FINITE GROUPS IN WHICH THE ZEROS OF EVERY NONLINEAR IRREDUCIBLE CHARACTER ARE CONJUGATE MODULO ITS KERNEL

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ABSTRACT. In this note we classify the groups $G$ in which the zeros of every nonlinear irreducible character $\chi$ are conjugate in $G/\ker(\chi)$. Our proof depends on the classification of finite simple groups. We prove a related result for monolithic characters (see the corollary below). Some open questions are posed and discussed.

Let $\text{Irr}(G)$ be the set of irreducible characters of a finite group $G$ (we consider only finite groups), $\text{Irr}_1(G)$ the set of nonlinear characters in $\text{Irr}(G)$. For $\chi \in \text{Irr}_1(G)$, let $T(\chi) = \{ x \in G \mid \chi(x) = 0 \}$. The elements of $T(\chi)$ are called zeros of $\chi$. By Burnside's Theorem (see [I, Theorem 3.15] or [K, Corollary 23.1.5]), $T(\chi) \neq \emptyset$ for every $\chi \in \text{Irr}_1(G)$. Obviously, $T(\chi)^x = T(\chi)$ for $x \in G$, i.e., $T(\chi)$ is a union of conjugacy classes of $G$ (= $G$-classes). For further information on the sets $T(\chi)$ and related subgroups see [K], Chapter 23.

E.M. Zhmud [Z1], [Z2] treated some properties of finite groups $G$ possessing a faithful irreducible character $\chi$ such that $T(\chi)$ is a $G$-class. The set of groups satisfying the Zhmud condition, is very big, and it is impossible to classify all such groups. In the other extreme, S.C. Gagola [G] studied the groups having an irreducible character vanishing on all but two classes. For further information on

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zeros of characters see [Ga], [Z3], [Z4]. Note that induced characters have many zeros, and we make use of this fact in what follows.

For $X \subseteq G$ and $N \trianglelefteq G$, let $XN/N = \{xN \mid x \in X\}$ be the subset in $G/N$. A subset $X$ is invariant in $G$ (or $G$-invariant) if $X^g = X$ for all $g \in G$. If $X$ is $G$-invariant, then $XN/N$ is $G/N$-invariant. In particular, if $\chi \in \text{Irr}_1(G)$, then by the above, $T(\chi)\ker(\chi)/\ker(\chi)$ is a nonempty (since $T(\chi) \cap \ker(\chi)$ is empty) $G/\ker(\chi)$-invariant subset.

**Definition 1.** A group $G$ is said to be a **CZ-group** if $T(\chi)$ is a conjugacy class of $G$ for every $\chi \in \text{Irr}_1(G)$. A group $G$ is said to be a **CZK-group** if $T(\chi)\ker(\chi)/\ker(\chi)$ is a conjugacy class in $G/\ker(\chi)$ for every $\chi \in \text{Irr}_1(G)$.

By definition, abelian groups are CZ-groups and CZ-groups are CZK-groups. Both the properties are inherited by epimorphic images.

Note that if $x \in T(\chi)$, $z \in \ker(\chi)$, then $xz \in T(\chi)$. Indeed, if $D$ is a representation of $G$ with character $\chi$, then $D(xz) = D(x)D(z) = D(x)$, and so $\chi(xz) = \text{tr}(D(x)) = \chi(x) = 0$. Therefore, $T(\chi)$ is a union of cosets of $\ker(\chi)$, and so $T(\chi)\ker(\chi)/\ker(\chi) = T(\chi)/\ker(\chi)$.

Obviously, $G$ is a CZ-group if and only if the character table of $G$ has a minimal possible number (namely, $|\text{Irr}_1(G)|$) zero entries. As a corollary of the main theorem, we obtain that a subgroups of CZ-groups are also CZ-groups. It is surprising that the symmetric group $S_4$ is the only CZK-group that is not a CZ-group. Note that $S_4$ has subgroups (namely, $A_4$ and Sylow 2-subgroups) that are not CZK-groups.

The proof of the main theorem in solvable case is based essentially on a corollary of the Isaacs-Passman Theorem [IP] on groups all of whose nonlinear irreducible characters have prime degrees (see Lemma 3 and Corollary 4 below). To prove the solvability of CZK-groups, we make use of the classification of finite simple groups and its consequence, due to Willems (see Lemma 1(a)).

Let $\{1\} < N \trianglelefteq G$, $\phi \in \text{Irr}_1(N)$ and $\chi$ an extension of $\phi$ to $G$. Since $\phi$ is $G$-invariant, it follows that $T(\phi)$ is $G$-invariant and $T(\phi) \subseteq T(\chi)$. In particular, if $T(\chi)$ is a $G$-class, then $T(\chi) = T(\phi)$. We make use of this remark in the proof of the theorem.

In the proof of the theorem we make use of the following

**Lemma 1.** (a) ([W1], [W2]) Every simple group of Lie type possesses an irreducible character $\chi$ such that $|G|/\chi(1)$ is odd ($\chi \in \text{Irr}(G)$ is said to be of $p$-defect 0 if $p \nmid |G|/\chi(1)$).

(b) A group $G$, containing a nilpotent subgroup of index 2, is supersolvable.
A group $G$ admitting a fixed-point-free automorphism of order 3 is nilpotent (of class at most 2).

Lemma 1(b) follows easily from [BZ, Exercise 3.19].

For $H < G$, set $H_G = \bigcap_{x \in G} H^x$, $D_H = G - \bigcup_{x \in H} H^x$. It is known that $H_G$ is the maximal normal subgroup of $G$ contained in $H$ and $D_H$ a nonempty $G$-invariant subset.

**Lemma 2.** Let $H$ be a nontrivial subgroup of a solvable group $G$ such that $D_H$ is a $G$-class. Then:

(a) If $H < G$, then $|G : H| = 2$ and $G$ is a Frobenius group with kernel $H$.

(b) If $H$ is nonnormal maximal subgroup of $G$, then $G/H_G$ is a Frobenius group with kernel $P/H_G$ of order $p^\alpha$ and complement $H/H_G$ of order $p^\alpha - 1$, where $p$ is a prime. If, in addition, $G$ is a CZK-group, then $G/H_G \cong S_3$, the symmetric group of degree 3.

(c) If $G$ is a nilpotent CZK-group, it is abelian.

**Proof.** (a) Let $H < G$. Then $D_H = G - H$ is a $G$-class, and so $(G/H)^\#$ is a conjugacy class so that $|G/H| = 2$. If $x \in G - H$, then $|G : C_G(x)| = |G - H| = \frac{1}{2}|G|$, and we obtain a Frobenius group with kernel $H$ of index 2.

(b) Suppose $H$ is nonnormal maximal subgroup of $G$. It suffices to consider the case when $H_G = \{1\}$. Let $P$ be a minimal normal subgroup of $G$. Then $N_G(P \cap H) \geq \langle P, H \rangle > H$, and so $P \cap H = \{1\}$, $G = P \cdot H$, a semidirect product. Set $|P| = p^\alpha$. Since $P^\# \subseteq D_H$ and $D_H$ is a $G$-class by assumption, it follows that $D_H = P^\#$ and $|D_H \cup \{1\}| = |P| = |G : H|$. On the other hand, it is easy to check that $|D_H| \geq |G : H| - 1$ with equality if and only if $H \cap H^x = \{1\}$ for all $x \in G - H$. Therefore, $H \cap H^x = \{1\}$ for all $x \in G - H$, i.e., $G$ is a Frobenius group with complement $H$ and kernel $P$. Since $P^\#$ is a $G$-class and $P$ is elementary abelian, it follows that $|H| = |P| - 1 = p^\alpha - 1$. Let, in addition, $G$ be a CZK-group. Every faithful irreducible character of $G$ vanishes outside $P$ by [I], Theorem 6.34, and so $G - P$ is a $G$-class. By (a), $|G : P| = 2$ so $p^\alpha - 1 = 2$, $p^\alpha = 3$ and $G \cong S_3$.

(c) is a corollary of (a) because a nonlinear irreducible character $\chi$ of $G$ always vanishes outside some proper normal subgroup (since $G$ is an M-group) and $G/\ker(\chi)$ is not a Frobenius group.

**Lemma 3.** [IP] Let $cd(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\} = \{1, p, q\}$, where $p, q$ are distinct primes. Then $G$ has one of the following normal series:

(a) $G > F > Z(F) = Z(G)$, where $|G : F| = p$, $G/Z(G)$ is a Frobenius group whose kernel $F/Z(G)$ of order $q^2$ is a minimal normal subgroup.


(b) $G > F > M = Z(G) \times R$, where $|G : F| = p$, $|F : M| = q$, $G/M$ and $F$ are nonabelian, $R$ is elementary abelian of order $r^m$ for a prime $r$, $F/M$ acts irreducibly on $R$, $\frac{r^m-1}{r^m/p-1} = q$.

Corollary 4. Let $cd(G) = \{1, 2, 3\}$ and $|G : G'| = 2$. Then $G \cong S_4$.

PROOF. By assumption (in the notation of Lemma 3), $F = G'$, $p = 2$, $q = 3$. Obviously, $G$ is a group of Lemma 3(b). Then $r^{m/2} + 1 - q - 3$, and so $r - 2$, $m = 2$, $|G/Z(G)| = 24$. Since Sylow subgroups of $G/Z(G)$ are not normal, it follows that $G/Z(G) \cong S_4$. By assumption, $Z(G) < G'$, and so $G$ is an epimorphic image of a covering group of $S_4$. Since covering groups of $S_4$ have irreducible character of degree 4, we get $Z(G) = \{1\}$, completing the proof.

Our principal result is the following

Theorem 5. A nonabelian group $G$ is a CZK-group if and only if it is either a Frobenius group with kernel of index 2 or $S_4$.

PROOF. It follows from the description of irreducible characters of Frobenius groups (see [I], Theorem 6.34) that a Frobenius group with kernel of index 2 is a CZK-group (moreover, it is a CZ-group). It is easy to check that $S_4$ is a CZK-group (however it is not a CZ-group).

Let $G$ be a CZK-group. Suppose that the theorem has proved for all CZK-groups of order $< |G|$. In what follows we assume that $G$ is not abelian.

(i) We claim that $G$ is not simple (in the case under consideration, $G$ is a CZ-group). Assume that this is false. To obtain a contradiction, we make use of the classification of finite simple groups. By [Atlas], the sporadic simple groups are not CZK-groups. Therefore, by the classification, it remains to show that the simple groups of Lie type and the alternating groups $A_n$ of degree $n > 4$ are not CZK-groups.

Assume that $G$ is a simple group of Lie type. Then by Lemma 1(a), there exists a character $\chi \in \text{Irr}(G)$ of 2-defect 0. By [I], Theorem 8.17, $\chi$ vanishes on all elements of even order. Since, by assumption, $T(\chi)$ is a conjugacy class, all elements of even order in $G$ have the same order, and so are involutions. This means that a Sylow 2-subgroup $S$ of $G$ is elementary abelian and $C_G(x) = S$ for every $x \in S^\#$. Hence by Brauer-Suzuki-Wall Theorem (see [HB], Theorem 11.2.7), $G \simeq L_2(2^n)$, $n > 1$. The group $G = L_2(2^n)$ ($n \geq 2$) has an irreducible character $\chi$ of degree $2^n + 1$ (Schur [S]; see also [D], §38). Note that $G$ has a cyclic Hall subgroup $Z$ of order $2^n + 1$. Since $(\chi(1), |G|/\chi(1)) = 1$ and $T(\chi)$ is a conjugacy
class, it follows that $\chi$ vanishes on $Z^#$ and all its conjugates by [I], Theorem 8.17, and so $2^n + 1$ is a prime number. Set $T = \bigcup_{x \in G}(Z^#)^x$. Obviously, $T$ is $G$-invariant subset, $T = T(\chi)$ (since $G$ is a CZK-group). Since $|N_G(Z) : Z| = 2$ and $Z$ is a TI-subgroup of $G$, we have $|T| = |Z^#| \cdot |G : N_G(Z)| = 2^{2n-1}(2^n - 1)$; however this number does not divide $|G| = 2^n(2^{2n} - 1)$ so $T$ is not a $G$-class. It follows that $L_2(2^n)$ is not a CZK-group.

Assume that $G = A_n$, the alternating group of degree $n > 4$. For $n \leq 7$ the result follows from the character tables of $A_n$ (see [Atlas]). In what follows we assume that $n > 7$. Define a function $\pi : A_n \to \mathbb{N} \cup \{0\}$ as follows: if $g \in G$, then $\pi(g)$ is the number of points fixed by $g$. Since $G = A_n$ is 2-transitive, we have $\pi = 1_G + \chi$, where $1_G$ is the principal character of $G$ and $\chi \in \text{Irr}(G)$.

Let $n = 2m$, $(m \geq 4)$ be even. Consider the following permutations in $G$:

$$a = (1,2,\ldots,2m-1), \quad b = (1,2)(3,4)(5,\ldots,2m-1).$$

Then $\chi(a) = 0 = \chi(b)$, but $a$ and $b$ are not conjugate in $G = A_n$, so that $A_{2m}$ is not a CZK-group.

Let $n = 2m + 1$, $(m \geq 4)$ be odd. Consider the following permutations in $G = A_{2m+1}$:

$$a = (1,2)(3,\ldots,2m), \quad b = ((1,2,3,4)(5,\ldots,2m).$$

As in the previous paragraph, $a$ and $b$ are nonconjugate zeros of $\chi$, and so $G = A_{2m+1}$ is not a CZK-group.

(ii) We claim that $G' < G$. Indeed, if $M$ is a maximal normal subgroup of $G$, then $G/M$ is a simple CZK-group. By (i), $G/M$ is abelian, and so $G' \leq M < G$, as desired.

(iii) Suppose that $G$ has a proper normal subgroup $M$ such that $\chi^G = \chi \in \text{Irr}(G)$ for some $\chi \in \text{Irr}(M)$; then $\chi$ is nonlinear. Since $M < G$, $\chi$ vanishes outside $M$; in particular, $\ker(\chi) < M$. Therefore, $G/\ker(\chi) - M/\ker(\chi) = T(\chi)/\ker(\chi)$ (since $G/\ker(\chi) - M/\ker(\chi)$ is $G/\ker(\chi)$-invariant and $T(\chi)/\ker(\chi)$ is a $G/\ker(\chi)$-class by assumption). In that case, $(G/M)^#$ is a $G/M$-class so that $|G : M| = 2$. By Lemma 2(a), $G/\ker(\chi)$ is a Frobenius group with kernel $M/\ker(\chi)$ (of index 2).

A. Let $G$ be solvable. We will use induction on $|G|$ to prove the theorem in this case.

(iv) We claim that if $G$ has an abelian subgroup $A$ of index 2, then $G$ is a Frobenius group with kernel $A$. By Lemma 1(b), $G$ is supersolvable. By [I], Lemma 12.12, $|G| = 2|G'||Z(G)|$. If $Z(G) = \{1\}$, then $A = G'$. In the case under consideration, $A$ is of odd order, and every involution from $G - A$ inverts
it follows that $G$ is a Frobenius group with kernel $A$. Assume that $Z(G) > \{1\}$. Since the intersection of kernels of the nonlinear irreducible characters of a nonabelian group is $\{1\}$ (see, for example, [BZ], Theorem 4.35), there exists $\chi \in \text{Irr}_1(G) - \text{Irr}(G/Z(G))$. If $\lambda \in \text{Irr}(\chi_A)$, then $\chi = \chi^G$ and $\ker(\chi) < A$. Then $T(\chi)/\ker(\chi) = G/\ker(\chi) - A/\ker(\chi)$ is a $G/\ker(\chi)$-class, and, by Lemma 2(a), $G/\ker(\chi)$ is a Frobenius group, which is impossible in view of $Z(G/\ker(\chi)) > \{1\}$ (by the choice of $\chi$).

(v) We will prove by induction on $|G|$ that $|G : G'| = 2$. We may assume that $G'$ is a minimal normal subgroup of $G$. By Lemma 2(c), $G$ is not nilpotent. Therefore, by [H], Satz 3.3.11, $G' \not\in \Phi(G)$ ($\Phi(G)$ is the Frattini subgroup of $G$), and so $G = H \cdot G'$, where $H$ is maximal in $G$; obviously, $H \cap G' = \{1\}$ and $H$ is abelian. Let $H_G = \{1\}$; then $G$ is a Frobenius group with kernel $G'$. Since all faithful irreducible characters of $G$ vanish off $G'$ (see [I], Theorem 6.34), $G - G'$ is a $G$-class, and we get $|G : G'| = 2$ by Lemma 2(a). Let $H_G > \{1\}$. Then $|G : G' \cdot H_G| = 2$ by induction, contrary to (iv) (since $H_G \leq Z(G)$ and $G' \cdot H_G$ is abelian of index 2 in $G$). This completes the proof of (v).

(vi) We claim that if $G$ has a nilpotent subgroup $A$ of index 2, then $A$ is abelian. Assume that $G$ is a counterexample of minimal order. By (v), $A = G'$. By Lemma 1(b), $G$ is supersolvable. By induction, $A$ is a nonabelian $p$-group, $p$ is a prime, $|A'| = p$. Since $G/A'$ is a Frobenius group by (iv) (in particular, $p > 2$), $A' \leq Z(A)$ and $G$ is not a Frobenius group (otherwise, $A$ is abelian by Burnside), we get $A' = Z(G)$. By induction, $A'$ is the only minimal normal subgroup of $G$. By Fitting’s Lemma (applied to $Z(A)$), we get $A' = Z(A)$. If $x, y \in A$, then $[x, y]^p = [x, y] - 1$ (since the nilpotence class of $A$ is 2) so $y^p \in Z(A) = A'$. It follows that $A/A'$ is elementary abelian so $A$ is extraspecial. Let $\theta \in \text{Irr}_1(A)$. Then $\theta^G$ is faithful, vanishes outside $A'$; therefore, since $G - A$ is not a conjugacy class of $G$, $\theta^G = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \text{Irr}(G)$ are two distinct extensions of $\theta$ (by Clifford theory and Lemma 2(a)). Then $T(\chi_1) = T(\theta)(= A - A')$ (see the remark preceding Lemma 1). If $x \in A - A' = T(\chi_1)$, then $2p = |G : C_G(x)| = |T(\chi_1)| = |A - A'|$. Setting $|A| = p^{1+2m}$, $m \in \mathbb{N}$, we get $2p = p^{2m+1} - p$, which is impossible. Thus, $A$ is abelian.

In what follows, we will assume that $G'$ is not nilpotent; then $G'' > \{1\}$ and $G'' \not\in \Phi(G)$ by [H], Satz 3.3.5.

(vii) We will prove that if $G''$ is the unique minimal normal subgroup of $G$, then $G \simeq S_4$. Set $|G''| = p^a$, $|G : G''| = 2a$, where $a > 1$ is odd; then $G/G''$ is a Frobenius group with kernel of order $a$ (see (iv)). Since $G'' \not\in \Phi(G)$ we get $G = H \cdot G''$, where $H$ is maximal in $G$ and $H \cap G'' = \{1\}$. By assumption,
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Let \( H = X \cdot A \), where \(|X| = 2\), \(|A| = a\), \(A\) is the abelian kernel of a Frobenius group \( H \). Assume that \( G' = A \cdot G'' \) is not a Frobenius group. Then \( yz = zy \) for some \( y \in A^\# \) and \( z \in (G'')^\# \). Since \( \langle y \rangle \triangleleft H \) and \( H \) is maximal in \( G \), it follows that \( \langle y \rangle \triangleleft G \), contrary to the uniqueness of \( G'' \). Thus, \( G' = A \cdot G'' \) is a Frobenius group (in particular, \( A \) is cyclic). Let a nonprincipal \( \mu \in \text{Irr}(G'') \). Then \( \theta = \mu^{G'} \in \text{Irr}(G') \) by [I], Theorem 6.34. Since \( G - G' \) is not a G-class (see Lemma 2(a)) and \( \theta^G \) vanishes outside \( G' \), it follows that \( \theta^G = \chi_1 + \chi_2 \), where \( \chi_1, \chi_2 \) are distinct extensions of \( \theta \) to \( G \). Since \( (\chi_1)^{G'} = \theta \), \( T(\chi_1) \) is a G-class and \( T(\theta) \) is a G-invariant subset (since \( \theta \) is a G-invariant character of \( G' \triangleleft G \)), it follows that \( T(\chi_1) = T(\theta) \). Note that \( T(\theta) = G' - G'' \) is the set of size \( ap' - p^a = (a - 1)p^a \).

If \( z \in G' - G'' \), then \( |G : C_G(z)| = 2p^a \). Hence \( (a - 1)p^a = 2p^a \), and so \( a = 3 \).

In particular, \( G/G'' \cong S_3 \). Assume that \( \text{Irr}(G) \) has a character \( \chi \) of degree 6. If \( \mu \in \text{Irr}(\chi_{G''}) \), then \( \chi = \mu^G \) (since \( |G : G''| = 6 \) and \( \mu \) is linear) and \( \chi \) is faithful. In the case considered, \( \chi \) vanishes outside \( G'' \). This is impossible since \( G - G'' \) is not a G-class in view of \( G/G'' \cong S_3 \). Thus, \( \text{cd}(G) = \{1, 2, 3\} \) by [I], Theorem 6.15. By (v) and Corollary 4, \( G \cong S_4 \).

(viii) We claim that if \( G'' \) is a minimal normal subgroup of \( G \), then \( G \cong S_4 \).

By (vii) we may assume that \( G \) has another minimal normal subgroup \( R \). By (v), \( R \triangleleft G' \). Moreover, \( R \times G'' \triangleleft G' \), by (iv) and (v). By induction, \( G/R \cong S_4 \), and so \( |G''| = 4 \). We have \( |G : R \times G''| = 6 \). As in (vii), \( \text{Irr}(G) \) has no character of degree 6 (if such a character exists, it is faithful, and then \( G - R \times G'' \) is a G-class by assumption, which is a contradiction). By Ito's Theorem ([I], Theorem 6.15), \( \text{cd}(G) = \{1, 2, 3\} \). By Corollary 4 and (v), \( G \cong S_4 \) (in particular, \( R \) does not exist).

(ix) We claim that if \( G'' \triangleright \{1\} \) is abelian, then \( G \cong S_4 \). As before, we will use induction on \( G \). By (viii), we may assume that \( G'' \) is not a minimal normal subgroup of \( G \). Let \( R \) be a minimal normal subgroup of \( G \) contained in \( G'' \). By induction, \( G/R \cong S_4 \). It follows that \( G'' \) is an (abelian) 2-subgroup of index 6 in \( G \). As in the proof of (vii) and (viii), \( \text{cd}(G) = \{1, 2, 3\} \). By Corollary 4 and (v), \( G \cong S_4 \) (in particular, \( R \) does not exist).

(x) We claim that if \( G'' \triangleright \{1\} \) is nilpotent, then \( G \cong S_4 \). In view of (ix), we may assume that \( G'' \) is nonabelian. By (ix), \( |G''/G'''| = 4 \), and so \( G'' \) is a 2-group of maximal class by Taussky's Theorem (see [H], Satz 3.11.9). By (vi), \( G' \) is not nilpotent. Therefore, \( G'' \) is the ordinary quaternion group (if \( P \) is a 2-group of maximal class such that \( \text{Aut}(P) \) is not a 2-group, then \( P \) is the ordinary quaternion group). In that case, \( G \) is a covering group of \( S_4 \) (by Schur's description of covering groups of the symmetric groups [S]; see also [Su], (3.2.21)).
Then $G$ has a faithful irreducible character $\chi$ of degree 4. Since $\text{cd}(G') = \{1, 2, 3\}$, it follows, by Clifford’s Theorem, that $\chi_{G'} = \phi_1 + \phi_2$, where $\phi_1, \phi_2 \in \text{Irr}(G')$ and $\phi_1^G = \chi$. Then $\chi$ vanishes outside $G'$ so $G$ is a Frobenius group with kernel $G'$ (Lemma 2(a)), which is not the case.

(x) We claim that $G^{''' \!} = \{1\}$ (in particular, if $G'' > \{1\}$, then $G \cong S_4$). Assume that this is false. Without loss of generality, we may assume that $N = G^{''' \!}$ is a minimal normal subgroup of $G$. By (x), $G''$ is not nilpotent, and so $N \not\leq \Phi(G)$ by [H], Satz 3.3.5. Therefore, $G = H \cdot N$, where $H \cap N = \{1\}$ and $H$ is maximal in $G$. Since $G''$ is not nilpotent, $H_G = \{1\}$, and so $C_{G'}(N) = N$. By (ix), $H \cong G/N \cong S_4$. Set $|N| = p^\alpha$. We have $p > 2$ (since $G''$ is not nilpotent). In particular, $\alpha > 1$. We have $4 \in \text{cd}(G'')$ (otherwise, by [A], $G''$ has an abelian subgroup of index 2, and then $C_{G'}(N) = N$, which is not the case). Let $\phi \in \text{Irr}(G'')$, $\phi(1) = 4$. If $\text{Irr}(G)$ has a character $\chi$ of degree 24, then $T(\chi) = G - N$ is a $G$-class (since $N$ is normal abelian of index 24 in $G$), contrary to Lemma 2(a). Let us consider the following two cases.

(1) Let $\phi \in \text{Irr}(G'')$. Then $\phi_{G''} = \theta + \phi_1 + \phi_2$, where $\phi, \phi_1, \phi_2 \in \text{Irr}(G')$ are distinct (faithful) extensions of $\phi$ to $G$. As above, $T(\theta) = T(\phi_1)$ (see the remark, preceding Lemma 1). But $\theta$ vanishes on $G' - N$ (since $|G' : N| = 12 = \theta(1)$ and $N$ is abelian), and this set is not a $G$-class since $G' / N \cong A_4$, and we obtain a contradiction.

(2) Let $\phi \in \text{Irr}(G'')$. Then $\phi_{G'} = \theta + \phi_1 + \phi_2$, where $\theta, \phi_1, \phi_2 \in \text{Irr}(G')$ are distinct extensions of $\phi$ to $G$ (by Clifford theory). Since $N \not\leq \ker(\theta)$, it follows that $\phi_{G'} = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \text{Irr}(G)$ are (faithful) distinct extensions of $\theta$ to $G$ (see Lemma 2(a)). We have $(\chi_1)_{G''} = \phi$, and so $T(\chi_1) = T(\phi)$ (since $\phi$ is $G$-invariant, the set $T(\phi)$ is invariant in $G$). Since $G$ is a CZK-group, $T(\phi)$ is a $G$-class, and, by [I], Theorem 8.17, it consists of elements of even order in $G''$, which are, consequently, involutions. But this is not true: $G''$ is not a Frobenius group (since its Sylow 2-subgroup is nonnormal abelian of type $(2, 2)$), and so $G''$ has an element of order $2p$. This contradiction completes the proof of (x).

Thus, the theorem is proved in the solvable case. It remains to prove that $G$ is solvable.

B. We claim that $G$ is solvable. Suppose that $G$ is a counterexample of minimal order. Then $G$ is not simple (by (i)) and has only one minimal normal subgroup, say $R$; $R$ is a direct product of isomorphic nonabelian simple groups, $G/R$ is solvable. By (ii) and (v), $|G : G'| = 2$. Let a nonprincipal $\phi \in \text{Irr}(R)$. Assume that $\phi^x \neq \phi$ for some $x \in G$. Then the inertia subgroup $I = I_G(\phi)$ is a proper subgroup of $G$. Obviously, $R \leq I$. If $\theta \in \text{Irr}(\phi^I)$, then $\theta_{G'} = \chi \in \text{Irr}(G)$ by
Theorem 6.11(a). Since $R$ is the only minimal normal subgroup of $G$ and $R \not\leq \ker(\chi)$, $\chi$ is faithful. The induced character $\chi$ vanishes on $D_I = G - \bigcup_{x \in G} I^x$, and so $T(\chi) = D_I$ (since $D_I$ is a nonempty invariant subset of $G$ and $T(\chi)$ is a $G$-class by assumption). If $I \triangleleft G$, then $G$ is a Frobenius group with kernel $I$, $|G : I| = 2$, by Lemma 2(a). In that case, $G$ is solvable, which is not the case. Assume that $I \not\leq G$. Let $I \leq H < G$, where $H$ is maximal in $G$. Since $D_H = G - \bigcup_{x \in G} H^x$ is a nonempty $G$-invariant subset and $D_H \subseteq T(\chi)$, it follows that $D_H = T(\chi)$ (since $T(\chi)$ is a $G$-class). By the induction hypothesis, $G/R$ is solvable. Therefore, by Lemma 2(b), $G/H_G \cong S_3$ (if $H < G$, we obtain a Frobenius group with kernel $H$ of index 2 by Lemma 2(a), which is not the case: $G$ is nonsolvable). In particular, $|G : H| = 3$. Let $H, H_1, H_2$ be all $G$-conjugates of $H$. Then $H \cap H_1 = H \cap H_2 = H_1 \cap H_2 = H_G$, and so $|H \cup H_1 \cup H_2| = |H| - 2|H_G| = \frac{2}{3}|G|$. We obtain $|D_H| = \frac{1}{3}|G|$. Therefore, if $x \in D_H$, then $|G : C_G(x)| = |D_H| = \frac{1}{3}|G|$ (since $D_H$ is a $G$-class), and so $C_G(x) = (x)$ is of order 3. Since $x \not\in H_G$, it follows that $x$ induces a fixed-point-free automorphism of $H_G$ of order $\sigma(x) = 3$. By Lemma 1(c), $H_G$ is nilpotent. Since $R \leq H_G$, it follows that $R$ is solvable, contrary to the assumption. Thus, all irreducible characters of $R$ are $G$-invariant. By the Brauer Permutation Lemma ([I], Theorem 6.32), every $R$-class is a $G$-class. Therefore, $R$ is simple. It follows that $R$ is a nonabelian simple CZK-group (in fact, if a nonprincipal $\phi \in \text{Irr}(R)$ and $\chi \in \text{Irr}(\phi^G)$, then $\chi_R = e\phi$; it follows that $T(\phi) = T(\chi)$ is a $G$-class, and so an $R$-class), contrary to (i). This completes the proof of the theorem.  

In particular, a nonabelian group is a CZ-group if and only if it is a Frobenius group with kernel of index 2. (According to the report of D. Chillag, he also classified CZ-groups.)

A character $\chi$ of $G$ is said to be monolithic if $\chi \in \text{Irr}(G)$ and $G/\ker(\chi)$ is a monolith. If $N \triangleleft G$ and $\chi$ is a monolithic character of $G/N$, then $\chi$ (considered as a character of $G$) is also a monolithic character of $G$. We consider the principal character $1_G$ of $G$ to be monolithic by definition. As a rule, the set of monolithic characters of $G$ is a proper subset of $\text{Irr}(G)$. As an easy consequence of the theorem we will prove the following

**Corollary 6.** If $T(\chi)$ is a conjugacy class for every nonlinear monolithic character of a nonabelian group $G$, then $G$ is a CZ-group.

**Proof.** Let $M$ be a maximal normal subgroup of $G$. Since all irreducible characters of $G/M$ are monolithic, it is a CZ-group. It follows from the theorem that $G/M$ is abelian. In particular, $G' < G$. Moreover, this reasoning shows that if
N < G, then (G/N)′ < G/N. Suppose that the corollary is proved for all groups of order < |G|. Let R be a minimal normal subgroup of G. By the induction hypothesis, G/R is solvable.

Assume that G/R is nonabelian. Let H/R be a normal subgroup of G/R such that G/H is nonabelian but every proper epimorphic image of G/H is abelian. All nonlinear irreducible characters of G/H are monolithic (see [1], Theorem 12.3). Therefore by the theorem, G/H is a Frobenius group with kernel L/H of index 2 (by the above, G/H is not nilpotent). Let λ be a nonprincipal character of L/H. Then λ^G = χ ∈ Irr_1(G) (see [1], Theorem 6.34), and χ vanishes outside L. By what we have said above, the character χ is monolithic. Therefore, G − L is a G-class, by assumption. By Lemma 2(a), G is a Frobenius group with kernel L of index 2.

Assume that G is not solvable. Then G/R is solvable and R is not solvable. By the result of the previous paragraph, G/R is abelian. Since this is true for every choice of R, it follows that R = G'. In that case, G is a monolith and all its nonlinear irreducible characters are monolithic, i.e., G is a CZ-group, contrary to the theorem.

Question 1. Classify the groups G such that the character table of G has \( |\text{Irr}_1(G)| + 1 \) zero entries (A_4, S_4 and A_5 satisfy this condition).

Question 2. Study the nonsolvable groups G such that T(χ) is a conjugacy class whenever χ ∈ Irr_1(G) and χ(1) is even (L_2(2^n) satisfies this condition, but Aut(L_2(2^3)) does not satisfy by [1], Theorem 8.17: it has elements of order 6).

Probably, L_2(2^n) are the only simple groups satisfying Problem 2 (see the reasoning in part (i) of the proof of the theorem). We do not know nonsolvable groups G such that T(χ) is a conjugacy class for all χ ∈ Irr_1(G) of odd degree.

Question 3. Classify the groups G such that T(χ) is a conjugacy class for all but one nonlinear irreducible characters χ of G (examples: SL(2,3) and, by [1], Theorem 3.15, all the groups of Question 1).

Question 4. Let G be a nonabelian group. For χ ∈ Irr_1(G), let z(χ) be k(χ) − 1, where T(χ) is a union of k(χ) conjugacy classes. Set z(G) = \( \sum_{\chi \in \text{Irr}_1(G)} z(\chi) \). Classify the simple groups G with small z(G).

Question 5. Classify the groups G such that T(χ)/ker(χ) is a conjugacy class for all nonlinear monolithic characters χ of G. It is easy to show that all such G are solvable.
Let $\chi \in \text{Irr}(G)$. Set $Z(\chi) = \{x \in G \mid |\chi(x)| = |\chi(1)|\}$. The set $Z(\chi)$ is a normal subgroup of $G$ (the quasikernel of $\chi$). It is easy to show that if $\chi \in \text{Irr}_1(G)$ then $T(\chi)Z(\chi) = T(\chi)$. A group $G$ is said to be CZQ-group if it is abelian or $T(\chi)/Z(\chi)$ is a $G/Z(\chi)$-class for every $\chi \in \text{Irr}_1(G)$. The property CZQ is inherited by epimorphic images.

**Question 6.** Classify CZQ-groups.

As in part B of the proof of the theorem, we can show that CZQ-groups are solvable. If $G/Z(G)$ is a CZK-group then $G$ is not necessary a CZQ-group (indeed, if $G$ is a covering group of the symmetric group $S_4$ of degree 4, then $G/Z(G) \cong S_4$ is a CZK-group but $G$ is not a CZQ-group. If a non-nilpotent group $G$ of order 12 has a cyclic subgroup of order 4, then $G$ is a CZQ-group. Probably, the derived length of a CZQ-group $G$ is at most two, unless $G = S_4 \times Z(G)$.

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