ESTIMATES OF SPANS OF A SIMPLE CLOSED CURVE INVOLVING MESH

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Communicated by Andrzej Lelek

Abstract. We show that the dual effectively monotone span of a simple closed curve $X$ in the plane does not exceed the infimum of the set of positive numbers $m$ such that a chain with mesh $m$ covers $X$. We also include a short direct proof of a known inequality $\sigma_0(0) \leq \epsilon(X)$, where $X$ is a continuum.

We begin with a brief review of the definitions introduced by A. Lelek in [1] and [2]. Let $X$ be a nonempty connected metric space. The span $\sigma(X)$ of $X$ is the least upper bound of the set of real numbers $r$, $r \geq 0$, that satisfy the following condition.

There exists a connected space $Y$ and a pair of continuous functions $f, g : Y \to X$ such that

$$f(Y) = g(Y)$$

and $\text{dist}[f(y), g(y)] \geq r$ for every $y \in Y$.

Relaxing the requirement posed by equality (1) to the inclusion $f(Y) \subseteq g(Y)$ produces the definition of the semispan $\sigma_0(X)$ of $X$. Requiring that $g$ be onto gives the definitions of the surjective span $\sigma^*(X)$ and the surjective semispan $\sigma_0^*(X)$.

It was pointed out in [2] that

$$0 \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam}(X).$$

In this paper we concentrate on the case when $X$ is a simple closed curve in the plane. Notice that in this case $\sigma^*(X) = \sigma(X)$ and $\sigma_0^*(X) = \sigma_0(X)$. We define the monotone span $\sigma_m(X)$ of $X$ as follows.

This paper was supported in part by a Mini Grant from Widener University.
**Definition 1.** If $X$ is a simple closed curve then

$$
\sigma_m(X) = \sup_{f,g} \inf_{t \in [0,1]} \|f(t) - g(t)\|
$$

where $f, g : [0,1] \to X$ are continuous on $[0,1]$, monotone on $[0,1)$, and $f([0,1]) = X = g([0,1])$.

Next we define the dual monotone span $\bar{\sigma}_m(X)$ of $X$.

**Definition 2.** If $X$ is a simple closed curve then

$$
\bar{\sigma}_m(X) = \inf_{h,k} \sup_{t \in [0,1]} \|h(t) - t(k)\|
$$

where $h, k : [0,1] \to X$ are continuous on $[0,1]$, monotone on $[0,1)$, $h([0,1]) = X = k([0,1])$, $h(0) = k(0)$, there exists a point $t' \in (0,1)$ such that $h([0,t']) \cap k([0,t']) = \{h(0)\}$ and neither $h([0,t'])$ nor $k([0,t'])$ is a singleton.

Finally, we define the dual effectively monotone span $\bar{\sigma}_{em}(X)$.

**Definition 3.** If $X$ is a simple closed curve then

$$
\bar{\sigma}_{em}(X) = \inf_{h,k} \sup_{t \in [0,1]} \|h(t) - t(k)\|
$$

where $h, k : [0,1] \to X$ are continuous, $h([0,1]) = X = k([0,1])$, $h(0) = k(0)$, there exists a point $t_0 \in (0,1)$ such that $h(t_0) = k(t_0) \neq h(0)$ and $h([0,t_0]) \cap k([0,t_0]) = \{h(0), h(t_0)\}$.

It follows from a more general result of A. Lelek [2, Th.2.1, p. 39] that when $X$ is a continuum then $\sigma_0(X) \leq \epsilon(X)$.\footnote{Another proof, due to E. Duda, appeared in H. Fernandez’s Doctoral Dissertation, U. of Miami, 1998} We include this estimate with a different direct proof.

**Theorem 1.1.** Let $X$ be a continuum and let $\epsilon(X)$ be the infimum of the set of positive numbers $m$ such that a chain with mesh $m$ covers $X$. Then $\sigma_0(X) \leq \epsilon(X)$.

**Proof.** Let $Y$ be a connected space and let $f, g : Y \to X$ be continuous functions such that $g(Y) \supset f(Y)$. Let $m$ be a number such that a chain $C$ with mesh $m$ covers $X$, and let $C_1, C_2, \ldots, C_n$ denote the links in the chain $C$ in their consecutive order. If there exist $y_0 \in Y$ and $i$, $1 \leq i \leq n$, such that $f(y_0), g(y_0) \in C_i$ then $\text{dist}\{f(y_0), g(y_0)\} \leq m$. In this case $\sigma_0(x) \leq m$, and the arbitrary choice of $m$ implies that $\sigma_0(X) \leq \epsilon(X)$.\footnote{Another proof, due to E. Duda, appeared in H. Fernandez’s Doctoral Dissertation, U. of Miami, 1998}
Suppose now that for each \( i, 1 \leq i \leq n \), and every \( y \in Y \), if \( f(y) \in \hat{C}_i \) then \( g(y) \notin \hat{C}_i \). This property, along with the continuity of \( f \) and \( g \), implies that the sets
\[
A = \{ y \in Y : f(y) \in C_i, \ g(y) \in C_j, \ i < j \}
\]
and
\[
B = \{ y \in Y : f(y) \in C_i, \ g(y) \in C_j, \ i > j \}
\]
are open and disjoint, and \( A \cup B = Y \). Furthermore, \( A \neq \emptyset \) and \( B \neq \emptyset \). Indeed, suppose that \( B = \emptyset \) and let \( k \) be the smallest number such that \( g(Y) \cap C_k \neq \emptyset \). Clearly, \( k > 1 \). Let \( y_1 \in Y \) and let \( g(y_1) \in C_k \). Then \( f(y_1) \in C_i \cap (X \setminus \hat{C}_k) \) for some \( i, i < k \). This contradicts the assumption that \( g(Y) \supset f(Y) \). Hence, \( B \neq \emptyset \). Similarly, we argue that \( A \neq \emptyset \). It follows that, in this case, \( A \) and \( B \) provide a separation of \( Y \). This contradicts the assumption that \( Y \) is connected. Therefore, only the first considered case holds, i.e. there exists \( y_0 \in Y \) and \( i, 1 \leq i \leq n \), such that \( f(y_0), g(y_0) \in \hat{C}_i \) and, hence, \( \sigma_0(X) \leq \epsilon(X) \).

It turns out that the same bound from above, \( \epsilon(X) \), holds for the dual effectively monotone space of a simple closed curve \( X \). For a pair of two distinct points \( A, B \in X \) we denote the counterclockwise arc on \( X \) from \( A \) to \( B \) by \( AB^\sim \).

**Theorem 1.2.** Let \( X \) be a simple closed curve. Then \( \bar{\sigma}_{em}(X) \leq \epsilon(X) \).

**Proof.** We need only assume that \( X \) is a polygon. Let \( \{C_j\}_{j=1}^N \) be a chain of closed sets with mesh \( \delta \) such that \( X \subseteq \bigcup_{j=1}^N C_j \). We choose a point \( E \in X \cap C_1 \) and a point \( F \in X \cap C_N \). Let \( g \) be the mapping that defines \( X \), \( g : [0, 1] \rightarrow X, g(0) = g(1), 1 : 1 \) on \([0, 1]\). Without loss of generality we assume that \( g(0) = E \). Let \( t_F \) be the point in \((0, 1)\) such that \( g(t_F) = F \). Define two homeomorphisms on \([0, 1]\) in the following way:
\[
g_1 : \ [0, 1] \rightarrow EF^\sim, \ g_1(t) = g(t_F t), \\
g_2 : \ [0, 1] \rightarrow FE^\sim, \ g_2(t) = g(1 - (1 - t_F) t).
\]
Note that \( g_1(0) = g_2(0) = E \), \( g_1(1) = g_2(1) = F \).

We shall construct two mappings \( h, k \) such that \( h : [0, 1] \rightarrow EF^\sim, \ k : [0, 1] \rightarrow FE^\sim \) and \( \forall t \in [0, 1] \exists j, k (j, k) \in \{1, \ldots, N\} \ni h(t), k(t) \in C_j \). First we assume, without loss of generality, that \( \partial C_j \) is a Jordan curve for each link \( C_j \) in the chain \( \{C_j\}_{j=1}^N \), and that \( C_j \cap C_{j+1} = \partial C_j \cap \partial C_{j+1}, \ldots, N - 1 \), while \( \text{diam}(C_j \cap X) \leq \delta \). Assume also that \( E \in \partial C_1 \) and there is an arc \( L_0 \), \( \partial C_1 \supset L_0 \), such that \( L_0 \cap X = E \). Similarly, \( F \in \partial C_N \) and there is an arc \( L_N, \partial C_N \supset L_N \), such that \( L_N \cap X = F \). Let \( L_j = \partial C_j \cap \partial C_{j+1} \) for \( j = 1, \ldots, N - 1 \), let \( i = \sqrt{-1} \), and let \( G : [0, 1] \times [0, i] \rightarrow \)
Define the definition of $h$ and $f$. Furthermore, due to Whittaker (see Theorem 3 in [4]) there exist two maps $\bar{h}$ and $\bar{f}$ on $[0,1]$. Let $H_1$ and $H_2$ be two homeomorphisms on $G^{-1}(E)$ and $G^{-1}(E^\sim)$, respectively, such that $H_1(G^{-1}(g_1(t))) = (t, f_1(t))$ and $H_2(G^{-1}(g_2(t))) = (t, f_2(t))$ for each $t \in [0,1]$.

We are now in a position to define the mappings $h, k$. For each $t \in [0,1]$ put

$$h(t) = G(H^{-1}_1(\bar{f}_1(t), f_1(\bar{f}_1(t)))),$$

$$k(t) = G(H^{-1}_2(\bar{f}_2(t), f_2(\bar{f}_2(t)))).$$

Note that $h(0) = k(0) = E$ since $H^{-1}_n(0,0) = G^{-1}(g_n(0)) = G^{-1}(E)$, $n = 1, 2$. Similarly, $h(1) = k(1) = F$ since $H^{-1}_n(1,1) = G^{-1}(g_n(1)) = G^{-1}(F)$, $n = 1, 2$. Furthermore, $h([0,1]) = EF^\sim$, $k([0,1]) = FE^\sim$, and $\forall t \in [0,1] \exists \exists s \in [0,1] : h(t), k(t) \in G([0,1] \times \{s_i\})$. Hence, $\forall t \in [0,1] \exists j, j \in \{1, \ldots, N\} : h(t), k(t) \in C_j$.

Finally, we define $h_1, k_1 : [0,1] \to X$ as follows:

$$h_1(t) = \begin{cases} h(2t), & \text{for } t \in [0,1/2] \\ h(2 - 2t), & \text{for } t \in [1/2,1] \end{cases}$$

$$k_2(t) = \begin{cases} k(2t), & \text{for } t \in [0,1/2] \\ k(2 - 2t), & \text{for } t \in [1/2,1] \end{cases}$$

It follows that $\forall t \in [0,1] h_1(t), k_1(t) \in C_j$ for the same $j$. Hence, $\forall t \in [0,1] ||h_1(t) - k_1(t)|| \leq \delta$. Since $h_1$ and $k_1$ satisfy the conditions imposed in the definition of $\overline{\sigma}_{em}$, this ends the proof of Theorem 1.2. 

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Note. The span of $X$ is equal to $\epsilon(X)$ when $X$ is the boundary of a convex region (see [3]).

References


Received February 25, 1997
Revised version received April 18, 2000

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