $Y_p$, and let $\gamma : E \to U$ be the obvious projection. Suppose that $Y$ is a vector space of bounded selections $\sigma : U \to E$ such that 1) $E = \bigcup \{\sigma(u) : u \in U, \sigma \in Y\}$ ("$Y$ is a full space of selections"); and 2) for each $\sigma \in Y$, the map $u \mapsto \|\sigma(u)\|$ is upper semicontinuous. Then there is a unique topology on $E$ making $\gamma : E \to U$ into a full bundle of Banach spaces, and such that each $\sigma \in Y$ is continuous.

As a special case of the above, we obtain

Proposition 2.2. ([4, Proposition 1.3]) Let $U$ be a topological space, and let $\{X^p : p \in U\}$ be a collection of closed subspaces of the Banach space $X$. Let $E = \bigcup \{X/X^p : p \in U\}$ be the disjoint union of the quotient spaces $X/X^p$. Then $E$ can be topologized in such a way that 1) $\pi : E \to U$ is a bundle of Banach spaces; and 2) for each $x \in X$, the selection $\tilde{x} : U \to E, \tilde{x}(p) = x + X^p$, is a bounded section of the bundle $\pi : E \to U$, iff the function $p \mapsto \|\tilde{x}(p)\|$ is upper semicontinuous on $U$ for each $x \in X$.

Lemma 2.3. Let $h \in \Delta$, and let $C$ be the bound on the approximate identity for $A$. Then $\|h\| \geq \frac{1}{C}$. Moreover, we may choose our collection $\{e_h : h \in \Delta\}$ so that $\{\|e_h\| : h \in \Delta\}$ is bounded.

PROOF. For a given $\delta > 0$, we may choose $u = u_{j_0}$ in the approximate identity so that $\hat{u}(h) > 1 - \delta$. Hence, $\|h\| \geq \frac{1}{C} \hat{u}(h) > \frac{1 - \delta}{C}$, and since $\delta$ was arbitrary, it follows that $\|h\| \geq \frac{1}{C}$. Now, for a fixed but arbitrary $h$, if we choose $u$ in the approximate identity so that $\hat{u}(h) > 1 - \delta$ and if we set $e_h = \frac{1}{\hat{u}(h)}u$, we see that $\|e_h\| \leq \frac{C}{1 - \delta}$.

Corollary 2.4. For any $T \in M(X)$, the selection $\tilde{T} : \Delta \to E$ is bounded.

PROOF. For $h \in \Delta$, we have $\|\tilde{T}(h)\| = \|T(e_h) + X^h\| \leq \|T(e_h)\| \leq \|T\|\|e_h\|$; and the collection of $e_h$'s can be chosen to be bounded, by the preceding.

Proposition 2.5. Given the data above, let $T \in M(X)$. Then the map $h \mapsto \|\tilde{T}(h)\| = \|T(e_h) + X^h\|$ is upper semicontinuous on $\Delta$. 

PROOF. Suppose that \( \varepsilon > 0 \) is given, and that \( \| \widetilde{T}(h) \| < \varepsilon \). Choose \( a_i \in K_h, y_i \in X \ (i = 1, \ldots, n), z \in X \), such that

\[
\| \widetilde{T}(h) \| \leq \| T(e_h) + \sum a_i y_i + (1 - e_h)z \| < \varepsilon,
\]

and set

\[
\varepsilon' = \varepsilon - \| T(e_h) + \sum a_i y_i + (1 - e_h)z \|.\]

From the upper semicontinuity of the map \( h' \mapsto \| a + K_{h'} \| \) (see [5]) and the continuity of the function \( \hat{e}_h \) on \( \Delta \), we can choose a neighborhood \( V \) of \( h \) such that when \( h' \in V \) all of the following hold:

\[
\sum \| a_i + K_{h'} \| < \frac{\varepsilon'}{3 \sum \| y_i \|};
\]

\[
\left| \frac{1}{\hat{e}_h(h)} - \frac{1}{\hat{e}_h(h')} \right| = \left| 1 - \frac{1}{\hat{e}_h(h')} \right| < \frac{\varepsilon'}{3 \| T(e_h) \|};
\]

and

\[
|1 - \hat{e}_h(h')| < \frac{\varepsilon'}{3C \| z \|}
\]

(where \( C \) is the bound on the approximate identity). Since the definition of \( X^{h'} \) is independent of the choice of \( e_{h'} \), for \( h' \in V \) we may just as well take \( e_{h'} = \frac{1}{\hat{e}_h(h')} e_h \). We also make use of the fact that, for \( a \in A \) and \( h \in \Delta \), we have

\[
\| a + K_h \| = \frac{1}{\| h \|} |\hat{a}(h)|; \text{ see [5, Lemma 1.3].}
\]

Then, for \( h' \in V \), we have the following:
The space $\tilde{X} = \{ \widetilde{T}_x : x \in X \}$ is a full space of selections in $\mathcal{C}^b(\mathcal{E})$, since, for each $x \in X$ and $h \in \Delta$ we have $\widetilde{T}_x(h) = T_x(e_h) + X^h = e_h x + X^h = x + X^h \in X_h$. The space $\widetilde{M}(X) = \{ \widetilde{T} : T \in M(X) \} \supset \tilde{X}$ is therefore also full. It follows by Proposition 1 or Proposition 2, above, that there is a unique topology on $\mathcal{E}$ which turns $\pi : \mathcal{E} \to \Delta$ into a bundle of Banach spaces, such that each $\widetilde{T}$ ($T \in M(X)$) is a member of the Banach $C_0(\Delta)$-module $\Gamma^b(\pi)$ of all continuous and bounded sections of the bundle $\pi : \mathcal{E} \to \Delta$. Moreover, we may regard $\Gamma^b(\pi)$ as a Banach $A$-module under the operation $(a \cdot \sigma)(h) = \widehat{a}(h)\sigma(h)$, as described above. In the language of [4], the map $\tilde{\pi} : M(X) \to \Gamma^b(\pi)$ is a sectional representation of $M(X)$ of Gelfand type.
We recall that, in this bundle topology on \( E \), neighborhoods of a point \( x + X^h \) are described by tubes: let \( \sigma \in \Gamma^h(\pi) \) be such that \( \sigma(h) = x + X^h \), let \( V \) be a neighborhood in \( \Delta \) of \( h \), and let \( \varepsilon > 0 \). Then \( \mathcal{T} = \mathcal{T}(V, \sigma, \varepsilon) = \{ z + X^{h'} : h' \in V, \| \sigma(h') - (z + X^{h'}) \| < \varepsilon \} \) is a neighborhood of \( \sigma(h) \), and in fact sets of this form, as \( V \) ranges over all neighborhoods of \( h \) and \( \varepsilon > 0 \) varies, form a fundamental system of neighborhoods of \( \sigma(h) \). Denote by \( E_2 \) the set \( E \) with its bundle topology generated by the \( \tilde{T} \) (\( T \in M(X) \)), and let \( p : \Delta \times X \to E_2 \) be the natural map \( (h, x) \mapsto \bar{x}(h) \).

**Proposition 2.6.** Let \( A \) be a commutative Banach algebra with bounded approximate identity, and let \( X \) be a Banach \( A \)-module. Then the spaces \( E_1 \) and \( E_2 \) are homeomorphic.

**Proof.** Since the topology on \( E_1 \) is the quotient topology on \( E \) induced by the map \( p : \Delta \times X \to E \), it suffices to show that the topology \( E_2 \) is also the quotient topology. We will show that the map \( p : \Delta \times X \to E_2 \) is continuous and open, and the desired result then follows from a standard topological argument.

a) \( p \) is continuous: Let \( \mathcal{T} = \mathcal{T}(V, \bar{x}, \varepsilon) \) be a bundle neighborhood of \( \bar{x}(h) = x + X^h = p(h, x) \) in \( E_2 \), as described above. If \( \mathcal{B}(x, \varepsilon) \) denotes the open ball around \( x \in X \) of radius \( \varepsilon > 0 \), then \( V \times B(x, \varepsilon) \) is a neighborhood of \( (h, x) \) in the product topology on \( \Delta \times X \). If \( (h', y) \in V \times B(x, \varepsilon) \), then \( h' \in V \), and \( \| \bar{x}(h') - \bar{y}(h') \| \leq \| x - y \| < \varepsilon \); i.e. \( \bar{y}(h') = p(h', y) \in \mathcal{T} \).

b) \( p \) is open: Consider a set of form \( V \times B(0, \varepsilon) \), which is nearly a typical open set in the product topology on \( \Delta \times X \). We claim that \( p(V \times B(0, \varepsilon)) \) is open in \( E_2 \). Let \( h \in V \) and \( y \in B(0, \varepsilon) \), and set \( \varepsilon' = \varepsilon - \| y \| \).

Consider the tube \( \mathcal{T} = \mathcal{T}(V, \bar{y}, \varepsilon') \) around \( \bar{y}(h) \), and let \( \bar{x}(h') \in \mathcal{T} \). Then \( h' \in V \), and \( \| \bar{x}(h') - \bar{y}(h') \| < \varepsilon' \). From the definition of the coset norm, we may choose \( q = \sum a_i w_i + (1 - e_{h'}) z \in K_{h'} X + (1 - e_{h'}) X \subset X^{h'} \) such that

\[
\| \bar{x}(h') - \bar{y}(h') \| \leq \| (x - y) + q \| < \varepsilon'.
\]

Then

\[
\| x + q \| \leq \| x - y + q \| + \| y \| < \varepsilon
\]

and

\[
(\bar{x} + q)(h') = \bar{x}(h') + \bar{q}(h') = \bar{x}(h').
\]

That is, \( \bar{x}(h') \in p(V \times B(0, \varepsilon)) \). Hence, \( \mathcal{T} \subset p(V \times B(0, \varepsilon)) \), and thus \( p(V \times B(0, \varepsilon)) \) is an open set. The final result follows by translation. \( \square \)
The space \( \widetilde{M(X)} = \{ \tilde{T} : T \in M(X) \} \) is a subspace of \( \Gamma^b(\pi) \) for the bundle \( \pi : E_1 = E_2 \to \Delta \). Given the homeomorphism of \( E_1 \) and \( E_2 \), for the remainder of the paper we will denote by \( E \) the fibered set with its bundle topology, and we will speak of the section space \( \Gamma(\pi) (\Gamma^b(\pi)) \) of all continuous (bounded) sections of the bundle \( \pi : E \to \Delta \). We will call this the multiplier bundle for \( X \).

We now examine for a moment the module \( Y = X_e = \text{closure in } X \) of \( \text{span}\{ax : a \in A, x \in X\} \). We note that, due to the Hewitt-Cohen factorization theorem, we actually have \( X_e = AX \).

As an \( A \)-module, \( Y \) has a representation (its sectional Gelfand representation, in the language of [4]) as a space \( \tilde{Y} \subset \Gamma(\pi') \) of a "canonical bundle" of Banach spaces \( \pi' : F \to \Delta \); the fibers of this bundle are the spaces \( Y_h = Y/Y^h \), where \( Y^h = \text{closure in } Y \) of \( \text{span}\{ay : a \in K_h, y \in Y\} \). It can be easily checked that, because \( Y \) is an essential \( A \)-module, we actually have \( Y^h = K_hY + (1 - e_h)Y \), as defined earlier. We note that we can also write \( Y^h = K_hX \), since \( A \) has a bounded approximate identity and is therefore factorable. The mapping \( \gamma : Y \to \Gamma(\pi') \) is given by \( \gamma(h) = y + Y^h \).

Consider the space \( M(Y) = \text{Hom}_A(A,Y) \) of multipliers from \( A \) to \( Y \). Let \( y \in Y \), and let \( T_y \in M(Y) \) be given by \( T_y(a) = ay \). Then \( \tilde{T}_y(h) = T_y(e_h) + Y^h = e_hy + Y^h = y + Y^h = \gamma(h) \). In other words, the representation of \( T_y \) as a section in the multiplier bundle for \( Y \) can be identified with the representation of \( y \) as a section in the canonical bundle for \( Y \). It turns out that the fibers of the multiplier bundle for \( Y = X_e \) and the multiplier bundle for \( X \) are related in general.

**Proposition 2.7.** There is a topological linear isomorphism of the fiber \( X_h = X/X^h \) of the multiplier bundle for \( X \) and of the fiber \( Y_h = Y/Y^h = X_e/(X_e)^h \) of the multiplier (canonical) bundle for \( Y = X_e \). If \( \|e_h\| \leq 1 \), then these fibers are isometrically isomorphic.

**Proof.** Let \( i : X_e \to X \) be the inclusion map, and let \( \rho_h : X \to X_h = X/X^h \) be the quotient map. Then \( \rho_h \circ i : X_e \to X_h \) is norm-decreasing and surjective (because, for each \( x \in X \), we have \( (\rho_h \circ i)(ehx) = e_hx + X^h = x + X^h \)). If \( ay \in K_hX = (X_e)^h = Y^h \), then \( ay \in X^h \), and so \( (\rho_h \circ i)(ay) = ay + X^h = X^h \); i.e. \( K_hX \subset \ker(\rho_h \circ i) \). Thus, there is a norm-decreasing map \( \phi_h : X_e/(X_e)^h \to X/X^h \) which carries \( ax + K_hX = ax + (X_e)^h \) to \( ax + X^h \).

On the other hand, consider the map of \( X \to X_e \) given by \( x \mapsto ehx \); this map clearly has norm \( \leq \|eh\| \). We compose this with the quotient map \( \rho'_h : X_e \to X_e/K_hX = X_e/(X_e)^h \); then \( \rho'_h \circ i : X_e \to X_e/K_hX = X_e/(X_e)^h = (X_e)_h \) and obtain a map \( \psi'_h : X \to (X_e)_h \) of norm \( \leq \|eh\| \). If \( w = ay + (1 - e_h)z \in X^h \), then...
\text{We offer the following without proof.}

\textbf{Corollary 2.9. Let } \pi : \mathcal{E} \to \Delta \text{ and } \pi' : \mathcal{F} \to \Delta \text{ be the multiplier bundle of } X \text{ and the multiplier bundle for } X_e, \text{ respectively. Then there is a topological linear isomorphism } \psi : \Gamma^b(\pi) \to \Gamma^b(\pi'), \text{ which is an isometry if the approximate identity for } A \text{ is bounded by } 1. \text{ Moreover, } \psi \text{ is a } C_0(\Delta)\text{-linear map. For } \sigma \in \Gamma^b(\pi), \text{ we have } \psi(\sigma) = \Psi \circ \sigma. \text{ The inverse map } \phi : \Gamma^b(\pi') \to \Gamma^b(\pi) \text{ is given by } \phi(\tau) = \Phi \circ \tau.
The following diagram illustrates the relationship among the maps constructed in this section. Here, $\rho_h$ and $\rho'_h$ will denote the quotient maps, $ev_h$ will denote the evaluation maps, and $\sim$ will denote the section maps.

\[
\begin{array}{cccccc}
\psi & X & \sim & \Gamma^b(\pi) & \Rightarrow & \Gamma^b(\pi') & \leftarrow X_e \\
\phi & \rho_h \swarrow \downarrow ev_h & \downarrow ev_h & \nearrow \rho'_h \\
\psi_h & X_h & \Rightarrow (X_e)_h \\
\phi_h
\end{array}
\]

3. The BSE Condition

An element $\sigma \in C(\mathcal{E})$ is said to be BSE (this refers to Bochner-Schoenberg-Eberlein; see [7] for an etymology of the term) if there exists some $\beta = \beta_\sigma > 0$ such that, for any choice of $h_i \in \Delta$, $f_i \in (X_{h_i})^*$ ($i = 1, \ldots, n$) we have

\[
\left| \sum_{i=1}^n (\sigma, f_i \circ ev_{h_i}) \right| = \left| \sum_{i=1}^n (\sigma(h_i), f_i) \right| \leq \beta \left\| \sum_{i=1}^n f_i \circ \rho_{h_i} \right\|_{X_*},
\]

where $\rho_h : X \to X_h$ is the quotient map and $ev_h : C(\mathcal{E}) \to X_h$, $\sigma \mapsto \sigma(h)$, is the evaluation map. Takahasi [7] shows that if $x \in X$, then $\overline{T_x}$ is BSE. If in addition $A$ is a regular Banach algebra, then $\overline{T}$ is BSE for each $T \in M(X)$. The fundamental question explored in [7] is, When is $\overline{M(X)}$ equal to the space of all continuous $\mathcal{E}$-valued BSE selections on $\Delta$, with $\mathcal{E}$ given its quotient topology? From the work done in the previous section, this is equivalent to the question of when $\overline{M(X)}$ is equal to $\Gamma_{BSE}(\pi)$, the space of BSE sections of the multiplier bundle $\pi : \mathcal{E} \to \Delta$. If $\overline{M(X)} = \Gamma_{BSE}(\pi)$, then the $A$-module $X$ is said to be BSE.

We first make an elementary observation, noted without proof in [7].

**Lemma 3.1.** Suppose that $\sigma \in C(\mathcal{E})$ is BSE. Then $\sigma$ is bounded.
PROOF. From the definition of the $BSE$ property, there exists $\beta = \beta_\sigma > 0$ such that for each $h \in \Delta$ and $f \in (X/X^h)^*$, we have

$$|\langle \sigma(h), f \rangle| \leq \beta \|f \circ \rho_h\|_{X^*} \leq \beta \|f\|,$$

since $\|\rho_h\| \leq 1$. We choose $f \in (X/X^h)^*$, with $\|f\| = 1$, such that $|\langle \sigma(h), f \rangle| = \|\sigma(h)\|$, and we obtain

$$\|\sigma(h)\| = |\langle \sigma(h), f \rangle| \leq \beta \|f\| = \beta,$$

i.e. $\|\sigma\| = \sup\{\|\sigma(h)\| : h \in \Delta\} \leq \beta$. \hfill $\square$

Thus, the question of when an $\mathcal{A}$-module $X$ is $BSE$ can now be studied by using only elements of $\Gamma^b(\pi)$, the bounded sections of the bundle $\pi : \mathcal{E} \to \Delta$. There is a relationship between $\mathcal{G}^b_S(\mathcal{E})$ and $\mathcal{G}^b_S(\mathcal{E}')$.

**Proposition 3.2.** Let $\psi : \Gamma^b(\pi) \to \Gamma^b(\pi')$, $\phi : \Gamma^b(\pi') \to \Gamma^b(\pi)$ be the topological linear isomorphisms described at the end of the previous section. If $\sigma \in \Gamma_{BSE}(\pi')$ (arising from $X_e$), then $\phi(\sigma) \in \Gamma_{BSE}(\pi)$ (arising from $X$). Conversely, if $A = C_0(\Delta)$ is a $C^*$-algebra, and if $\sigma \in \Gamma_{BSE}(\pi)$, then $\psi(\sigma) \in \Gamma_{BSE}(\pi')$.

**PROOF.** First, let $h \in \Delta$, let $f \in (X_h)^*$, and let $ax \in X_e$. Then

$$\langle ax, f \circ \rho_h \rangle = \langle ax + X^h, f \rangle$$

$$= \langle \phi_h(ax + (X_e)^h), f \rangle$$

$$= \langle ax + (X_e)^h, \phi_h^*(f) \rangle$$

$$= \langle ax, \phi_h^*(f) \circ \rho_h' \rangle.$$

That is, $\phi_h^*(f) \circ \rho_h' \in (X_e)^*$ is the restriction to $X_e \subset X$ of $f \circ \rho_h \in X^*$, where $\rho_h : X \to X_h$ and $\rho_h' : X_e \to (X_e)_h$ are the quotient maps.

With this in mind, we now let $h_i \in \Delta$, $f_i \in (X_{h_i})^*$, $i = 1, \ldots, n$, and suppose that $\sigma \in \Gamma_{BSE}(\pi')$. Then

$$|\sum |\phi(\sigma)(h_i), f_i)\rangle| = |\sum \langle \phi_{h_i}(\sigma(h_i)), f_i \rangle|$$

$$= |\sum \langle \sigma(h_i), \phi_{h_i}^*(f_i) \rangle|$$

$$\leq \beta_\sigma \|\sum \phi_{h_i}^*(f_i) \circ \rho_{h_i}'\|_{(X_e)^*}$$

$$\leq \beta_\sigma \|\sum f_i \circ \rho_{h_i}\|_{X^*}.$$
Thus, $\phi(\sigma) \in \Gamma_{BSE}(\pi)$.

Now, let $A$ be a $C^*$-algebra. We first note that, given $h_i \in \Delta$ $(i = 1, \ldots, n)$ we can choose our $e_{h_i}$ to have disjoint support. We also note that, if $ax \in X_\epsilon$ and $f \in (X_\epsilon)_h^*$, we have

$$
\langle ax, \psi_h^*(f) \circ \rho_h \rangle = \langle ax + X_h, \psi_h^*(f) \rangle
= \langle \psi_h(ax + X_h), f \rangle
= \langle ax + (X_h)^h, f \rangle
= \langle ax, f \circ \rho_h' \rangle.
$$

That is, $f \circ \rho_h' \in (X_\epsilon)^*$ is the restriction to $X_\epsilon$ of $\psi_h^*(f) \circ \rho_h \in X^*$.

Let $h_i \in \Delta, f_i \in (X_\epsilon)^*_h$, $(i = 1, \ldots, n)$. Let $\sigma \in \Gamma_{BSE}(\pi)$; we claim that $\psi(\sigma) \in \Gamma_{BSE}(\pi^*)$. We have

$$
|\sum (\langle \psi(\sigma)(h_i), f_i \rangle | - |\sum (\psi_{h_i}(\sigma(h_i)), f_i) |
\leq \beta_\sigma \| \sum \psi_{h_i}^*(f_i) \circ \rho_{h_i} \|_{X^*}.
$$

If $\epsilon > 0$ is given, we can choose $x \in X, \|x\| = 1$, such that

$$
\beta_\sigma \left\| \sum \psi_{h_i}(f_i) \circ \rho_{h_i} \right\|_{X^*} < \beta_\sigma \left\| \sum \langle x, \psi_{h_i}^*(f_i) \circ \rho_{h_i} \rangle \right\| + \epsilon.
$$

From our choice of $e_{h_i} (i = 1, \ldots, n)$ to have disjoint support, we see that $\| \sum_j e_{h_j} \| = 1$ and that $[\psi_{h_i}^*(f_i) \circ \rho_{h_i}] = \delta_{ij} [\psi_{h_i}^*(f_i) \circ \rho_{h_i}]$, where $\delta_{ij}$ is the Kronecker $\delta$.

It follows that

$$
\beta_\sigma \left\| \sum_i \langle x, \psi_{h_i}^*(f_i) \circ \rho_{h_i} \rangle \right\| + \epsilon
= \beta_\sigma \left\| \sum_i \langle x, [\psi_{h_i}^*(f_i) \circ \rho_{h_i}] \cdot (\sum_j e_{h_j}) \right\| + \epsilon
= \beta_\sigma \left\| \sum_i \langle (\sum_j e_{h_j})x, \psi_{h_i}^*(f_i) \circ \rho_{h_i} \rangle \right\| + \epsilon
= \beta_\sigma \left\| \sum_i \langle (\sum_j e_{h_j})x, f_i \circ \rho_{h_i}' \rangle \right\| + \epsilon
\leq \beta_\sigma \left\| \sum_i f_i \circ \rho_{h_i}' \right\|_{(X_\epsilon)^*} + \epsilon,
$$
because \((\sum_je_{h_j})x \in X_e\), by the restriction argument above and \(\|\sum_je_{h_j}\| \leq 1\). Thus, \(\psi(\sigma) \in \Gamma_{BSE}(\pi')\). 

\[\square\]

**Corollary 3.3.** If \(A\) is a \(C^*\)-algebra and if \(X_e\) is a BSE \(A\)-module, then so is \(X\).

**Proof.** Let \(X_e\) be BSE, and let \(\sigma \in \Gamma_{BSE}(\pi)\). Then \(\psi(\sigma) \in \Gamma_{BSE}(\pi')\), and so there exists \(T' \in M(X_e)\) such that \(\tilde{T'} = \psi(\sigma)\). If \(i : X_e \to X\) is the inclusion, then \(T = i \circ T' : A \to X\), and \(T \in M(X)\). We have

\[
(i \circ T')(h) = (i \circ T')(e_h) + X^h = T'(e_h) + X^h = \phi_h(T'(e_h) + (X_e)^h) = (\phi_h \circ \psi_h)(\sigma(h)) = \sigma(h),
\]

that is, \((i \circ T') = \sigma\). 

\[\square\]

We now turn to some special cases involving commutative \(C^*\)-algebras. If \(A = C_0(\Delta)\) is a commutative \(C^*\)-algebra, an \(A\)-module \(X\) is said to be \(C_0(\Delta)\)-locally convex if (among other equivalent formulations) we have \(\|ay_1 + by_2\| = \max\{\|ay_1\|, \|by_2\|\}\) for all \(a, b \in A\) with disjoint support, and for all \(y_1, y_2 \in X\). If \(X\) is \(C_0(\Delta)\)-locally convex, and if \(X\) is essential, then there is an isometric \(C_0(\Delta)\)-isomorphism of \(X\) and \(\Gamma_0(\pi)\), the space of sections of the multiplier bundle for \(X\) which disappear at infinity. (See [2] or [4] for details.)

**Proposition 3.4.** Suppose that \(A = C_0(\Delta)\) is a commutative \(C^*\)-algebra, and suppose that \(X\) is an \(A\)-module such that \(X_e\), the essential part of \(X\), is \(C_0(\Delta)\)-locally convex. Then 1) each element of \(\Gamma^b(\pi)\) is BSE; and 2) \(M(X) = \Gamma^b(\pi)\).

**Proof.** For 1), let \(\sigma \in \Gamma^b(\pi)\), and choose arbitrary \(h_i \in \Delta, f_i \in (X_{h_i})^* = (X/X_{h_i})^*\) \((i = 1, \ldots, n)\). Choose \(e_{h_i} \in A\) with disjoint support and such that \(\|e_{h_i}\| = e_{h_i}(h_i) = 1\), and choose \(x_i \in X\) such that \(\sigma(h_i) = x_i + X_{h_i} = \tilde{x}_i(h_i)\). Given \(\varepsilon > 0\), for each \(i = 1, \ldots, n\) choose \(z_i \in K_{h_i}X + (1 - e_{h_i})X \subseteq X_{h_i}\) such that

\[
\|\sigma(h_i)\| \leq \|x_i + z_i\| < \|\sigma\| + \varepsilon.
\]
Set \( w = \sum e_{h_i}(x_i + z_i) \). Then \( w \in X_e \), and so

\[
\|w\| = \|\sum e_{h_i}(x_i + z_i)\| = \max\{\|e_{h_i}\| \|x_i + z_i\|\}
\]

(because \( X_e \) is \( C_0(\Delta) \)-locally convex and the \( e_{h_i} \) have disjoint support)

\[
< \|\sigma\| + \varepsilon.
\]

Hence, it follows that

\[
|\sum_i \sigma(h_i), f_i) = |\sum_i (x_i + z_i, f_i \circ \rho_{h_i})|
\]

\[
= |\sum_i (e_{h_i}(x_i + z_i), f_i \circ \rho_{h_i})|
\]

\[
= \left| \sum_i \left( \sum_j e_{h_j}(x_j + z_j), f_i \circ \rho_{h_i} \right) \right|
\]

(since \( e_{h_i}(h_j) = \delta_{ij} = \) Kronecker \( \delta \))

\[
= \left| \sum_i \langle w, f_i \circ \rho_{h_i} \rangle \right|
\]

\[
\leq \|w\| \left\| \sum_i f_i \circ \rho_{h_i} \right\|_{X_*}
\]

\[
< (\|\sigma\| + \varepsilon) \left\| \sum_i f_i \circ \rho_{h_i} \right\|_{X_*}
\]

(We note that for \( x \in X \) and \( a \in A \), we have \((f \circ \rho_h)(ax) - f(\overline{ax}(h)) = f(a(h)\overline{x}(h)) = a(h)(f \circ \rho_h)(x)\).)

For part 2), suppose that \( \phi : \Gamma_0(\pi) \to X_e \) is the isometric \( C_0(\Delta) \)-isomorphism of the assumption. Among other properties of \( \phi \), we have \([\phi(\sigma)](h) = \sigma(h)\) for each \( \sigma \in \Gamma_0(\pi) \). Now, let \( \sigma \in \Gamma^b(\pi) \). We define \( T_\sigma : A \to X \) by \( T_\sigma(a) = \phi(a\sigma) \). Then, for \( b \in C_0(\Delta) \), we have

\[
bT_\sigma(a) = b\phi(a \cdot \sigma) = \phi(ba \cdot \sigma) = T_\sigma(ba).
\]

Clearly, \( T_\sigma \) is bounded, and so \( T \in M(X) \). Moreover, for \( h \in \Delta \), we have

\[
\overline{T_\sigma}(h) = [T_\sigma(e_h)]^-(h) = [\phi(e_h \cdot \sigma)]^-(h) = (e_h \cdot \sigma)(h) = \sigma(h).
\]

Two examples, worked out at some length in [7], then follow as corollaries:

**Corollary 3.5.** Let \( A \) be a commutative \( C^* \)-algebra, and let \( I \subset A \) be a closed ideal. Then \( I \) is BSE as an \( A \)-module, and \( A \) is BSE as an \( I \)-module.
PROOF. As an $A$-module, $I = I_e$ is essential, and since $I \subset C_0(\Delta_A)$ is $C_0(\Delta_A)$ locally convex, the result follows. On the other hand, as an $I$-module, $A_e = I = C_0(\Delta_I)$, which is $C_0(\Delta_I)$-locally convex.

**Corollary 3.6.** Let $A$ be a quasi-central $C^*$-algebra, with center $Z$. Then $A$ is BSE as a $Z$-module.

**Proof.** From the proof in ([7], Theorem 3.2), $A$ is essential as a $Z$-module. Note that $Z \simeq C_0(\Delta_Z)$. A variant of a result by Varela ([9], Theorem 3.5) shows that $A$ is isometrically isomorphic to the space $\Gamma_0(\pi)$ of sections of $\pi : E \to \Delta_Z$ which vanish at infinity, and hence that $A$ is $C_0(\Delta_Z)$-locally convex.

We now address questions asked by Takahasi in [7], as to whether $\Gamma_{BSE}(\pi) \subset \Gamma^b(\pi)$ is a Banach $A$-module.

**Proposition 3.7.** Let $A$ and $X$ be as generally given. Then $\Gamma_{BSE}(\pi)$ is an $A$-module.

**Proof.** Let $\sigma, \tau \in \Gamma_{BSE}(\pi)$. Choose $\beta_\sigma$ and $\beta_\tau$ as in the definition of $BSE$, and let $h_i \in \Delta, f_i \in (X_{h_i})^* (i = 1, \ldots, n)$. Then

$$\left| \sum \langle (\sigma + \tau)(h_i), f_i \rangle \right| \leq \left| \sum \langle \sigma(h_i), f_i \rangle \right| + \left| \sum \langle \tau(h_i), f_i \rangle \right| \leq (\beta_\sigma + \beta_\tau) \left\| \sum f_i \circ \rho_{h_i} \right\|_{X^*},$$

so that $\sigma + \tau \in \Gamma_{BSE}(\pi)$. Similarly, let $a \in A$. Then

$$\left| \sum \langle (a \cdot \sigma)(h_i), f_i \rangle \right| = \left| \sum \langle \widehat{\sigma(h_i)}, f_i \rangle \sigma(h_i) \right| = \left| \sum \langle \sigma(h_i), \widehat{\sigma(h_i)} f_i \rangle \right| \leq \beta_\sigma \left\| \sum \widehat{\sigma(h_i)}(f_i \circ \rho_{h_i}) \right\|_{X^*} \leq \beta_\sigma \left\| a \right\| \left\| \sum f_i \circ \rho_{h_i} \right\|_{X^*},$$

since $\sum (f_i \circ \rho_{h_i}) \cdot a = \sum (f_i \circ \rho_{h_i}) \widehat{\sigma(h_i)}$. Thus, $a \cdot \sigma \in \Gamma_{BSE}(\pi)$.

However, as the following example shows, $\Gamma_{BSE}(\pi)$ may not be a Banach space, even when $A$ is about as nice as it can be.

**Example 3.1:** Let $A = C([0, 1])$, and let $X = A^*$, the set of bounded Borel measures on $[0, 1]$. Since $A$ has an identity, we have $M(X) = X$, and it can be shown (see [10] or [4]) that, for $h \in \Delta = [0, 1]$, we have $X_h \simeq \mathbb{C}$. Under this
identification, for \( \mu \in X \) and \( h \in [0,1] \) we have \( \tilde{\mu}(h) = \mu(\{h\}) \), so that \( \ker(\tilde{\cdot}) \) is the space of continuous measures on \([0,1]\). Evidently, for any \( \mu \in X = M(X) \), \( \tilde{\mu} \) has only countable support in \([0,1]\), and we can identify \( \Gamma(\pi) - \Gamma^b(\pi) \) with \( c_0([0,1]) \), the closure under the sup-norm of the space of functions on \([0,1]\) which vanish off finite sets.

Now, \( A \) is a regular algebra, and so each element of \( \tilde{X} = \widetilde{M(X)} \) is \( BSE \). We will describe an element \( \sigma \in \Gamma(\pi) \) such that \( \sigma \neq \tilde{\mu} \) for any \( \mu \in X \) but such that there is a sequence \( \{\mu_n\} \subset X \) such that \( \sigma = \lim \mu_n \) in \( \Gamma(\pi) \); thus \( \Gamma_{BSE}(\pi) \) is not complete.

For each \( h \in [0,1] \), we have \( X_h \cong C \), so that for \( f \in (X_h)^* \) the action of \( f \) on \( X_h \) can be identified with multiplication by some \( \alpha = \alpha_f \in C \). We show that, given \( h_1, \ldots, h_n \in [0,1] \), we have \( \|\sum f_i \circ \rho_{h_i}\|_{X^*} = \max\{\|\alpha_{f_i}\| : i = 1, \ldots, n\} \). First, let \( \varepsilon > 0 \) be given. We can choose \( \tilde{\mu} \), \( \mu = 1 \) such that

\[
\|\sum f_i \circ \rho_{h_i}\|_{X^*} < \|\sum \langle \mu, f_i \circ \rho_{h_i} \rangle\| + \varepsilon = \|\sum \langle \alpha_f, \mu(\{h_i\})\rangle\| + \varepsilon < \sum |\alpha_f| \|\mu(\{h_i\})\| + \varepsilon \leq \max\{|\alpha_f|\} \sum |\mu(\{h_i\})| + \varepsilon \leq \max\{|\alpha_f|\} + \varepsilon,
\]

since \( \sum |\mu(\{h_i\})| \leq \|\mu\| = 1 \). Hence, \( \|\sum f_i \circ \rho_{h_i}\|_{X^*} \leq \max\{|\alpha_{f_i}|\} \). On the other hand, for each \( j = 1, \ldots, n \), we let \( \mu_j \in X \) be the unit point mass at \( h_j \). Then \( \|\mu_j\| = 1 \), and

\[
\left| \sum_i \langle \mu_j, f_i \circ \rho_{h_i} \rangle \right| = |\alpha_{f_j} \mu_j(\{h_j\})| = |\alpha_{f_j}| \leq \left| \sum_i f_i \circ \rho_{h_i} \right|_{X^*}.
\]

so that \( \max\{|\alpha_{f_j}|\} \leq \|\sum f_i \circ \rho_{h_i}\|_{X^*} \).

Now, consider \( \sigma \in \Gamma(\pi) \) given by \( \sigma(h) = h \), if \( h = 1/k \) for some \( k = 1,2, \ldots \), \( \sigma(h) = 0 \) otherwise. Let \( h_j = 1/j \) for \( j = 1, \ldots, n \), and let \( f_j \in (X_{h_j})^* \) be determined by \( \alpha_{f_j} = 1 \). Then

\[
\left| \sum_{j=1}^n \langle \sigma(h_j), f_j \rangle \right| = \sum_{j=1}^n 1/j.
\]
but \( \| \sum f_j \circ \rho_n \|_{X^*} = 1 \), so that \( \sigma \notin \Gamma_{BSE}(\pi) \). However, let \( \mu_n \in X \) be the discrete measure on \([0, 1]\) such that \( \mu_n(\{1/j\}) = 1/j \) for each \( j = 1, \ldots, n \). Then \( \widetilde{\mu}_n \in \Gamma_{BSE}(\pi) \) and \( \sigma = \lim \widetilde{\mu}_n \).

**Example 3.2:** It is also shown in [7] that when \( G \) is a compact abelian group, each of the convolution \( L^1(G) \)-modules \( C(G) \), \( L^p(G) \) (\( 1 \leq p \leq \infty \)) and \( M(G) \) is \( BSE \), and the question is asked whether the same is true for the case of non-compact \( G \). This is true, at least for \( L^p(G) \) when \( 1 < p < \infty \), but the reason turns out not to be especially interesting, as the following shows:

We have noted that for algebras \( A \) of the sort we are using, and \( A \)-modules \( X \), we have \( a(x + X^h) = \widehat{a}(h)(x + X^h) \). Thus, if \( f \in (X^h)^* = (X/X^h)^* \cong (X^h)^\perp \), we may write \( (f \cdot a)(x + X^h) = \widehat{a}(h)f(x + X^h) \), that is, \( f \cdot a = \widehat{a}(h)f \) in \( (X^h)^* \). Hence \( f \) (actually, its isomorphic image in \( (X^h)^\perp \)) generates a one-dimensional submodule in \( X^* \). Conversely, each element of \( X^* \) which generates a one-dimensional submodule in \( X^* \) clearly annihilates \( X^h \), and therefore has an isomorphic image in \( (X^h)^* \).

It is shown in [3] that for any locally compact abelian group \( G \), the one-dimensional submodules in \( L^p(G) \) (\( 1 \leq p \leq \infty \)) are scalar multiples of characters of \( G \). But when \( G \) is non-compact, these characters are not in \( L^p(G) \) for \( 1 \leq p < \infty \), and so \( L^p(G) \) has no one-dimensional submodules. It follows that if \( 1 < p < \infty \), \( X = L^p(G) \), and \( G \) is non-compact, then \( X^h = 0 \) for each character \( h \in \Delta_{L^1(G)} = \widehat{G} \). In this case, the only section of the multiplier bundle for \( L^p(G) \) as a module over \( L^1(G) \) is the zero section, which is trivially \( BSE \).

**Acknowledgment:** The author wishes to thank the referee for his several helpful suggestions and for calling an additional reference to his attention.

**References**


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