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ABSOLUTELY SUMMING OPERATORS ON NON COMMUTATIVE C^*-ALGEBRAS AND APPLICATIONS

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Abstract. Let E be a Banach space that does not contain any copy of ℓ^1 and $\mathcal{A}$ be a non commutative C^*-algebra. We prove that every absolutely summing operator from $\mathcal{A}$ into E^* is compact, thus answering a question of Pełczyński.

As application, we show that if G is a compact metrizable abelian group and Λ is a Riesz subset of its dual then every countably additive $\mathcal{A}^*$-valued measure with bounded variation and whose Fourier transform is supported by Λ has relatively compact range. Extensions of the same result to symmetric spaces of measurable operators are also presented.

1. Introduction

It is a well known result that every absolutely summing operator from a $C(K)$-space into a separable dual space is compact. More generally if F is a Banach space with the complete continuity property (CCP) then every absolutely summing operator from any $C(K)$-spaces into F is compact (see [10]).

It is the intention of the present note to study extensions of the above results in the setting of C^*-algebras, i.e., replacing the $C(K)$-spaces above by a general non commutative C^*- algebra. Typical examples of Banach spaces with the CCP are dual spaces whose preduals do not contain ℓ^1. Our main result is that if E is a Banach space that does not contain any copy of ℓ^1 and $\mathcal{A}$ is a C^*-algebra then every absolutely summing operator from $\mathcal{A}$ into E^* is compact. This answered positively the following question raised by Pełczynski (see [17] Problem 3. p. 20): Is every absolutely summing operator from a non commutative C^*-algebra into a Hilbert space compact? This result is also used to study relative compactness

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of range of countably additive vector measures with values in duals of non commutative C^*-algebras. In [9], Edgar introduced new types of Radon-Nikodym properties associated with Riesz subsets of countable discrete group (see the definition below) as generalization of the usual Radon-Nikodym property (RNP) and the Analytic Radon-Nikodym property (ARNP). These properties were extensively studied in [7] and [8]. In [7], it was shown that if Λ is a Riesz subset of a countable discrete group then $L^1[0,1]$ has the type II-Λ-RNP. In the other hand, Haagerup and Pisier showed in [14] that non commutative L^1-spaces have the ARNP so it is a natural question to ask if non commutative L^1-spaces have the type II-Λ-RNP for any given Riesz subset Λ. In this direction, we obtain (as a consequence of our main result) that if a countably additive vector measure of bounded variation is defined on the σ-field of Borel subsets of a compact metrizable abelian group and takes its values in a dual of a C^*-algebra then its range is relatively compact provided that its Fourier transform is supported by a Riesz subset of the dual group.

Our terminology and notation are standard. We refer to [3] and [4] for definitions from Banach space theory and [11], [16] and [21] for basic properties from the theory of operator algebras and non-commutative integrations.

2. Preliminary Facts and Notations

We recall some definitions and well known facts which we use in the sequel.

Let $\mathcal{A}$ be a C^*-algebra, we denote by $\mathcal{A}_h$ the set of Hermitian (self adjoint) elements of $\mathcal{A}$.

Definition 1. Let E and F be Banach spaces and $0 < p < \infty$. An operator $T : E \to F$ is said to be absolutely p-summing (or simply p-summing) if there exists C such that for any finite sequence $(e_1, e_2, \dots, e_n)$ of E, one has

$$\left(\sum_{i=1}^{n} \|Te_i\|^p\right)^{1/p} \leq C \max\Big\{\sum_{i=1}^{n} |\langle e_i, e^*\rangle|^p, \|e^*\| \leq 1\Big\}^{1/p}$$

The following class of operators was introduced by Pisier in [18] as extension of the p-summing operators in the setting of C^*-algebras.

Definition 2. Let $\mathcal{A}$ be a C^*-algebra and F be a Banach space, $0 < p < \infty$. An operator $T : \mathcal{A} \to F$ is said to be p-C^*-summing if there exists a constant C such that for any finite sequence $(A_1, \dots, A_n)$ of Hermitian elements of $\mathcal{A}$, one has

$$\left(\sum_{i=1}^{n} \|T(A_i)\|^p\right)^{1/p} \leq C \Big\| \Big(\sum_{i=1}^{n} |A_i|^p\Big)^{1/p} \Big\|_{\mathcal{A}} .$$

The smallest constant C for which the above inequality holds is denoted by $C_p(T)$.

It is clear that every p-summing operator is p-C^*-summing but the converse is false in general (we refer to [18] for a conterexample). It should be noted that if the C^*-algebra $\mathcal{A}$ is commutative then every p-C^*-summing operator from $\mathcal{A}$ into any Banach space is p-summing. The following extension of the classical Pietsch's factorization theorem ([3]) was obtained by Pisier (see Proposition 1.1 of [18]).

Proposition 2.1. *If $T : \mathcal{A} \to F$ is a p-C^*-summing operator then there exists a positive linear form f of norm less than 1 such that*

$$\|Tx\| \leq C_p(T)\{f(|x|^p)\}^{1/p} \ , \ \textit{for every } x \in \mathcal{A}_h.$$

Let $\mathcal{M}$ be a von-Neumann algebra and $\mathcal{M}_*$ be its predual. We recall that a functional f on $\mathcal{M}$ is called normal if it belongs to $\mathcal{M}_*$. In [19], it was shown that for the case of von-Neumann algebra and the operator T being weak* to weakly continuous then the positive linear form on the above proposition can be chosen to be normal; namely we have the following lemma (see Lemma 4.1 of [19]).

Lemma 2.2. *Let $T : \mathcal{M} \to F$ be a 1-C^*-summing operator. If T is weak* to weakly continuous then there exists a linear form $f \in \mathcal{M}_*$ with $\|f\| \leq 1$ such that*

$$\|Tx\| \leq C_1(T)f(|x|) \ , \ \textit{for every } x \in \mathcal{M}_h. \tag{1}$$

For the next lemma, we recall that for $x \in \mathcal{M}$ and $f \in \mathcal{M}^*$, xf (resp. fx) denotes the element of $\mathcal{M}^*$ defined by $xf(y) = f(yx)$ (resp. $fx(y) = f(xy)$) for all $y \in \mathcal{M}$.

Lemma 2.3. *Let f be a positive linear form on $\mathcal{M}$. For every $x \in \mathcal{M}$,*

$$f\left(|\operatorname{Re}(x)| + |\operatorname{Im}(x)|\right) \leq 2\|xf + fx\|_{\mathcal{M}^*}. \tag{2}$$

Proof. Assume first that $x \in \mathcal{M}_h$. The operator x can be decomposed as $x = x_+ - x_-$, where $x_+, x_- \in \mathcal{M}^+$ and $x_+x_- = 0$. There exists a projection $p \in \mathcal{M}$ such that $px_- = x_-p = x_-$ and $(1-p)x_+ = x_+(1-p) = x_+$. This yields the following estimates:

$$\begin{aligned}
f(|x|) &= f(x_+ + x_-) = f(x_+) + f(x_-) \\
&= \frac{1}{2}(xf + fx)(1-p) + \frac{1}{2}(xf + fx)(p) \\
&\leq \frac{1}{2}\left(\|xf + fx\|\|1-p\| + \|xf + fx\|\|p\|\right) \\
&\leq \|xf + fx\|.
\end{aligned}$$

For the general case, fix $x \in \mathcal{M}$. Let $a = \mathrm{Re}(x) = (x+x^*)/2$ and $b = \mathrm{Im}(x) = (x-x^*)/2i$. Clearly $x = a+ib$. Using the Hermitian case, we get:

$$\begin{aligned} f(|a|+|b|) &\leq \|af+fa\| + \|bf+fb\| \\ &\leq \|xf+fx\| + \|x^*f+fx^*\|; \end{aligned}$$

but since $f \geq 0$,

$$\begin{aligned} \|x^*f+fx^*\| &= \sup\{|f(sx^*+x^*s)|;\ s\in\mathcal{M}, \|s\|\leq 1\} \\ &= \sup\{|f^*(xs^*+s^*x)|;\ s\in\mathcal{M}, \|s\|\leq 1\} \\ &= \sup\{|f(xs^*+s^*x)|;\ s\in\mathcal{M}, \|s\|\leq 1\} \\ &= \|xf+fx\|, \end{aligned}$$

which completes the proof of the lemma. □

3. Main Theorem

Theorem 3.1. *Let $\mathcal{A}$ be a C^*-algebra, E be a Banach space that does not contain any copy of ℓ^1 and $T : \mathcal{A} \to E^* = F$ be a 1-summing operator then T is compact.*

We will divide the proof into two steps. First we will assume that the C^*-Algebra $\mathcal{A}$ is a σ-finite von-Neumann algebra and the operator T is weak* to weakly continuous; then we will show that the general case can be reduced to this case. We refer to [21] p. 78 for the definition and properties of σ-finite von-Neumann algebras.

Proposition 3.2. *Let $\mathcal{M}$ be a σ-finite von-Neumann algebra. Let $T : \mathcal{M} \to E^*$ be a weak* to weakly continuous 1-summing operator then T is compact.*

Proof. The operator T being weak* to weakly continuous and 1-summing, there exist a constant $C = C_1(T)$ and a normal positive functional f on $\mathcal{M}$ such that

$$\|Tx\| \leq Cf(|x|) \text{ for every } x \in \mathcal{M}_h.$$

Since the von-Neumann algebra $\mathcal{M}$ is σ-finite, there exists a faithful normal state f_0 in $\mathcal{M}_*$ (see [21] Proposition II-3.19). Replacing f by $f+f_0$, we can assume that the functional f on the inequality above is a faithful normal state and using Lemma 2.3, we get

$$\|Tx\| \leq 2C\|xf+fx\|_{\mathcal{M}^*} \text{ for every } x \in \mathcal{M}. \tag{3}$$

We may equip $\mathcal{M}$ with the scalar product by setting for every $x,\ y\ \in \mathcal{M}$,

$$\langle x, y\rangle = f\left(\frac{xy^*+y^*x}{2}\right).$$

Since f is faithful, $\mathcal{M}$ with $\langle .,. \rangle$ is a pre-Hilbertian. We denote the completion of this space by $L^2(\mathcal{M}, f)$ (or simply $L^2(f)$).

By construction, the inclusion map $J : \mathcal{M} \to L^2(\mathcal{M}, f)$ is bounded and is one to one (f is faithful). On the dense subspace $J(\mathcal{M})$ of $L^2(f)$, we define a map $\theta : J(\mathcal{M}) \to L^2(f)^*$ by $\theta(Jx) = \langle ., J(x^*) \rangle$. The map θ is clearly linear and is an isometry; indeed for every $x \in \mathcal{M}$, $\|\theta(Jx)\|^2 = \sup_{\|u\| \leq 1} \langle u, J(x^*) \rangle^2 = \langle J(x^*), J(x^*) \rangle = f(x^*x + xx^*) = \|Jx\|^2$. So it can be extended to a bounded map (that we will denote also by θ) from $L^2(f)$ onto $L^2(f)^*$.

Let $S = J^* \circ \theta \circ J$. The operator is defined from $\mathcal{M}$ into $\mathcal{M}^*$ and we claim that for every $x \in \mathcal{M}$, $Sx = xf + fx$. In fact for every $x,\ y \in \mathcal{M}$, we have:

$$\begin{aligned} Sx(y) &= J^* \circ \theta \circ Jx(y) \\ &= \theta \circ Jx(Jy) \\ &= \langle J(y), J(x^*) \rangle \\ &= f(xy + yx) = (xf + fx)(y). \end{aligned}$$

Notice also that since f is normal, the functionals xf and fx are both normal for every $x \in \mathcal{M}$; therefore $S(\mathcal{M}) \subset \mathcal{M}_*$. Also since J is one to one, J^* has weak* dense range. The latter with the facts that both J and θ have dense ranges imply that $S(\mathcal{M})$ is weak* dense in $\mathcal{M}^*$ so $S(\mathcal{M})$ is (norm) dense in $\mathcal{M}_*$.

Let us now define a map $L : S(\mathcal{M}) \to E^*$ by $L(xf + fx) = Tx$ for every $x \in \mathcal{M}$. The map L is clearly linear and one can deduce from inequality (3) above that L is bounded so it can be extended as a bounded operator (that we will denote also by L) from $\mathcal{M}_*$ into E^*. The above means that T can be factored as follows

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "T"] \arrow[dr, "S"'] & & E^* \\ & \mathcal{M}_* \arrow[ur, "L"'] & \end{tikzcd}$$

Taking the adjoints we get

$$\begin{tikzcd} E \arrow[rr, "T^*"] \arrow[dr, "L^*"'] & & \mathcal{M}_* \\ & \mathcal{M} \arrow[ur, "S^*"'] & \end{tikzcd}$$

To conclude the proof of the proposition, let $(e_n)_n$ be a bounded sequence in E. Since $E \not\hookrightarrow \ell^1$, we will assume (by taking a subsequence if necessary) that $(e_n)_n$ is weakly Cauchy. We will show that $(T^*(e_n))_n$ is norm-convergent. For that it is enough to prove that if $(e_n)_n$ is a weakly null sequence in E then $(\|T^*e_n\|)_n$ converges to zero.

Let $(e_n)_n$ be a weakly null sequence in E, $(L^*(e_n))_n$ is a weakly null sequence in $\mathcal{M}$. This implies that $((L^*(e_n))^*)_{n\geq 1}$ (the sequence of the adjoints of the $L^*(e_n)$'s) is weakly null in $\mathcal{M}$.

Since T is 1-summing, it is a Dunford-Pettis operator (i.e takes weakly convergent sequence into norm-convergent sequence). Hence

$$\lim_{n\to\infty} \|T((L^*e_n)^*)\|_{E^*} = 0 \ .$$

In particular, since $(e_n)_n$ is a bounded sequence in E, we have

$$\lim_{n\to\infty} \langle T((L^*e_n)^*), e_n\rangle = 0.$$

But

$$\begin{aligned}\langle T((L^*e_n)^*), e_n\rangle &= \langle LS((L^*e_n)^*), e_n\rangle \\ &= \langle S((L^*e_n)^*), L^*e_n\rangle \\ &= \langle \theta \circ J((L^*e_n)^*), J(L^*e_n)\rangle \\ &= \langle J(L^*e_n), J(L^*e_n)\rangle_{L^2(f)} \\ &= \|J(L^*e_n)\|^2_{L^2(f)}.\end{aligned}$$

So $\|J(L^*e_n)\|_{L^2(f)} \to 0$ as $n \to \infty$ and therefore since $T^* = S^* \circ L^* = J^* \circ \theta \circ J \circ L^*$, we get that $\lim_{n\to\infty} \|T^*e_n\| = 0$.

This shows that $\overline{T^*(B_E)}$ is compact and since B_E is weak* dense in $B_{E^{**}}$ and T^* is weak* to weakly continuous, $T^*(B_{E^{**}}) \subseteq \overline{T^*(B_E)}$ so T^* (and hence T) is compact. The proposition is proved. □

To complete the proof of the theorem, let $\mathcal{A}$ be a C^*-algebra and $T : \mathcal{A} \to E^*$ be a 1-summing operator. The double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is a von-Neumann algebra and $T^{**}:\ \mathcal{A}^{**} \to E^*$ is 1-summing. Let $(a_n)_n$ be a bounded sequence in $\mathcal{A}^{**}$. If we denote by $\mathcal{M}$ the von-Neumann algebra generated by $(a_n)_n$ then the predual $\mathcal{M}_*$ of $\mathcal{M}$ is separable and therefore the von-Neumann algebra $\mathcal{M}$ is σ-finite. Moreover, if we set $I : \mathcal{M} \to \mathcal{A}^{**}$ the inclusion map then I is weak* to weak* continuous. Hence $\mathcal{M}$ and $T^{**} \circ I$ satisfy the conditions of Proposition 3.2 so $T^{**} \circ I$ is compact and since the sequence $(a_n)_n$ is arbitrary, the operator T^{**} (and hence T) is compact. □

Remark. It should be noted that for the proof of Proposition 3.2, we only require the operator T to be C^*-summing and Dunford-Pettis so the conclusion of Proposition 3.2 is still valid for C^*-summing operators that are Dunford-Pettis.

4. Applications to vector measures

In this section we will provide some applications of the main theorem to study range of countably additive vector measures with values in duals of C^*-Algebras.

The letter G will denote a compact metrizable abelian group, $\widehat{G}$ its dual, $\mathcal{B}(G)$ is the σ-algebra of the Borel subsets of G, and λ the normalized Haar measure on G.

Let X be a Banach space and $1 \leq p \leq \infty$, we will denote by $L^p(G,X)$ the usual Bochner spaces for the measure space $(G, \mathcal{B}(G), \lambda)$; $M(G,X)$ the space of X-valued countably additive Borel measures of bounded variation; $C(G,X)$ the space of X-valued continuous functions and $M^\infty(G,X) = \{\mu \in M(G,X),\ |\mu| \leq C\lambda \text{ for some } C > 0\}$.

If $\mu \in M(G,X)$, we recall that the Fourier transform of μ is a map $\hat{\mu}$ from $\widehat{G}$ into X defined by $\widehat{\mu}(\gamma) = \int_G \bar{\gamma}\, d\mu$ for $\gamma \in \widehat{G}$.

For $\Lambda \subset \widehat{G}$, we will use the following notation:

$$L^p_\Lambda(G,X) = \{f \in L^p(G,X),\ \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$C_\Lambda(G,X) = \{f \in C(G,X),\ \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$M_\Lambda(G,X) = \{\mu \in M(G,X),\ \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$M^\infty_\Lambda(G,X) = \{\mu \in M^\infty(G,X),\ \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}.$$

We also recall that $\Lambda \subset \widehat{G}$ is called a Riesz subset if $M_\Lambda(G) = L^1_\Lambda(G)$. We refer to [20] and [13] for detailed discussions and examples of Riesz subsets of dual groups.

The following Banach space properties were introduced by Edgar in [9], and Dowling in [7].

Definition 3. Let Λ be a Riesz subset of $\widehat{G}$. A Banach space X is said to have the type I-Λ-Radon Nikodym Property (resp. type II-Λ-Radon Nikodym property) if $M^\infty_\Lambda(G,X) = L^\infty_\Lambda(G,X)$ (resp. $M_\Lambda(G,X) = L^1_\Lambda(G,X)$).

Our next result deals with property of dual of C^*-algebras related to the types of Radon-Nikodym properties defined above.

Theorem 4.1. *Let Λ be a Riesz subset of $\widehat{G}$ and $\mathcal{A}$ be a C^*-Algebra. If $F : \mathcal{B}(G) \to \mathcal{A}^*$ is a countably additive measure with bounded variation that satisfies $\widehat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$ then the range of F is a relatively compact subset of $\mathcal{A}^*$.*

Proof. Let $F : \mathcal{B}(G) \to \mathcal{A}^*$ be a measure with bounded variation and $\widehat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$. Let $S : C(G) \to \mathcal{A}^*$ be the operator defined by $Sf = \int f\, dF$. Since

F is of bounded variation, the operator S is integral (see [4] Theorem IV-3.3 and Theorem IV-3.12) and therefore $S^* : \mathcal{A}^{**} \to (C(G))^*$ is also integral. Now since $\widehat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$, if we denote by $\Lambda' = \{\gamma \in \widehat{G}, \bar{\gamma} \notin \Lambda\}$ then $S(\gamma) = 0$ for all $\gamma \in \Lambda'$ and therefore we have the following factorization:

$$\begin{array}{ccc} C(G) & \xrightarrow{S} & \mathcal{A}^* \\ {\scriptstyle Q}\big\downarrow & \nearrow{\scriptstyle L} & \\ C(G)/C_{\Lambda'}(G) & & \end{array}$$

where Q is the natural quotient map. Taking the adjoints, we get

$$\begin{array}{ccc} \mathcal{A}^{**} & \xrightarrow{S^*} & (C(G))^* \\ {\scriptstyle L^*}\searrow & & \nearrow{\scriptstyle Q^*} \\ & M_\Lambda(G) & \end{array}$$

Since Q^* is the formal inclusion and S^* is 1-summing, the operator L^* is 1-summing. The assumption Λ being a Riesz subset implies that $M_\Lambda(G) = L^1_\Lambda(G)$ is a separable dual (in particular its predual does not contain ℓ^1). So by Theorem 3.1, L^* (and hence S) is compact. This proves that the range of the representing measure F of S is relatively compact (see [4] Theorem II-2.18). $\square$

Our next result is a generalization of Theorem 4.1 for the case of symmetric spaces of measurable operators.

Let $\mathcal{M}$ be a semifinite von-Neumann algebra acting on a Hilbert space H. Let τ be a distinguished faithful normal semifinite trace on $\mathcal{M}$.

Let $\overline{\mathcal{M}}$ be the space of all measurable operators with respect to $(\mathcal{M}, \tau)$ in the sense of [16]; for $a \in \overline{\mathcal{M}}$ and $t > 0$, the t^{th}-s-number (singular number) of a is defined by

$$\mu_t(a) = \inf\{\|ae\| : e \in \mathcal{M} \text{ projection with } \tau(I - e) \leq t\} \ .$$

The function $t \mapsto \mu_t(a)$ defined on $(0, \tau(I))$ will be denoted by $\mu(a)$. This is a positive non-increasing function on $(0, \tau(I))$. We refer to [11] for complete detailed study of $\mu(a)$.

Let E be a rearrangement invariant Banach function space on $(0, \tau(I))$ (in the sense of [15]). We define the symmetric space $E(\mathcal{M}, \tau)$ of measurable operators by setting

$$E(\mathcal{M}, \tau) = \{a \in \overline{\mathcal{M}};\ \mu(a) \in E\}$$

and $\|a\|_{E(\mathcal{M},\tau)} = \|\mu(a)\|_E$.

It is well known that $E(\mathcal{M}, \tau)$ is a Banach space and if $E = L^p(0, \tau(I))$ $(1 \leq p \leq \infty)$ then $E(\mathcal{M}, \tau)$ coincides with the usual non-commutative L^p-space associated with the von-Neumann algebra $\mathcal{M}$. The space $E(\mathcal{M},\tau)$ is often referred to as the non-commutative version of the function space E. Some Banach space properties of these spaces can be found in [2], [6] and [22].

For the case where the trace τ is finite, we obtain the following generalization of Theorem 4.1 for symmetric spaces of measurable operators.

Corollary 4.2. *Assume that τ is finite. Let E be a rearrangement invariant function space on $(0,\tau(I))$ that does not contain c_0 and Λ be a Riesz subset of $\widehat{G}$. Let $F : \mathcal{B}(G) \to E(\mathcal{M},\tau)$ be a countably additive measure with bounded variation such that $\widehat{F}(\gamma) = 0$ for every $\gamma \notin \Lambda$ then the range of F is relatively compact.*

Proof. We will begin by reducing the general case to the case where $E(\mathcal{M},\tau)$ is separable. Since $\mathcal{B}(G)$ is countably generated, the range of F is separable. Choose $(A_n)_n \subset \mathcal{B}(G)$ so that $\{F(A_n), n \geq 1\}$ is dense in $\{F(A), A \in \mathcal{B}(G)\}$. Let $\widetilde{\mathcal{M}}$ be the von-Neumann algebra generated I and $F(A_n)$ $(n \geq 1)$ and $\tilde{\tau}$ the restriction of τ in $\widetilde{\mathcal{M}}$. Clearly $E(\widetilde{\mathcal{M}},\tilde{\tau})$ is a closed subspace of $E(\mathcal{M},\tau)$ and $F(A) \in E(\widetilde{\mathcal{M}},\tilde{\tau})$ for all $A \in \mathcal{B}(G)$. Moreover the space $E(\widetilde{\mathcal{M}},\tilde{\tau})$ is separable (see Lemma 5.6 of [22]). So without loss of generalities, we will assume that $E(\mathcal{M},\tau)$ is separable. It is a well known fact that $E(\mathcal{M},\tau)$ is contained in $L^1(\mathcal{M},\tau) + \mathcal{M}$ and since τ is finite, $E(\mathcal{M},\tau) \subset L^1(\mathcal{M},\tau)$. Let $J : E(\mathcal{M},\tau) \to L^1(\mathcal{M},\tau)$ be the formal inclusion. The measure $J \circ F$ is of bounded variation and $\widehat{J \circ F}(\gamma) = J(\widehat{F}(\gamma))$ for every $\gamma \in \widehat{G}$. One can conclude from Theorem 4.1 that the range of $J \circ F$ is relatively compact in $L^1(\mathcal{M},\tau)$.

To show that the range of F is relatively compact, fix $h : G \to E(\mathcal{M},\tau)^{**}$ a weak*-density of F with respect to the Haar measure λ (see [5]). We have for each $A \in \mathcal{B}(G)$,

$$F(A) = \text{weak*} - \int_A h(t)\ d\lambda(t)$$

and

$$|F|(A) = \int_A \|h(t)\|\ d\lambda(t).$$

For each $N \in \mathbb{N}$, let $A_N = \{t \in G, \|h(t)\| \leq N\}$ and F_N the measure defined by $F_N(A) = F(A \cap A_N)$ for all $A \in \mathcal{B}(G)$. Clearly $|F_N| \leq N\lambda$ for every $N \in \mathbb{N}$.

Define $T_N : L^1(G) \to E(\mathcal{M},\tau)$ by $T_N(f) = \int f(t)\, dF_N(t)$ for every $f \in L^1(G)$. The operator T_N is bounded and we claim that T_N is Dunford-Pettis; for that notice that since the range of $J \circ F$ is relatively compact so is the range of $J \circ F_N$ and therefore the operator $J \circ T_N$ is a Dunford-Pettis operator. The space $E(\mathcal{M},\tau)$ is separable and J is a semi-embedding (see Lemma 5.7 of [22]) so J is a G_δ-embedding (see [1] Proposition 1.8) and one can deduce from Theorem II.6

of [12], that T_N is a Dunford-Pettis operator. Hence the range of F_N is relatively compact. Now since

$$\lim_{N\to\infty} \|F - F_N\| = \lim_{N\to\infty} \int_{G\setminus A_N} \|h(t)\| \, d\lambda(t) = 0,$$

the range of F is relatively compact. □

Let us finish by asking the following question:

Question: Do non-commutative L^1-spaces have type II-Λ-RNP for any Riesz set Λ?

In light of Theorem 4.1, the result of Haagerup and Pisier ([14]) and so many properties that have been generalized from classical L^1-spaces to non-commutative L^1-spaces, one tends to conjecture that the answer of the above question is affirmative.

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