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ESTIMATES OF SPANS OF A SIMPLE CLOSED CURVE INVOLVING MESH

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Abstract. We show that the dual effectively monotone span of a simple closed curve X in the plane does not exceed the infimum of the set of positive numbers m such that a chain with mesh m covers X. We also include a short direct proof of a known inequality $\sigma_0(0) \leq \epsilon(X)$, where X is a continuum.

We begin with a brief review of the definitions introduced by A. Lelek in [1] and [2]. Let X be a nonempty connected metric space. The span $\sigma(X)$ of X is the least upper bound of the set of real numbers $r, r \geq 0$, that satisfy the following condition.

There exists a connected space Y and a pair of continuous functions $f, g : Y \to X$ such that

$$f(Y) = g(Y) \tag{1}$$

and dist$[f(y), g(y)] \geq r$ for every $y \in Y$.

Relaxing the requirement posed by equality (1) to the inclusion $f(Y) \subseteq g(Y)$ produces the definition of the semispan $\sigma_0(X)$ of X. Requiring that g be onto gives the definitions of the surjective span $\sigma^*(X)$ and the surjective semispan $\sigma_0^*(X)$.

It was pointed out in [2] that

$$0 \leq \sigma(X) \leq \sigma_0(X) \leq \text{ diam}(X).$$

In this paper we concentrate on the case when X is a simple closed curve in the plain. Notice that in this case $\sigma^*(X) = \sigma(X)$ and $\sigma_0^*(X) = \sigma_0(X)$. We define the monotone span $\sigma_m(X)$ of X as follows.

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Definition 1. If X is a simple closed curve then

$$\sigma_m(X) = \sup_{f,g} \inf_{t\in[0,1]} \|f(t) - g(t)\|\,,$$

where $f, g : [0,1] \to X$ are continuous on $[0,1]$, monotone on $[0,1)$, and $f([0,1]) = X = g([0,1])$.

Next we define the dual monotone span $\bar{\sigma}_m(X)$ of X.

Definition 2. If X is a simple closed curve then

$$\bar{\sigma}_m(X) = \inf_{h,k} \sup_{t\in[0,1]} \|h(t) - t(k)\|\,;$$

where $h, k : [0,1] \to X$ are continuous on $[0,1]$, monotone on $[0,1)$, $h([0,1]) = X = k([0,1])$, $h(0) = k(0)$, there exists a point $t' \in (0,1)$ such that $h([0,t']) \cap k([0,t']) = \{h(0)\}$ and neither $h([0,t'])$ nor $k([0,t'])$ is a singleton.

Finally, we define the dual effectively monotone span $\bar{\sigma}_{em}(X)$.

Definition 3. If X is a simple closed curve then

$$\overline{\sigma}_{em}(X) = \inf_{h,k} \sup_{t\in[0,1]} \|h(t) - k(t)\|\,,$$

where $h, k : [0,1] \to X$ are continuous, $h([0,1]) = X = k([0,1])$, $h(0) = k(0)$, there exists a point $t_0 \in (0,1)$ such that $h(t_0) = k(t_0) \neq h(0)$ and $h([0,t_0]) \cap k([0,t_0]) = \{h(0), h(t_0)\}$.

It follows from a more general result of A. Lelek [2, Th.2.1, p. 39] that when X is a continuum then $\sigma_0(X) \leq \epsilon(X)$.[1] We include this estimate with a different direct proof.

Theorem 1.1. *Let X be a continuum and let $\epsilon(X)$ be the infimum of the set of positive numbers m such that a chain with mesh m covers X. Then $\sigma_0(X) \leq \epsilon(X)$.*

Proof. Let Y be a connected space and let $f, g : Y \to X$ be continuous functions such that $g(Y) \supset f(Y)$. Let m be a number such that a chain C with mesh m covers X, and let $C_1, C_2, \ldots, C_n$ denote the links in the chain C in their consecutive order. If there exist $y_0 \in Y$ and $i,\ 1 \leq i \leq n$, such that $f(y_0), g(y_0) \in \bar{C}_i$ then $\operatorname{dist}\{f(y_0), g(y_0)\} \leq m$. In this case $\sigma_0(x) \leq m$, and the arbitrary choice of m implies that $\sigma_0(X) \leq \epsilon(X)$.

[1]Another proof, due to E. Duda, appeared in H. Fernandez's Doctoral Dissertation, U. of Miami, 1998

Suppose now that for each $i, 1 \leq i \leq n$, and every $y \in Y$, if $f(y) \in \bar{C}_i$ then $g(y) \notin \bar{C}_i$. This property, along with the continuity of f and g, implies that the sets

$$A = \{y \in Y : f(y) \in C_i,\ g(y) \in C_j,\ i < j\}$$

and

$$B = \{y \in Y : f(y) \in C_i,\ g(y) \in C_j, i > j\}$$

are open and disjoint, and $A \cup B = Y$. Furthermore, $A \neq \emptyset$ and $B \neq \emptyset$. Indeed, suppose that $B = \emptyset$ and let k be the smallest number such that $g(Y) \cap C_k \neq \emptyset$. Clearly, $k > 1$. Let $y_1 \in Y$ and let $g(y_1) \in C_k$. Then $f(y_1) \in C_i \cap (X \setminus \bar{C}_k)$ for some $i, i < k$. This contradicts the assumption that $g(Y) \supset f(Y)$. Hence, $B \neq \emptyset$. Similarly, we argue that $A \neq \emptyset$. It follows that, in this case, A and B provide a separation of Y. This contradicts the assumption that Y is connected. Therefore, only the first considered case holds, i.e. there exists $y_0 \in Y$ and $i,\ 1 \leq i \leq n$, such that $f(y_0), g(y_0) \in \bar{C}_i$ and, hence, $\sigma_0(X) \leq \epsilon(X)$. □

It turns out that the same bound from above, $\epsilon(X)$, holds for the dual effectively monotone space of a simple closed curve X. For a pair of two distinct points $A, B \in X$ we denote the counterclockwise arc on X from A to B by $AB^{\sim}$.

Theorem 1.2. *Let X be a simple closed curve. Then $\bar{\sigma}_{em}(X) \leq \epsilon(X)$.*

Proof. We need only assume that X is a polygon. Let $\{C_j\}_{j=1}^N$ be a chain of closed sets with mesh δ such that $X \subseteq \bigcup_{j=1}^N C_j$. We choose a point $E \in X \cap C_1$ and a point $F \in X \cap C_N$. Let g be the mapping that defines $X,\ g : [0,1] \to X, g(0) = g(1), 1:1$ on $[0,1)$. Without loss of generality we assume that $g(0) = E$. Let t_F be the point in $(0,1)$ such that $g(t_F) = F$. Define two homeomorphisms on $[0,1]$ in the following way:

$$\begin{aligned} g_1 : &= [0,1] \to EF^{\sim}, \quad g_1(t) = g(t_F t), \\ g_2 : &= [0,1] \to FE^{\sim}, \quad g_2(t) = g(1-(1-t_F)t). \end{aligned}$$

Note that $g_1(0) = g_2(0) = E,\ g_1(1) = g_2(1) = F$.

We shall construct two mappings h, k such that $h : [0,1] \to EF^{\sim},\ k : [0,1] \to FE^{\sim}$ and $\forall t \in [0,1] \exists j, j \in (1, \ldots, N) \ni h(t),\ k(t) \in C_j$. First we assume, without loss of generality, that ∂C_j is a Jordan curve for each link C_j in the chain $\{C_j\}_{j=1}^N$, and that $C_j \cap C_{j+1} = \partial C_j \cap \partial C_{j+1}, \ldots, N-1$, while $\operatorname{diam}(C_j \cap X) \leq \delta$. Assume also that $E \in \partial C_1$ and there is an arc $L_0,\ \partial C_1 \supset L_0$, such that $L_0 \cap X = E$. Similarly, $F \in \partial C_N$ and there is an arc $L_N, \partial C_N \supset L_N$, such that $L_N \cap X = F$. Let $L_j = \partial C_j \cap \partial C_{j+1}$ for $j = 1, \ldots, N-1$, let $i = \sqrt{-1}$, and let $G : [0,1] \times [0,i] \to$

$\bigcup_{j=1}^{N} C_j$ be a homeomorphism, mapping the unit square onto $\bigcup_{j=1}^{N} C_j$, with the following properties:

1) $\forall j = 0, \dots N \qquad G([0,1] \times \{ij/N\}) = L_j$

2) $\forall j = 1, \dots N \qquad \bigcup_{t \in [(j-1)/N, j/N]} G([0,1] \times \{ti\}) = C_j$

3) $\forall t, s \in [0,1] \qquad t \neq s \Rightarrow G([0,1] \times \{ti\}) \cap G([0,1] \times \{si\}) = \emptyset.$

Define $f_1, f_2 : [0,1] \to [0,1]$ as follows:

$$\forall_{t \in [0,1]}\ f_1(t) = \text{ Im } G^{-1}(g_1(t)), f_2(t) = \text{ Im } G^{-1}(g_2(t)).$$

Notice that $f_1(0) = f_2(0) = 0,\ f_1(1) = f_2(1) = 1$ and f_1, f_2 are continuous and piecewise weakly monotone. By the early version of the Mountain Climbers Theorem due to Whittaker (see Theorem 3 in [4]) there exist two maps $\bar{f}_1, \bar{f}_2 : [0,1] \to [0,1]$ such that $\bar{f}_1(0) = \bar{f}_2(0) = 0,\ \bar{f}_1(1) = \bar{f}_2(1) = 1$, and $f_1(\bar{f}_1(t)) = f_2(\bar{f}_2(t))$ for each $t \in [0,1]$. Let H_1 and H_2 be two homeomorphisms on $G^{-1}(EF^{\sim})$ and $G^1(FE^{\sim})$, respectively, such that $H_1(G^{-1}(g_1(t))) = (t, f_1(t))$ and $H_2(G^{-1}(g_2(t))) = (t, f_2(t))$ for each $t \in [0,1]$.

We are now in a position to define the mappings h, k. For each $t \in [0,1]$ put

$$\begin{aligned} h(t) &= G(H_1^{-1}(\bar{f}_1(t), f_1(\bar{f}_1(t)))), \\ k(t) &= G(H_2^{-1}(\bar{f}_2(t), f_2(\bar{f}_2(t)))). \end{aligned}$$

Note that $h(0) = k(0) = E$ since $H_n^{-1}(0,0) = G^{-1}(g_n(0)) = G^{-1}(E),\ n = 1,2.$ Similarly, $h(1) = k(1) = F$ since $H_n^{-1}(1,1) = G^{-1}(g_n(1)) = G^{-1}(F),\ n = 1,2.$ Furthermore, $h([0,1]) = EF^{\sim},\ k[(0,1]) = FE^{\sim}$, and $\forall t \in [0,1] \exists s \in [0,1] \ni h(t), k(t) \in G([0,1] \times \{si\})$. Hence, $\forall t \in [0,1] \exists j, j \in \{1, \dots N\} \ni h(t), k(t) \in C_j$.

Finally, we define $h_1, k_1 : [0,1] \to X$ as follows:

$$h_1(t) = \begin{cases} h(2t), & \text{for } t \in [0, 1/2] \\ h(2-2t), & \text{for } t \in [1/2, 1] \end{cases}$$

$$k_2(t) = \begin{cases} k(2t), & \text{for } t \in [0, 1/2] \\ k(2-2t), & \text{for } t \in [1/2, 1]. \end{cases}$$

It follows that $\forall_{t \in [0,1]} h_1(t),\ k_1(t) \in C_j$ for the same j. Hence, $\forall_{t \in [0,1]} \|h_1(t) - k_1(t)\| \leq \delta$. Since h_1 and k_1 satisfy the conditions imposed in the definition of $\overline{\sigma}_{em}$, this ends the proof of Theorem 1.2.

□

Note. The span of X is equal to $\epsilon(X)$ when X is the boundary of a convex region (see [3]).

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