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TOPOLOGY OF FIXED POINT SETS OF SURFACE HOMEOMORPHISMS

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Abstract. This paper investigates the topology of the fixed point set $\mathsf{Fix}(f)$ of orientation preserving homeomorphisms f of a connected surface M under the assumptions that M has finitely generated homology, $\mathsf{Fix}(f)$ is compact and nonempty with finitely many components, and no component of $M \setminus \mathsf{Fix}(f)$ is an open cell. This last condition holds if f preserves area, or nonwandering points are dense, or there is a nowhere dense global attractor. The main conclusion is that the Euler characteristic of $\mathsf{Fix}(f)$ for Čech cohomology is finite and no smaller than the Euler characteristic of M. Applications are made to attractors, analytic homeomorphisms, homoclinic points, prime power iterates, and commuting homeomorphisms.

1. Introduction

An enormous amount of research, originally inspired by the search for periodic solutions to differential equations, has been devoted to proving existence of fixed points for broad classes of maps. But the topological structure of fixed point sets has received little attention; notable exceptions include the program initiated by Smith on homeomorphisms of finite period (Smith [33], Borel *et al.* [5]) and the results of Neumann [25] and Carter [10] for area preserving twist maps of the annulus.

This paper investigates the Čech cohomology of the fixed point set $\mathsf{Fix}(f)$ for several kinds of orientation preserving surface homeomorphisms f. In various settings we establish the existence of acyclic components of $\mathsf{Fix}(f)$. The most useful assumption is that f preserves area; but many of the results hold in other

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common situations, e.g., every point is nonwandering, or there is a nowhere dense global attractor.

Our results are especially powerful when f is (real) analytic, for then $\mathsf{Fix}(f)$ is an analytic variety and thus has simple topology. In particular, compact acyclic components of $\mathsf{Fix}(f)$ are isolated fixed points. In some cases the existence of fixed points with index 1 is proved.

The most important hypothesis, (H0), is that no complementary component of $\mathsf{Fix}(f)$ having compact closure is simply connected. This holds when f preserves area and in many other cases.

Outline of contents First we use Brouwer's nonwandering theorem to motivate our chief assumption, then briefly review other basic results used throughout the paper. Sufficient conditions for (H0) are given in Proposition 1.1 and Theorem 2.1, followed by the fundamental Theorem 2.2, which gives inequalities relating the Euler characteristic of M to Betti numbers of components of $\mathsf{Fix}(f)$. Lower bounds are found for the number of acyclic components of $\mathsf{Fix}(f)$, followed by conditions under which such components have fixed point indices that are positive, or equal to 1. Many of the technical proofs are postponed to the final section. The basic results are applied to analytic homeomorphisms, homoclinic points, commuting homeomorphisms, and fixed points of prime power iterates. Examples 3.9 and 3.15 are simple calculations illustrating some of the theorems.

1.1. **Notation, conventions and background.** The sets of natural numbers, positive natural numbers and integers are denoted respectively by $\mathbf{N}$, $\mathbf{N}_+$ and $\mathbf{Z}$. All spaces are endowed with a metric, denoted by $d(x, y)$. Euclidean n-space is $\mathbf{R}^n$, the unit sphere in $\mathbf{R}^3$ is $\mathbf{S}^2$, and the closed unit disk is $\mathbf{D}^2$. By a disk or open cell sphere we mean a homeomorph of $\mathbf{D}^2$ or $\mathbf{R}^2$ respectively. Surfaces are metrizable; unless otherwise indicated, they are connected and oriented and have empty boundary. $\varnothing$ denotes the empty set.

Both $\overline{Y}$ and $\mathsf{clos}(Y)$ denote the closure of a subset. The frontier (set-theoretic boundary) of a subset $X \subset M$ is $\mathsf{clos}(X) \cap \mathsf{clos}(M \setminus X)$, denoted by $\dot{X}$ or $\mathsf{Fr}\,(X)$. If N is a manifold, its boundary is the manifold ∂N. Our manifolds are assumed to have empty boundary except where the contrary is obvious.

All maps are assumed continuous. A set X is *invariant* under a map g if it is nonempty and $g(X) = X$.

Homeomorphisms, indicated by $\approx$, are always bijective. Self homeomorphisms of a manifold are always orientation preserving unless the contrary is indicated.

- *Throughout this paper M denotes a connected, orientable metrizable surface without boundary, and f is an orientation preserving homeomorphism of such a surface.*

Wandering points A point is *wandering* for a map h if it has a neighborhood N disjoint from $h^n(N)$ for all $n > 0$. The set of wandering points is an open invariant set; its complement, the set of *nonwandering points*, is a closed invariant set containing all omega and alpha limit points and all recurrent points.

Chain recurrence We say x *chains to* y under a map $h\colon X \to X$ if for every $\epsilon > 0$ there exists $m \in \mathbf{N}_+$ and points $x = x_0, \ldots, x_m = y$ such that $d(x_j, h(x_{j-1})) < \epsilon,\ j = 1, \ldots m$. This notion is independent of the metric $d(\cdot, \cdot)$ when X is compact. x chains to every point of its omega limit set, while every point of its alpha limit set chains to x.

A point that chains to itself is *chain recurrent.* Nonwandering points are chain recurrent. $\mathsf{R}(h)$ denotes the closed invariant set of chain recurrent points. If $Q \subset X$ is invariant and dense, then $\mathsf{R}(h|Q) = \mathsf{R}(h) \cap Q$. When X is compact, $\mathsf{R}(h) = \mathsf{R}(h^n)$ for all $n \in \mathbf{N}_+$.

Chain recurrent points that chain to each other are *chain equivalent.* All points in a subset $Q \subset \mathsf{R}(h)$ are chain equivalent provided Q is connected, or Q is an alpha or omega limit set. Information on chain recurrence can be found in Akin [1], Conley [12].

Example. A nontrivial translation of the plane extends to a homeomorphism of the Riemann sphere for which every point is chain recurrent, but only the point at infinity is nonwandering.

Attractors An *attractor* for a homeomorphism $h\colon X \approx X$ is a proper, compact invariant set A that has a neighborhood N such that $\lim_{n\to\infty} \mathsf{dist}(h^n x, A) = 0$ uniformly for $x \in N$. The *basin* B of A is the open invariant set of points whose omega limit sets are in A. All chain recurrent points of B lie in A. An attractor is *global* if its basin is all of X.

Homology When X is a triangulable space (e.g., X is a surface or an analytic variety), the free part of $\check{H}^i(X)$ is naturally isomorphic to $\mathsf{Hom}(H_i(X), \mathbf{Z})$, and $\mathsf{b}_i(X) = \check{\mathsf{b}}^i(X)$. If X is a closed subset of a triangulable space S, then $\check{H}^i(X)$ is functorially isomorphic to the direct limit of the i'th singular cohomology groups of neighborhoods of X in S under the inclusion induced homomorphisms.

If $\check{\mathsf{b}}^0(X)$ (respectively, $\mathsf{b}_0(X)$) is finite, it equals the number of connected components (respectively, path components) of X. When X is compact, $\check{H}^1(X)$ is functorially isomorphic to the group of homotopy classes of maps from X to the circle (Eilenberg [15], Hu [22]).

Spanier [34] is a thorough survey of basic algebraic topology. For surface topology, Newman's elementary approach [26] is recommended.

1.2. **Basic tools for surface homeomorphisms.** Our point of departure is a well known corollary of Brouwer's plane translation theorem:

Brouwer's Nonwandering Theorem [6] *$f\colon \mathbf{R}^2 \approx \mathbf{R}^2$ has a fixed point provided it has a nonwandering point*

This is one of the oldest and deepest theorems in topology, and until the ingenious proof by Franks [17], one of the more difficult. (Even though Brouwer repudiated his fixed point theorems because the proofs were not constructive, we accept them.)

On the face of it, Brouwer's result appears to be merely an existence theorem, but it can be used to obtain information on the topology and location of the fixed point set. The basic idea is as follows:

Proposition 1.1. *Let W be a component of $M \setminus \mathsf{Fix}(f)$ that contains a nonwandering point. Then W is not simply connected.*

Proof. W is invariant by the theorem of Brown and Kister stated below. Therefore Brouwer's nonwandering theorem shows that W cannot be simply connected, because it contains a nonwandering point that is not fixed. $\square$

The next result, a useful addendum to Brouwer's nonwandering theorem, is an application of Theorem 5.7 of Brown [8], who ascribes it to Brouwer [6]:

Brouwer & Brown's Index Theorem [8] *If $f\colon \mathbf{R}^2 \approx \mathbf{R}^2$ has a nonwandering point that is not fixed, there is a fixed point free Jordan curve bounding a disk D such that $\mathsf{Ind}(f, D) = 1$.*

More information on indices is given in the following result:

Pelikan & Slaminka's Index Theorem [27] *If f preserves area, every isolated fixed point has index ≤ 1.*

For diffeomorphisms this was proved by Simon [31, 32]. Dold [14] can be consulted on the fixed point index.

The next theorem is valid in all dimensions:

Brown & Kister's Invariance Theorem [9] *An orientation preserving homeomorphism of a connected manifold with empty boundary preserves each complementary component of the fixed point set.*

Therefore if $K \neq \varnothing$ is a union of components K_α of $\mathsf{Fix}(f)$, then $M \setminus K$ is invariant because it is the intersection of the invariant sets $M \setminus K_\alpha$.

We will use the following classic to find fixed points in invariant acyclic continua of surface homeormorphisms:

Cartwright & Littlewood's Fixed Point Theorem [11] $f\colon \mathbf{R}^2 \approx \mathbf{R}^2$ *has a fixed point in every invariant continuum that doesn't separate the plane*

For an elegant proof see Brown [7].

2. Statement of the main theorems

Heret we state our basic hypotheses and theorems, postponing most proofs to Section 4.

Almost all our results use one or more of the following assumptions:

Hypothesis (H)

- **(H0):** no precompact component of $M \setminus \mathsf{Fix}(f)$ is simply connected
- **(H1):** $\mathsf{Fix}(f)$ is compact and nonempty, with only finitely many components
- **(H2):** the singular homology group $H_1(M)$ is finitely generated
- **(H3):** f is not the identity map

Note that (H1) holds whenever f is analytic or piecewise linear and $\mathsf{Fix}(f)$ is compact and nonempty.

The key condition is (H0). It is not hard to see that if it holds for f, it also holds for the restriction of f to any invariant open set. But (H0) is not automatically inherited by iterates of f. The following theorem gives conditions ensuring that (H0) holds not only for f, but also for its iterates.

A *proper measure* is a measure that is positive (possibly infinite) on every open set and finite on every compact set. We say f preserves (respectively: reduces) area if there is a proper measure $\mathbf{m}$ such that $\mathbf{m}(f(A)) = \mathbf{m}(A)$ for every measurable set A (respectively: $\mathbf{m}(f(A)) < \mathbf{m}(A)$ for every nonempty precompact open set).

Theorem 2.1. *Suppose f satisfies one of the following conditions:*

(a): *f preserves area, or reduces area*

(b): *the set of wandering chain recurrent points with compact orbit closures is nowhere dense*

Then (H0) *holds for every iterate $f^n, n > 0$.*

These conditions are found in diverse situations. When area is preserved, as in many mechanical systems, Poincaré's recurrence theorem implies (H0). When area is reduced, (H0) is vacuously true.

Condition (b), which is topologically invariant, implies (H0) thanks to Brouwer's nonwandering theorem. It is implied by simpler assumptions, including: every point is nonwandering; and also: chain recurrent points are nowhere dense. In particular, (b) holds for the restriction of f to the basin of a nowhere dense attractor or repellor.

Theorem 2.1 is proved in Section 4.

Many of our results are stated in terms of Betti numbers and Euler characteristics. The (singular) *Euler characteristic* $\chi(X)$ of a space X is defined as usual to be $\sum_j (-1)^j \mathsf{b}_j(X)$ provided this sum is finite, where $\mathsf{b}_j(X)$ denotes the rank of the singular homology group $H_j(X)$. The *Čech characteristic* is $\check{\chi}(X) = \sum_j (-1)^j \check{\mathsf{b}}^j(X)$ if this sum is finite, where $\check{\mathsf{b}}^j(X)$ denotes the j'th *Čech number*, i.e., the rank of the Čech cohomology group $\check{H}^j(X)$.

When X is a manifold or polyhedron, Čech cohomology is isomorphic to singular cohomology and $\check{\chi}(X) = \chi(X)$. If X is a connected proper subset of a connected surface, then $\check{\mathsf{b}}^0(X) = 1$, $\check{\mathsf{b}}^i(X) = 0$ for $i > 1$, and $\check{\chi}(X) = 1 - \check{\mathsf{b}}^1(X)$. If X is triangulated as a finite simplicial complex, $\chi(X) = \check{\chi}(X) = \sum_i (-1)^i \tau_i$ where τ_i denotes the number of i-simplices.

A space X is *acyclic* if it has the same Čech cohomology groups as a point, i.e., X is connected and $\check{\mathsf{b}}^j(X) = 0$ for $j > 0$; this implies $\check{\chi}(X) = 1$. Conversely, if $X \subset M$ is a continuum (compact, connected, nonempty set) of dimension 1 and $\check{\chi}(X) = 1$, then X is acyclic.

When (H1) holds, for each $j \in \mathbf{Z}$ we define

$$\kappa_j = \kappa_j(f) = \text{the number of components } K \text{ of } \mathsf{Fix}(f) \text{ such that } \check{\chi}(K) = j$$

Hypothesis (H) implies:

- $\kappa_j = 0$ for $j \geq 2$
- κ_1 is the number of acyclic components of $\mathsf{Fix}(f)$

The following inequalities are basic:

Theorem 2.2. *Assume Hypothesis* (H)*. Then:*

(i): $\check{\mathsf{b}}^1(\mathsf{Fix}(f)) < \infty$
(ii): $\chi(M) \leq \check{\chi}(\mathsf{Fix}(f)) < \infty$
(iii): $\kappa_1(f) \geq \chi(M) + \sum_{i>0} i \cdot \kappa_{-i}(f)$

The right hand side of (iii) equals the Čech characteristic of the union of those components K of $\mathsf{Fix}(f)$ such that $\check{\chi}(K) \leq 0$, which are the components that are not acyclic. Thus we have:

Corollary 2.3. *Assume Hypothesis* (H)*, and suppose no component of* $\mathsf{Fix}(f)$ *is acyclic. Then every component* K *satisfies* $0 \geq \chi(K) \geq \chi(M)$*.*

Conclusion (iii), above, gives a lower bound for the number of acyclic components of $\mathsf{Fix}(K)$, under Hypothesis (H). We show next that when $M = \mathbf{R}^2$ or $\mathbf{S}^2$, we can get some information under the weaker hypothesis (H0), which permits infinitely many components in $\mathsf{Fix}(f)$.

Let $K \subset \mathbf{R}^2$ be a continuum. Its *acyclic hull* $\mathsf{A}(K)$ is the union of K and the bounded complementary components of K. The following facts can be verified:

- $\mathsf{A}(K)$ is compact and acyclic with its frontier in K, and lies in every acyclic set containing K. In particular, $\mathsf{A}(K)$ is contained in the convex hull of K.
- K is acyclic $\Leftrightarrow \mathsf{A}(K) = K$.
- every bounded complementary component of $\mathsf{A}(K)$ is an invariant open cell whose frontier is in K.
- If K and L are disjoint continua, either $\mathsf{A}(K)$, $\mathsf{A}(L)$ are disjoint, or one of these sets contains the other.

Now let K be a continuum in an open cell E. The *acyclic hull of* K *in* E, denoted by $\mathsf{A}_E(K)$, is the union of K and the components of $E \backslash K$ having compact closure in E. Any homeomorphism $h\colon\, E \approx \mathbf{R}^2$ maps $\mathsf{A}_E(K)$ onto $\mathsf{A}(h(K))$.

Theorem 2.4. *Let* f *satisfy* (H0)*. Let* $E \subset M$ *be an open cell, not necessarily invariant, that contains a compact component* K *of* $\mathsf{Fix}(f)$*. Then* $\mathsf{A}_E(K)$ *is invariant and contains an acyclic component of* $\mathsf{Fix}(f)$*.*

This is proved in Section 4.

Example. Assumption (H0) cannot be dropped from Theorem 2.4. For a counterexample, take f to be the time one map of a flow in the plane generated by a vector field that vanishes only on the unit circle.

Theorem 2.5. *Let* $M = \mathbf{R}^2$ *or* $\mathbf{S}^2$ *and assume* (H0).

- **(i):** *Suppose* $M = \mathbf{R}^2$. *If* $K \subset \mathsf{Fix}(f)$ *is a compact component that is not acyclic,* $\mathsf{A}(K)$ *contains a compact acyclic component of* $\mathsf{Fix}(K)$.
- **(ii):** *Suppose* $M = \mathbf{S}^2$. *If* f *is not the identity map,* $\mathsf{Fix}(f)$ *has two acyclic components.*

Proof. Let $M = \mathbf{R}^2$ and consider a component E of $\mathsf{A}(K) \setminus K$. Then $\overline{E}$ is compact by definition of $\mathsf{A}(K)$, and E is an open cell (Newman [26], Chapter VI, Theorem 4.4)). Therefore E contains a compact component of $\mathsf{Fix}(f)$ by Axiom (H0), and Theorem 2.4 completes the proof of (i).

Under the assumptions in part (ii), $\mathsf{Fix}(f)$ is nonempty by Lefschetz's fixed point theorem. Let $K \subset \mathsf{Fix}(f)$ be a component. Suppose K is acyclic. Then $\mathbf{S}^2 \setminus K \approx \mathbf{R}^2$, therefore the restriction $f_0 = f|(\mathbf{S}^2 \setminus K)$ has a fixed point by (H0). Every component of $\mathsf{Fix}(f_0)$ is compact, since it is disjoint from K, hence from $\mathsf{Fr}\,(\mathbf{S}^2 \setminus K) \subset K$. Therefore $\mathsf{Fix}(f_0)$ has an acyclic component by Theorem 2.4, which is a second acyclic component of $\mathsf{Fix}(f)$.

If K is not acyclic, $\check{\mathsf{b}}^0(\mathbf{S}^2 \setminus K) = 1 + \check{\mathsf{b}}^1(K) \geq 2$ by Alexander duality (Spanier [34]), so $\mathbf{S}^2 \setminus K$ has at least two components. Each component is an open cell which is invariant by Brown-Kister, and contains a component of $\mathsf{Fix}(f)$ by (H0). Applying Theorem 2.4 to these components completes the proof. $\square$

We can now sharpen conclusion (iii) of Theorem 2.2:

Theorem 2.6. *Assume Hypothesis* (H). *Let* $X \subset \mathsf{Fix}(f)$ *be a nonempty union of components such that* $\check{\chi}(X) = \chi(M) - \nu$, $1 \leq \nu < \infty$. *Then there are* ν *invariant, precompact open cells that are components of* $M \setminus X$*, and each such cell contains an acyclic component of* $\mathsf{Fix}(f)$.

Proof. Each component C of $M \setminus X$ is a connected noncompact surface, so $\chi(C) > 0$ if and only if C is an open cell and $\chi(C) = 1$. As $\chi(M \setminus X) = \nu$ by Lemma 4.4, among these components are ν open cells E_i, necessarily disjoint, invariant by the Kister-Brown theorem, and precompact by Lemma 4.1. Each E_i contains a compact component of $\mathsf{Fix}(f)$ by (H0) and (H1); for each i, one such component is acyclic by Theorem 2.4. $\square$

A very strong assumption is that f reduces area:

Corollary 2.7. *Assume Hypothesis* (H) *and suppose f reduces area. Then $\check{\chi}(K) \geq \chi(M)$ for every component K of $\mathsf{Fix}(f)$.*

Proof. Otherwise Theorem 2.6 would yield a precompact invariant open set. $\square$

Theorem 2.8. *Assume $M \approx \mathbf{R}^2$ and f reduces area. Then every invariant continuum K is acyclic and contains a fixed point.*

Proof. Otherwise we reach the contradiction that $\mathsf{A}(K)$ has a precompact complementary component, whose area is invariant. Since K is acyclic, it contains a fixed point by the Cartwright-Littlewood theorem. $\square$

2.1. **Fixed point indices.** A *block* of fixed points is a nonempty compact set $B \subset \mathsf{Fix}(f)$ that is relatively open, that is, $B = \mathsf{Fix}(f) \cap U$ for some open set $U \subset M$. A connected block is also called an *isolated component* of $\mathsf{Fix}(f)$. A block of isolated fixed points is the same as a finite subset of $\mathsf{Fix}(f)$. The fixed point index (Dold [14]) of f at B (or U) is denoted by $\mathsf{Fix}(f, K) = \mathsf{Fix}(f, U) \in \mathbf{Z}$. When K is a singleton $\{p\}$ we may write its index as $\mathsf{Ind}(f, p)$, with the implicit declaration that p is isolated in $\mathsf{Fix}(f)$. If U is invariant and has finitely generated homology, $\mathsf{Ind}(f, U)$ equals the Lefschetz number of $f|U$.

The key property of the index is that it detects fixed points that persist under perturbation, in the following sense: If U is as above and $\mathsf{Ind}(f, U) = n \neq 0$, then every map g sufficiently close to f has a fixed point in U, and $\mathsf{Ind}(g, U) = n$.

Theorem 2.9. *Assume f preserves area. If T is an acyclic isolated component of $\mathsf{Fix}(f)$, then $\mathsf{Ind}(f, T) \leq 1$.*

Proof. T is the intersection of a sequence of disks $D_n \subset W$ such that $D_{n+1} \subset \operatorname{Int} D_n$ (Proposition 4.5). It is easy to see that $D_1 \setminus T \approx \mathbf{D}^2 \setminus \{0\}$. Therefore the identification space M/T, obtained by collapsing T to a point p_T, is a surface. f induces a homeomorphism f_T of M/T with an isolated fixed point at p_T, and $\mathsf{Ind}(f_T, p_T) = \mathsf{Ind}(f, T)$.

If $\mathbf{m}$ is a proper measure preserved by f, then $\mathbf{m}|(M \setminus T)$ extends to a unique proper measure on M/T, preserved by f_T. Therefore $\mathsf{Ind}(f, T) \leq 1$ by the Pelikan-Slaminka theorem. $\square$

The following is similar to Theorem 2.4, but has stronger hypotheses and conclusions. Note that the cell E need not be invariant.

Theorem 2.10. *Assume f preserves area. Let $E \subset M$ be an open cell such that $\mathsf{Fix}(f) \cap E$ is compact and $\mathsf{Ind}(f, E) = n > 0$. Then E contains n acyclic* *components of* $\mathsf{Fix}(f)$. *If* $\mathsf{Fix}(f) \cap E$ *consists of finitely many acyclic components, at least* n *of these components have index* 1.

PROOF. We first show that $\mathsf{Fix}(f) \cap E$ has an acyclic component. In fact, the acyclic hull of any component $T \subset \mathsf{Fix}(f) \cap E$ contains an acyclic component of $\mathsf{Fix}(f) \cap E$. For if T is not acyclic, $\mathsf{A}(T) \setminus T$ has a precompact component U, necessarily an invariant open cell in E. Theorem 2.4, applied to $f|U\colon U \to U$, shows that U contains an acyclic component of $\mathsf{Fix}(f)$.

We may assume the number of acyclic components in $\mathsf{Fix}(f) \cap E$ is m, $1 \leq m \leq n$. If there are no other components, each acyclic component has index ≤ 1 by Proposition 2.9, and as their indices sum to that of $\mathsf{Fix}(f) \cap E$, we conclude that $m = n$ and each component has index 1.

Suppose there exist $k \geq 1$ nonacyclic components T_i of $\mathsf{Fix}(f) \cap E$ whose acyclic hulls $\mathsf{A}_E(T_i)$ in E are pairwise disjoint. Each $\mathsf{A}_E(T_i)$ contains an acyclic component of $\mathsf{Fix}(f) \cap E$; therefore $k \leq m$. Take k as large as possible.

Claim: $\mathsf{Fix}(f) \cap E \subset \bigcup_i \mathsf{A}_E(T_i)$. For suppose a component L of $\mathsf{Fix}(f) \cap E$ meets $E \setminus \bigcup_{i=1}^k \mathsf{A}_E(T_i)$. As $L \cap T_i = \varnothing$, we see that L lies outside $\bigcup_i \mathsf{A}_E(T_i)$ because the frontier of the latter set is contained in $\bigcup T_i$. But then $\mathsf{A}_E(L)$ lies outside $\bigcup_i \mathsf{A}_E(T_i)$, contradicting maximality of k.

This shows that $\mathsf{Ind}(f, E) = \mathsf{Ind}(f, \bigcup_i \mathsf{A}_E(T_i)) = \sum_i \mathsf{Ind}(f, \mathsf{A}_E(T_i))$. By applying Proposition 2.9 to each $\mathsf{A}_E(T_i)$ and summing, we find $\mathsf{Ind}(f, E) = k$. This shows that $k = n$, and we also have $k \leq m \leq n$, so $m = n$.

The second part of the theorem follows from Proposition 2.9. $\square$

It seems likely that the same conclusion holds even if f is not area preserving, provided every point is nonwandering; but we can only prove weaker results, based on the Brouwer-Brown index theorem.

Lemma 2.11. *Let* $E \subset M$ *be an open cell. Each of the following conditions implies* E *contains an index* 1 *block of fixed points:*

(a): $f(E) \subset E$ *and some point of* E *is nonwandering but not fixed*
(b): E *contains an acyclic attractor*

PROOF. In treating (a) we may assume $f(E) = E$, otherwise replacing E by the invariant open set $W = \cup_{n \geq 0} f^{-n} E$, which is a cell because it is simply connected. Note that W is the basin of attraction of A. Conclusion (a) is now a consequence of the Brouwer-Brown index theorem. If $A \subset E$ is an acyclic attractor, Proposition 4.5 shows that A has a closed disk neighborhood D in its basin of attraction such that $f^m(D) \subset \mathrm{Int}(D)$ for a minimal $m \geq 1$. Then the component of A in the

$U = \bigcap_{i=0}^{m} \mathrm{Int}(f^i D))$ is an open cell such that $f(\overline{U}) \subset U$ and $\mathsf{Fix}(f) \cap \overline{U} \subset A$. Lefschetz's fixed point formula now shows $\mathsf{Ind}(f, A) = \mathsf{Ind}(f, U) = 1$; this implies (b). □

Theorem 2.12. *Let $E \subset M$ be an open cell such that:*

- **(a):** *$E \cap \mathsf{Fix}(f)$ is compact and has only finitely many components,*
- **(b):** *If U is an invariant open cell with compact closure in E, every point of U is nonwandering*
- **(c):** *E contains an index 1 block of fixed points*

Then E contains acyclic components of $\mathsf{Fix}(f)$ whose indices sum to 1. If f preserves area, E contains an acyclic component of index 1.

Proof. Let $B_1 \subset \mathsf{Fix}(f) \cap E$ be an index 1 block. If every component of B_1 is acyclic there is nothing more to prove.

Suppose some component K of B_1 is not acyclic; let E_1 be a precompact component of $\mathsf{A}_E(K) \setminus K$. Then E_1 is an invariant open cell; $\overline{E}_1$ is compact in E; every point of E_1 is nonwandering, by (b); and $E_1 \cap \mathsf{Fix}(f)$ is compact with only finitely many components by (a). The Brouwer-Brown index theorem shows E_1 contains an index 1 block B_2. Note that B_2 is disjoint from B_1.

We repeat this construction recursively, obtaining a sequence of pairwise distinct, index 1 blocks $B_1, B_2, \dots$ in $\mathsf{Fix}(f) \cap U$. By (a) there is a final term $B_r, 2 \le r < \infty$ in the sequence. Every component of B_r is acyclic, otherwise the sequence would continue. The components of B_r fulfill the first conclusion of the theorem, and the second follows from Theorem 2.9. □

Theorem 2.13. *Assume $f\colon \mathbf{R}^2 \approx \mathbf{R}^2$ is such that:*

- **(a):** *every component of $\mathsf{Fix}(f)$ is compact, and each compact set meets at most finitely many components of $\mathsf{Fix}(f)$*
- **(b):** *every point with compact orbit closure is nonwandering*
- **(c):** *some nonwandering point is not fixed*

Then there are acyclic components of $\mathsf{Fix}(f)$ whose indices sum to 1.

Proof. There is a set of finitely many components of $\mathsf{Fix}(f)$ whose indices sum to 1, by the Brouwer-Brown index theorem. If these components are acyclic there is nothing more to prove. In the contrary case, let K be a nonacyclic component, and apply Theorem 2.12 to any precompact component E of $\mathsf{A}(K) \setminus K$. □

Theorem 2.14. *Assume $f\colon \mathbf{S}^2 \approx \mathbf{S}^2$ is not the identity, $\mathsf{Fix}(f)$ has only finitely many components, and every point is nonwandering. Then $\mathsf{Fix}(f)$ has two acyclic components with positive index; these have index 1 if area is preserved.*

Proof. $\mathsf{Ind}(f, \mathbf{S}^2) = 2$ by Lefschetz's fixed point formula, so there exists $p \in \mathsf{Fix}(f)$. After identifying $\mathbf{S}^2 \setminus \{p\}$ with $\mathbf{R}^2$ by any homeomorphism, we can apply Theorem 2.13 to $E = \mathbf{S}^2 \setminus \{p\}$, obtaining an acyclic component $K \subset \mathsf{Fix}(f)$ of positive index. Similarly, $\mathbf{S}^2 \setminus K$ contains another such component. Use Theorem 2.9 to get the final conclusion of the theorem. □

The following is similar to Theorem 2.6:

Theorem 2.15. *Assume Hypothesis* (H)*, and let every point with compact orbit closure be nonwandering. Let $X \subset \mathsf{Fix}(f)$ be a nonempty union of components such that $\check{\chi}(X) = \chi(M) - \nu$, $1 \leq \nu < \infty$. Then there are ν invariant, precompact open cells U_i that are components of $M \setminus X$, and each U_i contains acyclic components of $\mathsf{Fix}(f)$ with indices summing to* 1*. When f preserves area, each U_i contains an acyclic component of $\mathsf{Fix}(f)$ having index* 1.

Proof. As in the proof of Theorem 2.6, among the components of $M \setminus X$ are ν precompact invariant open cells E_i with frontiers in $\mathsf{Fix}(f)$, and meeting $\mathsf{Fix}(f)$ in compact sets, necessarily components of $\mathsf{Fix}(f)$. Apply Theorem 2.13 to each E_i. □

3. Applications

We apply the preceding results to several dynamical situations.

3.1. **Attractors.** Let $A \subset M$ be an attractor for $f\colon M \approx M$, with basin B.

Proposition 3.1. *The Čech numbers $\check{\mathsf{b}}^j(A)$ are finite. The inclusion $A \to B$ induces an isomorphism in Čech cohomology.*

Proof. A has only a finite number k of components and these are permuted by f (Hirsch & Hurley [21]). Each component is an attractor under f^i for some $i \in \{1, \dots, k\}$. Therefore we assume there is only one component, otherwise replacing f by $f^{k!}$.

Fix a compact connected surface N such that $A \subset N \subset B$ and $f(N) \subset \mathrm{Int}(N)$; then $A = \bigcap_{n \geq 0} f^n(N)$. As the compact surfaces N and $f(N)$ are homeomorphic, the kernel of the homomorphism $H_1(f(N)) \to H_1(N)$ induced by inclusion is generated by the homology classes of the boundary curves of $f(N)$ that bound disks in N. By adjoining the preimages of these disks to N, we obtain a compact surface P having the same properties as N, that is, $f(P) \subset \mathrm{Int}\, P$ and $A = \bigcap_{n \geq 0} f^n(P)$, with additional property that inclusion $f(P) \to P$ induces isomorphisms in singular homology and Čech cohomology

The inclusion $A \to P$ induces Čech cohomology isomorphisms in dimensions ≥ 2 (the groups are trivial) and dimension 0. Continuity of Čech theory shows $\check{H}^1(A)$ is the direct limit of the sequence of isomorphisms

$$\check{H}^1(P) \to \check{H}^1(f(P)) \to \check{H}^1(f^2(P)) \to \cdots$$

induced by the inclusions. This proves the first statement of the theorem. The second follows because $\bigcup_{n\geq 0} f^{-n}(P) = B$, implying $H_1(B)$ is the direct limit of the sequence of inclusion induced isomorphisms $H_1(f^{-n}(P)) \to H_1(f^{-n-1}(P))$. □

Corollary 3.2. *If A is an acyclic attractor, its basin is an invariant open cell.*

Theorem 3.3. *Assume A is nowhere dense.*

- **(i):** $\check{\chi}(\mathsf{Fix}(f) \cap A) \geq \check{\chi}(A)$
- **(ii):** *Assume A is acyclic. Then:*
  - **(a):** *A contains an acyclic component of $\mathsf{Fix}(f)$.*
  - **(b):** *Suppose $\mathsf{Fix}(f) \cap A$ has only finitely many components, and every point of A is nonwandering. Then $\mathsf{Fix}(f) \cap A$ contains acyclic components whose indices sum to 1.*

Proof. Replacing M by the basin of A, we assume A is a global attractor. Therefore the set of chain recurrent points is nowhere dense, as it lies in A, so (H0) holds by Theorem 2.1(ii). Proposition 3.1 implies $\check{\mathsf{b}}^i(A) = \check{\mathsf{b}}^i(M)$ for all i, and (i) follows from Theorem 2.2(i).

Suppose A is acyclic. Lefschetz's formula shows $\mathsf{Ind}(f, M) = 1$ because A is an attractor, and M is an open cell by Corollary 3.2. Conclusion (ii)(a) now follows from Theorem 2.10. To prove (ii)(b), apply Theorem 2.12, noting that hypothesis (a) is vacuously satisfied because A is a nowhere dense set containing all compact invariant sets. □

3.2. **Analytic homeomorphisms.** In this subsection M has an analytic structure, and $f\colon M \approx M$ denotes an analytic homeomorphism (not necessarily a diffeomorphism). $\mathsf{Fix}(f)$ is an analytic variety, hence triangulable (Lojasiewicz [23]); therefore every component is isolated, and each compact subset of M meets at most finitely many components if $\mathsf{Fix}(f)$. Evidently $\dim X \leq 2$, and ≤ 1 if f is not the identity map. When $\mathsf{Fix}(f)$ is compact, it has only finitely many components and $\check{\chi}(F) = \chi(F) < \infty$.

The following result will frequently be used tacitly:

Proposition 3.4. *Let K be a triangulated compact connected analytic variety embedded as a proper subvariety of a connected analytic surface M without boundary. Then no simplex has dimension > 1, and the number of edges meeting at any vertex is even. Therefore $\chi(K) \le 1$ for each compact component K of $\mathsf{Fix}(f)$. Moreover:*

- **(i):** $\chi(K) = 1 \Leftrightarrow K$ *is acyclic* $\Leftrightarrow K$ *is a singleton.*
- **(ii):** $\chi(K) = 0 \Leftrightarrow K$ *is a Jordan curve*
- **(iii):** $\chi(K) < 0 \Leftrightarrow K$ *if and only if K is a homeomorphic to the connected union of two or more metric circles in $\mathbf{R}^2$.*

Proof. The statements about simplices follow from a special case of a theorem due to Deligne [13] and Sullivan [35]: The intersection of K with the boundary of any sufficiently small Riemannian disk centered at a point of K has even cardinality. This implies (i); parts (ii) and (iii) are proved by counting simplices. □

Corollary 3.5. *A compact acyclic component of $\mathsf{Fix}(f)$ is an isolated fixed point.*

Theorem 3.6. *Suppose f does not have an isolated fixed point.*

- **(i):** *If* (H0) *holds, no Jordan curve in $\mathsf{Fix}(f)$ is homotopically trivial in M.*
- **(ii):** *If Hypothesis* (H) *holds, $0 \ge \chi(K) \ge \chi(M)$ for every compact component K of $\mathsf{Fix}(f)$.*

Proof. (ii) is a consequence of Corollary 2.3. To prove (i), suppose *per contra* that $J \subset \mathsf{Fix}(f)$ is a homotopically trivial Jordan curve. There is an invariant disk $D \subset M$ with $\partial D = J$; this can be seen by lifting J to a Jordan curve in the universal covering. From the the structure of compact 1-dimensional analytic varieties (Theorem 3.4(iii)), we see that $D \cap \mathsf{Fix}(f)$ contains an invariant disk $D_1 \subset D$ such that $\partial D_1 = J_1$ and $\mathsf{Fix}(f) \cap D_1$ contains no other Jordan curve. Therefore the set $\mathsf{Fix}(f) \cap \mathrm{Int}(D_1)$, which is nonempty by (H0), consists of isolated points, contradicting the hypothesis. □

Theorem 3.7. *Assume Hypothesis* (H)*, and let every point with compact orbit closure be nonwandering. Let $X \subset \mathsf{Fix}(f)$ be a nonempty union of components such that $\check{\chi}(X) = \chi(M) - \nu$, $1 \le \nu < \infty$. Then:*

- **(i):** *$M \setminus X$ contains ν pairwise disjoint blocks of isolated fixed points, each block having index* 1.
- **(ii):** *If f preserves area, $M \setminus X$ contains ν isolated fixed points of index* 1.

Proof. There are ν components E_i of $M \setminus X$ that are invariant, precompact open cells by Theorem 2.6. Apply Theorem 2.12 to the E_i. □

From Theorem 2.14 we obtain:

Theorem 3.8. *Assume* $f\colon \mathbf{S}^2 \approx \mathbf{S}^2$ *is analytic and not the identity, and every point is nonwandering. Then there are two isolated fixed points of positive index; when area is preserved, each has index* 1.

Theorems 2.10 to 2.13 can be similarly adapted to analytic maps, with acyclic components of the fixed point set interpreted as isolated fixed points.

Example 3.9. Assume $\chi(M) \geq -9$. Let $f\colon M \approx M$ be analytic and satisfy Hypothesis (H). Suppose $\mathsf{Fix}(f)$ has 3 components, each with Euler characteristic -1, and 4 components, each with Euler characteristic -2, and perhaps other components. Then:

- **(i):** there are two isolated fixed points;
- **(ii):** if every point with compact orbit closure is nonwandering, there are two disjoint blocks (i.e., finite sets) of isolated fixed points, each block having index 1;
- **(iii):** if f preserves area, there are two isolated index 1 fixed points.

By Theorems 3.7(ii), the number of isolated fixed points is

$$\kappa_1 \;\geq\; -9 + 1\cdot\kappa_{-1} + 2\cdot\kappa_{-2} \;\geq\; -9 + 1\cdot 3 + 2\cdot 4 \;=\; 2$$

which proves (i). The same computation shows that we can take $\nu = 2$ in Theorem 3.7, yielding (ii) and (iii).

3.3. **Homoclinic points.** Let $p \in \mathsf{Fix}(f)$ be a *saddle*: f is C^1 in a neighborhood of p, and the linear operator df_p has eigenvalues λ, μ such that $0 < \lambda < 1 < \mu$. A *simple homoclinic loop* at p is a Jordan curve $J \subset M$ of the form $J_u \cup J_s$, where J_u and J_s are arcs in the unstable and stable curves at p, respectively, having common endpoints of which one is p. A *homoclinic cell* is an open cell whose boundary is a homoclinic loop.

It is easy to see that $\mathsf{Fix}(f) \cap E$ is compact. It is also nonempty; in fact, $\mathsf{Ind}(f^n, E) = 1$ (respectively, 2) for all $n \neq 0$ if there is smooth chart at p mapping a neighborhood of p in $\overline{E}$ onto a neighborhood of $(0,0)$ in the first quadrant (respectively, in the union of three quadrants); see Hirsch [20].

Theorem 3.10. *Assume* (H0). *Let* $E \subset M$ *be a homoclinic cell with* $\mathsf{Ind}(f, E) = \rho \in \{1, 2\}$.

- **(i):** *Let* $\rho = 1$. *Then* E *contains a compact acyclic component of* $\mathsf{Fix}(f)$. *If* f *is analytic and every point with compact orbit closure in* E *is nonwandering,* E *contains isolated fixed points whose indices sum to* 1.

(ii): *If f is analytic and f preserves area, E contains ρ isolated fixed points of index* 1.

Proof. (i) follows from Theorem 2.12, and (ii) from Theorem 2.10. $\square$

3.4. **Commuting homeomorphisms.** Lima [24] showed that any family of commuting flows on a compact surface with nonzero Euler characteristic have a common fixed point. A generalization to nilpotent Lie group actions is due to Plante [28].

There is a common fixed point for any family of commuting holomorphic homeomorphisms of the open unit disk $\{z \in \mathbf{C}\colon |z| < 1\}$ that extend continuously to the closed disk (A. Shields [30]). If $f_1, \ldots, f_r$ are analytic homeomorphisms of $\mathbf{R}^2$ that commute under composition, and $\mathsf{Fix}(f_1)$ is compact and nonempty, then f_1 has a fixed point that is periodic for $f_2, \ldots, f_r$ (Hirsch [19]). It is not known whether there must be a common fixed point.

Bonatti ([4]) proved there is a neighborhood U of the identity map in the orientation preserving diffeomorphism group $\mathsf{Diff}(\mathbf{S}^2)$ of the 2-sphere such that any commuting family in U has a common fixed point. For commuting $f, g \in \mathsf{Diff}(\mathbf{S}^2)$, Handel [18] defines an interesting invariant $W(f,g)$ in the fundamental group of $\mathsf{Diff}(\mathbf{S}^2)$, whose vanishing is equivalent to the existence of a common fixed point. This is also valid for commuting homeomorphisms with finite fixed point sets.

In this subsection $f\colon M \approx M$ satisfies Hypothesis (H), and $g\colon M \to M$ denptes a map such that $f \circ g = g \circ f$.

Proposition 3.11. *Let $K \subset \mathsf{Fix}(f)$ be a compact acyclic component invariant g. Then K contains a fixed point of g.*

This is proved in Section 4 as an application of the Cartwright-Littlewood theorem.

Corollary 3.12. *Assume g is a homeomorphism. If K is a compact acyclic component of $\mathsf{Fix}(f)$, there exists $q \in K$ and $n \in \{1, \ldots, \kappa_1(f)\}$ such that $g^n(q) = q$.*

Proof. g induces a permutation σ of the set of acyclic component of $\mathsf{Fix}(K)$. As this set has cardinality $\kappa_1(f)$, there exists $n \in \{1, \ldots, \kappa_1(f)\}$ such that $\sigma^n(K) = K$; now apply Proposition 3.11 to g^n. $\square$

Corollary 3.13. *Let $M = \mathbf{R}^2$ or $\mathbf{S}^2$ and assume g is a homeomorphism. Then $\mathsf{Fix}(g^n) \cap \mathsf{Fix}(f)$ is nonempty for some $n \in \{1, \ldots, \kappa_1(f)\}$, and contains at least two points when $M = \mathbf{S}^2$.*

Proof. This follows from Theorem 2.5 and Corollary 3.12. □

This cannot be sharpened to yield a common fixed point when $M = \mathbf{S}^2$: Take f and g to be the rotations defined by the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The full strength of Hypothesis (H) was used in the proof of Corollary 3.12 in order to ensure that $\mathsf{Fix}(f)$ has only finitely many acyclic components, but perhaps this condition is unnecessary. The following seems to be open:

Conjecture. Two commuting, orientation preserving homeomorphisms of $\mathbf{R}^2$ have a common fixed point provided one of them has compact nonempty fixed point set.

3.5. **Fixed points of prime power iterates.** Given $f\colon M \approx M$, let $\mathcal{P}$ denote the set of primes p for which there exists $n = p^k, k \in \mathbf{N}_+$ such that f^n satisfies Hypothesis (H) and $\check{\chi}(\mathsf{Fix}(f^n)) < 0$.

Theorem 3.14. *Assume* $\mathsf{Fix}(f) = \varnothing$. *Then* $\mathcal{P}$ *is finite. More precisely, if* $\mathcal{P}$ *is nonempty then* $\sum_{p\in\mathcal{P}} p \leq -\chi(M)$.

This is proved in Section 4.

Example 3.15. Assume M is compact with $\chi(M) \geq -2$. Let $f\colon M \approx M$ be analytic and fixed point free, with every point nonwandering. Then for every odd prime power $n = p^k$, either f^n has an isolated fixed point, or else every component of $\mathsf{Fix}(f^n)$ is a Jordan curve.

Hypothesis (H) holds by Theorem 2.1(b) and analyticity. $\chi(\mathsf{Fix}(f^n)) \geq 0$, for otherwise $\chi(M) \leq -p \leq -3$ by Theorem 3.14, contradicting the assumption that $\chi(M) \geq -2$. If there are no isolated fixed points, every component of $\mathsf{Fix}(f^n)$ has Euler characteristic ≤ 0; and as the sum of these characteristics is $\chi(\mathsf{Fix}(f^n)) \geq 0$, every component has characteristic zero, making it a Jordan curve.

For general $n \in \mathbf{N}_+$ Theorem 3.6 gives a weaker limitation on the topology of the set of n-periodic points.

4. Proofs

Our next goal is the proof of Theorem 2.1, which gives several conditions ruling out simply connected complementary components of the fixed point set.

We start with a purely topological fact:

Lemma 4.1. *Let $W \subset M$ be an open cell with compact nonempty frontier $\dot{W}$. Then $\overline{W}$ is compact and $\dot{W}$ is connected.*

Proof. Pick a neighborhood N of $\dot{W}$ that is a compact surface whose interior meets both W and $M \setminus \overline{W}$. Because W is connected, each of the finitely many components of ∂N is a Jordan curve in W or $S \setminus \overline{W}$. Define K to be the union of N and the finite, nonempty collection of disks in W whose boundaries are components of ∂N. Then K is a compact surface whose boundary is the union of the components of ∂N that are exterior to $\overline{W}$. Evidently K meets W and ∂K is disjoint from $\dot{W}$. As W is connected, this implies $\overline{W} \subset K$. Therefore $\overline{W}$ is compact.

Further analysis of N can be used to prove $\dot{W}$ connected. Alternatively, Lefschetz's dualtity theorem (Spanier [34]) implies $\check{H}^0(\overline{W}, \dot{W}) = H_2(W) = \{0\}$, whence exactness of the cohomology sequence

$$\mathbf{Z} = \check{H}^0(\overline{W}) \to \check{H}^0(\dot{W}) \to \check{H}^0(\overline{W}, \dot{W}) = \{0\}$$

of the pair $(\overline{W}, \dot{W})$ shows $\check{H}^0(\dot{W}) \approx \mathbf{Z}$, equivalent to $\dot{W}$ being connected. □

We return to the dynamics of $f\colon\ M \approx M$.

Lemma 4.2. *Let $V \subset M$ be an invariant open cell set containing no fixed point. Then:*

- **(i):** *Every point of V is wandering, with its alpha and omega limit sets contained in $\dot{V}$*
- **(ii):** *$V \subset \mathsf{R}(f|\overline{V})$ provided $\dot{V}$ is a continuum contained in $\mathsf{R}(f|\overline{V})$.*

Proof. (i) follows from Brouwer's Nonwandering Theorem. In (ii), $\overline{V}$ is compact and $\dot{V}$ connected by Lemma 4.1. Therefore the alpha and omega limit sets of every point in V are nonempty subsets of $\dot{V}$ by (i). As $\dot{V}$ is a connected set of chain recurrent points, all points in $\dot{V}$ are chain equivalent for $f|\overline{V}$; now (i) implies $x \in \mathsf{R}(f|\overline{V})$. □

Corollary 4.3. *Fix $n \geq 1$ and let W be a simply connected complementary component of $\mathsf{Fix}(f^n)$ having nonempty compact frontier. Then $\overline{W}$ is compact, and every point of W is wandering and chain recurrent for f, and has compact orbit closure.*

Proof. $f(W) = W$ by the Brown-Kister theorem. By Lemma 4.2, $\overline{W}$ is compact and every point of W is wandering and chain recurrent for f^n, which implies the conclusion of the theorem. □

4.1. **Proof of Theorem 2.1.** We argue by contradiction. Suppose $n \geq 1$ and W is a simply connected, precompact complementary component of $\mathsf{Fix}(f^n)$. Then invariance of W under f^n implies f cannot reduce area. If f preserves area, so does f^n; in this case Poincaré's recurrence theorem provides a dense subset of W consisting of points with compact orbit closure and recurrent for f^n, contradicting Corollary 4.3. This proves Theorem 2.1 under assumption (a), and assumption (b) also contradicts Corollary 4.3.

The following result is valid only in even dimensional manifolds; in the odd dimensional case it is valid modulo 2. We treat only the case of surfaces:

Lemma 4.4. *Assume $H_1(X)$ is finitely generated. Let $X \subset M$ be a compact subset. Then*

$$\chi(M) = \check{\chi}(X) + \chi(M \setminus X)$$

Proof. Represent X as the decreasing intersection of nested family of compact connected polyhedral surfaces (with boundary) $X_\alpha \subset M$, $\alpha \in \mathbf{N}$. Define Y_α as the (possibly noncompact) polyhedral surface $\mathsf{clos}(M \setminus X_\alpha)$. Then $X_\alpha \cap Y_\alpha$, being a compact 1-dimensional manifold without boundary, has zero Euler characteristic. Therefore

$$\chi(M) = \chi(X_\alpha) + \chi(Y_\alpha) - \chi(X_\alpha \cap Y_\alpha) = \chi(X_\alpha) + \chi(Y_\alpha)$$

Čech and singular theory are the same for polyhedra, and the inclusion map

$$M \setminus X_\alpha = Y_\alpha \setminus \partial Y_\alpha \hookrightarrow Y_\alpha$$

is a homotopy equivalence. Therefore

$$\chi(M) = \check{\chi}(X_\alpha) + \check{\chi}(M \setminus X_\alpha)$$

The continuity property for Čech cohomology shows $\check{H}^i(X)$ is the direct limit of the sequence of the inclusion induced homomorphisms

$$H^i(X_\alpha) \xrightarrow{\iota_\alpha} H^i(X_{\alpha+1}) \xrightarrow{\iota_{\alpha+1}} \cdots$$

Because all these groups are finitely generated, we can pass to a subsequence so that all the ι_α are isomorphisms, and the inclusion $X_\alpha \to X$ induces isomorphisms $H^i(X_\alpha) \approx \check{H}^i(X)$. This implies $\check{\chi}(X) = \chi(X_\alpha)$.

A similar direct limit argument with singular homology proves that $H_i(M \setminus X) \approx H_i(M \setminus X_\alpha)$. Therefore $\chi(M \setminus X) = \chi(M \setminus X_\alpha)$, ergo

$$\chi(M) = \check{\chi}(X) + \chi(M \setminus X)$$

$\square$

The preceding proof demonstrates a useful fact:

Proposition 4.5. *Let $X \subset M$ be a nonempty compact set with finitely generated Čech cohomology. Then every neighborhood of X contains a smaller neighborhood N of X such that N is a compact surface, and the inclusion $X \to N$ induces an isomorphism in Čech cohomology. Therefore every acyclic continuum in M is the intersection of a sequence family of disks $\{D_i\}$ with $D_{i+1} \subset \mathrm{Int}(D_i)$.*

The following theorem is the topological basis for Theorem 2.2:

Theorem 4.6. *Assume M has finitely generated homology. Let $X \subset M$ be a compact nonempty set with finitely many components, having no simply connected complementary component. Then*

- **(i):** *X has finitely generated Čech cohomology*
- **(ii):** *$M \setminus X$ has finitely generated singular homology*
- **(iii):** $\check{\chi}(X) \geq \chi(M)$

Proof. We first prove that $H_1(M \setminus X)$ is finitely generated. Consider the homomorphisms

$$\check{H}^0(X) \approx H_2(M, M \setminus X) \to H_1(M \setminus X) \to H_1(M)$$

The first map is a duality isomorphism (Spanier [34], Chap. 6, Sec.2, Theorem 16) and the other two are part of the exact homology sequence of the pair $(M, M \setminus X)$. Assumptions (i) and (ii) make all groups in the sequence other than $H_1(M \setminus X)$ finitely generated, so exactness proves the latter is also finitely generated.

Let $\{U_\lambda\}$ denote the family of complementary components of X. As each U_λ is a connected open surface that is not simply connected, $H_1(U_\lambda)$ is a free abelian group on $b_1(U_\lambda) \geq 1$ generators.

Now we show that $M \setminus X$ has finitely many components, i.e., $H_0(M \setminus X)$ is finitely generated. For the finitely generated group $H_1(M \setminus X)$ is the direct sum of the nontrivial groups $H_1(U_\lambda)$, so there can be only finitely many U_λ.

The exact homology sequence of $(M, M \setminus X)$ shows that the singular homology of $(M, M \setminus X)$ is finitely generated. The duality isomorphisms $\check{H}^i(X) \approx H_{2-i}(M, M \setminus X)$ shows that the Čech cohomology of X is finitely generated. Therefore $\check{\chi}(X)$, $\chi(M \setminus X)$ and $\chi(M \setminus X)$ are well defined and finite.

To prove $\check{\chi}(X) \geq \chi(M)$, in view of Lemma 4.4 it is enough to prove $\chi(M \setminus X) \leq 0$, or equivalently, $b_1(U_\lambda) \geq 1$ for each component U_λ of $M \setminus X$; and this was shown above. □

4.2. **Proof of Theorem 2.2.** Parts (i) and (ii) follow from Theorem 4.6. For (iii), let $\mathcal{C}$ denote be the set of components of $\mathsf{Fix}(f)$, and $\mathcal{C}(j)$ the subset of components K such that $\check{\chi}(K) = j$. Then $\mathcal{C}$ is finite, the cardinality of $\mathcal{C}(j)$ is $\kappa_j(f)$, and

$$\check{\chi}(\mathsf{Fix}(f)) = \sum_{K\in\mathcal{C}} \check{\chi}(K)$$

By Theorem 2.2:

$$\chi(M) \leq \sum_{K\in\mathcal{C}} \check{\chi}(K) = \sum_{j\leq 1}\sum_{K\in\mathcal{C}(j)} \check{\chi}(K) = \sum_{-\infty<j\leq 1} j\kappa_j = \kappa_1 - \sum_{i>0} i\kappa_{-i},$$

implying (iii).

4.3. **Proof of Theorem 2.4.** We first prove $\mathsf{A}_E(K)$ invariant. It suffices to prove that U is invariant for every component U of $E\setminus K$ with compact closure $\overline{U} \subset E$. Note that $\dot{U} \subset K \subset \mathsf{Fix}(f)$. Let V be a component of $U \setminus \mathsf{Fix}(f)$. Then $\dot{V} \subset \dot{U} \cup \mathsf{Fix}(f) \subset \mathsf{Fix}(f)$. Thus V is a nonempty connected open subset of $M \setminus \mathsf{Fix}(f)$ with $\dot{V} \subset \mathsf{Fix}(f)$, which implies V is a complementary component of $\mathsf{Fix}(f)$. Therefore V is invariant by the Brown-Kister theorem. This implies U is invariant, for U is the union of the invariant set $U \cap \mathsf{Fix}(f)$ and the components V of $U \setminus \mathsf{Fix}(f)$, which were proved invariant. This shows $\mathsf{A}_E(K)$ is invariant.

Denote by $\mathcal{C}$ the set of compact components of $\mathsf{Fix}(f) \cap E$. The order relation $\preceq$ on $\mathcal{C}$, defined by $L \preceq K \Leftrightarrow L \subset \mathsf{A}_E(K)$, is downward inductive, that is, every set-theoretically maximal totally ordered subset $\mathcal{K} \subset \mathcal{C}$ has a lower bound.

The nonempty compact set $Z = \bigcap_{K\in\mathcal{C}} \mathsf{A}_E(K)$ is acyclic (by the continuity axiom for Čech cohomology). Every point $z \in \mathsf{Fr}\,(Z)$ is the limit of a sequence of x_n where $x_n \in \mathsf{Fr}\,(\mathsf{A}_E(K_n))$ for some $K_n \in \mathcal{K}$. This implies $\mathsf{Fr}\, Z$ is connected. As $x_n \in K_n \subset \mathsf{Fix}(f) \cap E$, we have $z \in \mathsf{Fix}(f)$; and as the x_n lie in the compact set $\mathsf{A}_E(K_1) \subset E$, consequently $z \in \mathsf{Fix}(f) \cap E$.

The component of z in $\mathsf{Fix}(f)\cap E$ is compact, for $\mathsf{Fr}\,(\mathsf{A}_E(K_1)) \subset K_1$, a compact component of $\mathsf{Fix}(f)\cap E$. Thus every point of the connected set $\mathsf{Fr}\,(Z)$ lies in some element of $\mathcal{K}$. As these elements are disjoint sets, $\mathsf{Fr}\,(Z)$ lies in a unique $L \in \mathcal{K}$.

Consequently $Z \subset \mathsf{A}_E(L)$. Moreover, for all n we have $\dot{Z} \subset L \cap \mathsf{A}_E(L_n)$, consequently $L \cap \mathsf{A}_E(K_n) \neq \varnothing$. If L meets the frontier of $\mathsf{A}_E(K_n)$ then L meets K_n, entailing $L = K_n$; and otherwise $L \subset \mathsf{A}_E(K_n)$. This proves $L \preceq K_n$.

I claim L is $\preceq$-minimal in $\mathcal{K}$. For otherwise there exists $L_1 \in \mathcal{K}$ such that $L_1 \preceq L$, $L_1 \neq L$. Then L_1 lies in some bounded component U of $E \setminus L$, implying $Z \subset U$; but this yields the contradiction $Z \cap L = \varnothing$.

As L is $\preceq$-minimal in $\mathcal{K}$, it must be that $Z = \mathsf{A}_E(L)$. I claim that L, a compact component of $\mathsf{Fix}(f)$, is acyclic. Otherwise L has a bounded complementary component U. Note that U is simply connected because L is connected, and $\dot{U} \subset L \subset \mathsf{Fix}(f)$. Then U contains a point $p \in \mathsf{Fix}(f)$, for otherwise U is a simply connected complementary component of $\mathsf{Fix}(f)$, in violation of (H0). The component $J \subset \mathsf{Fix}(f)$ containing p cannot meet $\dot{U}$ because $\dot{U} \subset L$ and J is disjoint from L. Therefore $J \subset U$, whence J is bounded and thus compact. But this violates $\preceq$-minimality of L. Therefore L is acyclic, completing the proof.

4.4. **Proof of Proposition 3.11.** Consider first the case that $M \neq \mathbf{S}^2$, so that M has a universal covering space $\pi : \mathbf{R}^2 \to M$. Choose any $p \in K$ and $\hat{p} \in \pi^{-1}(p)$. There is a unique homeomorphism $\hat{f}\colon \mathbf{R}^2 \approx \mathbf{R}^2$ covering f such that $\hat{p} \in \mathsf{Fix}(\hat{f})$.

K lies in a disk $D \subset M$ (Proposition 4.5), so $\hat{p}$ lies in a disk $\hat{D} \subset \pi^{-1}(D)$ mapped homeomorphically onto D by π. Therefore π maps the set $\hat{K} = D \cap \pi^{-1}(K)$ homeomorphically onto K. The images of $\hat{K}$ under deck transformations $T : \mathbf{R}^2 \approx \mathbf{R}^2$ are the components of $\pi^{-1}(K)$.

Let $\hat{g} : \mathbf{R}^2 \approx \mathbf{R}^2$ be a lift of g. Then $\pi \circ \hat{g}^n(\hat{K}) = \hat{K}$. Therefore $\hat{g}(\hat{K})$ is a component of $\pi^{-1}(K)$, so there is a deck transformation T such that $T \circ \hat{g}(\hat{K}) = \hat{K}$.

Set $\hat{h} = T \circ \hat{g} : \mathbf{R}^2 \approx \mathbf{R}^2$; then $\hat{h}(\hat{K}) = \hat{K}$. I claim $\hat{h}$ commutes with $\hat{f}$. For there is a deck transformation T_1 such that $\hat{h} \circ \hat{f} = T_1 \circ \hat{f} \circ \hat{h}$. Applied to $\hat{K}$ these maps give $\hat{K} = T_1(\hat{K})$; since deck transformations act totally discontinuously, this shows T_1 is the identity.

As $\hat{K}$ is acyclic, $\hat{h}$ has a fixed point $\hat{q} \in \hat{K}$ by the Cartwright-Littlewood theorem. For the point $q = \pi(\hat{q})$ we therefore have $q = f(q) = g(q)$.

Now let $M = \mathbf{S}^2$. By Lefschetz's fixed point theorem, g has a fixed point q. If $q \in K$ there is nothing more to prove. If $q \notin K$, the Cartwright-Littlewood applied to $g|(\mathbf{S}^2 \setminus \{q\}) \approx \mathbf{R}^2$, establishes a fixed point in K.

4.5. **Proof of Theorem 3.14.** For $\nu \geq 1$ and $i = 1, \dots, \nu$, let $n_i = p_i^{k_i}$ be distinct prime powers such that:

(a): f^{n_i} satisfies Hypothesis (H).
(b): $\check{\chi}(\mathsf{Fix}(f^{n_i})) < 0$

We have to prove $\sum_i p_i \leq -\chi(M)$.

Set $\mathsf{Fix}(f^{n_i}) = \Gamma_i$. No complementary component U of the compact set $\cup_j \Gamma_j$ is simply connected. For suppose U is simply connected. $\dot{U}$ is compact, as it lies in $\cup_i \Gamma_i$, so $\dot{U}$ is connected by Lemma 4.1. The compact sets Γ_i are pairwise disjoint, for the intersection of any pair is contained in the empty set of fixed points. Therefore $\dot{U}$, being connected, lies in some Γ_j. This implies U is a complementary component of Γ_j; hence U is not simply connected, by (a).

By Theorem 4.6,

$$\check{\chi}(\cup_j \Gamma_j) \geq \chi(M)$$

Lemma (4.7) below shows that $\check{\chi}(\Gamma_i)$ is divisible by p_i. Therefore we may set

$$\check{\chi}(\Gamma_i) = -\mu_i p_i \leq -1,\ \mu_i \in \mathbf{N}_+$$

whence

$$-\sum_i p^i \geq -\sum_i \mu_i p_i = \sum_i \check{\chi}(\Gamma_i) = \check{\chi}(\cup_i \Gamma_i) \geq \chi(M)$$

The following result, an application of Floyd's theory of finite transformation groups [16], was used in the preceding proof:

Lemma 4.7. *Let Γ be a finite dimensional compact space with finitely generated Čech cohomology. Let $g\colon \Gamma \approx \Gamma$ generate a cyclic group G of homeomorphisms of Γ having prime power order $n = p^k$ where p is a prime and $k \in \mathbf{N}_+$. Then $\check{\chi}(\Gamma) \equiv \check{\chi}(\mathsf{Fix}(g)) \pmod p$.*

Proof. Let H_c^i denote the i'th cohomology functor with compact supports; denote the corresponding Euler characteristic by χ_c. For any compact pair (Y, B) there are natural isomorphisms $H_c^i(Y \setminus B) \approx \check{H}^i(Y, B)$ (Spanier [34], Chap. 6, Sec. 6, Lemma 11). Thus

$$\check{\chi}(Y, B) = \chi_c(Y \setminus B)$$

We proceed by induction on k. When $k = 1$ the conclusion follows from [16], Theorem 4.4. Take $k \geq 2$ and assume the conclusion of the lemma holds for smaller exponents. Set $\Gamma' = \mathsf{Fix}(g^n)$, which contains $\mathsf{Fix}(g)$. Then $g|\Gamma'$ generates a group of order p^{k-1}; therefore the inductive hypothesis implies

$$\check{\chi}(\Gamma') \equiv \check{\chi}(\mathsf{Fix}(g)) \pmod p$$

Now $g|(\Gamma \setminus \Gamma')$ generates a cyclic group of order p^{k-1} acting freely on $\Gamma \setminus \Gamma'$. Corollary 5.3 of [16] implies

$$\chi_c(\Gamma \setminus \Gamma') \equiv 0 \pmod{p^{k-1}}$$

and hence

$$\check{\chi}(\Gamma, \Gamma') \equiv 0 \pmod p$$

Exactness of the Čech cohomology sequence of the pair (Γ, Γ') now yields

$$\check{\chi}(\Gamma) \equiv \check{\chi}(\Gamma') + \check{\chi}(\Gamma, \Gamma') \equiv 0 + 0 \pmod p$$

□

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