HOUSTON JOURNAL OF MATHEMATICS

Volume 24, No. 4, 1998

THE ISOENERGY INEQUALITY FOR A HARMONIC MAP

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Communicated by Robert M. Hardt

ABSTRACT. Let u be a harmonic map from a unit ball B in $\mathbf{R}^n$ into a nonpositively curved manifold, $E(u)$ the energy of u, $E(u|_{\partial B})$ the energy of $u|_{\partial B}$. Then we obtain a relationship between $E(u)$ and $E(u|_{\partial B})$, called the isoenergy inequality, $(n-1)E(u) \leq E(u|_{\partial B})$. When the target manifold has no curvature assumption and u is stationary, it is shown that $(n-2)E(u) \leq E(u|_{\partial B})$. These isoenergy inequalities are sharp because equality is attained by some canonical harmonic maps.

1. INTRODUCTION

It is the isoperimetric inequality that relates the volume of a domain D in $\mathbf{R}^n$ with the volume of the boundary of D. Steiner [13] proved that for any dimension $n \geq 2$

$$n^n \omega_n Volume(D)^{n-1} \leq Volume(\partial D)^n,$$

where ω_n is the volume of a unit ball in $\mathbf{R}^n$ and equality holds if and only if D is a ball. Recently Kleiner [7] and Croke [2] obtained the same isoperimetric inequality for domains in a nonpositively curved n-dimensional simply connected Riemannian manifold for $n = 3$ and 4, respectively. But in a nonnegatively curved Riemannian manifold, the sharp isoperimetric inequality is known only for dimension two (see [1,5]).

In this paper we study an isoperimetric inequality for energy instead of volume. Let us consider a smooth harmonic map u from a closed unit ball $\bar{B} \subset \mathbf{R}^n$ to $\mathbf{R}^m, n \geq 2$. Define $E(u)$ and $E(u|_{\partial B})$ to be the energy of the map u and the energy of the restriction of u to ∂B, respectively. Then is there any relationship between $E(u)$ and $E(u|_{\partial B})$ that resembles the isoperimetric inequality? Here we answer this question affirmatively: we obtain a relationship in a sharp form, called

Supported in part by BSRI-94-1416,GARC,KOSEF(961-0104-020-2).

the isoenergy inequality, for a general target manifold N as well as $\mathbf{R}^m$. First, if N is nonpositively curved, then we show

$$(n-1)E(u) \ \leq\ E(u|_{\partial B}),$$

where equality holds when $N = \mathbf{R}^m$ and u is a linear map. Second, when N is any Riemannian manifold of dimension ≥ 3, we prove

$$(n-2)E(u) \ \leq\ E(u|_{\partial B}),$$

where equality holds if $N = S^{n-1} \subset \mathbf{R}^n$ and $u(x) = x/|x|$.

We would like to thank R. Schoen and L. Simon for their interest in this work.

2. The isoenergy inequality

Assume that M^n, N^k are Riemannian manifolds with N^k isometrically embedded in $\mathbf{R}^m$. We look at a bounded map $u : M \to N$ whose first derivatives are in L^2; such a map is thought of as a map $u = (u_1, \cdots, u_m) : M \to \mathbf{R}^m$ having image almost everywhere in N. Then the energy $E(u)$ of u is defined by

$$E(u) \ =\ \int_M |\nabla u|^2,$$

where $|\nabla u|^2 = \Sigma_{i=1}^m |\nabla u_i|^2$, ∇u_i being the gradient of u_i on M. $|\nabla u|^2$ is called the energy density of u. The critical points of $E(u)$ on the space of maps are referred to as harmonic maps. Thus $u \in C^2$ is harmonic if and only if

$$\Delta_M\, u \ \perp\ T_u N. \tag{1}$$

A harmonic map u is stationary if its energy is critical with respect to variations of the type $u \circ F_t$, where $F_t : M \to M$ is a smooth path of diffeomorphisms of M fixing the boundary. It can be shown that a C^0 harmonic map is stationary and that stationary harmonic maps satisfy the monotonicity property for the scale invariant energy in balls. We state an equivalent form of the monotonicity in the following lemma.

Lemma 2.1. *Let $B_\rho = \{x \in \mathbf{R}^n : |x| < \rho\}$ and $B = B_1$. Suppose $u : B_{1+\epsilon} \to N$, $\epsilon > 0$, is a stationary harmonic map. We have*

$$(n-2)\int_B |\nabla u|^2 \ =\ \int_{\partial B}\left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r}\right|^2\right), \quad r = |x|. \tag{2}$$

Proof. The monotonicity formula [9,11] says

$$\rho^{2-n}\int_{B_\rho} |\nabla u|^2 - \sigma^{2-n}\int_{B_\sigma} |\nabla u|^2 \ =\ 2\int_{B_\rho - B_\sigma} r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2,$$

for $0 < \sigma < \rho < 1+\epsilon$. Noting that $\int_{\partial B_\rho} f = \frac{d}{d\rho}\int_{B_\rho} f$ for almost all ρ, differentiate the formula with respect to ρ and set $\rho = 1$. □

The first isoenergy inequality of this paper holds for a stationary harmonic map of $B_{1+\epsilon}$ into an arbitrary target manifold.

Theorem 2.2. *Let $n \geq 3$ and suppose that $u : B_{1+\epsilon} \to N$, $\epsilon > 0$, is a stationary harmonic map. Then*

$$(n-2)E(u|_B) \;\leq\; E(u|_{\partial B}),$$

where equality can be attained if $N = S^{n-1} \subset \mathbf{R}^n$ and $u(x) = x/|x|$.

PROOF. Let $\overline{\nabla} u_i$ denote the gradient of u_i on ∂B. Observe that

$$E(u|_{\partial B}) \;=\; \int_{\partial B}\sum_i |\overline{\nabla} u_i|^2 \;=\; \int_{\partial B}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right).$$

It follows from (2) that

$$(n-2)E(u|_B) \;=\; E(u|_{\partial B}) - \int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2, \tag{3}$$

which gives the desired inequality. If $u(x) = x/|x|$, then $|\partial u/\partial r| = 0$ and hence equality holds. □

Remark 1. i) It should be mentioned that Lemma 2.1 and Theorem 2.2 fail to hold for nonstationary harmonic maps. See [8,10] for such maps.
ii) We should remark, in relation to Theorem 2.2, J.C.Wood's theorem that any smooth harmonic map u ($n \geq 2$) which is constant on ∂B is constant [14] (see also [15]); the case for weakly harmonic maps is still open.
iii) When $n = 3, 4, 5, 6$, there is a sequence $\{\phi_i\}$ of C^2 harmonic maps $\phi_i : \bar{B} \to S^n \subset \mathbf{R}^{n+1}$ (see [12]) such that $\phi_i(x) = (x, 0)$ for $x \in \partial B$, $E(\phi_i) < E(\phi_{i+1})$, and

$$n - 2 \;=\; \inf_i \frac{E(\phi_i|_{\partial B})}{E(\phi_i)}.$$

Now we prove the isoenergy inequality for a harmonic map from $\bar{B}$ into $\mathbf{R}^m$. Although it is a special case of the isoenergy inequality for harmonic maps into a nonpositively curved space (Theorem 2.4), we state it independently because the proof of the Euclidean case is different and interesting in its own right.

Theorem 2.3. *Suppose that u is a smooth harmonic map from $\bar{B} \subset \mathbf{R}^n, n \geq 2$, into $\mathbf{R}^m$. Then we have the isoenergy inequality*

$$(n-1)E(u) \;\leq\; E(u|_{\partial B}),$$

where equality holds if and only if u is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$.

PROOF. (1) implies

$$\Delta u_i = 0, \ i = 1, \cdots, m.$$

Hence

$$\begin{aligned} E(u) &= \frac{1}{2}\int_B \Delta \sum_i u_i^2 = \int_{\partial B} \sum_i u_i \frac{\partial u_i}{\partial r} \\ &\leq \left[\int_{\partial B} \sum_i u_i^2\right]^{1/2} \left[\int_{\partial B} \left|\frac{\partial u}{\partial r}\right|^2\right]^{1/2} = (*). \end{aligned}$$

Here, without loss of generality, let us assume

$$\int_{\partial B} u_i = 0, \ i = 1, \cdots, m.$$

Using (3) and the fact that $n-1$ is the first eigenvalue of the Laplacian on ∂B, one sees that

$$(*) \leq \left[\frac{1}{n-1}\int_{\partial B} \sum_i |\bar{\nabla} u_i|^2\right]^{1/2} \left[E(u|_{\partial B}) - (n-2)E(u)\right]^{1/2}.$$

Hence by combining the inequalities above one gets

$$E(u)^2 \leq \frac{1}{n-1} E(u|_{\partial B})\left[E(u|_{\partial B}) - (n-2)E(u)\right],$$

which gives the desired isoenergy inequality. Moreover equality holds if and only if u_i is a constant multiple of $\partial u_i / \partial r$ and

$$\Delta_{\partial B}\, u_i + (n-1)u_i = 0, \ i = 1, \cdots, m,$$

which holds if and only if u is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$. □

Theorem 2.4. *If u is a smooth harmonic map from $\bar{B} \subset \mathbf{R}^n, n \geq 2$, to N of nonpositive curvature, then*

$$(n-1)E(u) \leq E(u|_{\partial B}).$$

PROOF. The Bochner formula [3] says that if $u : M^n \to N^k$ is harmonic then

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 = ||\nabla' du||^2 &- \sum_{\alpha,\beta} R_N(u_* e_\alpha, u_* e_\beta, u_* e_\alpha, u_* e_\beta) \\ &+ \sum_i Ric_M(u^*\theta_i, u^*\theta_i) \end{aligned} \tag{4}$$

where ∇' is the pullback connection from TN, $e_1, \cdots, e_n$ is an orthonormal basis for TM and $\theta_1, \cdots, \theta_k$ is orthonormal for T^*N. Hence for $M = \bar{B}$ and N nonpositively curved, $|\nabla u|^2$ is subharmonic. Since the mean value of a subharmonic function on a sphere of radius r centered at the origin is monotonically nondecreasing as a function of r, one can deduce that

$$\frac{E(u)}{\omega_n} \leq \frac{1}{n\omega_n} \int_{\partial B} |\nabla u|^2 = \frac{1}{n\omega_n} \left[(n-2)E(u) + 2 \int_{\partial B} \left| \frac{\partial u}{\partial r} \right|^2 \right],$$

where equality follows from (2). So

$$E(u) \leq \int_{\partial B} \left| \frac{\partial u}{\partial r} \right|^2. \tag{5}$$

Then adding (3) to (5) gives the isoenergy inequality. □

Remark 2. In case u is a harmonic map from a ball B_ρ of radius ρ into N of nonpositive curvature, we obviously have

$$(n-1)E(u) \leq \rho E(u|_{\partial B_\rho}).$$

When the target manifold N is nonpositively curved we have an extension theorem by Eells-Sampson [4] and Hamilton [6]: given $\phi \in C^3(\bar{B}, N)$, there is a harmonic map $u \in C^2(\bar{B}, N)$ such that $u = \phi$ on ∂B, and u is homotopic to ϕ. This theorem allows us to impose a condition on $u|_{\partial B}$, e.g. conformality. Thus we can obtain a mixture of the isoenergy inequality and the isoperimetric inequality as follows.

Corollary 2.5. *Suppose N is nonpositively curved and let $B^l = \{x \in \mathbf{R}^l : |x| < 1\}$, $l = 2, 3$.*
i) *If $u : \bar{B}^2 \to N$ is harmonic and $u|_{\partial B}$ is a constant speed map, then*

$$4\pi Area(u(B^2)) \leq 2\pi E(u) \leq Length(u(\partial B^2))^2.$$

ii) *If $u : \bar{B}^3 \to N$ is harmonic and $u|_{\partial B}$ is conformal, then*

$$E(u) \leq Area(u(\partial B^3)).$$

iii) *If $u : \bar{B} \to N$ is harmonic and $u|_{\partial B}$ is conformal, then*

$$(n\omega_n)^{3-n} E(u)^{n-1} \leq Volume(u(\partial B))^2.$$

Proof. i) The first inequality is well known. For the second, use the constant speed condition and Theorem 2.4.

ii) A special case of iii).

iii) Let k be the length enlargement ratio of $u|_{\partial B}$. Then

$$\begin{aligned} E(u) &\leq \frac{1}{n-1} E(u|_{\partial B}) = \int_{\partial B} k^2 \leq \left[\int_{\partial B} 1\right]^{\frac{n-3}{n-1}} \left[\int_{\partial B} k^{n-1}\right]^{\frac{2}{n-1}} \\ &= (n\omega_n)^{\frac{n-3}{n-1}} Volume(\partial B)^{\frac{2}{n-1}}. \end{aligned}$$

□

References

[1] F. Bernstein, *Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene*, Math. Ann. **60**(1905), 117–136.

[2] C. Croke, *A sharp four dimensional isoperimetric inequality*, Comment. Math. Helv. **59**(1984), 187–192.

[3] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10**(1978), 1–68.

[4] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86**(1964), 109–160.

[5] F. Fiala, *Le problème des isopérimètres sur les surfaces ouvertes à courbure positive*, Comment. Math. Helv. **13**(1940/41), 293–346.

[6] R. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes 471, Springer, 1975.

[7] B. Kleiner, *An isoperimetric comparison theorem*, Invent. math. **108**(1992), 37–47.

[8] C.-C. Poon, *Some new harmonic maps from B^3 to S^2*, J. Diff. Geom., **34**(1991), 165–168.

[9] P. Price, *A monotonicity formula for Yang-Mills fields*, manuscripta math. **43**(1983), 131–166.

[10] T. Rivière, *Everywhere discontinuous harmonic maps into spheres*, Acta Math., **175**(1995), 197–226.

[11] R. Schoen, *Analytic aspects of the harmonic map problem*, Math. Sci. Res. Inst. Publ. Vol. **2**, Springer, Berlin, 1984, 321–358.

[12] R. Schoen and S.-T. Yau. *Lectures on harmonic maps*, International Press, 1997.

[13] J. Steiner, *Sur le maximum et le minimum des figures dans le plan sur la sphère et dans l'espace en général*, J. Reine Angew. Math. **24**(1842), 93–152.

[14] J.C. Wood, *Non-existence of solutions to certain Dirichlet problems for harmonic maps, I*, preprint.

[15] H. Karcher and J.C. Wood, *Non-existence results and growth properties for harmonic maps and forms*, J. Reine Angew. Math. **353**(1984), 165–180.

Received April 30, 1997

Revised version received January 10, 1998

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