A NOTE ON $\alpha$COZERO-COMPLEMENTED SPACES AND $\alpha$BOREL SETS

ANTHONY W. HAGER

Abstract. $\alpha$ is a regular cardinal number or the symbol $\infty$, and $X$ is a compact Hausdorff space. It is shown that (Theorem 2.1) $X$ is $\alpha$cozero-complemented iff each $\alpha$Borel set differs from an $\alpha$cozero set by a meagre set. This immediately yields (Corollary 2.2) $X$ is $\alpha$disconnected iff each $\alpha$Borel set differs from a clopen set by a meagre set. (The cases $\alpha = \infty$ in Theorem 2.1, and $\alpha = \omega_1$ in Corollary 2.2 are important theorems in Boolean Algebra). The Boolean algebra of $\alpha$Borel sets modulo meagre sets is considered as an extension of the clopen algebra on $X$ (when $X$ is Boolean), and compared with the $\alpha$cut-completion and the $\alpha$completion. (For $\alpha = \infty$, each of the three is the completion.)

1. Preliminaries

Throughout, $\alpha$ denotes a regular cardinal number or the symbol $\infty$ (the meaning of which will be clear from context), and topological spaces are compact Hausdorff. These assumptions are not always totally needed, but will simplify the discussion. Our main references will be [1, 3, 6].

In a space $X$, an $\alpha$cozero set is the union of $< \alpha$ cozero sets, and the collection of all these is denoted $\alpha$coz $X$. Since $X$ is compact Hausdorff, the cozero sets are the open $F_\sigma$’s. If clop$X$ (the collection of all clopen sets) is a basis (i.e., $X$ is a Boolean space), each cozero set is the union of $< \omega_1$ clopen sets, and each $\alpha$cozero set is the union of $< \alpha$ clopen sets. (One may see [1] about cozero sets.) We note that $\alpha$cozero = open, and $\omega_1$cozero = cozero.

1991 Mathematics Subject Classification. Primary 54H05, 06E10.

Key words and phrases. $\alpha$Borel set, $\alpha$cozero-complemented space, $\alpha$disconnected space, $\alpha$complete Boolean Algebra.

679
X is called \( \alpha \)-cozero-complemented, abbreviated \( \alpha \text{cc} \), if for each \( U \in \alpha \text{coz} X \) there is \( V \in \alpha \text{coz} X \) with \( U \cap V = \emptyset \) and \( U \cup V \) dense in \( X \); here we say that "\( V \) \( \alpha \)complements \( U \)". Note that any \( X \) is \( \alpha \text{cc} \): if \( U \) is \( \alpha \)-cozero, i.e., open, then \( V = X - \bar{U} \). The one point compactification of an infinite discrete space is not \( \omega_1 \text{cc} \).

The family of \( \alpha \text{Borel sets} \), denoted \( \alpha \mathcal{B} X \), is the \( \sigma \)-field in \( \mathcal{P} X \) (the power set of \( X \)) generated by \( \alpha \text{coz} X \). Note that \( \infty \text{Borel} = \text{Borel} \) and \( \omega_1 \text{Borel} = \text{Baire} \).

A subset of \( X \) which is the union of \( < \omega_1 \) nowhere dense sets is called \( \text{meagre} \) (or first category), and the collection of all these is denoted \( MX \), or \( M \). This is a \( \sigma \)-ideal in \( \alpha \mathcal{B} X \). We recall the Baire Category Theorem in the forms: the only meagre open set is \( \emptyset \), and the complement of a meagre set is dense (in a compact Hausdorff space).

Let \( A \) be a Boolean algebra with an ideal \( I \), and let \( B, G \in A \). The symmetric difference \( B \triangle G \) is \( (B - G) \lor (G - B) \), and \( B \triangle G \in I \) means \( B \) and \( G \) have the same image in the quotient \( A/I \).

\[ 2. \alpha \text{ccspaces} \]

**Theorem 2.1.** \( X \) is \( \alpha \)-cozero-complemented iff for each \( \alpha \text{Borel set} B \) there is an \( \alpha \text{cozero set} G \) with \( B \triangle G \) meagre.

**Proof.** Suppose the condition obtains. Let \( U \in \alpha \text{coz} X \), and apply the condition to \( X - U \): there is \( G \in \alpha \text{coz} X \) with \( (X - U) \triangle G \) meagre. Then \( (X - U) - G = X - (U \cup G) \) is meagre, so \( U \cup G \) is dense (by the Baire Category Theorem). And \( G - (X - U) = G \cap U \) is meagre, and also open thus empty (by the Baire Category Theorem).

Suppose \( X \) is \( \alpha \text{cc} \). Let \( \mathcal{C} = \{ S \subseteq X \mid \exists G \in \alpha \text{coz} X \text{ with } S \triangle G \text{ meagre} \} \). Clearly, \( \mathcal{C} \supseteq \alpha \text{coz} X \). We shall show that \( \mathcal{C} \) is a \( \sigma \)-field; it follows that \( \mathcal{C} \supseteq \alpha \mathcal{B} X \), as desired.

\( S \in \mathcal{C} \) implies \( X - S \in \mathcal{C} \): if \( S \in \mathcal{C} \), there is \( G \in \alpha \text{coz} X \) with \( S \triangle G \) meagre. Note that \( S \triangle G = (X - S) \triangle (X - G) \). Choose \( H \) \( \alpha \)complementing \( G \); then \( \mathcal{H} \triangle (X - G) \) is meagre. Since \( (X - S) \triangle (X - G) \), \( (X - G) \triangle H \) are both meagre, it follows that \( (X - S) \triangle H \) is meagre.

\( S_1, S_2, \ldots \in \mathcal{C} \) implies \( \bigcup S_n \in \mathcal{C} \): if \( S_n \in \mathcal{C} \), there is \( G_n \in \alpha \text{coz} X \) with \( S_n \triangle G_n \) meagre. Then, \( \bigcup G_n \in \alpha \text{coz} X \) since \( \alpha \) is regular (or just sequentially regular), and \( (\bigcup S_n) \triangle (\bigcup G_n) \subseteq \bigcup (S_n - G_n) \). The latter is meagre, so the former is meagre also. \( \square \)
X is called \( \alpha \) disconnected if \( \bar{U} \) is open for each \( U \in \text{ocoz} \ X \). (So, \( \infty \) disconnected means extremely disconnected, and \( \omega_1 \) disconnected means basically disconnected.)

**Corollary 2.2.** \( X \) is \( \alpha \) disconnected iff for each \( \alpha \) Borel set \( B \) there is a clopen set \( C \) with \( B \triangle C \) meagre.

**Proof.** Supposes the condition holds. Let \( U \in \text{ocoz} \ X \), and apply the condition to \( U \): there is clopen \( C \) with \( U \triangle C \) meagre. Then \( U \subseteq C \) (for if not, \( U - C \) is open and nonvoid, contradicting the Baire Category Theorem), and \( C \subseteq \bar{U} \) (for if not, \( C - \bar{U} \ldots \)). Since \( C \) is closed, \( C = \bar{U} \); so \( \bar{U} \) is open.

Suppose \( X \) is \( \alpha \) disconnected. If \( U \in \text{ocoz} \ X \), then \( X - \bar{U} \) complements \( U \), since \( \bar{U} \) is open. So \( X \) is \( \alpha \)cc, and we can use Theorem 2.1. If \( B \in \alpha \mathcal{B} X \), there is \( G \in \text{ocoz} \ X \) with \( B \triangle G \) meagre; and \( \bar{G} \) is clopen. Since \( B \triangle G \) and \( G \triangle \bar{G} \) are meagre; so is \( B \triangle \bar{G} \).

We quote Halmos [3, p.101] regarding Theorem 2.1 (\( \alpha = \infty \)) and Corollary 2.2 (\( \alpha = \omega_1 \)). "The (latter) resembles (the former) in many details, in both statement and proof. It is almost certain that the two results are special cases of a common generalization; it is far from certain whether the formulation and proof of such a generalization would yield any new information or save any time."

We may note, that in both Theorem 2.1 and Corollary 2.2 the implications "\( \Rightarrow \)" do not require assuming that \( X \) is compact Hausdorff, and about \( \alpha \), require only sequential regularity; and the implications "\( \Leftarrow \)" do not require assuming anything about \( \alpha \), and about \( X \), require only that the Baire Category Theorem holds. But, further observations below shall need regularity of \( \alpha \), and, that \( X \) be Boolean. So, we abandon such commentary.

We consider the meaning of Corollary 2.2 for Boolean algebras. Given the Boolean space \( X \), we have present two Boolean homomorphisms: the injection \( \text{clop} X \hookrightarrow \alpha \mathcal{B} X \) and the surjection \( \alpha \mathcal{B} X \xrightarrow{\mathcal{S}} \alpha \mathcal{B} X/\alpha M \); and the composite \( si \) is an injection, by the Baire Category Theorem. By Corollary 2.2 then \( X \) is \( \alpha \) disconnected iff \( si : \text{clop} X \to \alpha \mathcal{B} X/\alpha M \) is an isomorphism (onto). Given a Boolean algebra \( \mathcal{A} \), we have the Stone representation of \( \mathcal{A} \) as \( \text{clop} \mathcal{S} \mathcal{A} \), and \( \mathcal{A} \) is \( \alpha \)complete (meaning, \( \vee \mathcal{F} \) exists when \( |\mathcal{F}| < \alpha \)) if \( \mathcal{S} \mathcal{A} \) is \( \alpha \)disconnected [6, 22.4]. Thus,

**Corollary 2.3.** The Boolean algebra \( \mathcal{A} \) is \( \alpha \)complete iff and only if \( \mathcal{A} \) is isomorphic to \( \alpha \mathcal{B} \mathcal{S} \mathcal{A}/\alpha M \) via Stone representation and \( si \).
The case \( \alpha = \omega_1 \) in Corollary 2.3 implies, or is, the Loomis-Sikorski Theorem, usually stated as: each \( \omega_1 \)-complete Boolean algebra is \( \omega_1 \)-representable, i.e., isomorphic to an \( \omega_1 \)-field of sets modulo an \( \omega_1 \)-ideal. The corresponding statement for \( \alpha \)-complete algebras is very false; few \( \alpha \)-complete algebras are \( \alpha \)-representable. See [6, §29]. Whether Corollary 2.3 says anything useful in this regard, I don’t know.

We note (Proposition 2.4 below) that \( \alpha \)-complete Boolean algebras are \( \alpha \)-representable as \( \alpha \)-Frames: (See [5] for the various definitions, etc.) Given \( X \), \( \alpha \text{coz} \ X \) is an \( \alpha \)-Frame (since \( \alpha \) is regular). The codensity congruence on \( \alpha \text{coz} \ X \) is: \((U, V) \in \text{cdns} \) if \( \overline{U} = \overline{V} \). Evidently, \( X \) is \( \alpha \)-disconnected iff for each \( \alpha \)-cozero set \( B \) there is clopen \( C \) with \((B, C) \in \text{cdns} \) (cf. Corollary 2.2 above). Since \( \text{cdns} \) is an \( \alpha \)-Frame congruence, \( X/\text{cdns} \) is an \( \alpha \)-Frame, and the canonical surjection \( \alpha \text{coz} \ X \xrightarrow{\sim} \alpha \text{coz} \ X/\text{cdns} \) is an \( \alpha \)-Frame homomorphism. When \( X \) is a Boolean space, we have the lattice injection \( \text{clop} \ X \xhookrightarrow{\sim} \alpha \text{coz} \ X \), the composite \( tj \) is also a lattice injection, by the Baire Category Theorem, and \( tj \) is onto iff \( X \) is \( \alpha \)-disconnected (by the remark above). Then, as with Corollary 2.3, we obtain

**Proposition 2.4.** The Boolean algebra \( \mathcal{A} \) is \( \alpha \)-complete iff \( \mathcal{A} \) is lattice-, hence Boolean-, isomorphic to the \( \alpha \)-Frame \( \alpha \text{coz} \ SA/\text{cdns} \) via Stone representation and \( tj \).

We note two consequences of \( \alpha \text{cc} \), by way of Theorem 2.1, which are exactly similar to the known case of \( \alpha = \infty \).

A subalgebra \( \mathcal{A} \) of the Boolean algebra \( \mathcal{B} \) is \( \alpha \text{dense} \) if, for each \( b \in \mathcal{B} \) there is \( F \subseteq \mathcal{A} \) with \( |F| < \alpha \), with \( \bigvee F = b \). (“\( \alpha \text{dense} \)” is just called “dense”.)

**Theorem 2.5.**

1. If \( X \) is \( \alpha \text{cozero-complemented} \), then \( \alpha \mathcal{B} X/\alpha \mathcal{M} \) is \( \alpha \)-complete.
2. For Boolean \( X \), \( X \) is \( \alpha \text{cozero-complemented} \) if and only if \( \text{clop} X \) is \( \alpha \text{dense} \) in \( \alpha \mathcal{B} X/\alpha \mathcal{M} \) (via \( si \)).

**Lemma 2.6** (cf. [6, p.75]). Consider the Boolean surjection \( s : \alpha \mathcal{B} X \to \alpha \mathcal{B} X/\alpha \mathcal{M} \). Suppose \( \{G_i \mid i \in I\} \subseteq \alpha \text{coz} X \), with \( |I| < \alpha \). Then, \( s(\bigcup_i G_i) = \bigvee_i \{s(G_i)\} \).

**Proof.** This uses Banach’s Category Theorem [4, pp. 59-85]: In any space \( Y \), if \( \{X_i\}_i \) is a family of subsets such that for each \( i \), \( X_i \) is open in \( \bigcup_i X_i \), and \( X_i \) is meagre in \( Y \), then \( \bigcup X_i \) is meagre in \( Y \).

To prove the Lemma: First, for each \( G_i, G_i - \bigcup_i G_i \) is empty, hence meagre. So \( s(\bigcup_i G_i) \) is an upper bound for \( \{s(G_i)\}_i \). Second, suppose \( s(A) \) is another
upper bound: $A \in \alpha B X$, and $G_i - A \in \alpha M$ for each $G_i$. Since $G_i$ is open, and $G_i - A = (\bigcup_i G_i - A) \cap G_i$, $G_i - A$ is relatively open in $\bigcup_i G_i - A$. Thus, $\bigcup_i G_i - A = \bigcup_i (G_i - A) \in \alpha M$ (by Banach’s Theorem). So $s(\bigcup_i G_i) \leq s(A)$. □

PROOF OF THEOREM 2.5.

Point 1. Let $\{B_i \mid i \in I\} \subseteq \alpha B X$, $|I| < \alpha$. For each $i$, there is $G_i \in \text{acoz } X$ with $B_i \triangle G_i \in \alpha M$, by Theorem 2.1. Then, using Lemma 2.6, $s(\bigcup_i G_i) = \bigvee_i s(G_i) = \bigvee_i s(B_i)$, so this last exists.

Point 2. Suppose $X$ is $\alpha$-cc, and $B \in \alpha B(X)$. Choose $G \in \text{acoz } X$ with $B \triangle G \in \alpha M$, by Theorem 2.1. Now $G = \bigcup_i C_i$ for $< \alpha$ clopen sets $C_i$. We then have, using Lemma 2.6, $s(B) = s(\bigcup_i C_i) = \bigvee_i s(C_i)$ as desired.

Suppose clop $X$ is $\alpha$-dense in $\alpha B X/\alpha M$, and let $U \in \text{acoz } X$. By density, $s(X - U) = \bigvee_i s(C_i)$, for some $< \alpha$ clopen sets $C_i$. Let $V = \bigcup_i C_i$; of course $V \in \text{acoz } X$. By Lemma 2.6, $s(V) = \bigvee_i s(C_i)$, so that $s(X - U) = s(V)$, i.e., $(X - U) \triangle V \in \alpha M$. This means $V$ is an $\alpha$ complement for $U$ (using the Baire Category Theorem). □

The converse of Point 1 of Theorem 2.5 fails: An easy counterexample is $X$ the one-point compactification of a discrete space of cardinal $\geq \alpha$. I don’t know what property of $X$ is equivalent to $\alpha$-completeness of $\alpha B X/\alpha M$.

3. Comparison with other extensions

$A \leq B$ means that $A$ is a Boolean subalgebra of the Boolean algebra $B$. The completion of $A$ is, in its simplest definition, an extension $A \leq \bar{A}$, with

1. $\bar{A}$ complete, and
2. $A$ dense in $\bar{A}$. (Such an extension is unique up to isomorphism fixing $A$.)

We have just seen in Theorem 2.5 that (identifying $A$ with clop $S A$, by Stone Representation) $A \leq \alpha B S A/\alpha M$ is a model of $A \leq \bar{A}$. (This is standard, and Theorem 2.5 more-or-less replicates the proof in [6, §35]).

The $\alpha$-completion of $A$ (see [6, §35]) is an extension $A \leq \bar{A}$, with

1. $\bar{A}$ complete,
2. $A$ dense in $\bar{A}$, and
3. the only $C$ with $A \leq C \leq \bar{A}$ and $C$ $\alpha$-complete is $C = \bar{A}$.

Such an extension is unique as before, and it is easily seen that $\bigcap\{C \mid A \leq C \leq \bar{A}, C \text{ $\alpha$-complete}\}$ is a model $\bar{A}$ of $\bar{A}$. Of course, $\bar{A} = \bar{A}$, i.e., for $\alpha = \infty$, Condition 3 can be dropped (and, alternatively, Condition 2 can be dropped [2]).
Another construction of $\mathfrak{A}$ is by transfinite iteration of the passage $\mathcal{A} \to \mathcal{A}_1$, where $\mathcal{A}_1$ is the Boolean subalgebra of $\mathfrak{A}$ generated by the set $\mathcal{S}$ of all $\bigvee \mathcal{F}, \mathcal{F} \subseteq \mathcal{A}$ and $|\mathcal{F}| < \alpha$ ("$\bigvee$" meant in $\mathfrak{A}$).

**Theorem 3.1.** Let $\mathcal{A}$ be a Boolean algebra.

1. $\mathcal{A}/\mathcal{M}$ is the $\omega_1$-completion of $\mathcal{A}_1$.
2. $\mathcal{A}/\mathcal{M} \leq \mathfrak{A}$.

**Proof.** To compress notation, let $X \equiv \mathcal{S}$, and let $\mathcal{A}(\alpha) \equiv \mathcal{B}/\mathcal{M}$. With some obvious identifications and suppression of notation, we have

$$\mathcal{A} = \text{clop } X \leq \mathcal{A}(\alpha) \leq \mathcal{B}/\mathcal{M} = \mathfrak{A}$$

Here, $s(\text{coz } X)$ is exactly the generating set $\mathcal{S}$ for $\mathcal{A}_1$; this uses Lemma 2.6 and the fact that dense embeddings preserve all joins ([6, 23.1]). Since $\mathcal{A}$ is $\omega_1$-complete and $\mathcal{M}$ is an $\omega_1$ ideal ($\sigma$-ideal), $\mathcal{A}(\alpha)$ is $\omega_1$-complete and the quotient homomorphism $s$ preserves all countable joins ([6, 21.1]).

To prove Point 1, it suffices to show that, if $\mathcal{A} \leq \mathcal{C} \leq \mathcal{A}(\alpha)$ and $\mathcal{C}$ is $\omega_1$-complete, then $\mathcal{C} = \mathcal{A}(\alpha)$. (Since $\mathcal{A} \leq \mathcal{A}(\alpha) \leq \mathfrak{A}$, $\mathcal{A}$ is dense in $\mathcal{A}(\alpha)$.) Given such $\mathcal{C}$, consider $s^{-1}(\mathcal{C}) \subseteq \mathcal{B}$. We show that $s^{-1}(\mathcal{C})$ is a $\sigma$-field; then $s^{-1}(\mathcal{C}) = \mathcal{A}(\alpha)$. So let $A \in s^{-1}(\mathcal{C})$. Since $s$ is a homomorphism, $s(X - A) = s(A') \in \mathcal{C}$ ($A'$ being Boolean complement), so $X - A \in s^{-1}(\mathcal{C})$. Now let $A_1, A_2, \ldots \in s^{-1}(\mathcal{C})$. Since $s$ preserves countable joins, $s(\bigcup_n A_n) = \bigvee_n s(A_n)$ (the latter join in $\mathcal{A}(\alpha)$). Since $\mathcal{C}$ is $\omega_1$-complete, $\bigvee_n s(A_n)$ exists. The dense embedding $\mathcal{C} \leq \mathcal{A}(\alpha)$ preserves all joins, so $\bigvee_n s(A_n) = \bigvee_n s(A_n)$. Thus, $\bigvee_n s(A_n) \in \mathcal{C}$, so $\bigcup_n A_n \in s^{-1}(\mathcal{C})$. So $s^{-1}(\mathcal{C})$ is a $\sigma$-field, and Point 1 is shown.

To prove Point 2, we want to show that $(\mathcal{A} \leq \mathcal{C} \leq \mathfrak{A}$ with $\mathcal{C}$ $\omega_1$-complete implies $\mathcal{A}(\alpha) \leq \mathcal{C})$. Given such $\mathcal{C}$, clearly $\mathcal{A}_1 \subseteq \mathcal{C}$, and the result follows from Point 1.

**Corollary 3.2.** If $\mathcal{A}/\mathcal{M}$ is $\omega_1$-complete (e.g., if $\mathcal{A}$ is $\omega_1$-cc (Theorem 2.5)) then $\mathcal{A}/\mathcal{M}$ is the $\omega_1$-completion of $\mathcal{A}$.

Corollary 3.2 represents an unusual circumstance of $\mathfrak{A}$ being achieved in a concrete manner (i.e., rather than from without as a large intersection, or from within by transfinite induction, per the comments before Theorem 3.1).

On the other hand, there is another "$\omega_1$-generalization" of completeness which is better in this regard. See [2] for the details of what follows.
An $\alpha$-cut in the Boolean algebra $A$ is a pair $(F, H)$ of subsets of $A$ of cardinal $< \alpha$, with $F \leq H$ elementwise, and $\bigwedge \{ h - f \mid h \in H, f \in F \} = 0$. $A$ is said to be $\alpha$-cut-complete if, whenever $(F, H)$ is an $\alpha$-cut in $A$, then there is $a \in A$ with $\bigvee F = a = \bigwedge H$. Then, for each $A$, there is $A \leq A^\alpha$, with $A^\alpha$ $\alpha$-cut-complete, and $A$ $\alpha$-dense in $A^\alpha$. This $A^\alpha$ is called the $\alpha$-cut-completion of $A$, is unique up to isomorphism over $A$, and has this simple description as a subset of the completion $\overline{A}$:

$$A^\alpha = \{ \bigvee F \mid \text{there is } H \subseteq A \text{ for which } (F, H) \text{ is an } \alpha \text{-cut in } A \}.$$ (Clearly, $A^\infty$ is the completion $\overline{A}$). Obviously, we have

**Proposition 3.3.** For any Boolean algebra $A$, $A^\alpha$ is contained in the generating set $S$ for $\alpha A_1$, so that $A^\alpha \leq \alpha B S A / \alpha M$.

Observe that $S A$ is $\alpha$cc iff each subset of $A$ of cardinal $< \alpha$ is the left (lower) half of an $\alpha$-cut in $A$. The following is now obvious or routine.

**Proposition 3.4.** For a Boolean algebra $A$, the following are equivalent.

1. $A^\alpha = S$
2. $A^\alpha = \alpha B S A / \alpha M$.
3. $S$ is a Boolean algebra (hence an $\alpha$-complete algebra, hence $S = \overline{\alpha A}$).
4. $SA$ is $\alpha$cc.

**References**


Received February 10, 1998

Mathematics Department, Wesleyan University, Middletown, CT 06457

E-mail address: ahager@wesleyan.edu