Houston Journal of Mathematics

Volume 25, No. 4, 1999

VOLUME FORMS IN FINSLER SPACES

PAUL CENTORE
Communicated by the editors

Abstract. This paper considers two possible volume forms on a Finsler space and uses them to characterize Riemannian spaces and state a condition which Berwald spaces must satisfy. The first form is Busemann's previously known volume form, and the second is the volume form arising from a Riemannian metric canonically associated to the Finsler metric. The first form always exceeds the second; they agree if and only if the Finsler manifold actually is Riemannian. In a Berwald space, the "ratio" of the two forms is a constant.

1. Main Results

Finsler manifolds are a natural class of metric spaces; they generalize Riemannian spaces yet possess many similarities. For example, Finsler spaces possess geodesics and an exponential map, and [Busemann] has exhibited a natural Finsler volume form. After a preliminary exposition of these, this paper associates to any Finsler manifold a canonical Riemannian metric.

Definition 1. Let (M^n, F) be an n-dimensional Finsler manifold. In the coordinate system $(x^i, X^j = \frac{\partial}{\partial x^j})$ for some neighborhood in M, let

$$I_x = \{X \in T_xM | F(X) \leq 1\} \tag{1.1}$$

be the unit indicatrix in T_xM. Define the symmetric, positive-definite, twice-contravariant tensor

$$K^{ij}(x) = (n+2)\frac{\int_{I_x} X^i X^j dX}{\int_{I_x} dX}. \tag{1.2}$$

This tensor (or its inverse) is the *osculating Riemannian metric* for (M^n, F).

In addition to Busemann's form, this Riemannian metric gives another volume form and the comparison of the two yields an inequality which characterizes Riemannian spaces as "extremal cases" of Finsler spaces.

Theorem 1.1. *Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij}. Then $\omega(x) \geq k(x)$, and $\omega(x) = k(x)$ if and only if (M, F) actually is Riemannian, with metric K^{ij}.*

Furthermore, the "ratio" of these two volume forms yields a scalar invariant function $\mathcal{V}(x)$ for a Finsler manifold.

Definition 2. Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij}. Define

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}.$$

$\mathcal{V}(x)$ is clearly identically 1 only on a Riemannian space, and we show that, on a Berwald space (that is, a Finsler space whose geodesic equations

$$\ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k \tag{1.3}$$

are formally identical to the Riemannian equations in that the Christoffel symbols Γ^i_{jk} depend only on x), $\mathcal{V}(x)$ is a constant.

Theorem 1.2. *Let (M, F) be a Berwald manifold, with volume invariant $\mathcal{V}(x)$. Then $\mathcal{V}(x)$ is constant.*

The author obtained the results in this paper while writing his doctoral thesis at the University of Toronto, where they also appear.

2. Preliminaries

We will use the following notation. M or M^n is a manifold of dimension n with local co-ordinates x^i around a point x. If $X \in T_xM$, then $X = X^i \frac{\partial}{\partial x^i}$, where the X^i are co-ordinates for the tangent bundle canonically induced from the x^i for the base manifold. F is a Finsler metric, i.e. a function $F : TM \longrightarrow \mathbb{R}$ where

1. F is positive-definite: $F(x, X) \geq 0$, with equality iff $X = \vec{0}$.
2. F is smooth except on the zero-section: $F|_{TM\setminus\{(x,\vec{0})|x\in M\}}$ is C^∞.
3. F is strictly convex: at any (x, X), rank $\left[\frac{\partial^2 F}{\partial X^i \partial X^j}\right] = n - 1$.
4. F is homogeneous: $F(x, kX) = |k|F(x, X)$, for all $k \in \mathbb{R}$.

Finsler metrics are a natural generalization of Riemannian metrics. A Finsler space (M^n, F) is Riemannian if and only if F has the form

$$F^2(x, X) = g_{ij}(x)X^iX^j,$$

where the coefficients g_{ij} are independent of the tangent vector X. When this happens, every unit ball, or indicatrix I_x, at any point x,

$$I_x = \{X \in T_xM | F(X) \leq 1\},$$

is an ellipsoid.

Several commonly used Finsler quantities appear also in Riemannian space, and generally have the same interpretation there, though most Finsler quantities are functions of TM rather than M. Some frequently used quantities and relations:

$$\begin{aligned} g_{ij}(x,X) &:= \frac{1}{2}\frac{\partial^2 F^2(x,X)}{\partial X^i \partial X^j} \\ F^2(X^i \frac{\partial}{\partial x^i}) &= g_{ij}(x,X)X^iX^j \\ g^{jk}g_{ij} &= \delta_i^k \\ \Gamma^i_{jk}(x,X) &:= \frac{1}{2}g^{ir}(x,X)\left(\frac{\partial g_{jr}}{\partial x^k}(x,X) - \frac{\partial g_{jk}}{\partial x^r}(x,X) + \frac{\partial g_{rk}}{\partial x^j}(x,X)\right) \end{aligned}$$

As in Riemannian spaces, define, for a Finsler manifold, the distance between a pair of points to be the length of the shortest curve, or geodesic, joining them, where a curve's length is the integral of the "lengths" $F(X(t))$ of its tangent vectors $X(t)$. Let M be a differentiable manifold with a Finsler metric F. For $x \in M$, let $U \subset M$ be a co-ordinate chart with $x \in U$. Now define a function $\rho_x : U \longrightarrow \mathbb{R}$, where $\rho_x(y)$ is the Finsler distance from y to x. Then ρ_x is non-negative, 0 only at x, continuous at x, and smooth on $U \setminus \{x\}$.

We can also use F to define a nowhere-zero volume form [Busemann, §6]. To see how, note that, as a metric space, (M, F) has defined on it a natural Hausdorff measure. Choose the volume form which generates the n-dimensional measure, i.e. take, in the co-ordinate system (x^i, X^i),

$$\omega(x)dx = \left(\frac{\kappa_n}{\int_I dX}\right) dx,$$

where κ_n is the volume of the unit sphere in $\mathbb{R}^n$, and dx (resp. dX) abbreviates $dx^1dx^2 \dots dx^n$ (resp. $dX^1dX^2 \dots dX^n$). κ_n is chosen so that Busemann's form generalizes the Riemannian volume form.

Like a Riemannian manifold, a Finsler manifold also has an intrinsically defined exponential map, which sends one-dimensional subspaces of a tangent space isometrically onto geodesics. A geodesic in a Finsler manifold is a parametrized path $x(t) = (x^1(t), x^2(t), \dots x^n(t))$ which satisfies the differential equation

$$\ddot{x}^i = -\Gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k, \tag{2.1}$$

where a dot above the x indicates differentiation with respect to t. We note, in contradistinction to the Riemannian case, that the Christoffel symbols Γ^i_{jk} are functions on the tangent bundle rather than the manifold. This equation allows us to write exp in local co-ordinates around a point $p \in M$ as

$$\begin{aligned} \exp : T_pM &\longrightarrow M \\ x^i(\exp(X))) &= X^i - \frac{1}{2}\Gamma^i_{jk}(X)X^jX^k + O(|X|^3) \end{aligned}$$

In a Riemannian space, exp is C^∞ everywhere on T_pM; in a Finsler space, exp is C^∞ everywhere except the origin $(p, \vec{0})$, at which it is only C^1 [Rund, Chap. 3, §6].

One of the results of this paper involves a special class of spaces, Berwald spaces, which contains Riemannian spaces and is contained in Finsler spaces.

Definition 3 (AIM, §3.1.2)**.** Let (M, F) be a Finsler manifold. Then (M, F) is a Berwald space if, for any $x \in M$, and $X \in T_xM$,

$$\frac{\partial}{\partial X^l}(\Gamma^i_{jk})\bigg|_{(x,X)} = 0;$$

equivalently, we could say that the Christoffel symbols Γ^i_{jk} depend only on the base-point x and not on the tangent vector X.

In a Berwald space, the equations of geodesics (2.1) simplify to

$$\ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k, \tag{2.2}$$

which is formally equivalent to the Riemannian case. This simple expression ensures the smoothness of normal co-ordinates on a Berwald manifold. We obtain normal co-ordinates in a Finsler space just as we do in a Riemannian space: at a fixed point $p \in M$, we use the pull-back of the exponential map to move a neighborhood of $p \in M$ to a neighborhood of $\vec{0} \in T_pM$. The co-ordinates on T_pM are then used as normal co-ordinates for that neighborhood. In the Riemannian case, we typically make a further adjustment so that

$$F^2(p, X) = X^iX^i. \tag{2.3}$$

This normalization is not possible in a non-Riemannian Finsler space. Because exp is only C^1 at p, normal co-ordinates are also only C^1 at p (but C^∞ away from p) in a general Finsler space. In a Berwald space, however, 2.2 guarantees that both exp and normal co-ordinates are C^∞ *at* p as well as away from it. This added smoothness is essential in what follows.

3. The Osculating Riemannian Metric

Riemannian spaces are the best-known and most thoroughly studied class of Finsler spaces, so it makes sense to ask how the two are related. As a result of his investigation of Laplacians on Finsler spaces, [Centore] arrived at a Riemannian metric canonically associated to a Finsler metric.

Definition 4. Let (M^n, F) be an n-dimensional Finsler manifold. In the coordinate system (x^i, X^j) , with

$$I_x = \{X \in T_xM | F(X) \leq 1\} \tag{3.1}$$

the unit indicatrix in T_xM, define the symmetric, positive-definite, twice-contravariant tensor

$$K^{ij}(x) = (n+2)\frac{\int_{I_x} X^i X^j dX}{\int_{I_x} dX}. \tag{3.2}$$

This tensor (or its inverse) is the *osculating Riemannian metric* for (M^n, F).

This Riemannian metric was known earlier [BCS], and its appearance in this new context raised a natural question. We have two volume forms: Busemann's form, and, now, the volume form arising from the osculating Riemannian metric. If we know only these two volume forms, can we say anything about the Finsler space? The following theorem contains the answer (the "equality if and only if" part of this theorem was also known to [BL]).

Theorem 3.1. *Let $\mathcal{B} \subset \mathbb{R}^n = \{(X^1, X^2, \ldots, X^n) | X^i \in \mathbb{R}^n\}$ be a bounded, open, measurable set. Let $K^{ij} := (n+2)\frac{\int_{\mathcal{B}} X^i X^j dX}{\int_{\mathcal{B}} dX}$. We know that the components K^{ij} are the inverse components of that ellipsoid $\mathcal{E}$ which is the unit ball of the Euclidean metric $F^2(X) = K_{jk}X^jX^k$, where $K^{ij}K_{jk} = \delta^i_k$. Then*

$$\int_{\mathcal{B}} dX \leq \int_{\mathcal{E}} dX,$$

with equality if and only if

$$\mathcal{B} = \mathcal{E}.$$

To see the significance of this result, think of $\mathcal{B}$ as a unit indicatrix I_x of some Finsler metric F. Saying $\mathcal{B}$ is an ellipsoid $\mathcal{E}$ is saying that F is Riemannian at the point x. If I_x is an ellipsoid at *every* point x, then in fact the Finsler metric is Riemannian. The proof is a pointwise proof, that is, it uses solely the Finsler function restricted to the tangent space (which is isomorphic to $\mathbb{R}^n$) at one point, and takes no account of neighboring points or even infinitesimal changes in the metric. The only data needed for the proof is the unit Finsler ball $\mathcal{B}$ at a particular point, and we don't need either $\mathcal{B}$'s convexity or its symmetry, two hallmarks of a Finsler metric.

PROOF. Choose co-ordinates so that $K^{ij} = \delta^{ij}$, i.e

$$(n+2)\frac{\int_{\mathcal{B}} X^i X^j dX}{\int_{\mathcal{B}} dX} = \delta^{ij}. \tag{3.3}$$

The ellipsoid arising from these co-ordinates is just the unit sphere

$$E = \{(X^1, X^2, \ldots, X^n) \in \mathbb{R}^n | \Sigma(X^i)^2 = 1\},$$

and if we can obtain the result in this case, then we can obtain the result for any ellipsoid $\mathcal{E}$ simply by an appropriate linear transformation.

We first prove "equality if and only if," i.e. that $\int_{\mathcal{B}} dX = \int_E dX$ implies $\mathcal{B} = E$. Let $(r, \theta_1, \theta_2, \ldots, \theta_{n-1})$ be the usual spherical co-ordinates. Then

$$\begin{aligned} \int_{\mathcal{B}} r^2 dX &= \int_{\mathcal{B}} \Sigma(X^i)^2 dX \\ &= \frac{n}{n+2}\int_{\mathcal{B}} dX \qquad \text{(using (3.3))} \\ &= \frac{n}{n+2}\int_E dX \qquad \text{(by hypothesis)} \\ &= \int_E r^2 dX \text{(because } E \text{ is the unit sphere)} \\ \therefore \int_{\mathcal{B}} r^2 dX &= \int_E r^2 dX. \end{aligned} \tag{3.4}$$

We need only this equation and the hypothesis

$$\int_{\mathcal{B}} dX = \int_E dX \tag{3.5}$$

for the proof of the equality case.

Now consider that

$$\begin{aligned} \mathcal{B} \cup (E \backslash \mathcal{B}) &= E \cup (\mathcal{B} \backslash E); \\ \therefore \int_{\mathcal{B}} h(X) dX + \int_{E \backslash \mathcal{B}} h(X) dX &= \int_E h(X) dX + \int_{\mathcal{B} \backslash E} h(X) dX \end{aligned}$$

for any function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$. In particular, for the two cases $h(X) = r^2$ and $h(X) = 1$, expressions (3.4) and (3.5) allow us to cancel the first integral on each side:

$$\int_{E\backslash\mathcal{B}} r^2 dX, \quad = \quad \int_{\mathcal{B}\backslash E} r^2 dX, \tag{3.6}$$

$$\int_{E\backslash\mathcal{B}} dX \quad = \quad \int_{\mathcal{B}\backslash E} dX. \tag{3.7}$$

Note, however, that $r^2 \leq 1$ inside E and thus on $E\backslash\mathcal{B}$, while $r^2 \geq 1$ outside E and thus on $\mathcal{B}\backslash E$, i.e.

$$r^2|_{\mathcal{B}\backslash E} \geq r^2|_{E\backslash\mathcal{B}}. \tag{3.8}$$

(3.7) says that $E\backslash\mathcal{B}$ and $\mathcal{B}\backslash E$ have the same volume, yet (3.6) says that the positive function r^2, which is bigger on $\mathcal{B}\backslash E$ than on $E\backslash\mathcal{B}$ by (3.8), has the *same* integral over these two regions. This is possible only if $E\backslash\mathcal{B}$ and $\mathcal{B}\backslash E$ are empty, i.e.

$$E = \mathcal{B}.$$

To prove the inequality, again use the function r^2 in the set identity

$$\mathcal{B} \cup E\backslash\mathcal{B} \quad = \quad E \cup \mathcal{B}\backslash E :$$

$$\int_{\mathcal{B}} r^2 dX + \int_{E\backslash\mathcal{B}} r^2 dX \quad = \quad \int_{E} r^2 dX + \int_{\mathcal{B}\backslash E} r^2 dX. \tag{3.9}$$

As before,

$$r^2 \geq 1 \text{ on } \mathcal{B}\backslash E \Longrightarrow \int_{\mathcal{B}\backslash E} r^2 dX \quad \geq \quad \int_{\mathcal{B}\backslash E} dX, \tag{3.10}$$

$$r^2 \leq 1 \text{ on } E\backslash\mathcal{B} \Longrightarrow \int_{E\backslash\mathcal{B}} r^2 dX \quad \leq \quad \int_{E\backslash\mathcal{B}} dX, \tag{3.11}$$

$$\text{and} \int_{E} r^2 dX \quad = \quad \frac{n}{n+2} \int_{E} dX.$$

Use the hypothesis $(n+2)\frac{\int_B X^i X^j dX}{\int_B dX} = \delta^{ij}$ to get

$$\int_{\mathcal{B}} r^2 dX = \frac{n}{n+2} \int_{\mathcal{B}} dX \tag{3.12}$$

Substitute (3.10)-(3.12) into (3.9):

$$\frac{n}{(n+2)} \int_{\mathcal{B}} dX + \int_{E\backslash\mathcal{B}} dX \geq \frac{n}{(n+2)} \int_{E} dX + \int_{\mathcal{B}\backslash E} dX.$$

Use the set identities

$$\begin{aligned} \mathcal{B} &= (E \cup \mathcal{B}) \backslash (E \backslash \mathcal{B}) \\ E &= (E \cup \mathcal{B}) \backslash (\mathcal{B} \backslash E) \end{aligned}$$

on the first term on each side:

$$\begin{aligned} & \frac{n}{(n+2)} \left(\int_{E \cup \mathcal{B}} dX - \int_{E \backslash \mathcal{B}} dX \right) + \int_{E \backslash \mathcal{B}} dX \\ \geq \ & \frac{n}{(n+2)} \left(\int_{E \cup \mathcal{B}} dX - \int_{\mathcal{B} \backslash E} dX \right) + \int_{\mathcal{B} \backslash E} dX. \end{aligned}$$

Cancel the term $\frac{n}{(n+2)} \int_{E \cup \mathcal{B}} dx$, and divide by $\frac{2}{(n+2)}$ to get

$$\begin{aligned} \int_{E \backslash \mathcal{B}} dX &\geq \int_{\mathcal{B} \backslash E} dX \\ \int_{E \cap \mathcal{B}} dX + \int_{E \backslash \mathcal{B}} dX &\geq \int_{E \cap \mathcal{B}} dX + \int_{\mathcal{B} \backslash E} dX \\ \int_{E} dX &\geq \int_{\mathcal{B}} dX \ . \end{aligned} \tag{3.13}$$

□

This theorem characterizes Riemannian spaces as a subset of Finsler spaces solely in terms of volume functions. Typically we characterize Riemannian spaces as Finsler spaces whose Cartan tensor

$$C_{ijk}(x, X) := \frac{1}{4} \frac{\partial^3 F^2}{\partial X^i \partial X^j \partial X^k}(x, X)$$

vanishes. The Cartan tensor is a third-order tensor, whose components are functions of a manifold's $2n$-dimensional tangent bundle. The osculating Riemannian volume form is a zero-order tensor density, whose components are functions of the n-dimensional manifold. Thus the volume form criterion is simpler, and should be easier to apply in many circumstances. In technical terms, we state

Theorem 3.2. *Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij}. Then $\omega(x) \geq k(x)$, and $\omega(x) = k(x)$ if and only if (M, F) actually is Riemannian, with metric K^{ij}.*

Proof. If $\mathcal{B}(x)$ is the Finsler unit ball at a point x, then the coefficient of Busemann's volume form is

$$\omega(x) = \frac{\kappa_n}{\int_{\mathcal{B}} dX}.$$

The coefficient of the volume form from the osculating Riemannian metric, on the other hand, is

$$k(x) = \frac{\kappa_n}{\int_{\mathcal{E}} dX}.$$

where $\mathcal{E}$ is the unit ball of the osculating Riemannian metric at x. By Theorem 1, we always have

$$\int_{\mathcal{B}} dX \leq \int_{\mathcal{E}} dX.$$

Therefore $k(x) \leq \omega(x)$, with equality exactly when $\mathcal{B} = \mathcal{E}$, i.e. when (M, F) *is* the Riemannian space (M, K^{ij}). $\square$

4. The Volume Invariant $\mathcal{V}(x)$

Theorem 2 suggests the definition of a new quantity, an invariant volume function reminiscent of the one defined by Bao & Shen [BS]. Given two volume forms as above, we can always consider their "ratio," that is, the (sole) component of the first form in some co-ordinate system, divided by the (sole) component of the second form in that co-ordinate system. The result is clearly a scalar invariant.

Definition 5. Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij}. Define

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}, \tag{4.1}$$

or, substituting in expressions for k and ω:

$$= \quad \kappa_n (n+2)^n \left(\int_I dX \right)^{n+1} \det \left[\int_I X^i X^j dX \right]. \tag{4.2}$$

Because $k(x) \leq \omega(x)$, and both $k(x)$ and $\omega(x)$ are always positive, we have

$$0 < \mathcal{V}(x) \leq 1 \tag{4.3}$$

for any $x \in M$. Furthermore, by Theorem 2,$\mathcal{V}(x) \equiv 1$ if and only if the Finsler manifold is actually Riemannian. Already we see a difference between $\mathcal{V}(x)$ and Bao & Shen's invariant $\mathrm{Vol}(x)$: on a Riemannian space $\mathrm{Vol}(x) \equiv \kappa_{n-1}$ [BS, §1], but, as they remark in their second-last paragraph, some non-Riemannian spaces also take on the value κ_{n-1} identically. (The disagreement of the numbers 1 and

κ_{n-1} is not important here, because this could be remedied by a scaling factor; the important point is that the volume functions are *constant* over M.)

Apart from working out the derivative of $\mathrm{Vol}(x)$, Bao & Shen also prove the important result that $\mathrm{Vol}(x)$ is constant (with a constant value not generally κ_{n-1}) on any Landsberg space. We will prove the similar result that $\mathcal{V}(x)$ is constant on any Berwald space. Bao & Shen's method involved the Chern connection on points of the unit indicatrix. Our method will be radically different. We will use the fact that normal co-ordinates on a Berwald manifold are C^∞, instead of just C^1 as on a general Finsler manifold. We start by proving Finsler versions of the First-Variation Formula and the Gauss Lemma, then prove the essential fact that the derivatives of a Berwald metric vanish in normal co-ordinates, and finally prove $\mathcal{V}(x)$ is constant.

4.1. **First-Variation Formula.** . Let $\Sigma : (-\epsilon, \epsilon) \times [a, b] \to (M^n, F)$ be a smooth variation. Let $s \in (-\epsilon, \epsilon), t \in [a, b]$. In co-ordinates x^i, $(s, t) \mapsto \Sigma(s, t) = x^i(s, t)$, for all $i = 1..n$. Define

$$T := d\Sigma\left(\frac{\partial}{\partial t}\right) \quad ; \quad T^i := \frac{\partial x^i}{\partial t}$$
$$V := d\Sigma\left(\frac{\partial}{\partial s}\right) \quad ; \quad V^i := \frac{\partial x^i}{\partial s}$$

We require that $F(T) = c_s$ (a constant depending on s but not on t). Let

$$\begin{aligned} L(s) &:= \text{length of the } s\text{-curve in the variation } \Sigma \\ &= \int_a^b F(T)dt \\ &= \int_a^b F(\dot{x}^i(s,t))dt \end{aligned}$$

With the above definitions and conditions, we have the Finsler First-Variation Formula:

$$\left.\frac{\partial L(s)}{\partial s}\right|_{s=0} = \frac{1}{c_0}\left(\langle V, T\rangle_T\big|_a^b - \int_a^b \left\langle V, \left(\frac{\partial T^j}{\partial t} + \Gamma^j_{kl}(T)T^kT^l\right)\frac{\partial}{\partial x^j}\right\rangle_T dt\right)$$

Gauss Lemma. Let Σ be a variation as above, but now insist in addition that $\sigma_s(t) = \Sigma(s, t)$ is a geodesic for every s, and $L(s) = c$ for any s (i.e. every geodesic in the variation has the same length). Also require that $V(s, a) = 0$ for every s, so that all the geodesics originate from the same point $\Sigma(0,0)$. Thus Σ sweeps out a curve on the sphere of radius c around $\Sigma(0,0)$. Then

$$\langle V(s,b), T(s,b)\rangle_{T(s,b)} = 0$$

for any s.

Results like the two above, whose proofs we omit, have appeared elsewhere in various formulations; see for example [Shen, Lemma 2.4], [AP, §1.5], or [BC, §3].

Lemma 4.1. *Let (M,F) be a Finsler manifold with a normal co-ordinate system $\bar{x}^i$ around a point p. In this co-ordinate system, away from p, for any radial tangent vector $\bar{X} = \bar{X}^k \frac{\partial}{\partial \bar{x}^k}$, where $\bar{X}^k = a^k$, we have*

$$\bar{X}(F_r^2(\bar{x}, \bar{X})) = \frac{\partial F^2}{\partial \bar{x}^r}(\bar{x}, \bar{X}),$$

where $F_r^2 = \frac{\partial F^2}{\partial \bar{X}^r}$.

Proof. Because we are working away from p, normal co-ordinates are smooth, so we can take derivatives and define Christoffel symbols without any trouble. The geodesic equations for a path $\bar{x}(t)$ are

$$\ddot{\bar{x}}^i + \Gamma^i_{jk}(\dot{\bar{x}})\dot{\bar{x}}^j \dot{\bar{x}}^k = 0,$$

or, if $\bar{X}^i = \dot{\bar{x}}^i$,

$$\ddot{\bar{x}}^i + \Gamma^i_{jk}(\bar{X})\bar{X}^j \bar{X}^k = 0.$$

In normal co-ordinates, the paths

$$\bar{x}(t) = t(a_0, a_1, \ldots a_n)$$

are geodesics for any set of constants a_i. Along these geodesics, $\ddot{\bar{x}}^i = 0$, so

$$\Gamma^i_{jk}(\bar{X})\bar{X}^j \bar{X}^k = 0.$$

Expand:

$$\begin{aligned}
\frac{1}{2}\bar{g}^{ir}\left(\frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k} + \frac{\partial \bar{g}_{kr}}{\partial \bar{x}^j} - \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r}\right)\bar{X}^j \bar{X}^k &= 0 \\
\frac{1}{2}\bar{g}^{ir}\left(2\frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k}\bar{X}^j \bar{X}^k - \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r}\bar{X}^j \bar{X}^k\right) &= 0.
\end{aligned}$$

Since $\bar{g}^{ir}$ is invertible,

$$2\frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k}\bar{X}^j \bar{X}^k = \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r}\bar{X}^j \bar{X}^k.$$

Since $\bar{g}_{jr}(\bar{X})\bar{X}^j = \frac{1}{2}F_r^2$ by the homogeneity of F^2 on each tangent space, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial F^2}{\partial \bar{X}^r}\right)\bar{X}^k &= \frac{\partial}{\partial \bar{x}^r}(\bar{g}_{jk}\bar{X}^j\bar{X}^k) \\ \bar{X}(F_r^2) &= \frac{\partial}{\partial \bar{x}^r}(F^2(\bar{X})). \end{aligned} \tag{4.1.1}$$

□

Lemma 4.2. *Let (M, F) be a Finsler manifold with a normal co-ordinate system $\bar{x}^i$ around a point p. In these co-ordinates, if $\bar{T} \in T_pM$, and x is a point on the geodesic generated by $\bar{T}$, then*

$$\bar{g}_{ij}(p, \bar{T})\bar{T}^i = \bar{g}_{ij}(\bar{x}, \bar{T})\bar{T}^i.$$

Proof. Let $\rho : M \longrightarrow \mathbb{R}$ be the distance function from p. In normal co-ordinates, then,

$$F^2(p; y^1, y^2, \ldots, y^n) = \rho^2(y^1, y^2, \ldots, y^n),$$

for any set y^i. Let $\tilde{X} \in T_{\bar{T}}T_pM$ be a vertical vector such that $\tilde{X} = \bar{V}^i\frac{\partial}{\partial y^i}$. Then $\tilde{X}$ is tangent to the indicatrix at $\bar{T}$ if and only if

$$dF^2(\tilde{X}) = 0 \tag{4.1.2}$$

(consider F^2 as solely a function of the tangent vectors y^i, i.e. restrict F to T_pM). Then, because F^2 and ρ^2 are written with respect to the same co-ordinates,

$$d\rho^2(\bar{V}) = 0. \tag{4.1.3}$$

Now $\tilde{X}$ is tangent to the indicatrix at T if and only if

$$\bar{g}_{ij}(p, \bar{T})\bar{T}^iV^j = 0. \tag{4.1.4}$$

Furthermore, because normal co-ordinates give a geodesic variation about p, the Gauss Lemma tells us that

$$\bar{g}_{ij}(\bar{x}, \bar{T})\bar{T}^i\bar{V}^j = 0. \tag{4.1.5}$$

Statements (4.1.2)—(4.1.5) are equivalent, so in particular

$$\begin{aligned} \bar{g}_{ij}(p, \bar{T})\bar{T}^i\bar{V}^j &= \bar{g}_{ij}(\bar{x}, \bar{T})\bar{T}^iV^j. \\ (\bar{g}_{ij}(p, \bar{T})\bar{T}^i)\bar{V}^j &= (\bar{g}_{ij}(\bar{x}, \bar{T})\bar{T}^i)V^j. \end{aligned}$$

The vectors $\bar{V}^j$ for which both sides are 0 form an $(n-1)$-dimensional vector space, so the "vectors" $\bar{g}_{ij}(p,\bar{T})\bar{T}^i$ and $\bar{g}_{ij}(\bar{x},\bar{T})\bar{T}^i$ are determined and equal up to a multiplicative constant $k(x)$:

$$\bar{g}_{ij}(p,\bar{T})\bar{T}^i = k(x)\bar{g}_{ij}(\bar{x},\bar{T})\bar{T}^i.$$

Since, however, $F^2(p,\bar{T}) = F^2(\bar{x},\bar{T})$ (because $\bar{T}$ is the tangent to a geodesic parametrized by arclength), and since

$$F^2(\bar{T}) = \bar{g}_{ij}(\bar{T})\bar{T}^i\bar{T}^j,$$

the multiplicative constant $k(x)$ must be identically 1. $\therefore$

$$\bar{g}_{ij}(p,\bar{T})\bar{T}^j = \bar{g}_{ij}(\bar{x},\bar{T})\bar{T}^j.$$

□

The discussion so far has involved only a general Finsler manifold, and has always worked away from p. The final step specializes only to Berwald manifolds, and explicitly uses their extra differentiability for normal co-ordinates at p.

Theorem 4.3. *Let (M,F) be a Berwald manifold with a normal co-ordinate system $\bar{x}^i$ around a point p. In this co-ordinate system, for every $\bar{T} \in T_pM$,*

$$\frac{\partial F^2}{\partial \bar{x}^i}(p,\bar{T}) = 0.$$

Proof. Since normal co-ordinates on the tangent bundle of a Berwald manifold are at least C^1 everywhere, it follows that

$$\lim_{x\to p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x},\bar{T})$$

exists, and

$$\lim_{x\to p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x},\bar{T}) = \frac{\partial F^2}{\partial \bar{x}^i}(p,\bar{T}).$$

Furthermore, we can evaluate the limit along any path leading to p. Choose as a path the geodesic in the direction $\bar{T}$ (this will allow us to use Lemma 2). From Lemma 1,

$$\begin{aligned}
\lim_{x\to p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x},\bar{T}) &= \lim_{x\to p} \bar{T}(F_i^2(\bar{x},\bar{T})) \\
&= \lim_{x\to p} \bar{T}\left(\frac{\partial^2 F^2}{\partial \bar{X}^i \partial \bar{X}^j}(\bar{x},\bar{T})\bar{T}^j\right) \\
&= \lim_{x\to p} \bar{T}(2\bar{g}_{ij}(\bar{x},\bar{T})\bar{T}^j).
\end{aligned}$$

Lemma 2 says that the argument of $\bar{T}$ in the line above is constant along the geodesic generated by $\bar{T}$, so

$$\lim_{x\to p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x},\bar{T}) = 0. \tag{4.1.6}$$

□

Corollary 4.4. *Recall the volume form component* $\omega = \frac{\kappa_n}{\int_{I_x} dX}$ *on a Berwald manifold*(M,F)*. In normal co-ordinates* $\bar{x}^i$ *around* p*,*

$$\frac{\partial}{\partial \bar{x}^i} \frac{\kappa_n}{\int_{I_p} dX}\Big|_p = 0 \tag{4.1.7}$$

for any $i = 1, 2, \ldots, n$*.*

Proof. Examine the expression $\frac{\kappa_n}{\int_{I_p} dX}$, where we recall that

$$I_p = \{X \in T_pM | F^2(p,X) \le 1\}.$$

This is the only place in the expression that the metric function F appears, so the derivative of $\frac{\kappa_n}{\int_{I_p} dX}$ will involve only integrals with integrands containing $\frac{\partial F^2}{\partial \bar{x}^i}$. Since all these terms vanish, the expression (4.1.7) must vanish. □

Corollary 4.5. *Recall that* $K^{jk} = (n+2)\frac{\int_{I_x} X^j X^k dX}{\int_{I_x} dX}$ *is the expression for the osculating Riemannian metric to a Berwald manifold* (M,F)*. In normal co-ordinates* $\bar{x}^i$ *around* p*,*

$$\frac{\partial}{\partial \bar{x}^i}(K^{jk})\Big|_p = 0$$

for any $i = 1, 2, \ldots, n$*.*

The foregoing two corollaries are the only machinery we need to reach the final result.

Theorem 4.6. *Let* (M,F) *be a Berwald manifold, with volume invariant* $\mathcal{V}(x)$*. Then* $\mathcal{V}(x)$ *is constant, i.e.* $d\mathcal{V}(x) = 0$ *everywhere.*

Proof. Recall that

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}, \tag{4.1.8}$$

where $k(x)$ is the (coefficient in some co-ordinate system of the) volume form of the osculating Riemannian metric, and $\omega(x)$ is (the coefficient in the same co-ordinate system of) Busemann's volume form.

Choose for a co-ordinate system normal co-ordinates $\bar{x}^i$ around a point $p \in M$.

By Corollary 1, any first derivative of $\omega(x)$ vanishes at p in these co-ordinates, i.e.

$$\bar{d}(\omega(x))|_p = 0; \tag{4.1.9}$$

where $\bar{d}$ is the exterior derivative in normal co-ordinates of the function $\omega(x)$, where $\omega(x)dx$ is Busemann's volume form in normal co-ordinates.

By Corollary 2, any first derivative of $K^{jk}|_x$ vanishes at p in normal co-ordinates, where K^{jk} is the osculating Riemannian metric to (M, F). The K^{jk} generate a volume form, given in co-ordinates by

$$k(x)dx = \sqrt{\det[K_{kl}(x)]}dx, \tag{4.1.10}$$

where $K_{kl}K^{jk} = \delta^j_l$. Since

$$\bar{d}(K^{jk}(x))|_p = 0, \tag{4.1.11}$$

we have

$$\bar{d}(k(x))|_p = 0, \tag{4.1.12}$$

in normal co-ordinates.

Now consider $\bar{d}\mathcal{V}(x)$. By the quotient rule,

$$\begin{aligned} \bar{d}\mathcal{V}(x)|_p &= \frac{\bar{d}k(x)|_p\omega(p) - \bar{d}\omega(x)|_p k(p)}{\omega^2(p)} \\ &= \frac{0-0}{\omega^2(p)} \\ &= 0. \end{aligned} \tag{4.1.13}$$

(4.1.14)

Unlike $k(x)$ and $\omega(x)$, which are not really functions but rather components of volume forms, $\mathcal{V}(x)$ genuinely is a scalar function, so $\bar{d}\mathcal{V}(x)$ is genuinely its exterior derivative. Since $\bar{d}\mathcal{V}(x)$ vanishes in one co-ordinates system at p, it must vanish in any co-ordinate system at p, so

$$d\mathcal{V}(x)|_p = 0.$$

Since p was chosen arbitrarily, we could work through the above steps for *any* $p \in M$, and thus get

$$d\mathcal{V}(x) = 0$$

everywhere. □

References

[AIM] P. L. Antonelli, R. S. Ingarden, M. Matsumoto. *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology.* Kluwer Academic Publishers, Netherlands, 1993.

[AP] Marco Abate, Giorgio Patrizio. *Finsler Metrics—A Global Approach: with applications to geometric function theory.* Springer, Berlin, 1994.

[BC] David Bao, Shiing-Shen Chern. "On a Notable Connection in Finsler Geometry," Houston Jour. Math., Vol. 19, No. 1, 1993.

[BCS] David Bao, Shiing-Shen Chern, Zhongmin Shen. *An Introduction to Riemann-Finsler Geometry.* Springer, in preparation.

[BL] David Bao, Brad Lackey. Verbal communication at Finsler Laplacians conference, Edmonton, August, 1997.

[BS] David Bao, Zhongmin Shen. "On the Volume of Unit Tangent Spheres in a Finsler Manifold." Results in Mathematics, Vol. 26, 1994.

[Busemann] Herbert Busemann. "Intrinsic Area," Annals of Math., vol. 48 (1947) pp. 234-267.

[Centore] Paul Centore. "A Mean-Value Laplacian for Finsler Spaces," in *Proceedings of Conference on Finsler Laplacians,* Kluwer Academic Press, Netherlands, 1998.

[Rund] Hanno Rund. *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, 1959.

[Shen] Zhongmin Shen. "Volume Comparison and Its Applications in Riemann-Finsler Geometry." Advances in Mathematics, Vol. 128, Number 2, pp.306-328, 1997.

Received June 20, 1998

1546 Route 12, Gales Ferry, CT 06335, USA
E-mail address: centore@downcity.net