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THE BOUNDARY AND THE VIRTUAL COHOMOLOGICAL DIMENSION OF COXETER GROUPS

TETSUYA HOSAKA AND KATSUYA YOKOI
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Abstract. This paper consists of three parts: 1) We give some properties about the virtual cohomological dimension of Coxeter groups over principal ideal domains. 2) For a right-angled Coxeter group Γ with $\operatorname{vcd}_R \Gamma = n$, we construct a sequence $\Gamma_{W_0} \subset \Gamma_{W_1} \subset \cdots \subset \Gamma_{W_{n-1}}$ of parabolic subgroups with $\operatorname{vcd}_R \Gamma_{W_i} = i$. 3) We show that a parabolic subgroup of a right-angled Coxeter group is of finite index if and only if their boundaries coincide.

1. Introduction and Preliminaries

The purpose of this paper is to study Coxeter groups and their boundaries. Let V be a finite set and $m : V \times V \to \mathbb{N} \cup \{\infty\}$ a function satisfying the following conditions:

(1) $m(v,w) = m(w,v)$ for all $v, w \in V$,
(2) $m(v,v) = 1$ for all $v \in V$, and
(3) $m(v,w) \geq 2$ for all $v \neq w \in V$.

A *Coxeter group* is a group Γ having the presentation

$$\langle V \mid (vw)^{m(v,w)} = 1 \text{ for } v, w \in V \rangle,$$

where if $m(v,w) = \infty$, then the corresponding relation is omitted, and the pair (Γ, V) is called a *Coxeter system.* If $m(v,w) = 2$ or ∞ for all $v \neq w \in V$, then (Γ, V) is said to be *right-angled.* For a Coxeter system (Γ, V) and a subset $W \subset V$,

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Γ_W is defined as the subgroup of Γ generated by W. The pair (Γ_W, W) is also a Coxeter system. Γ_W is called a *parabolic subgroup*.

For a Coxeter system (Γ, V), the simplicial complex $K(\Gamma, V)$ is defined by the following conditions:

(1) the vertex set of $K(\Gamma, V)$ is V, and
(2) for $W = \{v_0, \ldots, v_k\} \subset V$, $\{v_0, \ldots, v_k\}$ spans a k-simplex of $K(\Gamma, V)$ if and only if Γ_W is finite.

A simplicial complex K is called a *flag complex* if any finite set of vertices, which are pairwise joined by edges, spans a simplex of K. For example, the barycentric subdivision of a simplicial complex is a flag complex.

For any finite flag complex K, there exists a right-angled Coxeter system (Γ, V) with $K(\Gamma, V) = K$. Namely, let V be the vertex set of K and define $m : V \times V \to \mathbb{N} \cup \{\infty\}$ by

$$m(v, w) = \begin{cases} 1 & \text{if } v = w, \\ 2 & \text{if } \{v, w\} \text{ spans an edge in } K, \\ \infty & \text{otherwise.} \end{cases}$$

The associated right-angled Coxeter system (Γ, V) satisfies $K(\Gamma, V) = K$. Conversely, if (Γ, V) is a right-angled Coxeter system, then $K(\Gamma, V)$ is a finite flag complex ([7, Corollary 9.4]).

For a group Γ and a ring R with identity, the *cohomological dimension of* Γ *over* R is defined as

$$\operatorname{cd}_R \Gamma = \sup\{i \mid H^i(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M\}.$$

If $R = \mathbb{Z}$ then $\operatorname{cd}_{\mathbb{Z}} \Gamma$ is simply called the cohomological dimension of Γ, and denoted $\operatorname{cd} \Gamma$. It is obvious that $\operatorname{cd}_R \Gamma \leq \operatorname{cd} \Gamma$ for a ring R with identity. It is known that $\operatorname{cd} \Gamma = \infty$ if Γ is not torsion-free ([5, Corollary VIII 2.5]). A group Γ is said to be *virtually torsion-free* if Γ has a torsion-free subgroup of finite index. For a virtually torsion-free group Γ the *virtual cohomological dimension of* Γ *over a ring* R is defined as $\operatorname{cd}_R \Gamma'$, where Γ' is a torsion-free subgroup of Γ of finite index, and denoted $\operatorname{vcd}_R \Gamma$. It is a well-defined invariant by Serre's Theorem: if G is a torsion-free group and G' is a subgroup of finite index, then $\operatorname{cd}_R G' = \operatorname{cd}_R G$ ([5, Theorem VIII 3.1]). If $R = \mathbb{Z}$ then $\operatorname{vcd}_{\mathbb{Z}} \Gamma$ is simply called the virtual cohomological dimension of Γ, and denoted $\operatorname{vcd} \Gamma$. It is known that every Coxeter group is virtually torsion-free and the virtual cohomological dimension of each Coxeter group is finite (cf. [6, Corollary 5.2, Proposition 14.1]).

For a simplicial complex K and a simplex σ of K, the *closed star* $\mathrm{St}(\sigma, K)$ of σ in K is the union of all simplexes of K having σ *as a face*, and the *link* $\mathrm{Lk}(\sigma, K)$ of σ in K is the union of all simplexes of K lying in $\mathrm{St}(\sigma, K)$ that are disjoint from σ.

In [10], Dranishnikov gave the following formula.

Theorem 1.1 (Dranishnikov [10])**.** *Let (Γ, V) be a Coxeter system and R a principal ideal domain. Then there exists the formula*

$$\mathrm{vcd}_R\,\Gamma = \mathrm{lcd}_R\,CK = \max\{\,\mathrm{lcd}_R\,K,\ \mathrm{cd}_R\,K + 1\},$$

where $K = K(\Gamma, V)$ and CK is the simplicial cone of K.

Here, for a finite simplicial complex K and an abelian group G, the *local cohomological dimension of K over G* is defined as

$$\mathrm{lcd}_G\,K = \max\nolimits_{\sigma\in K}\{i \mid H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G) \neq 0\},$$

and the *global cohomological dimension of K over G* is

$$\mathrm{cd}_G\,K = \max\{i \mid \tilde{H}^i(K; G) \neq 0\}.$$

When $\tilde{H}^i(K; G) = 0$ for each i, then we consider $\mathrm{cd}_G\,K = -1$. We note that $H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G)$ is isomorphic to $\tilde{H}^{i-1}(\mathrm{Lk}(\sigma, K); G)$. Hence, we have

$$\mathrm{lcd}_G\,K = \max\nolimits_{\sigma\in K}\{\mathrm{cd}_G\,\mathrm{Lk}(\sigma, K) + 1\}.$$

Remark. We recall Dranishnikov's remark in [11]. The definition of the local cohomological dimension in [10] is given by the terminology of the normal star and link. Since $\mathrm{Lk}(\sigma, K)$ is homeomorphic to the normal link of σ in K, their definitions are equivalent by the formula above.

Dranishnikov also proved the following theorem as an application of Theorem 1.1.

Theorem 1.2 (Dranishnikov [10])**.** *A Coxeter group Γ has the following properties:*

(a) $\mathrm{vcd}_{\mathbb{Q}}\,\Gamma \leq \mathrm{vcd}_R\,\Gamma$ *for any principal ideal domain R.*
(b) $\mathrm{vcd}_{\mathbb{Z}_p}\,\Gamma = \mathrm{vcd}_{\mathbb{Q}}\,\Gamma$ *for almost all primes p.*
(c) *There exists a prime p such that* $\mathrm{vcd}_{\mathbb{Z}_p}\,\Gamma = \mathrm{vcd}\,\Gamma$.
(d) $\mathrm{vcd}\,\Gamma \times \Gamma = 2\,\mathrm{vcd}\,\Gamma$.

In Section 2, we extend this theorem to one over principal ideal domain coefficients.

Let (Γ, V) be a Coxeter system and $K = K(\Gamma, V)$. Consider the product space $\Gamma \times |CK|$ of Γ with the discrete topology and the underlying space $|CK|$ of the cone of K. Define an equivalence relation $\sim$ on the space as follows: for $(\gamma_1, x_1), (\gamma_2, x_2) \in \Gamma \times |CK|$, $(\gamma_1, x_1) \sim (\gamma_2, x_2)$ if and only if $x_1 = x_2$ and $\gamma_1^{-1}\gamma_2 \in \Gamma_{V(x_1)}$, where $V(x) = \{v \in V | x \in \mathrm{St}(v, \beta^1 K)\}$. Here we consider that $|K|$ is naturally embedded in $|CK|$ as the base of the cone and $\beta^1 K$ denotes the barycentric subdivision of K. The natural left Γ-action on $\Gamma \times |CK|$ is compatible with the equivalence relation; hence, it passes to a left action on the quotient space $\Gamma \times |CK| / \sim$. Denote this quotient space by $A(\Gamma, V)$. The space $A(\Gamma, V)$ is contractible and Γ acts cocompactly and properly discontinuously on the space ([6, Theorem 13.5]).

We can also give the space $A(\Gamma, V)$ a structure of a piecewise Euclidean cell complex with the vertex set $\Gamma \times \{v_0\}$ ([7, §9]). $\Sigma(\Gamma, V)$ denotes this piecewise Euclidean cell complex. Refer to [7, Definition 2.2] for the definition of a piecewise Euclidean cell complex. In particular, if (Γ, V) is right-angled, then each cell of $\Sigma(\Gamma, V)$ is a cube, hence, $\Sigma(\Gamma, V)$ is a cubical complex. More precisely, for a *right-angled* Coxeter system (Γ, V), we can define the cubical complex $\Sigma(\Gamma, V)$ by the following conditions:

(1) the vertex set of $\Sigma(\Gamma, V)$ is Γ,
(2) for $\gamma, \gamma' \in \Gamma$, $\{\gamma, \gamma'\}$ spans an edge in $\Sigma(\Gamma, V)$ if and only if the length $l_V(\gamma^{-1}\gamma') = 1$, and
(3) for $\gamma \in \Gamma$ and $v_0, \ldots, v_k \in V$, the edges $|\gamma, \gamma v_0|, \ldots, |\gamma, \gamma v_k|$ form a $(k+1)$-cube in $\Sigma(\Gamma, V)$ if and only if $\{v_0, \ldots, v_k\}$ spans a k-simplex in $K(\Gamma, V)$.

We note the 1-skeleton of this cell complex is isomorphic to the Cayley graph of Γ with respect to V. For $\gamma \in \Gamma$ and a k-simplex $\sigma = |v_0, \ldots, v_k|$ of $K(\Gamma, V)$, let $C_{\gamma,\sigma}$ be the $(k+1)$-cube in $\Sigma(\Gamma, V)$ formed by $|\gamma, \gamma v_0|, \ldots, |\gamma, \gamma v_k|$. Then the vertex set of $C_{\gamma,\sigma}$ is $\gamma\Gamma_{\{v_0,\ldots,v_k\}}$. We note that

$$\gamma\Gamma_{\{v_0,\ldots,v_k\}} = \{\gamma v_0^{\epsilon_0} \cdots v_k^{\epsilon_k} \mid \epsilon_i \in \{0,1\},\ i = 0, \ldots, k\}.$$

For every Coxeter system (Γ, V), $\Sigma(\Gamma, V)$ is a $\mathrm{CAT}(0)$ geodesic space by a piecewise Euclidean metric (cf. [7, Theorem 7.8]). We define the boundary $\partial\Gamma$ as the set of geodesic rays in $\Sigma(\Gamma, V)$ emanating from the unit element $e \in \Gamma \subset \Sigma(\Gamma, V)$ with the topology of the uniform convergence on compact sets, i.e., $\partial\Gamma$ is the visual sphere of $\Sigma(\Gamma, V)$ at the point $e \in \Sigma(\Gamma, V)$. In general, for all points x, y in a $\mathrm{CAT}(0)$ space X, the visual spheres of X at points x and y are homeomorphic (cf. [9, Assertion 1]). This boundary is known to be a finite-dimensional compactum (i.e., metrizable compact space). Details of the boundaries of CAT(0) spaces can be found in [7] and [8].

It is still unknown whether the following conjecture holds.

Rigidity Conjecture (Dranishnikov [12]). *Isomorphic Coxeter groups have homeomorphic boundaries.*

We note that there exists a Coxeter group Γ with different Coxeter systems (Γ, V_1) and (Γ, V_2).

Let X be a compact metric space and G an abelian group. The *cohomological dimension of X over G* is defined as

$$\text{c-dim}_G X = \sup\{i \mid \check{H}^i(X, A; G) \neq 0 \text{ for some closed set } A \subset X\},$$

where $\check{H}^i(X, A; G)$ is the Čech cohomology of (X, A) over G.

In [2], Bestvina-Mess proved the following theorem for hyperbolic groups. An analogous theorem for Coxeter groups is proved by the same argument (cf. [9]).

Theorem 1.3 (Bestvina-Mess [2]). *Let Γ be a Coxeter group and R a ring with identity. Then there exists the formula*

$$\text{c-dim}_R \partial\Gamma = \text{vcd}_R \Gamma - 1.$$

In Section 3, for a right-angled Coxeter group Γ with $\text{vcd}_R \Gamma = n$, where R is a principal ideal domain, we construct a sequence $\Gamma_{W_0} \subset \Gamma_{W_1} \subset \cdots \subset \Gamma_{W_{n-1}}$ of parabolic subgroups with $\text{vcd}_R \Gamma_{W_i} = i$. Then we obtain, by Theorem 1.3, an analogous property to a theorem in the classical dimension theory (cf. [13, Theorem 1.5.1]), i.e., there exists a sequence $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups with $\text{c-dim}_R \partial\Gamma_{W_i} = i - 1$. Since the boundaries of Coxeter groups are always finite dimensional, we can also establish, for a right-angled Coxeter group Γ with $\dim \partial\Gamma = n$, the existence of a sequence $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups with $\dim \partial\Gamma_{W_i} = i$.

In Section 4, we show that, for a right-angled Coxeter system (Γ, V) and a subset $W \subset V$, a parabolic subgroup Γ_W is of finite index if and only if $\partial\Gamma_W = \partial\Gamma$.

Throughout this paper, a ring R means a commutative ring with identity $1_R \neq 0$. For a brief historical view of Coxeter groups and their boundaries, we refer the reader to [7]. Details of dimension/cohomological dimension theory can be found in [13] and [17].

2. The virtual cohomological dimension of Coxeter groups over principal ideal domains

In this section, we extend Dranishnikov's Theorem 1.2 to an analogous theorem over principal ideal domain coefficients by using an argument similar to one in [10]. We first prove the following lemma needed later.

Lemma 2.1. *Let R be a principal ideal domain. Let $t \geq 2$ be an integer. Then*

(i) *if the tensor product $\mathbb{Z}_t \otimes R$ is trivial, then the tensor product $\mathbb{Z}_t \otimes R/I$ and the torsion product $\mathrm{Tor}(\mathbb{Z}_t, R/I)$ are trivial for every ideal I in R, and*

(ii) *if R is not a field and the tensor product $\mathbb{Z}_t \otimes R/I$ is trivial for every non-trivial prime ideal I in R, then the tensor product $\mathbb{Z}_t \otimes R$ and the torsion product $\mathrm{Tor}(\mathbb{Z}_t, R)$ are trivial.*

Proof. Let $r_t \in R$ be the t sum $1_R + \cdots + 1_R$ of 1_R. Define the homomorphism $\varphi : R \to R$ by $\varphi(r) = r_t r$. Then there exists the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}(\mathbb{Z}_t, R) \longrightarrow R \xrightarrow{\varphi} R \longrightarrow \mathbb{Z}_t \otimes R \longrightarrow 0.$$

Hence the kernel of φ is isomorphic to $\mathrm{Tor}(\mathbb{Z}_t, R)$ and the cokernel of φ is isomorphic to $\mathbb{Z}_t \otimes R$.

(i) Suppose that $\mathbb{Z}_t \otimes R$ is trivial. It follows from $0 = \mathbb{Z}_t \otimes R \approx R/r_t R$ and the non-triviality of φ that r_t is a non-zero unit element of R. Since R is a principal ideal domain, φ is a monomorphism. It means that $\mathrm{Tor}(\mathbb{Z}_t, R) = 0$.

Let I be a non-trivial ideal in R. Consider the following exact sequence:

$$\mathrm{Tor}(\mathbb{Z}_t, R) \to \mathrm{Tor}(\mathbb{Z}_t, R/I) \to \mathbb{Z}_t \otimes I \to \mathbb{Z}_t \otimes R \to \mathbb{Z}_t \otimes R/I \to 0,$$

which is induced by the natural short exact sequence $I \hookrightarrow R \to R/I$. Then it is clear that $\mathbb{Z}_t \otimes R/I = 0$. We also see that $\mathrm{Tor}(\mathbb{Z}_t, R/I) \approx \mathbb{Z}_t \otimes I = 0$, since r_t is a unit element of R.

(ii) We note that there exists a non-trivial prime ideal I in R, because R is not a field.

Suppose that $\mathbb{Z}_t \otimes R/I$ is trivial for every non-trivial prime ideal I in R.

First, we show that $r_t \neq 0$ in R. If $r_t = 0$ in R, then for a non-trivial prime ideal I the homomorphism $R/I \to R/I$ defined by $r + I \mapsto r_t r + I$ is trivial. Hence $\mathbb{Z}_t \otimes R/I$ is isomorphic to $R/I \neq 0$. It contradicts $\mathbb{Z}_t \otimes R/I = 0$. Therefore, we have $r_t \neq 0$.

Then φ is a monomorphism, because R is an integral domain. Hence $\mathrm{Tor}(\mathbb{Z}_t, R)$ is trivial.

Next, we show that r_t is a unit. Suppose that r_t is not a unit. Since R is a principal ideal domain, r_t is presented as $r_t = p_1 \cdots p_k$ by some prime elements $p_1, \ldots, p_k$ of R. Then $I = p_1 R$ is a non-trivial prime ideal in R. The homomorphism $R/I \to R/I$ defined by $r + I \mapsto r_t r + I$ is trivial, because $r_t r + I = p_1(p_2 \cdots p_k r) + I = I$. Hence $\mathbb{Z}_t \otimes R/I$ is isomorphic to $R/I \neq 0$. It contradicts $\mathbb{Z}_t \otimes R/I = 0$. Therefore r_t is a unit.

Then φ is an epimorphism. It means that $\mathbb{Z}_t \otimes R$ is trivial. □

Theorem 2.2. *Let* Γ *be a Coxeter group and* R *a principal ideal domain. Then* Γ *has the following properties:*

(a) $\operatorname{vcd}_{\mathbb{Q}} \Gamma \leq \operatorname{vcd}_{R/I} \Gamma \leq \operatorname{vcd}_R \Gamma \leq \operatorname{vcd} \Gamma$ *for any prime ideal* I *in* R.

(b) $\operatorname{vcd}_{R/I} \Gamma = \operatorname{vcd}_{\mathbb{Q}} \Gamma$ *for almost all prime ideals* I *in* R, *if* R *is not a field.*

(c) *There exists a non-trivial prime ideal* I *in* R *such that* $\operatorname{vcd}_{R/I} \Gamma = \operatorname{vcd}_R \Gamma$, *if* R *is not a field.*

(d) $\operatorname{vcd}_R \Gamma \times \Gamma = 2 \operatorname{vcd}_R \Gamma$.

Proof. Let (Γ, V) be a Coxeter system, R a principal ideal domain, and $K = K(\Gamma, V)$. We note that R/I is a field for every non-trivial prime ideal I in R, and R has the only trivial prime ideal if R is a field.

(a) For any prime ideal I in R, we have $\operatorname{vcd}_{\mathbb{Q}} \Gamma \leq \operatorname{vcd}_{R/I} \Gamma$ by Theorem 1.2 (a), and $\operatorname{vcd}_R \Gamma \leq \operatorname{vcd} \Gamma$. We show $\operatorname{vcd}_{R/I} \Gamma \leq \operatorname{vcd}_R \Gamma$.

If I is trivial, then it is obvious. We suppose that I is a non-trivial prime ideal in R. Let $\operatorname{vcd}_{R/I} \Gamma = n$. Then $\operatorname{lcd}_{R/I} CK = n$ by Theorem 1.1. Hence there exists a simplex σ of CK such that $\tilde{H}^{n-1}(\operatorname{Lk}(\sigma, CK); R/I) \neq 0$. By the universal coefficient formula, either $\tilde{H}^{n-1}(\operatorname{Lk}(\sigma, CK)) \otimes R/I$ or $\operatorname{Tor}(\tilde{H}^n(\operatorname{Lk}(\sigma, CK)), R/I)$ is non-trivial. Since $\tilde{H}^{n-1}(\operatorname{Lk}(\sigma, CK))$ and $\tilde{H}^n(\operatorname{Lk}(\sigma, CK))$ are finitely generated abelian groups, we have $\tilde{H}^{n-1}(\operatorname{Lk}(\sigma, CK)) \otimes R \neq 0$ or $\tilde{H}^n(\operatorname{Lk}(\sigma, CK)) \otimes R \neq 0$ by Lemma 2.1 (i). By the universal coefficient formula, $\tilde{H}^{n-1}(\operatorname{Lk}(\sigma, CK); R) \neq 0$ or $\tilde{H}^n(\operatorname{Lk}(\sigma, CK); R) \neq 0$. In both cases, we have $\operatorname{vcd}_R \Gamma = \operatorname{lcd}_R CK \geq n$ by Theorem 1.1.

(b) Let $\operatorname{vcd}_{\mathbb{Q}} \Gamma = n$. We define $\mathcal{A}$ as the set of non-trivial prime ideals I in R such that $\tilde{H}^i(\operatorname{Lk}(\sigma, CK)) \otimes R/I \neq 0$ for some simplex σ of CK and integer $i \geq n$. We show that $\mathcal{A}$ contains every non-trivial prime ideal I in R with $\operatorname{vcd}_{R/I} \Gamma \neq n$.

Suppose I is a non-trivial prime ideal in R with $\operatorname{vcd}_{R/I} \Gamma \neq n$. Then $\operatorname{lcd}_{R/I} CK = \operatorname{vcd}_{R/I} \Gamma > n$ by Theorem 1.1 and Theorem 1.2 (a). Hence there exist a simplex σ of CK and an integer $i \geq n$ such that $\tilde{H}^i(\operatorname{Lk}(\sigma, CK); R/I) \neq 0$. By the universal coefficient formula, either $\tilde{H}^i(\operatorname{Lk}(\sigma, CK)) \otimes R/I$ or $\operatorname{Tor}(\tilde{H}^{i+1}(\operatorname{Lk}(\sigma, CK)), R/I)$ is non-trivial. Here we note that for a field F and an integer $t \geq 2$, the tensor product $\mathbb{Z}_t \otimes F$ is trivial if and only if the torsion product $\mathrm{Tor}(\mathbb{Z}_t, F)$ is trivial. Therefore $\tilde{H}^i(\mathrm{Lk}(\sigma, CK)) \otimes R/I \neq 0$ or $\tilde{H}^{i+1}(\mathrm{Lk}(\sigma, CK)) \otimes R/I \neq 0$ because R/I is a field. In both cases, I is an element of $\mathcal{A}$. Therefore to prove our desired property, it is sufficient to show that $\mathcal{A}$ is finite.

Let T be the set of all torsion coefficients of $\tilde{H}^i(\mathrm{Lk}(\sigma, CK))$ for every simplex σ of CK and integer $i \geq n$. Since CK is a finite simplicial complex and $\tilde{H}^i(\mathrm{Lk}(\sigma, CK))$ is a finitely generated torsion group for each simplex σ of CK and $i \geq n$, which is by $\mathrm{lcd}_{\mathbb{Q}}\, CK = \mathrm{vcd}_{\mathbb{Q}}\, \Gamma = n$, we have that T is finite. For each $t \in T$, we define $\mathcal{B}_t$ as the set of non-trivial prime ideals I such that $\mathbb{Z}_t \otimes R/I \neq 0$. Then we note that $\mathcal{A} = \bigcup_{t \in T} \mathcal{B}_t$.

We show that $\mathcal{B}_t$ is finite for each $t \in T$. Let $r_t \in R$ be the t sum $1_R + \cdots + 1_R$ of 1_R. Since R is a principal ideal domain, R is a unique factorization domain. Hence r_t is presented as $r_t = p_1 \cdots p_k$ by some prime elements $p_1, \ldots, p_k$. Let I be a non-trivial prime ideal in R such that $\mathbb{Z}_t \otimes R/I$ is non-trivial. For the homomorphism $\bar{\varphi} : R/I \to R/I$ defined by $r + I \mapsto r_t r + I$, the cokernel of $\bar{\varphi}$ is isomorphic to $\mathbb{Z}_t \otimes R/I$. Since R/I is a field, $\bar{\varphi}$ is trivial. Hence I is a member of $\{p_1 R, \ldots, p_k R\}$, because $p_1, \cdots, p_k$ are prime elements. Therefore we have that the cardinality of $\mathcal{B}_t$ is at most k. Hence $\mathcal{A}$ is finite, because T is finite.

(c) Let $\mathrm{vcd}_R\, \Gamma = n$. Then there exists a simplex σ of CK such that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK); R) \neq 0$ by Theorem 1.1. By the universal coefficient formula, either $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R$ or $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CK)), R)$ is non-trivial.

First, we show that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R$ is non-trivial. To show the fact, we suppose that $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CK)), R)$ is non-trivial. Let the numbers $s_1, \ldots, s_l$ be the torsion coefficients of $\tilde{H}^n(\mathrm{Lk}(\sigma, CK))$. Then there exists a number s_j such that $\mathrm{Tor}(\mathbb{Z}_{s_j}, R) \neq 0$. By Lemma 2.1 (ii), there exists a non-trivial prime ideal I in R such that $\mathbb{Z}_{s_j} \otimes R/I \neq 0$. Then $\tilde{H}^n(\mathrm{Lk}(\sigma, CK)) \otimes R/I$ is non-trivial. By the universal coefficient formula, $\tilde{H}^n(\mathrm{Lk}(\sigma, CK); R/I)$ is non-trivial. Hence we have $\mathrm{vcd}_{R/I}\, \Gamma = \mathrm{lcd}_{R/I}\, CK \geq n + 1$ by Theorem 1.1. On the other hand, we have $\mathrm{vcd}_{R/I}\, \Gamma \leq \mathrm{vcd}_R\, \Gamma = n$ by Theorem 2.2 (a). This is a contradiction. Thus $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CK)), R)$ is trivial. Therefore $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R$ must be non-trivial.

Next, we show that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R/I$ is non-trivial for some non-trivial prime ideal I in R. Let β be the Betti number and the numbers $t_1, \ldots, t_k$ the torsion coefficients of $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK))$. If β is non-zero, then it is clear that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R/I$ is non-trivial for any non-trivial prime ideals I in R. If β is zero, then there exists a number t_i such that $\mathbb{Z}_{t_i} \otimes R \neq 0$. By Lemma 2.1 (ii), there exists a non-trivial prime ideal I in R such that $\mathbb{Z}_{t_i} \otimes R/I \neq 0$. Then $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CK)) \otimes R/I$ is non-trivial.

By the universal coefficient formula and Theorem 1.1, we have $\mathrm{vcd}_{R/I}\,\Gamma \geq n$. Hence, $\mathrm{vcd}_{R/I}\,\Gamma = n$ by Theorem 2.2 (a).

(d) In general, for groups Γ_1, Γ_2 the inequality $\mathrm{vcd}_R\,\Gamma_1 \times \Gamma_2 \leq \mathrm{vcd}_R\,\Gamma_1 + \mathrm{vcd}_R\,\Gamma_2$ holds, where the equality holds, if R is a field ([3, Theorem 4 c)]). Hence, in our case, the equality $\mathrm{vcd}_R\,\Gamma \times \Gamma = 2\,\mathrm{vcd}_R\,\Gamma$ holds, if R is a field. We suppose that R is not field. Then the inequality $\mathrm{vcd}_R\,\Gamma \times \Gamma \leq 2\,\mathrm{vcd}_R\,\Gamma$ holds. We show that $\mathrm{vcd}_R\,\Gamma \times \Gamma \geq 2\,\mathrm{vcd}_R\,\Gamma$. By Theorem 2.2 (c), there exists a non-trivial prime ideal I in R such that $\mathrm{vcd}_{R/I}\,\Gamma = \mathrm{vcd}_R\,\Gamma$. We note that R/I is a field. Then we have $2\,\mathrm{vcd}_R\,\Gamma = 2\,\mathrm{vcd}_{R/I}\,\Gamma = \mathrm{vcd}_{R/I}\,\Gamma \times \Gamma$. Since $\Gamma \times \Gamma$ is also a Coxeter group, $\mathrm{vcd}_{R/I}\,\Gamma \times \Gamma \leq \mathrm{vcd}_R\,\Gamma \times \Gamma$ by Theorem 2.2 (a). Therefore we have $\mathrm{vcd}_R\,\Gamma \times \Gamma = 2\,\mathrm{vcd}_R\,\Gamma$. □

3. A sequence of parabolic subgroups of a right-angled Coxeter group

In this section, we prove the following theorem.

Theorem 3.1. *Let (Γ, V) be a right-angled Coxeter system with $\mathrm{vcd}_R\,\Gamma = n$, where R is a principal ideal domain. Then there exists a sequence $W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset V$ such that $\mathrm{vcd}_R\,\Gamma_{W_i} = i$ for $i = 0, \ldots, n-1$. In particular, we can obtain a sequence of simplexes $\tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1}$ such that W_i is the vertex set of $\mathrm{Lk}(\tau_i, K(\Gamma, V))$ and $K(\Gamma_{W_i}, W_i) = \mathrm{Lk}(\tau_i, K(\Gamma, V))$.*

We note that Theorem 3.1 is not always true for general Coxeter groups. Indeed, there exists the following counter-example.

Example. We consider the Coxeter system (Γ, V) defined by $V = \{v_1, v_2, v_3\}$ and

$$m(v_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

Then Γ is not right-angled, and $K(\Gamma, V)$ is not a flag complex. Indeed, $\Gamma_{\{v_i, v_j\}}$ is finite for each $i, j \in \{1, 2, 3\}$, but Γ is infinite (cf. [4, p.98, Proposition 8]). Since $\mathrm{cd}\,K(\Gamma, V) = 1$ and $\mathrm{lcd}\,K(\Gamma, V) = 1$, we have $\mathrm{vcd}\,\Gamma = 2$ by Theorem 1.1. For any proper subset $W \subset V$, $\mathrm{vcd}\,\Gamma_W = 0$, because Γ_W is a finite group. Hence there does not exist a subset $W \subset V$ such that $\mathrm{vcd}\,\Gamma_W = 1$.

We first show some lemmas.

Lemma 3.2. *Let K be a simplicial complex. If τ is a simplex of K and τ' is a simplex in the link* $\mathrm{Lk}(\tau, K)$*, then the join $\tau * \tau'$ is a simplex of K and* $\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K)) = \mathrm{Lk}(\tau * \tau', K)$.

Proof. Let τ be a simplex of K and τ' in $\mathrm{Lk}(\tau, K)$. Since τ' is in $\mathrm{Lk}(\tau, K)$, the join $\tau * \tau'$ is a simplex of K and $\tau \cap \tau' = \emptyset$. For a simplex σ of K, σ is in $\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K))$ if and only if $\sigma * \tau'$ is in $\mathrm{Lk}(\tau, K)$ and $\sigma \cap \tau' = \emptyset$, i.e., $\sigma * \tau' * \tau$ is a simplex of K and $\sigma \cap (\tau * \tau') = \emptyset$. Therefore σ is in $\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K))$ if and only if σ is in $\mathrm{Lk}(\tau * \tau', K)$. Hence we have $\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K)) = \mathrm{Lk}(\tau * \tau', K)$. □

Lemma 3.3. *Let K be a simplicial complex and G an abelian group. For a simplex τ of K, there exists the inequality* $\mathrm{lcd}_G \mathrm{Lk}(\tau, K) \leq \mathrm{lcd}_G K$.

Proof. Let $\mathrm{lcd}_G \mathrm{Lk}(\tau, K) = n$. Then there exists a simplex τ' in $\mathrm{Lk}(\tau, K)$ such that $\tilde{H}^{n-1}(\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K)); G) \neq 0$. By Lemma 3.2, $\mathrm{Lk}(\tau', \mathrm{Lk}(\tau, K)) = \mathrm{Lk}(\tau * \tau', K)$. Hence we have $\mathrm{lcd}_G K \geq n$. □

In [10], Dranishnikov showed the following relation of $\mathrm{lcd}_G K$ and $\mathrm{cd}_G K$.

Theorem 3.4 (Dranishnikov [10])**.** *For every abelian group G and every finite simplicial complex K, there exists the inequality* $\mathrm{lcd}_G K \geq \mathrm{cd}_G K$.

Using this theorem and the lemmas above, we show the following key lemma.

Lemma 3.5. *Let (Γ, V) be a right-angled Coxeter system with $\mathrm{vcd}_R \Gamma = n$, where V is nonempty and R is a principal ideal domain. Then there exists a proper subset W of V such that $\mathrm{vcd}_R \Gamma_W = n$ or $n - 1$. In particular, we can obtain a simplex σ of $K(\Gamma, V)$ such that W is the vertex set of $\mathrm{Lk}(\sigma, K(\Gamma, V))$ and $K(\Gamma_W, W) = \mathrm{Lk}(\sigma, K(\Gamma, V))$.*

Proof. Since $\mathrm{vcd}_R \Gamma = n$, we have $\mathrm{lcd}_R K(\Gamma, V) = n$ or $\mathrm{cd}_R K(\Gamma, V) = n - 1$ by Theorem 1.1. If $\mathrm{lcd}_R K(\Gamma, V) \leq n - 1$, then $\mathrm{cd}_R K(\Gamma, V) = n - 1$, and $\mathrm{lcd}_R K(\Gamma, V) = n - 1$ by Theorem 3.4. Therefore $\mathrm{lcd}_R K(\Gamma, V) = n$ or $n - 1$.

We set $m := \mathrm{lcd}_R K(\Gamma, V)$. Then there exists a simplex σ of $K(\Gamma, V)$ such that $\tilde{H}^{m-1}(\mathrm{Lk}(\sigma, K(\Gamma, V)); R) \neq 0$ and $\tilde{H}^{i}(\mathrm{Lk}(\sigma, K(\Gamma, V)); R) = 0$ for any $i \geq m$. Hence we have $\mathrm{cd}_R \mathrm{Lk}(\sigma, K(\Gamma, V)) = m - 1$. Let W be the vertex set of $\mathrm{Lk}(\sigma, K(\Gamma, V))$. We note that W is a proper subset of V.

Then we show that

$$K(\Gamma_W, W) = \mathrm{Lk}(\sigma, K(\Gamma, V)). \tag{1}$$

It is clear that the vertex set of $K(\Gamma_W, W)$ is the vertex set of $\mathrm{Lk}(\sigma, K(\Gamma, V))$. Let $\{v_0, \ldots, v_k\}$ be a subset of W which spans a simplex of $K(\Gamma_W, W)$. Since

$\{v_0, \dots, v_k\}$ generates a finite subgroup of $\Gamma_W \subset \Gamma$, $\{v_0, \dots, v_k\}$ spans a simplex of $K(\Gamma, V)$. It follows from $v_i \in W = \mathrm{Lk}(\sigma, K(\Gamma, V))^{(0)}$ that the join $v_i * \sigma$ forms a simplex of $K(\Gamma, V)$ and $v_i \notin \sigma$ for each $i = 0, \dots, k$. We note that $K(\Gamma, V)$ is a flag complex, since Γ is right-angled. Hence the join $|v_0, \dots, v_k| * \sigma$ forms a simplex of $K(\Gamma, V)$ and $|v_0, \dots, v_k| \cap \sigma = \emptyset$, i.e., $|v_0, \dots, v_k|$ is a simplex in $\mathrm{Lk}(\sigma, K(\Gamma, V))$. Conversely, let $\{v_0, \dots, v_k\}$ be a subset of W which spans a simplex in $\mathrm{Lk}(\sigma, K(\Gamma, V))$. Then $\{v_0, \dots, v_k\}$ generates a finite subgroup of Γ. Since $\{v_0, \dots, v_k\} \subset W$, $\{v_0, \dots, v_k\}$ generates a finite subgroup of Γ_W. Hence $\{v_0, \dots, v_k\}$ spans a simplex of $K(\Gamma_W, W)$. Thus it follows that $K(\Gamma_W, W) = \mathrm{Lk}(\sigma, K(\Gamma, V))$.

We note that $\mathrm{cd}_R\, K(\Gamma_W, W) = m - 1$, and $\mathrm{lcd}_R\, K(\Gamma_W, W) \leq m$ by (1) and Lemma 3.3. Hence $\mathrm{vcd}_R\, \Gamma_W = m$ by Theorem 1.1. Thus we have $\mathrm{vcd}_R\, \Gamma_W = n$ or $n - 1$. □

Using this lemma, we prove Theorem 3.1.

Proof of Theorem 3.1. Let (Γ, V) be a right-angled Coxeter system with $\mathrm{vcd}_R\, \Gamma = n$, where R is a principal ideal domain.

By Lemma 3.5, we can obtain subsets $\{V_i\}_i$ of V and simplexes $\{\sigma_i\}_i$ of $K(\Gamma, V)$ satisfying the following conditions:

(1) $V_0 = V$,
(2) V_{i+1} is a proper subset of V_i,
(3) $K(\Gamma_{V_{i+1}}, V_{i+1}) = \mathrm{Lk}(\sigma_{i+1}, K(\Gamma_{V_i}, V_i))$, and
(4) $\mathrm{vcd}_R\, \Gamma_{V_{i+1}} = \mathrm{vcd}_R\, \Gamma_{V_i}$ or $\mathrm{vcd}_R\, \Gamma_{V_i} - 1$.

Then we note, by conditions (1), (3) and Lemma 3.2, that

$$\begin{aligned} K(\Gamma_{V_i}, V_i) &= \mathrm{Lk}(\sigma_i, K(\Gamma_{V_{i-1}}, V_{i-1})) \\ &= \mathrm{Lk}(\sigma_i, \mathrm{Lk}(\sigma_{i-1}, K(\Gamma_{V_{i-2}}, V_{i-2}))) \\ &= \mathrm{Lk}(\sigma_{i-1} * \sigma_i, K(\Gamma_{V_{i-2}}, V_{i-2})) \\ &= \cdots \\ &= \mathrm{Lk}(\sigma_1 * \cdots * \sigma_i, K(\Gamma_{V_0}, V_0)) \\ &= \mathrm{Lk}(\sigma_1 * \cdots * \sigma_i, K(\Gamma, V)). \end{aligned}$$

Since V is finite, there exists a number m such that V_m is the empty set by condition (2). Then $\mathrm{vcd}_R\, \Gamma_{V_m} = 0$, because Γ_{V_m} is the trivial group. Hence we can have a subsequence $\{V_{i_j}\}_j$ of $\{V_i\}_i$ such that $\mathrm{vcd}_R\, \Gamma_{V_{i_j}} = n - j$ for each $j = 1, \dots, n$ by condition (4).

We set $W_j := V_{i_{n-j}}$ and $\tau_j := \sigma_1 * \cdots * \sigma_{i_{n-j}}$ for $j = 0, \dots, n-1$. Then we have $W_j \subset W_{j+1}$, $\tau_j \succ \tau_{j+1}$, $\mathrm{vcd}_R \Gamma_{W_j} = j$ and $K(\Gamma_{W_j}, W_j) = \mathrm{Lk}(\tau_j, K(\Gamma, V))$ for each j by our construction. □

By Theorem 1.3, we can obtain the following corollary.

Corollary 3.6. *For a right-angled Coxeter system (Γ, V) with $\text{c-dim}_R\, \partial\Gamma = n$, where R is a principal ideal domain, there exists a sequence $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of (Γ, V) such that $\text{c-dim}_R\, \partial\Gamma_{W_i} = i$ for each $i = 0, 1, \dots, n-1$.*

In general, for a finite dimensional compactum X, the equality $\text{c-dim}_{\mathbb{Z}} X = \dim X$ holds ([17, §2, Remark 4]). Since the boundaries of Coxeter groups are always finite dimensional, we obtain the following corollary.

Corollary 3.7. *For a right-angled Coxeter system (Γ, V) with $\dim \partial\Gamma = n$, there exists a sequence $\partial\Gamma_{W_0} \subset \partial\Gamma_{W_1} \subset \cdots \subset \partial\Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of (Γ, V) such that $\dim \partial\Gamma_{W_i} = i$ for each $i = 0, 1, \dots, n-1$.*

4. The boundaries of parabolic subgroups of a right-angled Coxeter group

In this section, we show that a parabolic subgroup of a right-angled Coxeter group is of finite index if and only if their boundaries coincide.

If X and Y are topological spaces, let us define $X * Y$ to be the quotient space of $X \times Y \times [0, 1]$ obtained by identifying each set $x \times Y \times 0$ to a point and each set $X \times y \times 1$ to a point.

The following proposition is known.

Proposition 4.1 (Dranishnikov [10]). *Let (Γ_1, V_1) and (Γ_2, V_2) be Coxeter systems. Then we have*

$$\partial(\Gamma_1 \times \Gamma_2) = \partial\Gamma_1 * \partial\Gamma_2.$$

Sketch of proof. We give a proof only if Γ_1 and Γ_2 are right-angled.

We recall the construction of the cubical complex $\Sigma(\Gamma, V)$ in Section 1. A cube of $\Sigma(\Gamma_1, V_1)$ has a form of $C_{\gamma,\sigma}$ for some $\gamma \in \Gamma_1$ and simplex σ of $K(\Gamma_1, V_1)$, where $C_{\gamma,\sigma}$ is the cube defined by $\gamma \in \Gamma_1$ and $\sigma \in K(\Gamma_1, V_1)$. In the same way, let $D_{\delta,\tau}$ and $E_{\epsilon,\upsilon}$ be the cubes of $\Sigma(\Gamma_2, V_2)$ and $\Sigma(\Gamma_1 \times \Gamma_2, V_1 \cup V_2)$ defined by $\delta \in \Gamma_2$ and $\tau \in K(\Gamma_2, V_2)$, and $\epsilon \in \Gamma_1 \times \Gamma_2$ and $\upsilon \in K(\Gamma_1 \times \Gamma_2, V_1 \cup V_2)$, respectively.

Then

$$\varphi : \Sigma(\Gamma_1, V_1) \times \Sigma(\Gamma_2, V_2) \to \Sigma(\Gamma_1 \times \Gamma_2, V_1 \cup V_2)$$

defined by $\varphi(C_{\gamma,\sigma} \times D_{\delta,\tau}) = E_{(\gamma,\delta),\sigma*\tau}$ is the natural isomorphism of cubical complexes, where $\gamma \in \Gamma_1$, $\delta \in \Gamma_2$, $\sigma \in K(\Gamma_1, V_1)$ and $\tau \in K(\Gamma_2, V_2)$. Here we note that $K(\Gamma_1 \times \Gamma_2, V_1 \cup V_2) = K(\Gamma_1, V_1) * K(\Gamma_2, V_2)$. Hence we have

$$\Sigma(\Gamma_1 \times \Gamma_2, V_1 \cup V_2) = \Sigma(\Gamma_1, V_1) \times \Sigma(\Gamma_2, V_2).$$

Then the natural identification

$$\psi : \partial\Gamma_1 * \partial\Gamma_2 \to \partial(\Gamma_1 \times \Gamma_2)$$

is defined by $\psi([\xi_1, \xi_2, \theta]) = (\theta\xi_1, \theta\xi_2)$, where

$$(\theta\xi_1, \theta\xi_2)(t) = (\xi_1(t\cos(\theta\pi/2)), \xi_2(t\sin(\theta\pi/2)))$$

for $t \geq 0$. Thus we have

$$\partial(\Gamma_1 \times \Gamma_2) = \partial\Gamma_1 * \partial\Gamma_2.$$

This formula for every Coxeter group is also proved by the same argument. In this case, $C_{\gamma,\sigma}$, $D_{\delta,\tau}$, and $E_{\epsilon,\upsilon}$ play cells of the piecewise Euclidean cell complexes $\Sigma(\Gamma_1, V_1)$, $\Sigma(\Gamma_2, V_2)$, and $\Sigma(\Gamma_1 \times \Gamma_2, V_1 \cup V_2)$, respectively. □

Theorem 4.2. *Let (Γ, V) be a right-angled Coxeter system and W a subset of V. Then the following conditions are equivalent:*

(1) *The parabolic subgroup $\Gamma_W \subset \Gamma$ is of finite index.*
(2) *$\{v, v'\}$ spans an edge of $K(\Gamma, V)$ for any $v \in V \setminus W$ and $v' \in V$.*
(3) *$\Gamma = \Gamma_W \times \Gamma_{V\setminus W}$ and $\Gamma_{V\setminus W} \approx \mathbb{Z}_2^{|V\setminus W|}$.*
(4) *$\partial\Gamma = \partial\Gamma_W$.*

Proof. (1) ⇒ (2): Suppose that there exist $v \in V \setminus W$ and $v' \in V$ such that $\{v, v'\}$ does not span an edge of $K(\Gamma, V)$. Since $\Gamma_{\{v,v'\}}$ is infinite, $\Gamma_{\{v,v'\}}$ is the free product $\Gamma_{\{v\}} * \Gamma_{\{v'\}}$. For each integer t, let $\gamma_t = (vv')^t$. If $s \neq t$, then $\gamma_t^{-1}\gamma_s = (vv')^{s-t} \notin \Gamma_W$, since $v \notin W$. Hence $\{\gamma_t\Gamma_W | t = 1, 2, \dots\}$ is infinite, and $\{\gamma\Gamma_W | \gamma \in \Gamma\}$ is infinite. Therefore Γ_W is not a subgroup of finite index.

(2) ⇒ (3): Suppose (2) holds. Since (Γ, V) is right-angled, $\{v, v'\}$ spans an edge of $K(\Gamma, V)$ if and only if $vv' = v'v$. Hence we have that $\Gamma = \Gamma_W \times \Gamma_{V\setminus W}$ and $\Gamma_{V\setminus W} \approx \mathbb{Z}_2^{|V\setminus W|}$.

(3) ⇒ (1): Suppose (3) holds. Then

$$[\Gamma : \Gamma_W] = [\Gamma_W \times \Gamma_{V\setminus W} : \Gamma_W] = |\Gamma_{V\setminus W}| = |\mathbb{Z}_2^{|V\setminus W|}| = 2^{|V\setminus W|}.$$

(3) ⇒ (4): Suppose (3) holds. Since $\Gamma_{V\setminus W} \approx \mathbb{Z}_2^{|V\setminus W|}$ is finite, the boundary $\partial\Gamma_{V\setminus W}$ is empty. Hence $\partial\Gamma = \partial\Gamma_W$ by Proposition 4.1.

(4) $\Rightarrow$ (2): Suppose that there exist $v \in V \setminus W$ and $v' \in V$ such that $\{v, v'\}$ does not span an edge of $K(\Gamma, V)$. Then $\Gamma_{\{v,v'\}} = \Gamma_{\{v\}} * \Gamma_{\{v'\}}$. Consider the geodesic ray $\xi : [0, \infty) \to \Sigma(\Gamma_{\{v,v'\}}, \{v, v'\}) \subset \Sigma(\Gamma, V)$ such that

$$\xi(t) = \begin{cases} (vv')^{t/2} & \text{if } t = 0, 2, 4, \ldots \\ (vv')^{(t-1)/2} v & \text{if } t = 1, 3, 5, \ldots \end{cases}$$

Then $\xi \in \partial\Gamma$, but $\xi \notin \partial\Gamma_W$, because $v \in V \setminus W$. Thus we have $\partial\Gamma \neq \partial\Gamma_W$. □

References

[1] M. Bestvina, The virtual cohomological dimension of Coxeter groups, Geometric Group Theory Vol 1, LMS Lecture Notes **181** (1993), 19–23.

[2] M. Bestvina and G. Mess, *The boundary of negatively curved groups*, Jour. of Amer. Math. Soc. **4** (3) (1991), 469–481.

[3] R. Bieri, *Homological dimension of discrete groups*, Queen Mary College, London (1976).

[4] N. Bourbaki, Groupes et Algebrès de Lie, Chapters IV-VI, Masson, Paris (1981).

[5] K. S. Brown, Cohomology of groups, Springer-Verlag, New York, Heidelberg, Berlin (1982).

[6] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Annals of Math. **117** (1983), 293–324.

[7] M. W. Davis, *Nonpositive curvature and reflection groups*, Preprint (1994).

[8] M. W. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geometry **34** (1991), 347–388.

[9] A. N. Dranishnikov, *Boundaries and cohomological dimensions of Coxeter groups*, Preprint (1994).

[10] A. N. Dranishnikov, *On the virtual cohomological dimensions of Coxeter groups*, Proc. Amer. Math. Soc. **125** (7) (1997), 1885–1891.

[11] A. N. Dranishnikov, *Boundaries of Coxeter groups and simplicial complexes with given links*, Preprint (1997).

[12] A. N. Dranishnikov, *On boundaries of hyperbolic Coxeter groups*, Preprint (1998).

[13] R. Engelking, *Theory of dimensions finite and infinite*, Heldermann-Verlag, Sigma Series in Pure Math. **10** (1995).

[14] M. Gromov, Hyperbolic groups, in Essays in group theory, S. M. Gersten (ed.), M.S.R.I. **8** (1987), 75-264.

[15] T. W. Hungerford, Algebra, Springer-Verlag, Berlin (1974).

[16] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press (1990).

[17] V. I. Kuzminov, *Homological dimension theory*, Russian Math. Surveys **23** (5) (1968), 1–45.

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(T. Hosaka) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan

E-mail address: thosaka@math.tsukuba.ac.jp

(K. Yokoi) Department of Mathematics, Interdisciplinary faculty of Science and Engineering, Shimane University, Matsue, 690-8504, Japan
E-mail address: yokoi@math.shimane-u.ac.jp