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AN EXPONENTIAL MAPPING OVER SET-VALUED MAPPINGS

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Abstract. The paper presents an approach to "selection homotopy extension" properties of set-valued mappings showing that they become equivalent to usual selection extension properties of exponential set-valued mappings associated to them. As a result, several "controlled" homotopy extension theorems are obtained like consequences of ordinary selection theorems. Also, involving set-valued mappings, a simple proof of the Borsuk homotopy extension theorem is given.

1. Introduction

Let Z and Y be topological spaces, and let 2^Y be the family of the non-empty subsets of Y. Also, let

$$\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}.$$

A single-valued map $f : Z \to Y$ is a *selection* for a set-valued mapping $\varphi : Z \to 2^Y$ if $f(z) \in \varphi(z)$ for every $z \in Z$.

In this paper, we deal with extending of partial selections for set-valued mappings which domain is the product $X \times \mathbb{I}$ of a space X with the closed unit interval $\mathbb{I} = [0,1]$. More precisely, the purpose of the paper is to present reasonable conditions on X, $A \subset X$ and $\varphi : X \times \mathbb{I} \to \mathcal{F}(Y)$ under which every continuous selection $h : X \times \{0\} \cup A \times \mathbb{I} \to Y$ for $\varphi | X \times \{0\} \cup A \times \mathbb{I}$ can be extended to a continuous selection for φ. A central role in this will be played by a class of

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mappings $\varphi : X \times \mathbb{I} \to \mathcal{F}(Y)$ for which the above problem is, in fact, equivalent to an usual selection problem for set-valued mappings defined only on X.

So far, concerning continuous selections, the best *continuity* condition on set-valued mappings is that of lower semi-continuity. Let us recall that a mapping $\varphi : Z \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set

$$\varphi^{-1}(U) = \{z \in Z : \varphi(z) \cap U \neq \emptyset\}$$

is open in Z for every open $U \subset Y$. Let us also recall a property which is a weaker condition for the existence of continuous selections. Let (Y, d) be a metric space and let, for $S \in 2^Y$ and $\varepsilon > 0$, $B^d_\varepsilon(S)$ denote $\{y \in Y : d(y, S) < \varepsilon\}$. A mapping $\varphi : Z \to 2^Y$ is *quasi-lower semi-continuous*, or q.l.s.c., [8] if for every $z \in Z$, every neighbourhood V of z, and every $\varepsilon > 0$ there exists a point $z' \in V$ such that $z \in \operatorname{int}(\varphi^{-1}(B^d_\varepsilon(y)))$ for every $y \in \varphi(z')$. It should be mentioned that every l.s.c. mapping is q.l.s.c. one while the converse is not true (see, for instance, [8, 9, 21]). Also, note that the definition of q.l.s.c. mappings depends on the metric d on the range Y.

In the present paper, we are interested of set-valued mappings $\varphi : X \times \mathbb{I} \to 2^Y$ which are "equi-l.s.c." with respect to the first coordinate and each restriction on the second coordinate has nice selection properties. Towards this end, let us agree to say that $\varphi : X \times T \to 2^Y$ is *equi-l.s.c. at* X if for every open set $U \subset Y$ and a point $(x_0, t_0) \in \varphi^{-1}(U)$ there exists a neighbourhood $V \times W$ of (x_0, t_0) such that

$$V \subset \{x \in X : \varphi(x, t) \cap U \neq \emptyset\}, \quad \text{whenever } t \in W \text{ with } \varphi(x_0, t) \cap U \neq \emptyset.$$

Concerning set-valued mappings whose domain is $\mathbb{I}$, an important place is occupied by special q.l.s.c. mappings. Namely, we shall say that $\psi : \mathbb{I} \to 2^Y$ is *lower quasi-monotone*, or l.q.m., if for every $t \in \mathbb{I}$, every neighbourhood V of t, and every $\varepsilon > 0$ there exists $\delta \in [0, 1)$ such that

(i) $\delta \cdot t \in V$;
(ii) $[\delta \cdot t, t] \subset \operatorname{int}(\psi^{-1}(B^d_\varepsilon(y)))$ for every $y \in \psi(\delta \cdot t)$.

As it follows from the definition, every l.q.m. $\psi : \mathbb{I} \to 2^Y$ is also q.l.s.c. but the converse is not true. In fact, as we will see, there are l.s.c. mappings $\psi : \mathbb{I} \to 2^Y$ which fail to be l.q.m. ones. On the other hand, one can easy construct l.q.m. mappings $\psi : \mathbb{I} \to 2^Y$ which are not l.s.c.

Turning to the basic component of our reduction, we shall use $C(\mathbb{I}, Y)$ to denote the space of all continuous maps $k : \mathbb{I} \to Y$ with the topology of uniform convergence, i.e. the topology generated by the metric $\boldsymbol{d}$ defined as

$$\boldsymbol{d}(k,\ell) = \sup\{d(k(t),\ell(t)) : t \in \mathbb{I}\}, \quad k,\ell \in C(\mathbb{I},Y).$$

The following are well-known (see, for instance, [7, Chapter XII; Theorems 3.1, 5.3 and 8.2]).

(1.1) $w(C(\mathbb{I},Y)) \le w(Y)\cdot\aleph_0 = w(Y)$.

(1.2) $(C(\mathbb{I},Y),\boldsymbol{d})$ is complete if so is (Y,d).

(1.3) For every space X, the *exponential* map $\Lambda : Y^{X\times\mathbb{I}} \to C(\mathbb{I},Y)^X$, defined for $x\in X$ and $t\in\mathbb{I}$ by $[\Lambda(f)(x)](t) = f(x,t)$, establishes an one-to-one correspondence between continuous maps $f : X\times\mathbb{I}\to Y$ and continuous maps $g : X\to C(\mathbb{I},Y)$.

Now, we consider the following extension of Λ over set-valued mappings:

(1.4) To every pair (φ,f) of a mapping $\varphi : X\times\mathbb{I}\to\mathcal{F}(Y)$ and its partial selection $f : X\times\{0\}\to Y$ we associate another set-valued mapping

$$\Lambda(\varphi,f) : X \to 2^{C(\mathbb{I},Y)}\cup\{\emptyset\}$$

defined for $x\in X$ by

$$\Lambda(\varphi,f)(x) = \{k\in C(\mathbb{I},Y) : k(t)\in\varphi(x,t),\ t\in\mathbb{I},\ \text{and}\ k(0)=f(x,0)\}.$$

To state our main result, we need also the following terminology. For technical reasons only, let us agree that an image of a "(-1)-sphere" is contractible in $S\subset Y$ provided S is non-empty. Suppose that $n\ge -1$. A family $\mathcal{S}$ of subsets of (Y,d) is uniformly equi-LC^n [15] if to every $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, for every $S\in\mathcal{S}$, every continuous image of a k-sphere ($k\le n$) in S of diameter $<\delta(\varepsilon)$ is contractible over a subset of S of diameter $<\varepsilon$. A space S is C^n if every continuous image of a k-sphere ($k\le n$) in S is contractible in S.

The following theorem will be proved.

Theorem 1.5. *Let (Y,d) be a complete metric space, $\mathcal{S}\subset\mathcal{F}(Y)$ be uniformly equi-LC^n for some $n\ge 0$, $\varphi : X\times\mathbb{I}\to\mathcal{S}$ be such that each $\varphi|\{x\}\times\mathbb{I}$, $x\in X$, is l.q.m., and let $f : X\times\{0\}\to Y$ be a selection for $\varphi|X\times\{0\}$. Then, $\{\Lambda(\varphi,f)(x) : x\in X\}$ is a family of closed subsets of $C(\mathbb{I},Y)$ which is uniformly equi-LC^{n-1} and $\Lambda(\varphi,f)(x)$ is C^{n-1} for every $x\in X$. If, in addition, f is continuous and φ is equi-l.s.c. at X, then $\Lambda(\varphi,f)$ is l.s.c.*

Motivated by Theorem 1.5, we shall say that $\varphi : X \times \mathbb{I} \to 2^Y$ is Λ-*l.s.c.* if it is equi-l.s.c. at X and each $\varphi|\{x\} \times \mathbb{I}$, $x \in X$, is l.q.m.

Theorem 1.5 has several interesting applications. In Sections 4 and 6 of the paper we use this theorem to prove some "controlled" homotopy extension results. Thus, we also improve some known results. In fact, once Theorem 1.5 is known, the approach to these applications becomes almost transparent. To illustrate this, let us mention here only the following generalization of [16, Theorem 3.4]. Below, $\dim_X(X\backslash A) \leq n$ means that $\dim(S) \leq n$ for every $S \subset X\backslash A$ which is closed in X.

Theorem 1.6. *Let (Y, d) be a complete metric space, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n in Y for some $n \geq 0$, X be paracompact, and let A be a closed subset of X with $\dim_X(X\backslash A) \leq n$. Also, let $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be a Λ-l.s.c. mapping. Then, every continuous selection $h : X \times \{0\} \cup A \times \mathbb{I} \to Y$ for $\varphi|X \times \{0\} \cup A \times \mathbb{I}$ can be extended to a single-valued continuous selection for φ.*

PROOF. Let $h : X\times\{0\}\cup A\times\mathbb{I} \to Y$ be a continuous selection for $\varphi|X\times\{0\}\cup A\times\mathbb{I}$. Set $f = h|X \times \{0\}$ and $g = h|A \times \mathbb{I}$. By Theorem 1.5, $\Lambda(\varphi, f) : X \to \mathcal{F}(C(\mathbb{I}, Y))$ is an l.s.c. mapping such that $\{\Lambda(\varphi, f)(x) : x \in X\}$ is uniformly equi-LC^{n-1} and each $\Lambda(\varphi, f)(x)$ is C^{n-1}. Also, $\Lambda(g) : A \to C(\mathbb{I}, Y)$ is a continuous selection for $\Lambda(\varphi, f)|A$. Then, by [15, Theorem 1.2], $\Lambda(g)$ can be extended to a single-valued continuous selection q for $\Lambda(\varphi, f)$. Finally, the map $\ell = \Lambda^{\leftarrow}(q)$ is a continuous extension of h which is a selection for φ. □

Concerning the right place of Theorem 1.6, we refer the interested reader to Section 4. In the same section, the reader can find some natural examples of Λ-l.s.c. mappings. In effect, the *constant* set-valued mappings stand for the most trivial one. Relying on them, in Section 5 we obtain a simple proof of the famous Borsuk homotopy extension theorem and next, in Section 6, further its generalizations in terms of selections.

The proof of Theorems 1.5 will be finally accomplished in Section 3. It is based on special "homotopy extension" properties of uniformly equi-LC^n families of sets which are established in the next Section 2 of the paper.

A word should be said also about the statements of Theorem 1.5 and especially about the requirement on the restrictions $\varphi|\{x\} \times \mathbb{I}$, $x \in X$, to be l.q.m. As [16, Example 7.1] shows, Theorem 1.5 fails if this condition is replaced by the one each $\varphi|\{x\} \times \mathbb{I}$, $x \in X$, to be merely l.s.c. On the other hand, [16, Example 7.2] shows that the requirement on the restrictions $\varphi|\{x\} \times \mathbb{I}$, $x \in X$, to be l.q.m. with respect to the fixed metric d is essential and cannot be replaced by that the restrictions to be l.q.m. with respect to an admissible metric ρ on Y. In fact, this also demonstrates that the metric property of $\mathcal{S}$ to be "uniformly equi-LC^n" is essential and cannot be replaced by the topological one of "equi-LC^n in Y".

2. Homotopy extension properties of uniformly equi- LC^n families of sets

Let $\varphi : X \times \mathbb{I} \to 2^Y$. We shall say that a mapping $h : X \times \mathbb{I} \to Y$ is a *φ-homotopy* if it is a continuous selection for φ, and that h is a *homotopy* of $f : X \times \{0\} \to Y$ if it is a continuous extension of f.

Suppose that A is a subset of a space X. We shall say that $\varphi : X \times \mathbb{I} \to 2^Y$ has the *Selection Homotopy Extension Property* with respect to A, or the *SHEP* at A, if, whenever $f : X \times \{0\} \to Y$ is a continuous selection for $\varphi|X \times \{0\}$, every $\varphi|A \times \mathbb{I}$- homotopy $g : A \times \mathbb{I} \to Y$ of $f|A \times \{0\}$ can be extended to a φ-homotopy $h : X \times \mathbb{I} \to Y$ of f.

In this section, we will establish a "controlled" variant of the SHEP. To prepare for this, we need the following "controlled" extension property of uniformly equi-LC^n families which was actually stated in [15, Corollary 4.2].

(2.1) Let (Y, d) be a complete metric space, and let $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi- LC^n. Then, to every $\varepsilon > 0$ there corresponds a $\gamma(\varepsilon) > 0$ with the following property: If X is paracompact with $\dim(X) \le n+1$, $A \subset X$ is closed, $\Phi : X \to \mathcal{S}$ is l.s.c., $\ell : A \to Y$ is a continuous selection for $\Phi|A$, and $p : X \to Y$ is a continuous $\gamma(\varepsilon)$-selection for Φ such that $\boldsymbol{d}(p|A, \ell) < \gamma(\varepsilon)$, then ℓ can be extended to a continuous selection $h : X \to Y$ for Φ such that $\boldsymbol{d}(p, h) < \varepsilon$.

Here, a map $p : X \to Y$ is a *μ-selection* for Φ if $d(p(x), \Phi(x)) < \mu$ for every $x \in X$.

In what follows, we consider only T_1-spaces. To every mapping $\Psi : X \to 2^Y$ we associate another one $\Psi' : X \to 2^Y \cup \{\emptyset\}$ defined by

$$\Psi'(x) = \{y \in \Psi(x) : x \in \operatorname{int}(\Psi^{-1}(W)) \text{ for every neighbourhood } W \text{ of } y\},$$

which is known as a *derived* mapping of Ψ [5]. Concerning derived mappings, we will need the following result which was actually proved in [10, Theorems 1.2 and 3.1] (see the remark after the proof of [10, Theorem 1.2]).

(2.2) Let (Y, d) be a complete metric space, $\mathcal{T} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n, and let

$$\mathcal{T}' = \bigcup \Big\{ \{\Psi'(x) : x \in X\} : \quad \Psi : X \to \mathcal{T} \text{ is a q.l.s.c. mapping} \Big\}.$$

Then, $\mathcal{T}'$ is uniformly equi-LC^n too.

To state our result, we need also the following notation. Let $\mathbb{S}^k$ denote the k-sphere, and $\mathbb{B}^k$ the corresponding k-ball For technical reasons only, we assume that $\mathbb{S}^{-1} = \emptyset$ as well as $\mathbb{S}^{-1} \times X = \emptyset$ for every space X. Finally, to every $\Psi : \mathbb{I} \to \mathcal{F}(Y)$ and every natural $n \geq 0$ we associate another set-valued mapping $\Psi^n : \mathbb{B}^n \times \mathbb{I} \to \mathcal{F}(Y)$ defined by $\Psi^n(b, t) = \Psi(t)$, $(b, t) \in \mathbb{B}^n \times \mathbb{I}$.

Lemma 2.3. *Let (Y, d) be a complete metric space, and let $\mathcal{T} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n. Then, to every $\varepsilon > 0$ there corresponds an $\alpha(\varepsilon) > 0$ with the following two properties*:

(a) *If $0 \leq k < n$, $\Psi : \mathbb{I} \to \mathcal{T}$ is q.l.s.c., and $f_0 : \mathbb{B}^{k+1} \times \{0\} \to Y$ is a constant selection for $\Psi^{k+1}|\mathbb{B}^{k+1} \times \{0\}$, then every $\Psi^{k+1}|\mathbb{S}^k \times \mathbb{I}$-homotopy $g : \mathbb{S}^k \times \mathbb{I} \to Y$ of f_0, with $\boldsymbol{d}(g|\{s_1\} \times \mathbb{I}, g|\{s_2\} \times \mathbb{I}) < \alpha(\varepsilon)$ for every $s_1, s_2 \in \mathbb{S}^k$, can be extended to a Ψ^{k+1}-homotopy $h : \mathbb{B}^{k+1} \times \mathbb{I} \to Y$ of f_0 such that $\boldsymbol{d}(h|\{b_1\} \times \mathbb{I}, h|\{b_2\} \times \mathbb{I}) < \varepsilon$ for every $b_1, b_2 \in \mathbb{B}^{k+1}$.*

(b) *If $-1 \leq k < n$ and $\Psi : \mathbb{I} \to \mathcal{T}$ is l.q.m., then Ψ^{k+1} has the SHEP at $\mathbb{S}^k$. In particular, the statement of* (a) *is true with $\alpha(+\infty) = +\infty$.*

Proof. By (2.2), the family $\mathcal{S} = \mathcal{T}'$ is uniformly equi-LC^n. Then, let $\delta(\varepsilon) \leq \varepsilon$ be as in the definition of uniformly equi-LC^n of $\mathcal{S}$. Next, let $\alpha(\varepsilon) = \delta(\gamma(\varepsilon/3)/3)$, where $\gamma(\varepsilon) \leq \varepsilon$ is as in (2.1) applied with $\mathcal{S}$, and let us show that this $\alpha(\varepsilon)$ works for the case of (a). So, let $0 \leq k < n$ and let $\Psi : \mathbb{I} \to \mathcal{T}$ be q.l.s.c. Also, let f_0 and g be as in (a). Set $\Phi = \Psi' : \mathbb{I} \to \mathcal{S}$. According to [9, Theorem 2.1], Φ is an l.s.c. mapping such that g is a selection for $\Phi^{k+1}|\mathbb{S}^k \times \mathbb{I}$. Since g is uniformly continuous, there exists a positive integer m such that, for every $s \in \mathbb{S}^k$ and $t', t'' \in \mathbb{I}$,

$$|t' - t''| \leq \frac{1}{m} \qquad \text{implies} \qquad d(g(s, t'), g(s, t'')) < \alpha(\varepsilon). \tag{1}$$

Whenever $0 \leq i \leq m$, we now set

$$\mathbb{S}^k_i = \mathbb{S}^k \times \left\{\tfrac{i}{m}\right\} \quad \text{and} \quad \mathbb{B}^{k+1}_i = \mathbb{B}^{k+1} \times \left\{\tfrac{i}{m}\right\}.$$

Let $0 \leq j < m$. Since δ is as in the definition of uniformly equi-LC^n of $\mathcal{S}$, $k+1 \leq n$, $\operatorname{diam}(g(\mathbb{S}^k_{j+1})) < \delta(\gamma(\varepsilon/3)/3)$ and $\Phi^{k+1}\big|\mathbb{B}^{k+1}_{j+1}$ is a constant mapping, there exists a continuous selection $f_{j+1} : \mathbb{B}^{k+1}_{j+1} \to Y$ for $\Phi^{k+1}\big|\mathbb{B}^{k+1}_{j+1}$ such that f_{j+1} is an extension of $g\big|\mathbb{S}^k_{j+1}$ with

$$\operatorname{diam}\left(f_{j+1}\left(\mathbb{B}^{k+1}_{j+1}\right)\right) < \gamma(\varepsilon/3)/3\,. \tag{2}$$

Next, we set $X_{j+1} = \mathbb{B}^{k+1}\times\left[\frac{j}{m}, \frac{j+1}{m}\right]$ and $A_{j+1} = \mathbb{B}^{k+1}_j \cup \mathbb{S}^k \times \left[\frac{j}{m}, \frac{j+1}{m}\right] \cup \mathbb{B}^{k+1}_{j+1}$, and then we define a continuous $\ell_{j+1} : A_{j+1} \to Y$ by $\ell_{j+1}\big|\mathbb{B}^{k+1}_i = f_i$ for $i = j, j+1$, and $\ell_{j+1}\big|\mathbb{S}^k \times \left[\frac{j}{m}, \frac{j+1}{m}\right] = g\big|\mathbb{S}^k \times \left[\frac{j}{m}, \frac{j+1}{m}\right]$. Note that ℓ_{j+1} is a selection for $\Phi^{k+1}\big|A_{j+1}$ such that, by (1) and (2),

$$\operatorname{diam}\left(\ell_{j+1}\left(A_{j+1}\right)\right) < \gamma(\varepsilon/3). \tag{3}$$

For convenience, let $\Phi_{j+1} = \Phi^{k+1}\big|X_{j+1}$. Now, we shall extend ℓ_{j+1} to a continuous selection $h_{j+1} : X_{j+1} \to Y$ for Φ_{j+1} such that

$$\operatorname{diam}\left(h_{j+1}\left(X_{j+1}\right)\right) < \varepsilon\,. \tag{4}$$

Towards this end, we define a continuous $p_{j+1} : X_{j+1} \to Y$ by $p_{j+1}(b,t) = \ell_{j+1}(b, j/m)$, $(b,t) \in X_{j+1}$. Note that, by (3), p_{j+1} is a continuous $\gamma(\varepsilon/3)$-selection for Φ_{j+1} such that $\boldsymbol{d}\big(p_{j+1}\big|A_{j+1}, \ell_{j+1}\big) < \gamma(\varepsilon/3)$. Hence, by (2.1), ℓ_{j+1} can be extended to a continuous selection $h_{j+1} : X_{j+1} \to Y$ for Φ_{j+1} such that $\boldsymbol{d}(p_{j+1}, h_{j+1}) < \varepsilon/3$. This h_{j+1} is as required.

Finally, we define our $h : \mathbb{B}^{k+1}\times\mathbb{I} \to Y$ by $h|X_{j+1} = h_{j+1}$, $0 \le j < m$, which is possible because $j > 0$ implies $X_j \cap X_{j+1} = \mathbb{B}^{k+1}_j$ and $h_{j+1}\big|\mathbb{B}^{k+1}_j = f_j = h_j\big|\mathbb{B}^{k+1}_j$. By virtue of (4), this completes the proof of (a).

We now proceed to the proof of (b). Suppose that $-1 \le k < n$ and $\Psi : \mathbb{I} \to \mathcal{T}$ is l.q.m. Take a continuous selection $f_0 : \mathbb{B}^{k+1}\times\{0\} \to Y$ for $\Psi^{k+1}|\mathbb{B}^{k+1}\times\{0\}$. In case $k \ge 0$, let $g : \mathbb{S}^k \times \mathbb{I} \to Y$ be a $\Psi^{k+1}|\mathbb{S}^k\times\mathbb{I}$-homotopy of f_0. Otherwise, let g be the restriction of any map, say $g = f_0|\mathbb{S}^k \times \mathbb{I}$. We apply the same reduction as before. Namely, let $\mathcal{S} = \mathcal{T}'$ and let $\Phi = \Psi' : \mathbb{I} \to \mathcal{S}$. Then, Φ is an l.s.c. mapping such that, by [10, Lemma 2.1], for every $t \in \mathbb{I}$, every neighbourhood V of t, and every $\varepsilon > 0$ there exists $\delta \in [0,1)$ such that

$$\delta\cdot t \in V \quad \text{and} \quad \Psi(\delta\cdot t) \subset \bigcap\left\{B^d_\varepsilon(\Phi(s)) : s \in [\delta\cdot t, t]\right\}. \tag{5}$$

In particular, this implies that f_0 is a selection for $\Phi^{k+1}|\mathbb{B}^{k+1}\times\{0\}$ because $\Phi(0) = \Psi(0)$. Hence, as in the previous case, g is a $\Phi^{k+1}|\mathbb{S}^k\times\mathbb{I}$-homotopy of f_0. Then, let us denote by A the set of all points $t \in \mathbb{I}$ for which there exists a continuous extension $h_t : \mathbb{B}^{k+1}\times[0,t] \to Y$ of f_0 such that h_t is a selection for $\Phi^{k+1}|\mathbb{B}^{k+1}\times[0,t]$ and $h_t|\mathbb{S}^k\times[0,t] = g|\mathbb{S}^k\times[0,t]$. Obviously, $0 \in A$. First, we show that A is open. Let $t_0 \in A$, and let $h_0 = h_{t_0}$ be as above. Since Φ^{k+1} is l.s.c., $\dim(\mathbb{B}^{k+1}\times\mathbb{I})\le n+1$ and $\mathbb{B}^{k+1}\times\mathbb{I}$ is compact, by (2.2) and [15, Theorem 1.2], there exists $t_1>t_0$ and a continuous selection $h_1:\mathbb{B}^{k+1}\times[0,t_1]\to Y$ for $\Phi^{k+1}|\mathbb{B}^{k+1}\times[0,t_1]$ such that $h_1|\mathbb{S}^k\times[0,t_1]=g|\mathbb{S}^k\times[0,t_1]$ and $h_1|\mathbb{B}^{k+1}\times[0,t_0]=h_0$. Therefore, $t_0\in[0,t_1)\subset A$.

Now, we show that A is closed. Take a point $t_1\in\overline{A}$ such that $t_1>0$. It follows from the definition of A that $[0,t_1)\subset A$. Let $\gamma(1)$ be as in (2.1) applied to the elements of $\mathcal{S}$. In case $k\ge 0$, use the continuity of g and fix a point $t_0\in[0,t_1)$ such that $d(g(s,t),g(s,t_1))<\gamma(1)/2$ for every $s\in\mathbb{S}^k$ and $t\in[t_0,t_1]$. Otherwise, let $t_0=0$. Then, let $\delta\in[0,1)$ be as in (5) applied with $t=t_1$, $V=(t_0,1]$ and $\varepsilon=\gamma(1)$. Since $t=\delta\cdot t_1\in A$, there now exists a continuous extension $h:\mathbb{B}^{k+1}\times[0,t]\to Y$ of f_0 which is a selection for $\Phi^{k+1}|\mathbb{B}^{k+1}\times[0,t]$ and $h|\mathbb{S}^k\times[0,t]=g|\mathbb{S}^k\times[0,t]$. Then, define $p_1:\mathbb{B}^{k+1}\times[0,t_1]\to Y$ by $p_1(b,s)=h(b,s)$ if $s\le t$ and $p_1(b,s)=h(b,t)$ otherwise. In this way, by (5), we get a continuous $\gamma(1)$-selection p_1 for $\Phi^{k+1}|\mathbb{B}^{k+1}\times[0,t_1]$ such that $p_1|\mathbb{B}^{k+1}\times[0,t]=h$. Because of the choice of t_0 and t, we also have that $d(p_1(z),g(z))<\gamma(1)$ for every $z\in\mathbb{S}^k\times[t,t_1]$. Therefore, by (2.1), h can be extended to a continuous selection $h_1:\mathbb{B}^{k+1}\times[0,t_1]\to Y$ for $\Phi^{k+1}|\mathbb{B}^{k+1}\times[0,t_1]$ such that $h_1|\mathbb{S}^k\times[0,t_1]=g|\mathbb{S}^k\times[0,t_1]$. That is, $t_1\in A$. As a result, A is a non-empty clopen subset of $\mathbb{I}$. Hence, $A=\mathbb{I}$ which complete the proof. □

Let us explicitly state the following partial case of Lemma 2.3(b) which we will use in the sequel.

Corollary 2.4. *Let (Y,d) be a complete metric space, $\mathcal{S}\subset\mathcal{F}(Y)$ be uniformly equi-LC^0, and let $\Phi:\mathbb{I}\to\mathcal{S}$ be an l.q.m. mapping. Then, for every point $y\in\Phi(0)$ there exists a continuous selection $h:\mathbb{I}\to Y$ for Φ such that $h(0)=y$.*

3. Proof of Theorem 1.5

We are now ready for the proof of Theorem 1.5. In fact, it consists of the following three separate statements.

Proposition 3.1. *Let (Y,d) be a complete metric space, $\mathcal{S}\subset\mathcal{F}(Y)$ be a uniformly equi-LC^0, $\varphi:X\times\mathbb{I}\to\mathcal{S}$ be such that each $\varphi|\{x\}\times\mathbb{I}$, $x\in X$, is l.q.m., and let $f:X\times\{0\}\to Y$ be a selection for $\varphi|X\times\{0\}$. Then, $\Lambda(\varphi,f):X\to\mathcal{F}(C(\mathbb{I},Y))$.*

Proof. Take a point $x\in X$. Since $\varphi(x,t)$ is closed for every $t\in\mathbb{I}$, to prove that $\Lambda(\varphi,f)(x)\in\mathcal{F}(C(\mathbb{I},Y))$ it suffices to show that $\Lambda(\varphi,f)(x)\ne\emptyset$. That this is so, it follows from Corollary 2.4 because $\varphi|\{x\}\times\mathbb{I}$ is l.q.m. □

Our next proposition states just the principal effect which the exponential mapping Λ of (1.4) yields when it transforms a pair (φ, f) into a set-valued mapping $\Lambda(\varphi, f)$.

Proposition 3.2. *Let (Y, d) be a complete metric space, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n for some $n \geq 1$, $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be such that each $\varphi|\{x\} \times \mathbb{I}$, $x \in X$, is l.q.m., and let $f : X \times \{0\} \to Y$ be a selection for $\varphi|X \times \{0\}$. Then, $\{\Lambda(\varphi, f)(x) : x \in X\}$ is uniformly equi-LC^{n-1} and each $\Lambda(\varphi, f)(x)$ is C^{n-1}.*

Proof. Let $\alpha(\varepsilon)$ be as in Lemma 2.3 applied to $\mathcal{S}$. It suffices to show that the family $\{\Lambda(\varphi, f)(x) : x \in X\}$ is uniformly equi-LC^{n-1} with $\delta(\varepsilon) = \alpha(\varepsilon)$ because one can take $\alpha(+\infty) = +\infty$. So, let $0 \leq k < n$, $x \in X$, and let $q : \mathbb{S}^k \to \Lambda(\varphi, f)(x)$ be continuous with $\operatorname{diam}(q(\mathbb{S}^k)) < \alpha(\varepsilon)$. We look for a continuous extension $p : \mathbb{B}^{k+1} \to \Lambda(\varphi, f)(x)$ of q such that $\operatorname{diam}(p(\mathbb{B}^{k+1})) < \varepsilon$. Towards this end, we consider the l.q.m. mapping $\Phi : \mathbb{I} \to \mathcal{S}$ defined by $\Phi(t) = \varphi(x, t)$, $t \in \mathbb{I}$. Also, we fix a constant selection f_0 for $\Phi^{k+1}|\mathbb{B}^{k+1} \times \{0\}$ by $f_0(\mathbb{B}^{k+1} \times \{0\}) = \{f(x, 0)\}$, and its $\Phi^{k+1}|\mathbb{S}^k \times \mathbb{I}$-homotopy g by $g = \Lambda^{\leftarrow}(q)$. Then, by Lemma 2.3, g can be extended to a Φ^{k+1}-homotopy $h : \mathbb{B}^{k+1} \times \mathbb{I} \to Y$ of f_0 such that $\boldsymbol{d}(h|\{b_1\} \times \mathbb{I}, h|\{b_2\} \times \mathbb{I}) < \varepsilon$ for $b_1, b_2 \in \mathbb{B}^{k+1}$. Finally, $p = \Lambda(h)$ is as required. □

We complete the proof of Theorem 1.5 showing that $\Lambda(\varphi, f)$ is l.s.c. provided f is continuous and φ is Λ-l.s.c.

Proposition 3.3. *Let (Y, d) be a complete metric space, $\mathcal{T} \subset \mathcal{F}(Y)$ be uniformly equi-LC^0, $\varphi : X \times \mathbb{I} \to \mathcal{T}$ be Λ-l.s.c., and let $f : X \times \{0\} \to Y$ be a continuous selection for $\varphi|X \times \{0\}$. Then, $\Lambda(\varphi, f)$ is l.s.c.*

Proof. Let $\varepsilon > 0$, $x_0 \in X$, and let $k_0 \in \Lambda(\varphi, f)(x_0)$. By (2.2), the family $\mathcal{S} = \mathcal{T}'$ is uniformly equi-LC^0. Then, let $\gamma(\varepsilon)$ be as in (2.1) applied to $\mathcal{S}$. Since f and k_0 are continuous and φ is equi- l.s.c. at X, for every $t \in \mathbb{I}$ there exists a neighbourhood $V_t \times W_t$ of (x_0, t) such that $W_t \subset k_0^{\leftarrow}(B^d_{\gamma(\varepsilon)/4}(k_0(t)))$ and, for every $s \in W_t$,

$$V_t \subset \{x \in X : d(f(x, 0), f(x_0, 0)) < \gamma(\varepsilon) \text{ and } \varphi(x, s) \cap B^d_{\gamma(\varepsilon)/4}(k_0(t)) \neq \emptyset\}.$$

Since $\mathbb{I}$ is compact, there exists a finite $T \subset \mathbb{I}$ such that $\mathbb{I} = \bigcup\{W_t : t \in T\}$. Then, let us check that $V = \bigcap\{V_t : t \in T\}$ works. Take a point $x \in V$, and let us consider the l.q.m. mapping $\Phi = \varphi|\{x\} \times \mathbb{I} : \mathbb{I} \to \mathcal{T}$. On the one hand, by the choice of V, the map k_0 is a $\gamma(\varepsilon)/2$-selection for Φ. Since Φ is q.l.s.c., by [10, Lemma 2.1], this implies that k_0 is a $\gamma(\varepsilon)$-selection for $\Phi' : \mathbb{I} \to \mathcal{S}$. On the other hand, $f(x, 0) \in \Phi'(0) = \Phi(0)$ because Φ is, in fact, l.q.m. Finally, let us observe that $d(f(x,0), f(x_0,0)) < \gamma(\varepsilon)$ while Φ' is l.s.c. Thus, by (2.1), Φ' has a continuous selection $k : \mathbb{I} \to Y$ such that $\boldsymbol{d}(k, k_0) < \varepsilon$ and $k(0) = f(x,0)$. This completes the proof because $k \in \Lambda(\varphi, f)(x)$. $\square$

For later use, let us mention the following consequence of Proposition 3.3.

Corollary 3.4. *Let (Y, d) be a complete metric space, $\mathcal{T} \subset \mathcal{F}(Y)$ be uniformly equi-LC^0, and let $\varphi : X \times \mathbb{I} \to \mathcal{T}$ be a Λ-l.s.c. mapping. Then, the derived mapping φ' of φ has the property that $\varphi'|\{x\} \times \mathbb{I} = (\varphi|\{x\} \times \mathbb{I})'$ for every $x \in X$.*

4. Selection homotopy extension theorems I

In this section, we present some possible applications of Theorem 1.5. Our first result is the following improvement of Theorem 1.6 from the Introduction.

Theorem 4.1. *Let Y be a Banach space, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n, X be paracompact, and let $Z \subset X$ with $\dim_X(Z) \le n$. Also, let $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be a Λ-l.s.c. mapping such that $\varphi(z)$ is convex for every $z \notin Z \times \mathbb{I}$. Then, φ has the SHEP at A for every $A \subset X$ closed.*

Proof. Take a continuous selection $f : X \times \{0\} \to Y$ for $\varphi|X \times \{0\}$, a closed $A \subset X$ and a $\varphi|A \times \mathbb{I}$-homotopy g of f. We consider the Banach space $C(\mathbb{I}, Y)$ and the mapping $\Lambda(\varphi, f)$ of (1.4). By Theorem 1.5, $\Lambda(\varphi, f) : X \to \mathcal{F}(C(\mathbb{I}, Y))$ is l.s.c., $\{\Lambda(\varphi, f)(z) : z \in Z\}$ is uniformly equi-LC^{n-1} and $\Lambda(\varphi, f)(z)$ is C^{n-1} for every $z \in Z$. Note that $\Lambda(\varphi, f)(x)$ is convex for every $x \notin Z$ because $\varphi|\{x\} \times \mathbb{I}$ is convex-valued for $x \notin Z$. Also, note that $\Lambda(g) : A \to C(\mathbb{I}, Y)$ is a continuous selection for $\Lambda(\varphi, f)|A$. Then, by [17, Theorem 1.2], $\Lambda(g)$ can be extended to a single-valued continuous selection ℓ for $\Lambda(\varphi, f)$. Finally, the map $h = \Lambda^{\leftarrow}(\ell)$ is a φ- homotopy of f which extends g. $\square$

In the special case of a d-continuous φ and $Z \subset X$ open, Theorem 4.1 implies [16, Theorem 3.4] because every metric space can be embedded isometrically into a Banach space. Let us recall that $\varphi : X \times \mathbb{I} \to 2^Y$ is *d-continuous* if, given $\varepsilon > 0$, every $(x_0, t_0) \in X \times \mathbb{I}$ admits a neighbourhood W such that, for every $(x_1, t_1) \in W$,

$$\varphi(x_1, t_1) \subset \bigcap \{B^d_\varepsilon(\varphi(x,t)) : (x,t) \in W\}.$$

It should be mentioned that the Michael's arguments for proving [16, Theorem 3.4] work if the restriction on the *continuity* of φ is weakened to that of *quasi-continuity*. A mapping $\varphi : X \times \mathbb{I} \to 2^Y$ is called *quasi-continuous* [16] if it is l.s.c.

and, for every $\varepsilon > 0$, every point $(x_0, t_0) \in X \times \mathbb{I}$ has a neighbourhood $U \times V$ such that, for every $(x_1, t_1) \in U \times V$,

$$\varphi(x_1, t_1) \subset \bigcap \{B^d_\varepsilon(\varphi(x_1, t)) : t \in V \cap [t_1, 1]\}.$$

Every d-continuous $\varphi : X \times \mathbb{I} \to 2^Y$ is quasi-continuous while the converse is not true, see [16, Example 3.2]. On the other hand, as it follows from the definitions, every quasi-continuous φ is Λ-l.s.c. The following is a simple example showing that our Theorem 4.1 improves the case of quasi-continuity as well.

Example 4.2. A compact space X and a Λ-l.s.c. mapping $\varphi : X \times \mathbb{I} \to \mathcal{F}(\mathbb{R})$ into the convex compact subsets of the real line $\mathbb{R}$ such that φ is not quasi-continuous.

Proof. Let $X = \{0, 1/n : n = 1, 2, \dots\}$. Define $\varphi : X \times \mathbb{I} \to \mathcal{F}(\mathbb{R})$ by letting for $(x, t) \in X \times \mathbb{I}$ that $\varphi(x, t) = [0, (1-t) \cdot n]$ if $x = 1/n$ and $\varphi(x, t) = \{0\}$ otherwise. This φ satisfies all our requirements. □

Theorem 1.5 applies also to "continuous" mappings. For the purpose, let us recall some concepts concerning *continuity* of set-valued mappings. Let Y be a space. A mapping $\Phi : X \to 2^Y$ is *upper semi-continuous*, or *u.s.c.*, if

$$\Phi^{\#}(U) = \{x \in X : \Phi(x) \subset U\}$$

is open in X for every $U \subset Y$ open. If (Y, d) is a metric space, then Φ is d-u.s.c. provided $\Phi^{\#}(B^d_\varepsilon(\Phi(x)))$ is a neighbourhood of x for every $x \in X$ and $\varepsilon > 0$. Finally, we say that Φ is *continuous* if it is both l.s.c. and u.s.c., and we say that Φ is *d-proximal continuous* if it is both l.s.c. and d-u.s.c. It should be mentioned that a continuous Φ is not necessarily d-continuous and vice versa (see, e.g., [13, Proposition 2.6]), while every continuous or d-continuous Φ is certainly d-proximal continuous. On the other hand, there are d-proximal continuous mappings Φ which are neither continuous nor d-continuous (see, [13, Propositions 2.5]). In view of that, we shall henceforth restrict our attention only to *d-proximal continuity*. Finally, let us agree to say that a mapping $\varphi : X \times \mathbb{I} \to 2^Y$ is *d-proximal Λ-continuous* if φ is d-proximal continuous and $\varphi|\{x\} \times \mathbb{I}$ is d-continuous for every $x \in X$.

Proposition 4.3. *Let (Y, d) be a complete metric space, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^0, $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be d-proximal Λ-continuous, and let $f : X \times \{0\} \to Y$ be a continuous selection for $\varphi|X \times \{0\}$. Then, $\Lambda(\varphi, f)$ is $\boldsymbol{d}$-proximal continuous.*

Proof. By Theorem 1.5, we have only to check that $\Lambda(\varphi, f)$ is $\boldsymbol{d}$-u.s.c. To this end, let $x_0 \in X$ and $\varepsilon > 0$. Also, let $\gamma(\varepsilon)$ be as in (2.1) applied to $\mathcal{S}$. Since

$\varphi|\{x_0\} \times \mathbb{I}$ is d-continuous and φ is d-u.s.c., there exists a neighbourhood U of x_0 in X such that $x \in U$ implies

$$d(f(x,0), f(x_0,0)) < \gamma(\varepsilon) \quad \text{and} \quad \varphi(x,t) \subset B^{\boldsymbol{d}}_{\gamma(\varepsilon)}(\varphi(x_0,t)),\ t \in \mathbb{I}. \tag{6}$$

Now, let $k \in \Lambda(\varphi, f)(x)$ for some $x \in U$. Then, by (6), k is a continuous $\gamma(\varepsilon)$-selection for $\varphi|\{x_0\} \times \mathbb{I}$ such that $d(k(0), f(x_0,0)) < \gamma(\varepsilon)$. Hence, by (2.1), there is $k_0 \in \Lambda(\varphi, f)(x_0)$ with $\boldsymbol{d}(k, k_0) < \varepsilon$. So, $U \subset \Lambda(\varphi, f)^{\#}(B^{\boldsymbol{d}}_{\varepsilon}(\Lambda(\varphi, f)(x_0)))$. $\square$

To state our results for d-proximal Λ-continuous mappings, we need to recall some other concepts. For a subset Z of a space X, we use $\text{r-dim}_X(Z) \leq m$ to denote that $\dim(V) \leq m$ for every $V \subset Z$ which is cozero-set in X (see, [12]). Here, $\dim(V) \leq m$ means that every finite cozero-set cover $\mathcal{V}$ of V admits a finite cozero-set refinement $\mathcal{U}$ of order $\text{Ord}(\mathcal{U}) \leq m+1$, i.e. the *covering dimension* of V in the sense of Morita [19]. Let us mention that $\text{r-dim}_X(Z) \leq m$ is valid if either $\dim(X) \leq m$ ([12, Lemma 5.1]) or $\dim(Z) \leq m$ ([12, Corollary 5.2]). For more detailed information about $\text{r-dim}_X(Z)$, see [12]. Finally, let us recall that a subset A of a space X is P^{τ}*-embedded*, where τ is an infinite cardinal number, if for every locally finite cozero-set cover $\mathcal{W}$ of A of cardinality $|\mathcal{W}| \leq \lambda$ there exists a locally finite cozero-set cover $\mathcal{U}$ of X such that $\mathcal{W}$ is refined by $\mathcal{U} \cap A = \{U \cap A : U \in \mathcal{U}\}$. The notion "$P^{\tau}$-embedded" in this sense is the same as "P^{τ}-embedded" in the sense of Shapiro [24] which was introduced by Arens [2] under the name "τ-normally embedded" (see, [24]).

Theorem 4.4. *Let Y be a Banach space, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n, X be a space, and let $Z \subset X$ with* $\text{r-dim}_X(Z) \leq n$. *Also, let $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be a d-proximal Λ-continuous mapping such that $\varphi(z)$ is convex for every $z \notin Z \times \mathbb{I}$. Then, φ has the SHEP at A for every $P^{w(Y)}$- embedded subset A of X.*

Proof. We repeat the proof of Theorem 4.1. Briefly, let f be a continuous selection for $\varphi|X \times \{0\}$, $A \subset X$ be $P^{w(Y)}$-embedded, and let g be a $\varphi|A \times \mathbb{I}$-homotopy of f. By Proposition 4.3, the mapping $\Lambda(\varphi, f)$ is $\boldsymbol{d}$-proximal continuous. By Theorem 1.5, the family $\{\Lambda(\varphi, f)(z) : z \in Z\}$ is uniformly equi-LC^{n-1} and $\Lambda(\varphi, f)(z)$ is C^{n-1} for every $z \in Z$. Finally, $\Lambda(\varphi, f)(x)$ is convex for every $x \notin Z$. Then, by [11, Theorem 5.1], the continuous selection $\Lambda(g) : A \to C(\mathbb{I}, Y)$ for $\Lambda(\varphi, f)|A$ can be extended to a single-valued continuous selection ℓ for $\Lambda(\varphi, f)$. The map $h = \Lambda^{\leftarrow}(\ell)$ is a φ-homotopy of f which extends g. $\square$

Theorem 4.4 presents another improvement of [16, Theorem 3.4] showing that this theorem remains valid for arbitrary X and under weaker restrictions on the *continuity* of φ.

5. The Borsuk homotopy extension theorem as a selection problem

The technique developed in this paper allows one to read the famous Borsuk homotopy extension theorem [4] as an usual problem of the selection theory. Namely, using only the construction of an exponential mapping $\Lambda(\varphi, .)$ stated in (1.4), the properties of the space $C(\mathbb{I}, Y)$ listed in (1.1), (1.2) and (1.3), and the well-known Michael selection theorem [14, Theorem 3.2″], we reduce the proof of this theorem to extension properties of P^τ-embedded sets of topological spaces. In this way, we obtain a simple proof of the following general version of the Borsuk homotopy extension theorem.

Theorem 5.1. *Let X be a space, Y be an ANR for metrizable spaces which is Čech complete, and let $f : X \to Y$ be continuous. Then, for every $P^{w(Y)}$-embedded subset A of X, every homotopy $\ell : A \times \mathbb{I} \to Y$ of $f|A$ can be extended to a homotopy $h : X \times \mathbb{I} \to Y$ of f.*

It should be mentioned that, in this form, Theorem 5.1 coincides with [20, Theorem 5]. In case X is normal and Y is separable, it coincides with [25, Theorem 3] (for countably paracompact normal X, see [6]). Also, let us mention the case of countably paracompact and collectionwise normal X and $A \subset X$ closed, [6].

In what follows, for a Banach space E, we denote

$$\mathcal{F}_c(E) = \{S \in \mathcal{F}(E) : S \text{ is convex}\}.$$

Our proof of Theorem 5.1 follows an elegant approach to this theorem that was announced in [22, Theorem 4.3]. Now, however, we rely on the following simple consequence of the famous Michael selection theorem [14, Theorem 3.2″] which supplies a more direct way to the fact of interest.

Proposition 5.2. *Let X be a topological space, (E, d) be a Banach space, and let $\Phi : X \to \mathcal{F}_c(E)$ be d-continuous. Also, let $A \subset X$ be $P^{w(E)}$-embedded, and let $g : A \to E$ be a continuous selection for $\Phi|A$. Then, g can be extended to a single-valued continuous selection for Φ.*

Proof. By a result of [1, 19, 23][1], there exists a continuous $f : X \to E$ which extends g. Considering $\mathcal{F}_c(E)$ as a space with the topology generated by the *Hausdorff distance* $H(d)$ on $\mathcal{F}(E)$, we let $Z = \mathcal{F}_c(E) \times E$. Also, let $\Psi : Z \to \mathcal{F}_c(E)$ and $k : Z \to E$ be the projections. Now, define a map $h : X \to Z$ by $h(x) = (\Phi(x), f(x)),\ x \in X$, which is continuous because so are Φ and f. Finally,

[1] for a simple proof of this fact, see [23]

define $B = \{z \in Z : k(z) \in \Psi(z)\}$ which is closed because k is continuous and Ψ is d-continuous (as a set-valued mapping). Since Z is metrizable and Ψ is l.s.c., by [14, Theorem 3.2″], $k|B$ can be extended to a continuous selection $\ell : Z \to E$ for Ψ. Therefore $\ell \circ h$ is a continuous selection for Φ which extends g because $h(A) \subset B$. □

In fact, Proposition 5.2 is a partial case of [13, Theorem 6.1]. However, the use of [13, Theorem 6.1] is not justifiable because its own proof is quite complicated.

PROOF OF THEOREM 5.1. Let A, f and ℓ be as in this theorem and let d be a compatible complete metric on Y. Embed (Y, d) isometrically in a Banach space (E, d) with $w(E) = w(Y)$. Next, define $\varphi : X \times \mathbb{I} \to \mathcal{F}_c(E)$ by $\varphi(z) = E$, $z \in X \times \mathbb{I}$, and let $\Phi = \Lambda(\varphi, f)$ be as (1.4). Note that, by (1.2), $C(\mathbb{I}, E)$ is a Banach space and $\Phi : X \to \mathcal{F}_c(C(\mathbb{I}, E))$. Also, note that Φ is $\boldsymbol{d}$-continuous. Indeed, let $\varepsilon > 0$, $x_0 \in X$, and let U be a neighbourhood of x_0 such that $\operatorname{diam}(f(U)) < \varepsilon/2$. Take points $x_1, x_2 \in U$ and $k_1 \in \Phi(x_1)$. Next, define an l.s.c. mapping $\phi : \mathbb{I} \to \mathcal{F}_c(E)$ by $\phi(t) = \overline{B^d_{\varepsilon/2}(k_1(t))}$ if $t > 0$ and $\phi(0) = \{f(x_2)\}$. By [14, Theorem 3.2″], ϕ admits a continuous selection $k_2 : \mathbb{I} \to E$. Then $\boldsymbol{d}(k_1, k_2) \le \varepsilon/2 < \varepsilon$ and $k_2 \in \Phi(x_2)$, and therefore $\Phi(x_1) \subset B^{\boldsymbol{d}}_{\varepsilon}(\Phi(x_2))$. So, Φ is d-continuous. Finally, note that $\Lambda(\ell)$ is a continuous selection for $\Phi|A$. Hence, by (1.1) and Proposition 5.2, $\Lambda(\ell)$ extends to a continuous selection $g : X \to C(\mathbb{I}, E)$ for Φ. Now, let us recall that Y is an ANR. Therefore, there exists a neighbourhood V of Y in E and a retraction $r : V \to Y$. Note that $C(\mathbb{I}, V) \subset C(\mathbb{I}, E)$ is open, $C(\mathbb{I}, Y) \subset C(\mathbb{I}, E)$ is closed, and $\Lambda(\ell)(A) \subset C(\mathbb{I}, Y) \subset C(\mathbb{I}, V)$. Hence, $F = g^{\leftarrow}(C(\mathbb{I}, Y))$ is a zero-set of X, $W = g^{\leftarrow}(C(\mathbb{I}, V))$ is a cozero-set of X, and $A \subset F \subset W$. Then, take a continuous function $\alpha : X \to \mathbb{I}$ such that $\alpha^{\leftarrow}(0) = X \backslash W$ and $\alpha^{\leftarrow}(1) = F$. Finally, the map $h : X \times \mathbb{I} \to Y$ defined for $x \in X$ and $t \in \mathbb{I}$ by $h(x, t) = r(\Lambda^{\leftarrow}(g)(x, \alpha(x) \cdot t))$ is as required. □

6. Selection homotopy extension theorems II

We conclude this paper presenting some further applications of Theorem 1.5 which are related to Theorem 5.1. In fact, our first result is the following generalization of Theorem 5.1 in terms of selections.

Theorem 6.1. *Let (Y, d) be a complete metric ANR, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n, X be a space, and let $Z \subset X$ with* $\text{r-dim}_X(Z) \le n$. *Also, let $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be a d-proximal Λ- continuous mapping such that $\varphi(z) = Y$ for every $z \notin Z \times \mathbb{I}$. Then, φ has the SHEP at A for every $P^{w(Y)}$-embedded subset A of X.*

PROOF. Let $A \subset X$ be $P^{w(Y)}$-embedded, f be a single-valued continuous selection for $\varphi|X \times \{0\}$, and let g be a $\varphi|A \times \mathbb{I}$-homotopy of f. By Theorem 5.1, g can be extended to a homotopy $h : X \times \mathbb{I} \to Y$ of f. By Proposition 4.3, $\Lambda(\varphi, f)$ is $\boldsymbol{d}$-proximal continuous. Hence, by [13, Proposition 2.2] and [3, Theorem 3.2], the set $D = \{x \in X : \boldsymbol{d}(\Lambda(h)(x), \Lambda(\varphi, f)(x)) = 0\}$ is a zero-set of X. On the other hand, $V = X \backslash D \subset Z$. So, by condition, $\dim(V) \leq n$. Finally, let us observe that, by Theorem 1.5, $\{\Lambda(\varphi, f)(x) : x \in V\}$ is uniformly equi-LC^{n-1} and $\Lambda(\varphi, f)(x)$ is C^{n-1} for every $x \in V$. Therefore, by [11, Theorem 1.1], the map $\Lambda(h)|D$ can be extended to a continuous selection $\ell : X \times \mathbb{I} \to C(\mathbb{I}, Y)$ for $\Lambda(\varphi, f)$. Then, $\Lambda^{\leftarrow}(\ell)$ is a φ-homotopy of f which is an extension of g. $\square$

The last result of this paper is a further improvement of Theorem 5.1 in the special case of a paracompact X. To state it, let us recall that a metric space (Y, d) is a *uniform ANR* if for every $\varepsilon \in (0, +\infty]$ there exists $\delta(\varepsilon) \in (0, +\infty]$ such that, whenever (Y, d) is isometrically embedded into a Banach space (E, d) as a closed subset, there is a retraction $r : B_{\delta(+\infty)}(Y) \to Y$ such that

$$(7) \qquad z \in E \quad \text{and} \quad d(z, Y) < \delta(\varepsilon) \qquad \text{imply} \quad d(z, r(z)) < \varepsilon.$$

Theorem 6.2. *Let (Y, d) be a complete metric uniform ANR, $\mathcal{S} \subset \mathcal{F}(Y)$ be uniformly equi-LC^n, X be a paracompact space, and let $Z \subset X$ with* $\dim_X(Z) \leq$ *n. Also, let $\varphi : X \times \mathbb{I} \to \mathcal{S}$ be a Λ- l.s.c. mapping such that $\varphi(z) = Y$ for every $z \notin Z \times \mathbb{I}$. Then, φ has the SHEP at A for every $A \subset X$ closed.*

PROOF. Let $f : X \times \{0\} \to Y$ be a continuous selection for $\varphi|X \times \{0\}$, $A \subset X$ be closed, and let g be a $\varphi|A \times \mathbb{I}$-homotopy of f. Let $\psi = \varphi'$ and $\mathcal{T} = \mathcal{S}'$. According to Corollary 3.4, $\psi : X \times \mathbb{I} \to \mathcal{T}$ is an l.s.c. mapping such that f is a selection for $\psi|X \times \{0\}$, g is a $\psi|A \times \mathbb{I}$-homotopy of f and $\psi(z) = Y$ for every $z \notin Z \times \mathbb{I}$. Then, define a continuous selection $\ell : X \times \{0\} \cup A \times \mathbb{I} \to Y$ for $\psi|X \times \{0\} \cup A \times \mathbb{I}$ by $\ell|X \times \{0\} = f$ and $\ell|A \times \mathbb{I} = g$. As a result, our proof is now reduced to the verification that ℓ can be extended to a continuous selection for ψ. To this end, we regard [17, Property (3.2)]. Namely, we consider this property for $B = X \times \{0\} \cup A \times \mathbb{I}$, ℓ and ψ as follows:

(6.3) To every $\varepsilon > 0$ there corresponds $\beta(\varepsilon) > 0$ such that for every continuous $\beta(\varepsilon)$-selection h for ψ, with $\boldsymbol{d}(h|B, \ell) < \beta(\varepsilon)$, and every $\mu > 0$ there exists a continuous μ-selection k_μ for ψ such that $\boldsymbol{d}(k_\mu, h) < \varepsilon$ and $\boldsymbol{d}(k_\mu|B, \ell) < \mu$.

By virtue of [17, Proposition 3.3], it will be now sufficient to show that ψ satisfies (6.3), and that

(6.4) for every $\varepsilon > 0$ (hence, for some $\beta(\varepsilon) > 0$ as well) there exists a continuous ε-selection $h_\varepsilon : X \times \mathbb{I} \to Y$ for ψ such that $\boldsymbol{d}(h_\varepsilon|B, \ell) < \varepsilon$.

This is what we shall do till the end of this proof. First, to show that ψ satisfies (6.3), let us observe that $\dim_{X\times\mathbb{I}}(Z \times \mathbb{I}) \le n+1$. Indeed, take a subset $T \subset Z \times \mathbb{I}$ which is closed in $X \times \mathbb{I}$. Since the projection $\pi : X \times \mathbb{I} \to X$ is perfect, $S = \pi(T) \subset X$ is a closed. Hence, $\dim S \le n$ because $S \subset Z$, and therefore, by a result of [18], $\dim(S \times \mathbb{I}) \le n+1$. Then, $\dim T \le n+1$ because $T \subset S \times \mathbb{I}$ is closed. That is, $\dim_{X\times\mathbb{I}}(Z \times \mathbb{I}) \le n+1$. On the other hand, by (2.2), the family $\mathcal{T}$ is uniformly equi-LC^n. Hence, according to the proof of [17, Theorem 1.3], ψ satisfies [17, Property (3.2)] and, in particular, (6.3) as well.

We complete the proof showing that ψ satisfies (6.4). To this end, note that, by Corollary 3.4, $\Lambda(\psi, f) = \Lambda(\varphi, f)$. Hence, by Theorem 1.5, $\Lambda(\psi, f) : X \to \mathcal{F}(C(\mathbb{I}, Y))$ is l.s.c., the family $\{\Lambda(\psi, f)(x) : x \in X\}$ is uniformly equi-LC^{n-1} and each $\Lambda(\psi, f)(x)$ is C^{n-1}. On the other hand, the map $\Lambda(g) : A \to Y$ is a continuous selection for $\Lambda(\psi, f)|A$. Then, define another l.s.c. mapping $\Psi : X \to \mathcal{F}(C(\mathbb{I}, Y))$ by $\Psi(x) = \{\Lambda(g)(x)\}$ if $x \in A$ and $\Psi(x) = \Lambda(\psi, f)(x)$ otherwise, see [14, Example 1.3*].

Embed (Y, d) isometrically into a Banach space (E, d). Next, take an $\varepsilon > 0$, and let $\delta(\varepsilon)$ be as in the definition of uniform ANR of (Y, d). According to [17, Lemma 4.1], there exists a locally finite open cover $\mathcal{U}$ of X and a continuous map $u : |\mathcal{N}^n(\mathcal{U})| \to C(\mathbb{I}, E)$ such that

$$u(|\sigma|) \subset B^{\boldsymbol{d}}_{\delta(\varepsilon)}(\Psi(x)) \quad \text{for every } \sigma \in \mathcal{N}^n(\mathcal{U}) \text{ and } x \in \bigcap \sigma. \tag{8}$$

Here, $\mathcal{N}(\mathcal{U})$ is the *nerve* of $\mathcal{U}$, i.e. the simplicial complex

$$\mathcal{N}(\mathcal{U}) = \{\sigma \subset \mathcal{U} : \bigcap \sigma \neq \emptyset\}.$$

Also, $\mathcal{N}^n(\mathcal{U})$ is the n- *skeleton* of $\mathcal{N}(\mathcal{U})$ while $|\mathcal{N}^n(\mathcal{U})|$ is the *polytope* on $\mathcal{N}^n(\mathcal{U})$.

Now, as in [17, Theorem 1.2], we extend u to a continuous map $v : |\mathcal{N}(\mathcal{U})| \to E$ such that

$$v(|\sigma|) \subset \operatorname{conv}(u(|\sigma \cap \mathcal{N}^n(\mathcal{U})|)) \quad \text{for every simplex } \sigma \in \mathcal{N}(\mathcal{U}). \tag{9}$$

Next, consider the continuous map $\Lambda^{\leftarrow}(v) : |\mathcal{N}(\mathcal{U})| \times \mathbb{I} \to E$. According to the definition of Ψ, (8) and (9), we get that

$$\Lambda^{\leftarrow}(v)(|\sigma| \times \{0\}) \subset B^d_{\delta(\varepsilon)}(f(x, 0)), \quad \sigma \in \mathcal{N}(\mathcal{U}) \text{ and } x \in \bigcap \sigma, \tag{10}$$

and

$$\text{(11)} \quad \Lambda^{\leftarrow}(v)(|\sigma| \times \{t\}) \subset B^{d}_{\delta(\varepsilon)}(g(x,t)), \quad \sigma \in \mathcal{N}(\mathcal{U}),\ x \in A \cap \left(\bigcap \sigma\right) \text{ and } t \in \mathbb{I}.$$

Then, let $r : B^{d}_{\delta(+\infty)}(Y) \to Y$ be the corresponding retraction. Thus, by (10) and (11), we get a continuous map

$$q = r \circ \Lambda^{\leftarrow}(v)\Big|(|\mathcal{N}(\mathcal{U})| \times \{0\} \cup |\mathcal{N}(\mathcal{U})| \times A) : |\mathcal{N}(\mathcal{U})| \times \{0\} \cup |\mathcal{N}(\mathcal{U})| \times A \to Y.$$

Hence, by Theorem 5.1, it can be extended to a continuous map $p : |\mathcal{N}(\mathcal{U})| \times \mathbb{I} \to Y$. According to the properties of r (see (7)), the definition of Ψ, (10) and (11), this implies that

$$\text{(12)} \qquad \Lambda(p)(|\sigma|) \subset B^{\boldsymbol{d}}_{\delta(\varepsilon)}(\Psi(x)) \quad \text{for every } \sigma \in \mathcal{N}(\mathcal{U}) \text{ and } x \in \left(\bigcap \sigma\right) \backslash Z.$$

We accomplish the construction of the required h_ε just like in the proof of [17, Theorem 1.2]. Namely, take an open cover $\{V_U : U \in \mathcal{U}\}$ of X with $F_U = \overline{V_U} \subset U$ for every $U \in \mathcal{U}$. Next, to every $x \in X$ we associate the simplex $\sigma(x) = \{U \in \mathcal{U} : x \in F_U\}$. Finally, for every $x \in X$, set $H_x = \bigcup\{F_U : U \in \mathcal{U} \text{ and } x \notin F_U\}$.

Now, for every $s \in S = X \backslash Z$, we consider the set

$$G_s = \left\{x \in X : \Lambda(p)(|\sigma(s)|) \subset B^{\boldsymbol{d}}_{\varepsilon}(\Psi(x))\right\} \backslash H_s.$$

Thus, by (12), we get a neighbourhood G_s of s with

$$\text{(13)} \qquad \sigma(x) \subset \sigma(s), \quad \text{for every } x \in G_s.$$

Hence, the set $M = X \backslash \bigcup\{G_s : s \in S\}$ is closed in X and $M \subset Z$. Therefore, $\dim(M) \leq n$ which implies the existence of an open cover $\{W_U : U \in \mathcal{U}\}$ of X such that

$$W_U \subset V_U \quad \text{for every } U \in \mathcal{U},$$

and

$$|\{U \in \mathcal{U} : x \in W_U\}| \leq n + 1 \quad \text{for every } x \in M.$$

Finally, take a partition of unity $\{\xi_U : U \in \mathcal{U}\}$ on X with $X \backslash W_U \subset \xi_U^{\leftarrow}(0)$, $U \in \mathcal{U}$, and then define a continuous map $\xi : X \to |\mathcal{N}(\mathcal{U})|$ by

$$\xi(x) = \sum\{\xi_U(x) \cdot U : U \in \mathcal{U}\}, \quad x \in X.$$

The map $h_\varepsilon : X \times \mathbb{I} \to Y$, defined by $h_\varepsilon = \Lambda^{\leftarrow}(\Lambda(p) \circ \xi)$, is as required in (6.4). □

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