ABSOLUTELY SUMMING OPERATORS ON NON COMMUTATIVE $C^*$-ALGEBRAS AND APPLICATIONS

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Abstract. Let $E$ be a Banach space that does not contain any copy of $\ell^1$ and $A$ be a non commutative $C^*$-algebra. We prove that every absolutely summing operator from $A$ into $E^*$ is compact, thus answering a question of Pełczyński.

As application, we show that if $G$ is a compact metrizable abelian group and $\Lambda$ is a Riesz subset of its dual then every countably additive $A^*$-valued measure with bounded variation and whose Fourier transform is supported by $\Lambda$ has relatively compact range. Extensions of the same result to symmetric spaces of measurable operators are also presented.

1. Introduction

It is a well known result that every absolutely summing operator from a $C(K)$-space into a separable dual space is compact. More generally if $F$ is a Banach space with the complete continuity property (CCP) then every absolutely summing operator from any $C(K)$-spaces into $F$ is compact (see [10]).

It is the intention of the present note to study extensions of the above results in the setting of $C^*$-algebras, i.e., replacing the $C(K)$-spaces above by a general non commutative $C^*$- algebra. Typical examples of Banach spaces with the CCP are dual spaces whose preduals do not contain $\ell^1$. Our main result is that if $E$ is a Banach space that does not contain any copy of $\ell^1$ and $A$ is a $C^*$-algebra then every absolutely summing operator from $A$ into $E^*$ is compact. This answered positively the following question raised by Pełczyński (see [17] Problem 3. p. 20): Is every absolutely summing operator from a non commutative $C^*$-algebra into a Hilbert space compact? This result is also used to study relative compactness

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of range of countably additive vector measures with values in duals of non commutative $C^*$-algebras. In [9], Edgar introduced new types of Radon-Nikodym properties associated with Riesz subsets of countable discrete group (see the definition below) as generalization of the usual Radon-Nikodym property (RNP) and the Analytic Radon-Nikodym property (ARNP). These properties were extensively studied in [7] and [8]. In [7], it was shown that if $\Lambda$ is a Riesz subset of a countable discrete group then $L^1[0,1]$ has the type II-$\Lambda$-RNP. In the other hand, Haagerup and Pisier showed in [14] that non commutative $L^1$-spaces have the ARNP so it is a natural question to ask if non commutative $L^1$-spaces have the type II-$\Lambda$-RNP for any given Riesz subset $\Lambda$. In this direction, we obtain (as a consequence of our main result) that if a countably additive vector measure of bounded variation is defined on the $\sigma$-field of Borel subsets of a compact metrizable abelian group and takes its values in a dual of a $C^*$-algebra then its range is relatively compact provided that its Fourier transform is supported by a Riesz subset of the dual group.

Our terminology and notation are standard. We refer to [3] and [4] for definitions from Banach space theory and [11], [16] and [21] for basic properties from the theory of operator algebras and non-commutative integrations.

2. Preliminary Facts and Notations

We recall some definitions and well known facts which we use in the sequel.

Let $\mathcal{A}$ be a $C^*$-algebra, we denote by $\mathcal{A}_h$ the set of Hermitian (self adjoint) elements of $\mathcal{A}$.

**Definition 1.** Let $E$ and $F$ be Banach spaces and $0 < p < \infty$. An operator $T : E \to F$ is said to be absolutely $p$-summing (or simply $p$-summing) if there exists $C$ such that for any finite sequence $(e_1, e_2, \ldots, e_n)$ of $E$, one has

$$\left( \sum_{i=1}^{n} \|T e_i\|^p \right)^{1/p} \leq C \max \left\{ \sum_{i=1}^{n} |\langle e_i, e^* \rangle|^p, \|e^*\| \leq 1 \right\}^{1/p}.$$  

The following class of operators was introduced by Pisier in [18] as extension of the $p$-summing operators in the setting of $C^*$-algebras.

**Definition 2.** Let $\mathcal{A}$ be a $C^*$-algebra and $F$ be a Banach space, $0 < p < \infty$. An operator $T : \mathcal{A} \to F$ is said to be $p$-$C^*$-summing if there exists a constant $C$ such that for any finite sequence $(A_1, \ldots, A_n)$ of Hermitian elements of $\mathcal{A}$, one has

$$\left( \sum_{i=1}^{n} \|T(A_i)\|^p \right)^{1/p} \leq C \left( \sum_{i=1}^{n} |A_i|^p \right)^{1/p} \|\mathcal{A}\|.$$
The smallest constant $C$ for which the above inequality holds is denoted by $C_p(T)$.

It is clear that every $p$-summing operator is $p$-$C^*$-summing but the converse is false in general (we refer to [18] for a counterexample). It should be noted that if the $C^*$-algebra $A$ is commutative then every $p$-$C^*$-summing operator from $A$ into any Banach space is $p$-summing. The following extension of the classical Pietsch’s factorization theorem ([3]) was obtained by Pisier (see Proposition 1.1 of [18]).

**Proposition 2.1.** If $T : A \to F$ is a $p$-$C^*$-summing operator then there exists a positive linear form $f$ of norm less than 1 such that

$$\|Tx\| \leq C_p(T)\{f(|x|^p)\}^{1/p}, \text{ for every } x \in A_h.$$  

Let $\mathcal{M}$ be a von-Neumann algebra and $\mathcal{M}_*$ be its predual. We recall that a functional $f$ on $\mathcal{M}$ is called normal if it belongs to $\mathcal{M}_*$. In [19], it was shown that for the case of von-Neumann algebra and the operator $T$ being weak* to weakly continuous then the positive linear form on the above proposition can be chosen to be normal; namely we have the following lemma (see Lemma 4.1 of [19]).

**Lemma 2.2.** Let $T : \mathcal{M} \to F$ be a $1$-$C^*$-summing operator. If $T$ is weak* to weakly continuous then there exists a linear form $f \in \mathcal{M}_*$ with $\|f\| \leq 1$ such that

$$\|Tx\| \leq C_1(T)f(|x|), \text{ for every } x \in \mathcal{M}_h.$$  

For the next lemma, we recall that for $x \in \mathcal{M}$ and $f \in \mathcal{M}_*$, $xf$ (resp. $fx$) denotes the element of $\mathcal{M}_*$ defined by $xf(y) = f(yx)$ (resp. $fx(y) = f(xy)$) for all $y \in \mathcal{M}$.

**Lemma 2.3.** Let $f$ be a positive linear form on $\mathcal{M}$. For every $x \in \mathcal{M},$

$$f(|\text{Re}(x)| + |\text{Im}(x)|) \leq 2\|xf + fx\|_{\mathcal{M}^*}.\ (2)$$  

**Proof.** Assume first that $x \in \mathcal{M}_h$. The operator $x$ can be decomposed as $x = x_+ - x_-$, where $x_+, x_- \in \mathcal{M}^+$ and $x_+x_- = 0$. There exists a projection $p \in \mathcal{M}$ such that $px_- = x_-p = x_-$ and $(1-p)x_+ = x_+(1-p) = x_+$. This yields the following estimates:

$$f(|x|) = f(x_+ + x_-) = f(x_+) + f(x_-)$$

$$= \frac{1}{2}(xf + fx)(1-p) + \frac{1}{2}(xf + fx)(p)$$

$$\leq \frac{1}{2}(\|xf + fx\||1-p\| + \|xf + fx\||p|)$$

$$\leq \|xf + fx\|.\]
For the general case, fix \( x \in \mathcal{M} \). Let \( a = \text{Re}(x) = (x + x^*)/2 \) and \( b = \text{Im}(x) = (x - x^*)/2i \). Clearly \( x = a + ib \). Using the Hermitian case, we get:

\[
 f(|a| + |b|) \leq \|af + fa\| + \|bf + fb\|
\]

\[
 \leq \|xf + fx\| + \|x^*f + fx^*\|
\]

but since \( f \geq 0 \),

\[
\|x^*f + fx^*\| = \sup\{|f(sz^* + z^*s)|; \ s \in \mathcal{M}, \|s\| \leq 1\}
\]

\[
= \sup\{|f^*(zs^* + s^*z)|; \ s \in \mathcal{M}, \|s\| \leq 1\}
\]

\[
= \sup\{|f(xs^* + s^*x)|; \ s \in \mathcal{M}, \|s\| \leq 1\}
\]

\[
= \|xf + fx\|
\]

which completes the proof of the lemma.

\[
\square
\]

3. Main Theorem

**Theorem 3.1.** Let \( A \) be a \( C^* \)-algebra, \( E \) be a Banach space that does not contain any copy of \( \ell^1 \) and \( T : A \to E^* = F \) be a 1-summing operator then \( T \) is compact.

We will divide the proof into two steps. First we will assume that the \( C^* \)-Algebra \( A \) is a \( \sigma \)-finite von-Neumann algebra and the operator \( T \) is weak* to weakly continuous; then we will show that the general case can be reduced to this case. We refer to [21] p. 78 for the definition and properties of \( \sigma \)-finite von-Neumann algebras.

**Proposition 3.2.** Let \( \mathcal{M} \) be a \( \sigma \)-finite von-Neumann algebra. Let \( T : \mathcal{M} \to E^* \) be a weak* to weakly continuous 1-summing operator then \( T \) is compact.

**Proof.** The operator \( T \) being weak* to weakly continuous and 1-summing, there exist a constant \( C = C_1(T) \) and a normal positive functional \( f \) on \( \mathcal{M} \) such that

\[
\|Tx\| \leq Cf(|x|) \text{ for every } x \in \mathcal{M}_h.
\]

Since the von-Neumann algebra \( \mathcal{M} \) is \( \sigma \)-finite, there exists a faithful normal state \( f_0 \) in \( \mathcal{M}_* \) (see [21] Proposition II-3.19). Replacing \( f \) by \( f + f_0 \), we can assume that the functional \( f \) on the inequality above is a faithful normal state and using Lemma 2.3, we get

\[
(3) \quad \|Tx\| \leq 2C\|xf + fx\|_{\mathcal{M}^*} \text{ for every } x \in \mathcal{M}.
\]

We may equip \( \mathcal{M} \) with the scalar product by setting for every \( x, y \in \mathcal{M} \),

\[
\langle x, y \rangle = f \left( \frac{xy^* + y^*x}{2} \right).
\]
Since $f$ is faithful, $\mathcal{M}$ with $\langle \cdot, \cdot \rangle$ is a pre-Hilbertian. We denote the completion of this space by $L^2(\mathcal{M}, f)$ (or simply $L^2(f)$).

By construction, the inclusion map $J : \mathcal{M} \to L^2(\mathcal{M}, f)$ is bounded and is one to one ($f$ is faithful). On the dense subspace $J(\mathcal{M})$ of $L^2(f)$, we define a map $\theta : J(\mathcal{M}) \to L^2(f)^*$ by $\theta(Jx) = \langle \cdot, J(x^*) \rangle$. The map $\theta$ is clearly linear and is an isometry; indeed for every $x \in \mathcal{M}$, $\|\theta(Jx)\|^2 = \sup_{\|u\| \leq 1} \langle u, J(x^*) \rangle^2 = f(x^*x + xx^*) = \|Jx\|^2$. So it can be extended to a bounded map (that we will denote also by $\theta$) from $L^2(f)$ onto $L^2(f)^*$.

Let $S = J^* \circ \theta \circ J$. The operator is defined from $\mathcal{M}$ into $\mathcal{M}^*$ and we claim that for every $x \in \mathcal{M}$, $Sx = xf + fx$. In fact for every $x, y \in \mathcal{M}$, we have:

$$\begin{align*}
xS(y) &= J^* \circ \theta \circ Jx(y) \\
&= Jx(Jy) \\
&= \langle J(y), J(x^*) \rangle \\
&= f(xy + yx) = (xf + fx)(y).
\end{align*}$$

Notice also that since $f$ is normal, the functionals $xf$ and $fx$ are both normal for every $x \in \mathcal{M}$; therefore $S(\mathcal{M}) \subset \mathcal{M}_*$. Also since $J$ is one to one, $J^*$ has weak* dense range. The latter with the facts that both $J$ and $\theta$ have dense ranges imply that $S(\mathcal{M})$ is weak* dense in $\mathcal{M}^*$ so $S(\mathcal{M})$ is (norm) dense in $\mathcal{M}_*$.

Let us now define a map $L : S(\mathcal{M}) \to E^*$ by $L(xf + fx) = Tx$ for every $x \in \mathcal{M}$. The map $L$ is clearly linear and one can deduce from inequality (3) above that $L$ is bounded so it can be extended as a bounded operator (that we will denote also by $L$) from $\mathcal{M}_*$ into $E^*$. The above means that $T$ can be factored as follows

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{S} & \mathcal{M}_* \\
\downarrow & & \uparrow L \\
E^* & \xrightarrow{T} & \mathcal{M}_*
\end{array}$$

Taking the adjoints we get

$$\begin{array}{ccc}
E & \xrightarrow{T^*} & \mathcal{M}^* \\
\downarrow & & \uparrow S^* \\
L^* & \xrightarrow{L} & \mathcal{M}
\end{array}$$

To conclude the proof of the proposition, let $(e_n)_n$ be a bounded sequence in $E$. Since $E \not\hookrightarrow \ell^1$, we will assume (by taking a subsequence if necessary) that $(e_n)_n$ is weakly Cauchy. We will show that $(T^*(e_n))_n$ is norm-convergent. For that it
is enough to prove that if \((e_n)_n\) is a weakly null sequence in \(E\) then \(\|T^*e_n\|\) converges to zero.

Let \((e_n)_n\) be a weakly null sequence in \(E\), \((L^*(e_n))_n\) is a weakly null sequence in \(\mathcal{M}\). This implies that \(((L^*(e_n))^*)_n\) (the sequence of the adjoints of the \(L^*(e_n)\)’s) is weakly null in \(\mathcal{M}\).

Since \(T\) is 1-summing, it is a Dunford-Pettis operator (i.e takes weakly convergent sequence into norm-convergent sequence). Hence

\[
\lim_{n \to \infty} \|T((L^*e_n)^*)_n\|_{E^*} = 0.
\]

In particular, since \((e_n)_n\) is a bounded sequence in \(E\), we have

\[
\lim_{n \to \infty} \langle T((L^*e_n)^*)_n, e_n \rangle = 0.
\]

But

\[
\langle T((L^*e_n)^*)_n, e_n \rangle = \langle LS((L^*e_n)^*)_n, e_n \rangle = \langle S((L^*e_n)^*)_n, L^*e_n \rangle = \langle \theta \circ J((L^*e_n)^*)_n, J(L^*e_n) \rangle = \langle J(L^*e_n), J(L^*e_n) \rangle_{L^2(f)} = \|J(L^*e_n)\|_{L^2(f)}^2.
\]

So \(\|J(L^*e_n)\|_{L^2(f)} \to 0\) as \(n \to \infty\) and therefore since \(T^* = S^* \circ L^* = J^* \circ \theta \circ J \circ L^*\), we get that \(\lim_{n \to \infty} \|T^*e_n\| = 0\).

This shows that \(\overline{T^*(B_E)}\) is compact and since \(B_E\) is weak* dense in \(B_{E^{**}}\) and \(T^*\) is weak* to weakly continuous, \(T^*(B_{E^{**}}) \subseteq \overline{T^*(B_E)}\) so \(T^*\) (and hence \(T\)) is compact. The proposition is proved.

\[\square\]

To complete the proof of the theorem, let \(\mathcal{A}\) be a \(C^*\)-algebra and \(T: \mathcal{A} \to E^*\) be a 1-summing operator. The double dual \(\mathcal{A}^{**}\) of \(\mathcal{A}\) is a von-Neumann algebra and \(T^{**}: \mathcal{A}^{**} \to E^*\) is 1-summing. Let \((a_n)_n\) be a bounded sequence in \(\mathcal{A}^{**}\). If we denote by \(\mathcal{M}\) the von-Neumann algebra generated by \((a_n)_n\) then the predual \(\mathcal{M}^*\) of \(\mathcal{M}\) is separable and therefore the von-Neumann algebra \(\mathcal{M}\) is \(\sigma\)-finite. Moreover, if we set \(I: \mathcal{M} \to \mathcal{A}^{**}\) the inclusion map then \(I\) is weak* to weak* continuous. Hence \(\mathcal{M}\) and \(T^{**} \circ I\) satisfy the conditions of Proposition 3.2 so \(T^{**} \circ I\) is compact and since the sequence \((a_n)_n\) is arbitrary, the operator \(T^{**}\) (and hence \(T\)) is compact.

\[\square\]

Remark. It should be noted that for the proof of Proposition 3.2, we only require the operator \(T\) to be \(C^*\)-summing and Dunford-Pettis so the conclusion of Proposition 3.2 is still valid for \(C^*\)-summing operators that are Dunford-Pettis.
4. Applications to vector measures

In this section we will provide some applications of the main theorem to study range of countably additive vector measures with values in duals of \( C^* \)-Algebras.

The letter \( G \) will denote a compact metrizable abelian group, \( \hat{G} \) its dual, \( \mathcal{B}(G) \) is the \( \sigma \)-algebra of the Borel subsets of \( G \), and \( \lambda \) the normalized Haar measure on \( G \).

Let \( X \) be a Banach space and \( 1 \leq p \leq \infty \), we will denote by \( L^p(G, X) \) the usual Bochner spaces for the measure space \( (G, \mathcal{B}(G), \lambda); \ M(G, X) \) the space of \( X \)-valued countably additive Borel measures of bounded variation; \( C(G, X) \) the space of \( X \)-valued continuous functions and \( M^\infty(G, X) = \{ \mu \in M(G, X), |\mu| \leq C\lambda \text{ for some } C > 0 \} \).

If \( \mu \in M(G, X) \), we recall that the Fourier transform of \( \mu \) is a map \( \hat{\mu} \) from \( \hat{G} \) into \( X \) defined by \( \hat{\mu}(\gamma) = \int_G \bar{\gamma} \, d\mu \) for \( \gamma \in \hat{G} \).

For \( \Lambda \subset \hat{G} \), we will use the following notation:

\[
L^p_\Lambda(G, X) = \{ f \in L^p(G, X), \ \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \} \\
C_\Lambda(G, X) = \{ f \in C(G, X), \ \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \} \\
M_\Lambda(G, X) = \{ \mu \in M(G, X), \ \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \} \\
M^\infty_\Lambda(G, X) = \{ \mu \in M^\infty(G, X), \ \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}.
\]

We also recall that \( \Lambda \subset \hat{G} \) is called a Riesz subset if \( M_\Lambda(G) = L^1_\Lambda(G) \). We refer to [20] and [13] for detailed discussions and examples of Riesz subsets of dual groups.

The following Banach space properties were introduced by Edgar in [9], and Dowling in [7].

**Definition 3.** Let \( \Lambda \) be a Riesz subset of \( \hat{G} \). A Banach space \( X \) is said to have the type I-\( \Lambda \)-Radon Nikodym Property (resp. type II-\( \Lambda \)-Radon Nikodym property) if \( M^\infty_\Lambda(G, X) = L^\infty_\Lambda(G, X) \) (resp. \( M_\Lambda(G, X) = L^1_\Lambda(G, X) \)).

Our next result deals with property of dual of \( C^* \)-algebras related to the types of Radon-Nikodym properties defined above.

**Theorem 4.1.** Let \( \Lambda \) be a Riesz subset of \( \hat{G} \) and \( A \) be a \( C^* \)-Algebra. If \( F : \mathcal{B}(G) \to A^* \) is a countably additive measure with bounded variation that satisfies \( \hat{F}(\gamma) = 0 \) for \( \gamma \notin \Lambda \) then the range of \( F \) is a relatively compact subset of \( A^* \).

**Proof.** Let \( F : \mathcal{B}(G) \to A^* \) be a measure with bounded variation and \( \hat{F}(\gamma) = 0 \) for \( \gamma \notin \Lambda \). Let \( S : C(G) \to A^* \) be the operator defined by \( Sf = \int f \, dF \). Since
Let $F$ be of bounded variation, the operator $S$ is integral (see [4] Theorem IV-3.3 and Theorem IV-3.12) and therefore $S^*:\mathcal{A}^{**}\to (C(G))^*$ is also integral. Now since $\hat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$, if we denote by $\Lambda' = \{ \gamma \in \hat{G}, \bar{\gamma} \notin \Lambda \}$ then $S(\gamma) = 0$ for all $\gamma \in \Lambda'$ and therefore we have the following factorization:

\[
\begin{array}{c}
C(G) \\ \downarrow q \\
C(G)/C_{\Lambda'}(G) \\ \downarrow L
\end{array} \xrightarrow{S^*} \begin{array}{c}
\mathcal{A}^{**} \\ \downarrow L^{**} \\
(C(G))^* \\ \downarrow Q^*
\end{array}
\]

where $Q$ is the natural quotient map. Taking the adjoints, we get

\[
\begin{array}{c}
\mathcal{A}^{**} \\ \downarrow L^{**} \\
(C(G))^* \\ \downarrow Q^*
\end{array} \xrightarrow{S} \begin{array}{c}
M_{\Lambda}(G) \\
M_{\Lambda}(G)
\end{array}
\]

Since $Q^*$ is the formal inclusion and $S^*$ is 1-summing, the operator $L^*$ is 1-summing. The assumption $\Lambda$ being a Riesz subset implies that $M_{\Lambda}(G) = L_{\Lambda}^1(G)$ is a separable dual (in particular its predual does not contain $\ell^1$). So by Theorem 3.1, $L^*$ (and hence $S$) is compact. This proves that the range of the representing measure $F$ of $S$ is relatively compact (see [4] Theorem II-2.18).

Our next result is a generalization of Theorem 4.1 for the case of symmetric spaces of measurable operators.

Let $\mathcal{M}$ be a semifinite von-Neumann algebra acting on a Hilbert space $H$. Let $\tau$ be a distinguished faithful normal semifinite trace on $\mathcal{M}$.

Let $\mathcal{M}$ be the space of all measurable operators with respect to $(\mathcal{M}, \tau)$ in the sense of [16]; for $a \in \mathcal{M}$ and $t > 0$, the $t^{th}$-s-number (singular number) of $a$ is defined by

\[
\mu_t(a) = \inf\{ \|ae\| : e \in \mathcal{M} \text{ projection with } \tau(I - e) \leq t \}.
\]

The function $t \mapsto \mu_t(a)$ defined on $(0, \tau(I))$ will be denoted by $\mu(a)$. This is a positive non-increasing function on $(0, \tau(I))$. We refer to [11] for complete detailed study of $\mu(a)$.

Let $E$ be a rearrangement invariant Banach function space on $(0, \tau(I))$ (in the sense of [15]). We define the symmetric space $E(\mathcal{M}, \tau)$ of measurable operators by setting

\[
E(\mathcal{M}, \tau) = \{ a \in \mathcal{M} : \mu(a) \in E \}
\]

and $\|a\|_{E(\mathcal{M}, \tau)} = \|\mu(a)\|_E$.

It is well known that $E(\mathcal{M}, \tau)$ is a Banach space and if $E = L^p(0, \tau(I))$ ($1 \leq p \leq \infty$) then $E(\mathcal{M}, \tau)$ coincides with the usual non-commutative $L^p$-space associated
with the von-Neumann algebra $\mathcal{M}$. The space $E(\mathcal{M}, \tau)$ is often referred to as the non-commutative version of the function space $E$. Some Banach space properties of these spaces can be found in [2], [6] and [22].

For the case where the trace $\tau$ is finite, we obtain the following generalization of Theorem 4.1 for symmetric spaces of measurable operators.

**Corollary 4.2.** Assume that $\tau$ is finite. Let $E$ be a rearrangement invariant function space on $(0, \tau(I))$ that does not contain $c_0$ and $\Lambda$ be a Riesz subset of $\hat{G}$. Let $F : \mathcal{B}(G) \to E(\mathcal{M}, \tau)$ be a countably additive measure with bounded variation such that $\hat{F}(\gamma) = 0$ for every $\gamma \notin \Lambda$ then the range of $F$ is relatively compact.

**Proof.** We will begin by reducing the general case to the case where $E(\mathcal{M}, \tau)$ is separable. Since $\mathcal{B}(G)$ is countably generated, the range of $F$ is separable. Choose $(A_n)_n \subset \mathcal{B}(G)$ so that $\{F(A_n), n \geq 1\}$ is dense in $\{F(A), A \in \mathcal{B}(G)\}$. Let $\hat{\mathcal{M}}$ be the von-Neumann algebra generated $I$ and $F(A_n) \,(n \geq 1)$ and $\hat{\tau}$ the restriction of $\tau$ in $\hat{\mathcal{M}}$. Clearly $E(\mathcal{M}, \hat{\tau})$ is a closed subspace of $E(\mathcal{M}, \tau)$ and $F(A) \in E(\hat{\mathcal{M}}, \hat{\tau})$ for all $A \in \mathcal{B}(G)$. Moreover the space $E(\hat{\mathcal{M}}, \hat{\tau})$ is separable (see Lemma 5.6 of [22]). So without loss of generalities, we will assume that $E(\mathcal{M}, \tau)$ is separable. It is a well known fact that $E(\mathcal{M}, \tau)$ is contained in $L^1(\mathcal{M}, \tau) + \mathcal{M}$ and since $\tau$ is finite, $E(\mathcal{M}, \tau) \subset L^1(\mathcal{M}, \tau)$. Let $J : E(\mathcal{M}, \tau) \to L^1(\mathcal{M}, \tau)$ be the formal inclusion. The measure $J \circ F$ is of bounded variation and $J \circ F(\gamma) = J(\hat{F}(\gamma))$ for every $\gamma \in \hat{G}$. One can conclude from Theorem 4.1 that the range of $J \circ F$ is relatively compact in $L^1(\mathcal{M}, \tau)$.

To show that the range of $F$ is relatively compact, fix $h : G \to E(\mathcal{M}, \tau)^{**}$ a weak*-density of $F$ with respect to the Haar measure $\lambda$ (see [5]). We have for each $A \in \mathcal{B}(G)$,

$$F(A) = \text{weak}^* - \int_A h(t) \, d\lambda(t)$$

and

$$|F|(A) = \int_A \|h(t)\| \, d\lambda(t).$$

For each $N \in \mathbb{N}$, let $A_N = \{t \in G, \|h(t)\| \leq N\}$ and $F_N$ the measure defined by $F_N(A) = F(A \cap A_N)$ for all $A \in \mathcal{B}(G)$. Clearly $|F_N| \leq N\lambda$ for every $N \in \mathbb{N}$.

Define $T_N : L^1(G) \to E(\mathcal{M}, \tau)$ by $T_N(f) = \int f(t) \, dF_N(t)$ for every $f \in L^1(G)$. The operator $T_N$ is bounded and we claim that $T_N$ is Dunford-Pettis; for that notice that since the range of $J \circ F$ is relatively compact so is the range of $J \circ F_N$ and therefore the operator $J \circ T_N$ is a Dunford-Pettis operator. The space $E(\mathcal{M}, \tau)$ is separable and $J$ is a semi-embedding (see Lemma 5.7 of [22]) so $J$ is a $G$-embedding (see [1] Proposition 1.8) and one can deduce from Theorem II.6.
of \cite{12}, that $T_N$ is a Dunford-Pettis operator. Hence the range of $F_N$ is relatively compact. Now since
\[ \lim_{N \to \infty} \|F - F_N\| = \lim_{N \to \infty} \int_{G \setminus A_N} \|h(t)\| \, d\lambda(t) = 0, \]
the range of $F$ is relatively compact. \qed

Let us finish by asking the following question:

**Question:** Do non-commutative $L^1$-spaces have type II-$\Lambda$-RNP for any Riesz set $\Lambda$?

In light of Theorem 4.1, the result of Haagerup and Pisier (\cite{14}) and so many properties that have been generalized from classical $L^1$-spaces to non-commutative $L^1$-spaces, one tends to conjecture that the answer of the above question is affirmative.

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