SOME HOMOTOPY PROPERTIES OF SPACES OF FINITE SUBSETS OF TOPOLOGICAL SPACES

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Abstract. For $X$ a non-empty topological space and $k$ a positive integer, we denote by $\text{Sub}(X, k)$ the set of non-empty subsets of $X$ having cardinality $\leq k$, suitably topologized. The $\text{Sub}(\cdot, k)$ are homotopy functors and their properties are studied. We prove that if $X$ is Hausdorff and path-connected, then for all $k \geq 1$ and $n \geq 0$, the maps $\pi_n(\text{Sub}(X, k)) \to \pi_n(\text{Sub}(X, 2k + 1))$ induced by the inclusion are the 0-maps. In the direction of non-triviality, we prove that if $X$ is a non-empty closed manifold of dimension $\geq 2$, then for each $k \geq 1$, $\text{Sub}(X, k)$ is homologically non-trivial.

1. Introduction

Let $X$ be a non-empty topological space and $k$ a positive integer. We denote by $\text{Sub}(X, k)$ the set of non-empty subsets of $X$ having cardinality $\leq k$. As a set, $\text{Sub}(X, k)$ contains the configuration spaces $C(X, i)$ for $1 \leq i \leq k$ where $C(X, i)$ is the space of unordered $i$-tuples of distinct points of $X$. The $C(X, i)$ have proved important in homotopy theory (e.g. [1], [4]) and certain geometric applications (e.g. [3], [5], [7], [8], [9], [10], [11]). Our topologization of $\text{Sub}(X, k)$ will be such that for $1 \leq i \leq k$, $C(X, i)$ with its standard topology will be a subspace of $\text{Sub}(X, k)$, and will take into account the fact that finite subsets of different cardinalities may nevertheless be close. In contrast with the $C(\cdot, k)$, the $\text{Sub}(\cdot, k)$ are functors (in fact, homotopy functors).

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Our first main result is that if $X$ is a non-empty path-connected Hausdorff space, then for each $k \geq 1$ and $n \geq 0$ the map
\[
\pi_n(\text{Sub}(X,k)) \to \pi_n(\text{Sub}(X,2k+1))
\]
induced by the inclusion is the 0-map. In contrast with this, our second main result is that if $X$ is a non-empty closed manifold of dimension $\geq 2$, then $\text{Sub}(X,k)$ is homologically non-trivial for all $k \geq 1$.

In §2 we topologize the $\text{Sub}(X,k)$ and establish some general topological properties. In §3 we establish some homotopy properties of the $\text{Sub}(X,k)$, and our main results are proved in §4.

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2. General Topology of $\text{Sub}(X,k)$

Let $X$ be a non-empty Hausdorff space and $k$ a positive integer. Write $X^k$ for the $k$-fold Cartesian product $X \times \cdots \times X$ and let $q_k^X : X^k \to \text{Sub}(X,k)$ be given by $q_k^X(x_1, \ldots, x_k) = \{x_1, \ldots, x_k\}$. Thus, for example, if $x, y \in X$, then $q_3^X(x, x, y) = q_3^X(x, y, y) = \{x, y\}$. We give $\text{Sub}(X,k)$ the quotient topology relative to the surjection $q_k^X$, and will henceforth call this topology the standard topology on $\text{Sub}(X,k)$. Note that $q_k^X$ factors through the $k^\text{th}$ symmetric product $Sp^k(X)$. We sometimes abbreviate $q_k^X$ by leaving off the subscript $k$ and/or the superscript $X$ when there is no danger of confusion. Trivially, the quotient map $q : X^1 \to \text{Sub}(X,1)$ is a homeomorphism.

If $k$ is a positive integer, let $\underline{k}$ denote $\{1, \ldots, k\}$. For positive integers $k, l$ and any function $\alpha : \underline{k} \to \underline{l}$, we obtain, for any topological space $X$, a continuous map $\alpha_* : X^l \to X^k$ given by $\alpha_* (x_1, \ldots, x_l) = (x_{\alpha(1)}, \ldots, x_{\alpha(k)})$. Let $\mathcal{N}$ denote the full sub-category of the category of sets whose objects are the $\underline{k}$, $k \geq 1$. The following Lemma is immediate:

**Lemma 2.1.** For each fixed topological space $X$, the assignments $\underline{k} \mapsto X^k$ and $\alpha \mapsto \alpha_*$ constitute a contravariant functor from $\mathcal{N}$ to the category of topological spaces. Furthermore, if $\alpha : \underline{k} \to \underline{l}$, then for each non-empty topological space $X$, the image of $\alpha_*$ is $\{(x_1, \ldots, x_k) \mid x_i = x_j \text{ whenever } \alpha(i) = \alpha(j)\}$. \qed

**Lemma 2.2.** Let $\alpha : \underline{k} \to \underline{l}$ and suppose $X$ is a non-empty Hausdorff space. Then the image of $\alpha_*$ is closed in $X^k$.

**Proof.** Suppose $x = (x_1, \ldots, x_k) \in X^k - \text{Im } \alpha_*$. Then there exist $i, j \in \underline{k}$ such that $\alpha(i) = \alpha(j)$ but $x_i \neq x_j$. Choose disjoint neighborhoods $U, V$ in $X$ of $x_i$, $x_j$.\n
For \( k \geq l \geq 1 \) let \( \text{surj}(k, l) \) denote the set of all surjections \( k \rightarrow l \).

**Lemma 2.3.** Let \( X \) be a non-empty Hausdorff space. Then for integers \( k \geq l \geq 1 \) and \( \alpha \in \text{surj}(k, l) \), \( \alpha \circ_{X} : X^{l} \rightarrow X^{k} \) is a homeomorphism onto a closed subspace of \( X^{k} \).

**Proof.** Since \( \alpha \) is surjective, we can choose a function \( \beta : l \rightarrow k \) such that \( \alpha \beta \) is the identity on \( l \). It follows that \( \beta \circ_{X} \alpha \) is the identity on \( X^{l} \) and so \( \alpha \circ_{X} \) is a homeomorphism onto its image. The latter is closed in \( X^{k} \) by Lemma 2.2.

We have set inclusions
\[
\text{Sub}(X, 1) \subset \text{Sub}(X, 2) \subset \text{Sub}(X, 3) \subset \cdots
\]
The question arises as to whether the standard topology on \( \text{Sub}(X, k) \) agrees with the subspace topology derived from the standard topology on \( \text{Sub}(X, k + 1) \). Fortunately the two topologies agree:

**Proposition 2.4.** Let \( X \) be a non-empty Hausdorff space and \( k \) a positive integer. Then the standard topology on \( \text{Sub}(X, k) \) agrees with the subspace topology derived from the standard topology on \( \text{Sub}(X, k + 1) \). Moreover, \( \text{Sub}(X, k) \) is closed in \( \text{Sub}(X, k + 1) \).

**Proof.** For each \( \alpha \in \text{surj}(k + 1, k) \), it follows from Lemma 2.3 that \( \alpha \circ_{X} : X^{k} \rightarrow X^{k+1} \) is a closed map. For each such \( \alpha \) the diagram
\[
\begin{array}{ccc}
X^{k} & \xrightarrow{\alpha \circ_{X}} & X^{k+1} \\
\downarrow & & \downarrow \\
\text{Sub}(X, k) & \xrightarrow{i} & \text{Sub}(X, k+1)
\end{array}
\]
commutes where \( i \) is the inclusion map. Thus \( i \) is continuous with respect to the standard topologies on \( \text{Sub}(X, k) \) and \( \text{Sub}(X, k + 1) \). Thus all assertions will follow if we show that \( i \) is a closed map with respect to the standard topologies.

Let \( A \) be closed in \( \text{Sub}(X, k) \). Then \( q_{k}^{-1}(A) \) is closed in \( X^{k} \) and hence each \( \alpha \circ_{X} \circ q_{k}^{-1}(A) \) is closed in \( X^{k+1} \). We have
\[
q_{k+1}^{-1}(i(A)) = \bigcup_{\alpha \in \text{surj}(k+1, k)} \alpha \circ_{X} \circ q_{k}^{-1}(A).
\]
Since \( \text{surj}(k+1, k) \) is finite, \( q_{k+1}^{-1}(i(A)) \) is closed in \( X^{k+1} \), and hence \( i(A) \) is closed in \( \text{Sub}(X, k+1) \). \( \square \)

**Proposition 2.5.** Let \( X \) be a non-empty Hausdorff space and \( k \) a positive integer. Then the quotient map \( q_k : X^k \to \text{Sub}(X, k) \) is a closed map.

**Proof.** We proceed by induction on \( k \), the result being trivial for \( k = 1 \). Suppose \( k > 1 \) and that \( q_{k-1} : X^{k-1} \to \text{Sub}(X, k - 1) \) is a closed map. Note that for any subset \( A \) of \( X^k \),

\[
q_k^{-1}(A) = \left( \bigcup_{\alpha \in \text{surj}[k, k-1]} \alpha_x q_{k-1}^{-1}(q_k(A) \cap \text{Sub}(X, k - 1)) \right) \cup \bigcup_{\beta \in \text{surj}[k, k]} \beta_x(A).
\]

Since, by Lemma 2.3, the \( \alpha_x \) and \( \beta_x \) are all closed maps, it remains only to show that if \( A \) is closed in \( X^k \), then \( q_k(A) \cap \text{Sub}(X, k - 1) \) is closed in \( \text{Sub}(X, k - 1) \).

This follows immediately from the fact that

\[
q_k(A) \cap \text{Sub}(X, k - 1) = \bigcup_{\alpha \in \text{surj}[k, k-1]} q_{k-1} \alpha_x^{-1}(A)
\]

and the inductive hypothesis that \( q_{k-1} \) is a closed map. \( \square \)

In general, \( q_k^X \) need not be an open map. For example, if \( X = \mathbb{R} \) and \( U = (0, 2) \times (0, 2) \times (2, 4) \), then \( (1, 3, 3) \in q_3^{-1}q_2(U) \) but \( (1, 3, 3) \) is not an interior point of \( q_3^{-1}q_2(U) \). However, we do have the following:

**Lemma 2.6.** Let \( X \) be a non-empty Hausdorff space, \( k \) a positive integer, and suppose \( U \) is open in \( X \). Then \( q(U \times X^{k-1}) \) and \( q(U^k) \) are open in \( \text{Sub}(X, k) \).

**Proof.** Each \( \sigma \in \text{surj}(k, k) = \Sigma_k \) yields a self-homeomorphism \( \sigma_x : X^k \to X^k \). Note that

\[
q^{-1}(U \times X^{k-1}) = \bigcup_{\sigma \in \Sigma_k} \sigma_x(U \times X^{k-1}),
\]

a union of open sets, proving the openness of \( q(U \times X^{k-1}) \). Since \( q^{-1}(U^k) = U^k \), the openness of \( q(U^k) \) follows. \( \square \)

**Proposition 2.7.** If \( X \) is a non-empty Hausdorff space and \( k \) a positive integer, then \( \text{Sub}(X, k) \) is Hausdorff.

**Proof.** Let \( S \) and \( T \) be distinct points in \( \text{Sub}(X, k) \). We can suppose that there exists an \( x \in S \) such that \( x \not\in T \). Since \( X \) is Hausdorff we can choose disjoint open subsets \( U \) and \( V \) of \( X \) such that \( x \in U \) and \( T \subset V \). By Lemma 2.6, \( q(U \times X^{k-1}) \) and \( q(V^k) \) are open in \( \text{Sub}(X, k) \). Note that they are disjoint and that \( S \in q(U \times X^{k-1}), T \in q(V^k) \). \( \square \)
Suppose $X$ is a pointed Hausdorff space with basepoint $x_0$. For any positive integer $k$, let $\text{Sub}_0(X, k)$ denote the subspace of $\text{Sub}(X, k)$ consisting of those subsets which contain $x_0$. Then $\text{Sub}_0(X, k)$ is a pointed Hausdorff space with basepoint $\{x_0\}$.

**Proposition 2.8.** Let $X$ be a pointed Hausdorff space and $k$ a positive integer. Then $\text{Sub}_0(X, k)$ is a closed subspace of $\text{Sub}(X, k)$.

**Proof.** $q^{-1}(\text{Sub}_0(X, k))$ consists of all $k$-tuples of points of $X$ with at least one coordinate equal to $x_0$, and this is closed in $X^k$. 

Suppose $X$ and $Y$ are non-empty Hausdorff spaces and $f : X \to Y$ continuous. For $k \geq 1$ define $\text{Sub}(f, k) : \text{Sub}(X, k) \to \text{Sub}(Y, k)$ by $\text{Sub}(f, k)(S) = f(S)$ for each $S \in \text{Sub}(X, k)$. If $X$ and $Y$ are pointed and $f$ is a pointed map, define $\text{Sub}_0(f, k) : \text{Sub}_0(X, k) \to \text{Sub}_0(Y, k)$ to be the restriction of $\text{Sub}(f, k)$.

**Proposition 2.9.** For each $k \geq 1$, $\text{Sub}(\cdot, k)$ is a covariant functor from the category of non-empty Hausdorff spaces to itself. If $f : X \to Y$ is a continuous map of non-empty Hausdorff spaces, the diagram

$$
\begin{array}{ccc}
\text{Sub}(X, k) & \xrightarrow{\text{Sub}(f, k)} & \text{Sub}(Y, k) \\
\downarrow & & \downarrow \\
\text{Sub}(X, k + 1) & \xrightarrow{\text{Sub}(f, k + 1)} & \text{Sub}(Y, k + 1)
\end{array}
$$

commutes, where the vertical maps are the inclusions.

**Proof.** The only issue is continuity of $\text{Sub}(f, k)$ when $f : X \to Y$ is continuous. This is immediate from commutativity of the diagram

$$
\begin{array}{ccc}
X^k & \xrightarrow{f^k} & Y^k \\
\downarrow{q^X} & & \downarrow{q^Y} \\
\text{Sub}(X, k) & \xrightarrow{\text{Sub}(f, k)} & \text{Sub}(Y, k)
\end{array}
$$

the continuity of the top and two vertical maps, and the fact that $q^X$ is a quotient map. 

By restriction we obtain:
Proposition 2.10. For each $k \geq 1$, $Sub_0(\cdot, k)$ is a covariant functor from the category of pointed Hausdorff spaces to itself. If $f : X \to Y$ is a pointed continuous map of pointed Hausdorff spaces, the diagram

$$
\begin{array}{c c c}
Sub_0(X, k) & \xrightarrow{Sub_0(f, k)} & Sub_0(Y, k) \\
\downarrow & & \downarrow \\
Sub_0(X, k + 1) & \xrightarrow{Sub_0(f, k + 1)} & Sub_0(Y, k + 1)
\end{array}
$$

commutes, where the vertical maps are the inclusions.

We next describe a base for the standard topology on $Sub(X, k)$. Suppose $U_1, \ldots, U_r$ are pairwise-disjoint, non-empty open subsets of $X$ with $r \leq k$. Write $(U_1, \ldots, U_r)^X_k = \{ A \in Sub(X, k) \mid A \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq r \text{ and } A \subset U_1 \cup \cdots \cup U_r \}$. Let $B$ be any base for the topology on $X$ and let

$$
B_k = \{ (U_1, \ldots, U_r)^X_k \mid U_i \in B \text{ for all } i \}.
$$

Proposition 2.11. Let $X$ be a non-empty Hausdorff space, $B$ a base for the topology on $X$, and $k$ a positive integer. Then $B_k$ is a base for the standard topology on $Sub(X, k)$. Moreover, if $V$ is open in $Sub(X, k)$ and $A = \{ x_1, \ldots, x_r \} \in V$ where the $x_i$ are distinct, there exist pairwise-disjoint neighborhoods $U_1, \ldots, U_r$ of $x_1, \ldots, x_r$, respectively, in $B$ such that $A \in (U_1, \ldots, U_r)^X_k \subset V$.

Proof. Let $(U_1, \ldots, U_r)^X_k \in B_k$. We have

$$
q^{-1}((U_1, \ldots, U_r)^X_k) = \bigcup_{\alpha \in \text{surj}(k, r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)},
$$

a union of open rectangles and hence open in $X^k$, whence $B_k$ consists of open sets in $Sub(X, k)$.

Now let $V$ be any open subset of $Sub(X, k)$ and suppose $A = \{ x_1, \ldots, x_r \} \in V$ where the $x_i$ are distinct. Then $q^{-1}(V)$ is open in $X^k$ and $q^{-1}(A) \subset q^{-1}(V)$. Note that $q^{-1}(A) = \{ (x_{\alpha(1)}, \ldots, x_{\alpha(k)}) \mid \alpha \in \text{surj}(k, r) \}$. For each $\alpha \in \text{surj}(k, r)$ we can choose pairwise-disjoint open neighborhoods $U_{\alpha(1)}^0, \ldots, U_{\alpha(k)}^0$ of $x_1, \ldots, x_r$, respectively, such that $U_{\alpha(1)}^0 \times \cdots \times U_{\alpha(k)}^0 \subset q^{-1}(V)$. For $1 \leq i \leq r$ we can choose $U_i \in B$ such that

$$
x_i \in U_i \subset \bigcap_{\alpha \in \text{surj}(k, r)} U_{\alpha(i)}^0.
$$

Then $A \in (U_1, \ldots, U_r)^X_k \subset V$. 

\qed
Corollary 2.12. If $X$ is a non-empty second-countable Hausdorff space, then so are the $Sub(X,k)$ for all $k \geq 1$. 

Let $X$ be a non-empty Hausdorff space and $A$ a non-empty subspace of $X$. Then for each $k \geq 1$, $Sub(A,k)$ is a subset of $Sub(X,k)$. If $i : A \rightarrow X$ denotes the inclusion map, then $Sub(i,k) : Sub(A,k) \rightarrow Sub(X,k)$ is the inclusion map. The question arises as to whether or not the subspace topology on $Sub(A,k)$ derived from the standard topology on $Sub(X,k)$ coincides with the standard topology on $Sub(A,k)$. Fortunately, the two are the same:

Proposition 2.13. Let $A$ be a non-empty subspace of the Hausdorff space $X$ and $k \geq 1$. Then:

(a) The subspace topology on $Sub(A,k)$ derived from the standard topology on $Sub(X,k)$ coincides with the standard topology on $Sub(A,k)$.

(b) If $A$ is open (respectively closed) in $X$, then $Sub(A,k)$ is open (respectively closed) in $Sub(X,k)$.

Proof of (a). Since the inclusion map $Sub(A,k) \rightarrow Sub(X,k)$ is $Sub(i,k)$ where $i : A \rightarrow X$ is the inclusion, it follows that the inclusion of $Sub(A,k)$ into $Sub(X,k)$ is continuous with respect to the standard topologies. Thus it remains only to show that each subset $U$ of $Sub(A,k)$ which is open in the standard topology on $Sub(A,k)$ is also open in the subspace topology derived from the standard topology on $Sub(X,k)$. It suffices to show that whenever $S = \{x_1, \ldots, x_r\} \in U \subseteq Sub(A,k)$ where the $x_i$ are distinct and $U$ is open in the standard topology on $Sub(A,k)$, then there exist pairwise-disjoint open neighborhoods $V_1, \ldots, V_r$ in $X$ of $x_1, \ldots, x_r$, respectively, such that $(V_1, \ldots, V_r)^X \cap Sub(A,k) \subseteq U$. We can choose pairwise-disjoint open neighborhoods $U_1, \ldots, U_r$ in $A$ of $x_1, \ldots, x_r$, respectively, such that $(U_1, \ldots, U_r)^A \subseteq U$. Since $A$ has the subspace topology derived from $X$, there exist open subsets $T_1, \ldots, T_r$ of $X$ such that $U_i = T_i \cap A$ for each $i$. Using the Hausdorffness of $X$, we can choose pairwise-disjoint open sets $V_1, \ldots, V_r$ in $X$ such that $x_i \in V_i \subseteq T_i$ for each $i$. It is immediate that

$$(V_1, \ldots, V_r)^X \cap Sub(A,k) \subseteq (U_1, \ldots, U_r)^A \subseteq U.$$ 

Proof of (b). Let $A$ be open in $X$ and write $T(A)$, $T(X)$ for the topologies on $A$ and $X$, respectively. Then the inclusion of bases $T(A)_k \subseteq T(X)_k$ yields that $Sub(A,k)$ is open in $Sub(X,k)$.

Suppose $A$ is closed in $X$ and $S \subseteq Sub(X,k) - Sub(A,k)$. Say $S = \{x_1, \ldots, x_r\}$ where the $x_i$ are distinct and $x_1 \notin A$. We can choose pairwise-disjoint open
neighborhoods $U_1, \ldots, U_r$ in $X$ of $x_1, \ldots, x_r$, respectively, with $U_1 \subseteq X - A$. Then $S \in (U_1, \ldots, U_r)^X_{k+1} \subseteq \text{Sub}(X, k) - \text{Sub}(A, k)$. \hfill \Box$

**Proposition 2.14.** Let $X$ be a non-empty Hausdorff space and $k, l$ positive integers. Then the union map

$$\mu : \text{Sub}(X, k) \times \text{Sub}(X, l) \rightarrow \text{Sub}(X, k + l)$$

given by $\mu(S, T) = S \cup T$ is continuous.

**Proof.** Let $\mathcal{T}$ denote the topology on $X$. Suppose $V = (U_1, \ldots, U_r)^X_{k+l} \in \mathcal{T}_{k+l}$ and that $(S, T) \in \mu^{-1}(V)$. Suppose $U_{m_1}, \ldots, U_{m_p}$ are the distinct $U_i$ which meet $S$, and $U_{n_1}, \ldots, U_{n_q}$ the distinct $U_i$ which meet $T$. Then $S \in (U_{m_1}, \ldots, U_{m_p})^X_k \in \mathcal{T}_k$, $T \in (U_{n_1}, \ldots, U_{n_q})^X_l \in \mathcal{T}_l$, and each $U_i$ occurs either among the $U_{m_j}$ or $U_{n_j}$ (possibly both). It is immediate that

$$(S, T) \in (U_{m_1}, \ldots, U_{m_p})^X_k \times (U_{n_1}, \ldots, U_{n_q})^X_l \subseteq \mu^{-1}(V),$$

establishing the openness of $\mu^{-1}(V)$. \hfill \Box$

**Proposition 2.15.** Suppose $X$ is a non-empty, locally compact, Hausdorff space. Then for each $k \geq 1$, $\text{Sub}(X, k)$ is locally compact.

**Proof.** Let $S = \{x_1, \ldots, x_r\} \subseteq \text{Sub}(X, k)$ where the $x_i$ are distinct. Since $X$ is locally compact and Hausdorff we can find pairwise-disjoint open neighborhoods $U_1, \ldots, U_r$ of $x_1, \ldots, x_r$, respectively, whose closures $\overline{U}_i$ are all compact. Then

$$S \in (U_1, \ldots, U_r)^X_k \subseteq \bigcup_{a \in \text{supp}(k, r)} q(\overline{U}_{a(1)} \times \cdots \times \overline{U}_{a(k)}).$$

The latter union is a finite union of compact spaces and hence compact, providing a compact neighborhood of $S$ in $\text{Sub}(X, k)$. \hfill \Box$

**Proposition 2.16.** Let $X$ and $Y$ be non-empty Hausdorff spaces. Let $k$ and $l$ be positive integers. Suppose either $X$ and $Y$ are both locally compact, or that $Y$ is locally compact and $l = 1$. Then the cartesian product map

$$cp : \text{Sub}(X, k) \times \text{Sub}(Y, l) \rightarrow \text{Sub}(X \times Y, kl)$$

given by $cp(S, T) = S \times T$ is continuous.
Proof. We have the commutative diagram:

\[
\begin{array}{ccc}
X^k \times Y^l & \xrightarrow{f} & (X \times Y)^{kl} \\
q_X^k \times q_Y^l & \downarrow & \downarrow q_X^{kl} \\
Sub(X, k) \times Sub(Y, l) & \xrightarrow{cp} & Sub(X \times Y, kl)
\end{array}
\]

where, regarding \((X \times Y)^{kl}\) as the space of \(k \times l\) matrices with entries in \(X \times Y\), \(f\) is given by

\[f(x_1, \ldots, x_k, y_1, \ldots, y_l)_{ij} = (x_i, y_j).\]

Under the hypotheses, either all the spaces involved are locally compact and Hausdorff, or \(q_X^k\) is the identity map on a locally compact Hausdorff space. Under either hypothesis, \(q_X^k \times q_Y^l\) is a quotient map. The continuity of \(cp\) now follows from the continuity of the other maps in the above diagram.

**Proposition 2.17.** Suppose \(X\) is a non-empty regular space. Then for each \(k \geq 1\), \(Sub(X, k)\) is regular.

**Proof.** Let \(S = \{x_1, \ldots, x_r\} \in Sub(X, k)\) where the \(x_i\) are distinct. By Proposition 2.11, it suffices to show that if \(U_1, \ldots, U_r\) are pairwise-disjoint open neighborhoods of \(x_1, \ldots, x_r\), respectively, in \(X\), then there exists an open neighborhood \(V\) of \(S\) in \(Sub(X, k)\) such that \(\nabla V \subset (U_1, \ldots, U_r)_k^X\). By regularity of \(X\) there exist open neighborhoods \(V_1, \ldots, V_r\) in \(X\) of \(x_1, \ldots, x_r\), respectively, such that for each \(i\), \(V_i \subset U_i\). Let

\[A = \bigcup_{\alpha \in \text{surj}(k, r)} \prod_{\alpha(1)} V_1 \times \cdots \times \prod_{\alpha(k)} V_r.\]

Then \(A\) is closed in \(X^k\) and so by Proposition 2.5, \(q(A)\) is closed in \(Sub(X, k)\). Taking \(V = (V_1, \ldots, V_r)_k^X\) we have

\[S \in V \subset q(A) \subset (U_1, \ldots, U_r)_k^X.\]

Corollary 2.12 and Proposition 2.17, together with the Urysohn Metrization Theorem, yield:

**Theorem 2.18.** Let \(X\) be a non-empty second-countable metric space. Then for all \(k \geq 1\), \(Sub(X, k)\) is metrizable.

**Proposition 2.19.** Let \(X\) and \(Y_1, \ldots, Y_k\) be non-empty topological spaces with \(X\) Hausdorff. Suppose \(f_i : Y_i \to X\) are continuous, \(1 \leq i \leq k\). Then the map \(f : Y_1 \times \cdots \times Y_k \to Sub(X, k)\) given by \(f(y_1, \ldots, y_k) = \{f_1(y_1), \ldots, f_k(y_k)\}\) is continuous.
Proof. $f$ is the composition

$$Y_1 \times \cdots \times Y_k \xrightarrow{f_1 \times \cdots \times f_k} X \times \cdots \times X \xrightarrow{g} \operatorname{Sub}(X, k).$$

Proposition 2.20. Let $X$ and $Y$ be non-empty topological spaces with $X$ Hausdorff. Suppose $f_1, \ldots, f_k : Y \to X$ are continuous. Then $g : Y \to \operatorname{Sub}(X, k)$ given by $g(y) = \{f_1(y), \ldots, f_k(y)\}$ is continuous.

Proof. $g$ is the composition

$$Y \xrightarrow{\Delta} Y^k \xrightarrow{f} \operatorname{Sub}(X, k)$$

where $\Delta$ is the $k$-fold diagonal map and $f$ is as in Proposition 2.19.

Suppose we are given a topological group $G$, a non-empty Hausdorff space $X$, and a continuous group action $\alpha : G \times X \to X$. For each $k \geq 1$, $\alpha$ induces a group action $\alpha_k : G \times \operatorname{Sub}(X, k) \to \operatorname{Sub}(X, k)$ in the evident way.

Proposition 2.21. Let $\alpha : G \times X \to X$ be a continuous group action where $X$ is a non-empty Hausdorff space and $G$ a locally compact Hausdorff topological group. Then for each $k \geq 1$, $\alpha_k : G \times \operatorname{Sub}(X, k) \to \operatorname{Sub}(X, k)$ is a continuous group action.

Proof. The only issue is continuity of $\alpha_k$. We have the commutative diagram

$$
\begin{array}{ccc}
G \times X^k & \xrightarrow{f} & X^k \\
\downarrow{1_G \times q_k} & & \downarrow{q_k} \\
G \times \operatorname{Sub}(X, k) & \xrightarrow{\alpha_k} & \operatorname{Sub}(X, k)
\end{array}
$$

where $f(g, x_1, \ldots, x_k) = (g x_1, \ldots, g x_k)$. $1_G \times q_k$ is a quotient map since $G$ is locally compact and Hausdorff. Since $f$ and $q_k$ are continuous, continuity of $\alpha_k$ follows.

Let $X$ be a non-empty Hausdorff space. Recall that $C(X, k)$, the configuration space of unordered $k$-tuples of distinct points of $X$, is the quotient space $F(X, k)/\Sigma_k$ where

$$F(X, k) = \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}$$
and the symmetric group $\Sigma_k$ acts by permuting coordinates. As a set, $C(X, k) \subset \text{Sub}(X, k)$. Note that $F(X, k)$ is open in $X^k$, the diagram

\[
\begin{array}{ccc}
F(X, k) & \subset & X^k \\
\downarrow q' & & \downarrow q \\
C(X, k) & \subset & \text{Sub}(X, k)
\end{array}
\]

is commutative where $q$ and $q'$ are the respective quotient maps, and $F(X, k) = q^{-1}(C(X, k))$. Thus:

**Proposition 2.22.** Let $X$ be a non-empty Hausdorff space and $k \geq 1$. Then the topology on $C(X, k)$ as a quotient of $F(X, k)$ coincides with the subspace topology derived from the standard topology on $\text{Sub}(X, k)$. Moreover, $C(X, k)$ is open in $\text{Sub}(X, k)$.

For $X$ a locally compact Hausdorff space, let $X^+$ denote the one-point compactification of $X$. We follow the convention that if $X$ is already compact, then $X^+$ is the union of $X$ with a new isolated point.

For any non-empty Hausdorff space $X$ and any $k \geq 2$, the composition

\[
C(X, k) \xrightarrow{i} \text{Sub}(X, k) \xrightarrow{p} \text{Sub}(X, k)/\text{Sub}(X, k-1)
\]

where $p$ is the collapsing map, is a continuous injection onto a subspace whose complement consists of a single point $\ast$.

**Proposition 2.23.** Let $X$ be a non-empty regular space and $k \geq 2$. Then:

(a) The injection $C(X, k) \to \text{Sub}(X, k)/\text{Sub}(X, k-1)$ is a homeomorphism of $C(X, k)$ onto an open subspace of $\text{Sub}(X, k)/\text{Sub}(X, k-1)$.

(b) If, additionally, $X$ is compact, then $\text{Sub}(X, k)/\text{Sub}(X, k-1)$ is $C(X, k)^+$, the one-point compactification of $C(X, k)$.

**Proof.** $\text{Sub}(X, k)$ is regular by Proposition 2.17, and $\text{Sub}(X, k-1)$ is closed in $\text{Sub}(X, k)$ by Proposition 2.4. Thus $\text{Sub}(X, k)/\text{Sub}(X, k-1)$ is Hausdorff. Write $i : C(X, k) \to \text{Sub}(X, k)/\text{Sub}(X, k-1)$ for the above injection. It follows easily from the openness of $C(X, k)$ in $\text{Sub}(X, k)$ (Proposition 2.22) that $i$ is an open map. Part (a) now follows, and we henceforth identify $C(X, k)$ with an open subspace of $\text{Sub}(X, k)/\text{Sub}(X, k-1)$.

Suppose, additionally, $X$ is compact. Then $C(X, k)$ is locally compact and Hausdorff, and so $C(X, k)^+$ is defined. Since

\[
\text{Sub}(X, k)/\text{Sub}(X, k-1) = C(X, k) \cup \{\ast\}
\]
and the former is compact Hausdorff, part (b) follows. □

**Theorem 2.24.** For each \( n \geq 1 \) and \( k \geq 1 \), \( \text{Sub}(R^n, k) \) is topologically embeddable in some finite-dimensional Euclidean space.

**Proof.** We proceed by induction on \( k \), the result being immediate for \( k = 1 \). Suppose \( k > 1 \) and that \( f : \text{Sub}(R^n, k-1) \to R^k \) is a topological embedding for some positive integer \( a \). By Theorem 2.18, \( \text{Sub}(R^n, k) \) is normal and so, by Proposition 2.4 and the Tietze Extension Theorem, \( f \) extends to a continuous map \( g : \text{Sub}(R^n, k) \to R^k \). Again, using Theorem 2.18 and Proposition 2.4, there exists a non-negative-valued continuous map \( \alpha : \text{Sub}(R^n, k) \to R \) such that \( \alpha^{-1}(0) = \text{Sub}(R^n, k-1) \). Since \( C(R^n, k) \) is a smooth manifold, there exists a topological embedding \( h : C(R^n, k) \to S^b \) for some positive integer \( b \). Define \( i : \text{Sub}(R^n, k) \to R^{k+1} \) by

\[
i(x) = \begin{cases} 
\alpha(x)h(x) & \text{if } x \in C(R^n, k), \\
0 & \text{if } x \in \text{Sub}(R^n, k-1).
\end{cases}
\]

Continuity of \( i \) follows easily from the facts that \( h \) and \( i \) are continuous, \( h \) is bounded, and \( \alpha \) vanishes on \( \text{Sub}(R^n, k-1) \). Define \( j : \text{Sub}(R^n, k) \to R^{n+b+2} \) by \( j(x) = (g(x), \alpha(x), i(x)) \). Then \( j \) is continuous. Since \( g \) distinguishes different points of \( \text{Sub}(R^n, k-1) \), \( \alpha \) distinguishes points of \( \text{Sub}(R^n, k-1) \) from points of \( C(R^n, k) \), and \( i \) distinguishes different points of \( C(R^n, k) \), \( j \) is injective. Thus, writing \( D^n \) for the closed unit disk in \( R^n \), compactness of \( \text{Sub}(D^n, k) \) implies that the restriction of \( j \) to \( \text{Sub}(D^n, k) \) is a topological embedding. Since the interior of \( D^n \) is homeomorphic to \( R^n \), the result now follows from the functoriality of \( \text{Sub}(\cdot, k) \) and Proposition 2.13. □

**Corollary 2.25.** Suppose \( X \) is homeomorphic to a non-empty subspace of some finite-dimensional Euclidean space. Then for each \( k \geq 1 \), \( \text{Sub}(X, k) \) is topologically embeddable in some finite-dimensional Euclidean space. □

3. **Homotopy Properties of \( \text{Sub}(X, k) \)**

**Proposition 3.1.** Let \( (X, x_0) \) be a path-connected pointed Hausdorff space. Then for all \( k \geq 1 \), \( \text{Sub}(X, k) \) and \( \text{Sub}_0(X, k) \) are path-connected.

**Proof.** Since \( X^k \) and \( \{x_0\} \times X^{k-1} \) are path-connected, so are their images under the quotient map \( q \). □

**Proposition 3.2.** Let \( h : X \times I \to Y \) be a homotopy from \( f \) to \( g \) where \( X \) and \( Y \) are non-empty Hausdorff spaces. Then:
(a) $h_k : \text{Sub}(X, k) \times I \to \text{Sub}(Y, k)$ given by
$$h_k(\{x_1, \ldots, x_r\}, t) = \{h(x_1, t), \ldots, h(x_r, t)\}$$
is a homotopy from $\text{Sub}(f, k)$ to $\text{Sub}(g, k)$. Moreover, the diagram
$$\begin{array}{ccc}
\text{Sub}(X, k) \times I & \xrightarrow{h_k} & \text{Sub}(Y, k) \\
\downarrow & & \downarrow \\
\text{Sub}(X, k+1) \times I & \xrightarrow{h_{k+1}} & \text{Sub}(Y, k+1)
\end{array}$$
commutes, where the vertical maps are the inclusions.
(b) If $A$ is a non-empty subset of $X$ and $h$ is a homotopy rel $A$ (i.e. $h(a, t)$ is independent of $t$ for each $a \in A$), then $h_k$ is a homotopy rel $\text{Sub}(A, k)$.

**Proof.** The only issue is the continuity of $h_k$. We have the commutative diagram
$$\begin{array}{ccc}
X^k \times I & \xrightarrow{1_k \times \Delta} & X^k \times I^k \\
\downarrow & & \downarrow \\
\text{Sub}(X, k) \times I & \xrightarrow{h_k} & \text{Sub}(Y, k)
\end{array}$$
where $\Delta : I \to I^k$ is the $k$-fold diagonal map, and $\iota$ is the permutation map which interleaves the coordinates of $X^k$ with those of $I^k$. Since $I$ is locally compact and Hausdorff, $q^X_k \times 1_I$ is a quotient map. The continuity of $h_k$ now follows.

**Corollary 3.3.** Let $h : X \times I \to Y$ be a pointed homotopy from $f$ to $g$ where $X$ and $Y$ are pointed Hausdorff spaces. Then $h_k : \text{Sub}_0(X, k) \times I \to \text{Sub}_0(Y, k)$ given by
$$h_k(\{x_1, \ldots, x_r\}, t) = \{h(x_1, t), \ldots, h(x_r, t)\}$$
is a pointed homotopy from $\text{Sub}_0(f, k)$ to $\text{Sub}_0(g, k)$.

**Corollary 3.4.** For non-empty Hausdorff spaces $X$ and $k \geq 2$, the homotopy type of $\text{Sub}(X, k)/\text{Sub}(X, k-1)$ depends only on the homotopy type of $X$.

In general, the homotopy type of $C(X, k)$ is not determined by the homotopy type of $X$ (see [4]). However, by Proposition 2.23 and Corollary 3.4, for non-empty compact Hausdorff spaces $X$, the homotopy type of $C(X, k)$ depends only on the homotopy type of $X$. 

Proposition 3.5. Suppose \( i : A \xrightarrow{\sim} X \) is a cofibration where \( X \) is Hausdorff, and \( A \) non-empty. Then for each \( k \geq 1 \), \( \text{Sub}(i, k) : \text{Sub}(A, k) \to \text{Sub}(X, k) \) is a cofibration. If \( X \) is pointed with basepoint in \( A \), then \( \text{Sub}_0(i, k) : \text{Sub}_0(A, k) \to \text{Sub}_0(X, k) \) is a cofibration.

**Proof.** Let \( r : X \times I \to X \times I \) be a retraction of \( X \times I \) onto \( (X \times \{0\}) \cup (A \times I) \). Let \( \pi_1 : X \times I \to X \) and \( \pi_2 : X \times I \to I \) denote the respective projections. Let \( f : \text{Sub}(X, k) \times I \to \text{Sub}(X, k) \times I \) be the composition

\[
\text{Sub}(X, k) \times I = \text{Sub}(X, k) \times \text{Sub}(I, 1) \xrightarrow{ep} \text{Sub}(X \times I, k) \\
\text{Sub}(X, k) \times \text{Sub}(I, k) \\
\text{Sub}(X, k) \times I
\]

The cartesian product map \( ep \) is continuous by Proposition 2.16. The composition \( \min \circ q_1^f \) is clearly continuous, and so \( \min \) is continuous. Hence \( f \) is continuous.

It is easily checked that \( f \) is a retraction onto \( (\text{Sub}(X, k) \times \{0\}) \cup (\text{Sub}(A, k) \times I) \).

The required retraction in the pointed case is obtained by restriction of this \( f \). \( \square \)

**Proposition 3.6.** Suppose \( X \) is a non-empty locally contractible Hausdorff space. Then for each \( k \geq 1 \), \( \text{Sub}(X, k) \) is locally contractible.

**Proof.** It suffices to show that whenever \( U_1, \ldots, U_r \) are mutually disjoint open subsets of \( X \) with \( h_i : U_i \times I \to U_i \) a strong deformation retraction to a one-point space \( \{x_i\} \), \( 1 \leq i \leq r \leq k \), then \( \{x_1, \ldots, x_r\} \) is a strong deformation retract of \( (U_1, \ldots, U_r)^X \). For each \( \alpha \in \text{surj}(k, r) \) let

\[
h^\alpha : U_{\alpha(1)} \times \cdots \times U_{\alpha(k)} \times I \to U_{\alpha(1)} \times \cdots \times U_{\alpha(k)}
\]

be given by

\[
h^\alpha ((u_1, \ldots, u_k), t) = (h_{\alpha(1)}(u_1, t), \ldots, h_{\alpha(k)}(u_k, t))
\]

and let

\[
h : \left( \bigcup_{\alpha \in \text{surj}(k, r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)} \right) \times I \to \bigcup_{\alpha \in \text{surj}(k, r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)}
\]

be the disjoint union of the \( h^\alpha \). Passage to quotients yields a continuous

\[
\bar{h} : (U_1, \ldots, U_r)^X_k \times I \to (U_1, \ldots, U_r)^X_k
\]
which is the desired strong deformation retraction.

**Lemma 3.7.** Let $X$ be a non-empty compact locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Then:
(a) For each $k \geq 1$, $\text{Sub}(X, k)$ is an ANR.
(b) For each $k \geq 2$, $\text{Sub}(X, k)/\text{Sub}(X, k - 1)$ is an ANR.

**Proof.** By Corollary 2.25 and Proposition 3.6, $\text{Sub}(X, k)$ is a compact, locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Part (a) now follows from [2, p. 240].

Applying [12, Theorem 8.2] to the case $X_1 = \text{Sub}(X, k)$, $A_1 = \text{Sub}(X, k - 1)$ and $X_2 = \text{a~one-point space}$, part (b) follows.

**Theorem 3.8.** Let $X$ be a non-empty compact locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Then for all $k > 1$, the inclusion $\text{Sub}(X, k - 1) \to \text{Sub}(X, k)$ is a cofibration.

**Proof.** By Lemma 3.7(a), the spaces $\text{Sub}(X, i)$ are locally compact, separable metric ANRs, and hence ENRs. The assertion is now a consequence of [6, p. 84, Problem 3].

### 4. Main Theorems

**Theorem 4.1.** Let $X$ be a path-connected pointed Hausdorff space. Then for each $k \geq 1$ and $n \geq 0$, the map $\pi_n(\text{Sub}_0(X, k)) \to \pi_n(\text{Sub}_0(X, 2k - 1))$ induced by the inclusion is the 0-map.

**Proof.** The result is immediate for $n = 0$ by Proposition 3.1. Let $n \geq 1$. Thus $\pi_n$ is group-valued. We use additive notation even though the group operation might be non-commutative in case $n = 1$. Let $f : S^n \to \text{Sub}_0(X, k)$ be a pointed map. We have the homotopy-commutative diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{f \times f} & \text{Sub}_0(X, k) \times \text{Sub}_0(X, k) \\
\downarrow & & \downarrow \\
S^n & \xrightarrow{\Delta} & S^n \times S^n \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sub}_0(X, k) \times \text{Sub}_0(X, k) & \xrightarrow{\mu} & \text{Sub}_0(X, 2k - 1) \\
\downarrow & & \downarrow \\
\text{Sub}_0(X, k) & \xrightarrow{i} & \\
\end{array}
\]

(in fact, all regions are strictly commutative except for the triangle involving the diagonal and comultiplication on $S^n$). In this diagram, $\varphi$ is the folding map, $\mu$
the restriction of the union map, and $i$ the inclusion. Thus, writing $i$ for the identity map on $S^n$,

$$i_*[f] = i_*f_*[i] = i_*\varphi_* (f \vee f)_* \mu_*[i] = i_*([f] + [f])$$

and so $i_*[f] = 0$. 

\[\]

**Theorem 4.2.** Let $(X, x_0)$ be a path-connected pointed Hausdorff space. Then for each $k \geq 1$ and $n \geq 0$, the map $\pi_n(Sub(X, k)) \to \pi_n(Sub(X, 2k + 1))$ induced by the inclusion is the 0 map.

**Proof.** The case $n = 0$ is immediate from Proposition 3.1. Let $n \geq 1$. Then $\pi_n$ is group-valued and as in the proof of Theorem 4.1 we use additive notation. We have the commutative diagrams

\[
\begin{array}{ccc}
\text{Sub}(X, k) & \xrightarrow{\alpha} & \text{Sub}_0(X, k + 1) \\
\downarrow i_1 & & \downarrow i \\
\text{Sub}_0(X, 2k + 1) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sub}(X, k) \times \text{Sub}(X, k) & \xrightarrow{\mu} & \text{Sub}(X, 2k) \\
\downarrow \Delta & & \downarrow j \\
\text{Sub}(X, k) \\
\end{array}
\]

where $\alpha$ adjoins $x_0$ to each set, $\Delta$ is the diagonal map, $i_1$ and $i_2$ are the axial inclusions, $\mu$ the union map, and the other maps are inclusions.

Let $f : S^n \to \text{Sub}(X, k)$ be a pointed map. Then from general homotopy theory, $\Delta_*[f] = l_1[f] + l_2[f]$. Thus

$$j_*[f] = j_*\mu_* \Delta_*[f] = j_*\mu_* (l_1[f] + l_2[f]) = 2l_*i_*\alpha_*[f] = 0$$

since $i_* = 0$ by Theorem 4.1. 

For any non-empty Hausdorff space $X$, let $\text{Sub}(X) = \bigcup_{k \geq 1} \text{Sub}(X, k)$ with the weak topology. Thus $\text{Sub}(X)$ is the space of all non-empty finite subsets of $X$. From Theorem 4.2 we have:

**Corollary 4.3.** Let $X$ be a non-empty path-connected Hausdorff space. Then $\text{Sub}(X)$ is weakly contractible.
Theorem 4.4. Let $M$ be a non-empty compact connected $n$-dimensional manifold without boundary, $n \geq 2$. Then for each $k \geq 1$, the mod 2 singular cohomology group $H^{nk}(\text{Sub}(M, k); \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$, and $H^i(\text{Sub}(M, k); \mathbb{Z}/2) = 0$ for $i > nk$.

Proof. All homology and cohomology groups below are with $\mathbb{Z}/2$ coefficients, and for brevity we write $M_k$ for $\text{Sub}(M, k)$. We proceed by induction on $k$, the result being immediate for $k = 1$. Suppose $k > 1$ and, inductively, that
\begin{equation}
H^i(M_{k-1}) = 0 \text{ for } i > n(k - 1).
\end{equation}

Since $M_k - M_{k-1}$ is the configuration space $C(M, k)$, an $nk$-dimensional manifold, the pair $(M_k, M_{k-1})$ is a compact relative $nk$-manifold and so by Lefschetz duality (see, e.g., [13, p. 297, Theorem 19]), we have isomorphisms
\begin{equation}
\tilde{H}^j(M_k, M_{k-1}) \cong H_{nk-j}(C(M, k))
\end{equation}
for all $j$, where $\tilde{H}$ denotes Alexander cohomology. Since $M_k$ and $M_{k-1}$ are compact ANR’s by Lemma 3.7, it follows from [13, p. 290, Theorem 10] that the above Alexander cohomology groups are isomorphic to the corresponding singular cohomology groups. Thus
\begin{equation}
H^j(M_k, M_{k-1}) = 0 \text{ for } j > nk
\end{equation}
and
\begin{equation}
H^{nk}(M_k, M_{k-1}) \cong H_0(C(M, k)) \cong \mathbb{Z}/2.
\end{equation}

Let $i > nk$. Then exactness of
\[
H^i(M_k, M_{k-1}) \to H^i(M_k) \to H^i(M_{k-1})
\]
and the vanishing of the extreme groups by (2) and (1), we have $H^i(M_k) = 0$.

From exactness of
\[
H^{nk-1}(M_{k-1}) \to H^{nk}(M_k, M_{k-1}) \to H^{nk}(M_k) \to H^{nk}(M_{k-1}),
\]
it follows from (1) that the extreme groups vanish, and so by (3), $H^{nk}(M_k) \cong \mathbb{Z}/2$, completing the proof. \hfill \Box

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