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SPECIAL UNIONS OF UNICOHERENT CONTINUA

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Abstract. It is proved that a Hausdorff continuum is unicoherent if it is the union of two unicoherent continua whose intersection is connected and locally connected.

1. Introduction and First Definitions

A separable metric space X is said to be *acyclic* if every continuous map from X into S^1 is homotopic to a constant map. Classical results of Eilenberg assert that a metric continuum is acyclic, and hence unicoherent, if it is the union of two acyclic continua whose intersection is connected; the fact is seen as an amalgam of (7.3), (6.22), and (5.2) in Chapter XI of [Wh]. It follows from this theorem and the theory of Peano continua that a metric continuum is unicoherent if it is the union of two locally connected unicoherent continua whose intersection is connected ((7.6) in Chapter XI of [Wh]). Special types of unicoherence are preserved under similar unions, as in [Ow], even when Eilenberg's homotopy theorems are unavailable. Such is the case in the present paper, where it is shown that a Hausdorff continuum P is unicoherent if it is the union of unicoherent continua J and K such that $J \cap K$ is connected and locally connected. Acyclic graphs play an important part in the proof as nerves of covers (tools also used in [St]). Our proof is sketched in the next paragraph for a metric continuum (P, ρ).

If such a continuum P is not unicoherent then it is the union of continua P_{-1} and P_1 with $P_{-1} \cap P_1 = A \,|\, B$. Choose $\varepsilon > 0$ so that $\rho(a, b) \geq \varepsilon$ for all $a \in A$, $b \in B$. By unicoherence of J and K, the locally connected continuum $J \cap K$ is seen to be a separating subcontinuum of P that intersects each component of P_{-1}, each component of P_1, and each component of $P_{-1} \cap P_1$. Let $V = J \cap K$,

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and express V as the union of a finite collection of continua, $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$, with ρ-*diameter*$(V_j) < \varepsilon/2$ for $1 \leq j \leq s$. For each $i \in \{-1, 1\}$, let $\mathcal{V}(P_i) = \{V_j \in \mathcal{V} : V_j \cap P_i \neq \emptyset\}$ and define compacta

$$J_i = (J \cap P_i) \cup \bigcup \mathcal{V}(P_i)$$
$$K_i = (K \cap P_i) \cup \bigcup \mathcal{V}(P_i)$$

Then $J = J_{-1} \cup J_1$ and $K = K_{-1} \cup K_1$. Also, the sets $P'_{-1} = J_{-1} \cup K_{-1}$ and $P'_1 = J_1 \cup K_1$ are continua containing P_{-1} and P_1, respectively, and $P'_{-1} \cap P'_1$ is still not connected. Let $\Sigma J = \{S : S$ is a component of J_{-1} or of $J_1\}$. Boundary bumping gives $|\Sigma J| \leq |\mathcal{V}(P_{-1})| + |\mathcal{V}(P_1)|$. There is a connected graph associated with ΣJ having one vertex for each element of ΣJ and one edge for each pair of distinct intersecting elements of ΣJ. This graph, the *nerve* of ΣJ, is denoted $T_{[\,]}$. Since J is unicoherent, $T_{[\,]}$ is acyclic, i. e., $T_{[\,]}$ is a tree. Similarly define $\Sigma K = \{S : S$ is a component of K_{-1} or of $K_1\}$, and the nerve $T_{\langle\,\rangle}$ of ΣK is again a tree. Also, if $\Sigma V = \{S : S$ is a component of $\bigcup \mathcal{V}(P_{-1})$ or of $\bigcup \mathcal{V}(P_1)\}$, then the nerve $T_{(\,)}$ of ΣV is connected, though not necessarily a tree. The proof is finished when boundary-bumping of components in ΣJ and ΣK, together with a graph-theoretic result relating $T_{[\,]}$, $T_{\langle\,\rangle}$ and $T_{(\,)}$, force the conclusion that $P'_1 \cap P'_{-1}$ is connected, contradicting the previously cited disconnectedness of this set.

For most fundamental definitions the reader is referred to [Ho]. A *compactum* is a nonempty compact topological space, and a *continuum* is a connected compactum. In this paper, only Hausdorff continua are considered. P always denotes a Hausdorff continuum. We let $C(P)$ denote the set of all subcontinua of P. P is said to be *unicoherent* if $M \cap N$ is connected whenever $M, N \in C(P)$ and $M \cup N = P$. Given sets A and B, the symbol $A \backslash B$ denotes the set of all elements of A that are not elements of B. We let $Cl_P(A)$, or just $Cl(A)$ when P is understood by context, denote the closure of a set $A \subseteq P$. $Bd_P(A)$ and $Int_P(A)$ denote the boundary and interior of A in P, and the subscript P is again frequently omitted. If A and B are nonempty *separated* subsets of P (i.e., $Cl(A) \cap B$ and $A \cap Cl(B)$ are empty), we write $A \cup B = A\,|B$.

The cardinality of a collection $\Omega \subseteq C(P)$ is denoted by $|\Omega|$. Throughout this paper the subscripting convention for any collection $\{\Omega_i : i \in I\}$ is that $\Omega_i = \Omega_j$ **if and only if** $i = j$. For finite collections one thus has $|\{\Omega_1, ..., \Omega_w\}| = w$. For a nonvoid $\Omega \subseteq C(P)$ and $A \subseteq P$, we define

$$\Omega^* = \bigcup \Omega, \quad \Omega(A) = \{H \in \Omega : H \cap A \neq \emptyset\}\,, \quad \text{and } \Omega(A)^* = \bigcup \Omega(A).$$

$\emptyset^*$ is defined to be $\emptyset$. The symbols $\subset$ and $\supset$ denote **proper** inclusion and containment, respectively. We say that a subcollection Ω of $C(P))$ *covers B irreducibly* if $\Omega^* = B$ and $\Phi^* \neq B$ for each $\Phi \subset \Omega$. One observes that Ω covers Ω^* irreducibly if and only if each member of Ω contains at least one point of Ω^* that no other member of Ω contains.

Lemma 1.1. *If Ω $(\subseteq C(P))$ covers Ω^* irreducibly and $\Psi, \Omega_1, \Omega_2, \dots, \Omega_w \subseteq \Omega$, then*

$$\begin{aligned} &\Omega_1^* \subseteq \Omega_2^* \Longleftrightarrow \Omega_1 \subseteq \Omega_2 \ , \\ &\Omega_1^* = \Omega_2^* \Longleftrightarrow \Omega_1 = \Omega_2 \ , \text{ and} \\ &\Omega_1^* \cup \ldots \cup \Omega_w^* = \Psi^* \Longleftrightarrow \ \Omega_1 \cup \ldots \cup \Omega_w = \Psi. \end{aligned} \tag{1}$$

Proof. The first line follows from the observation made just preceding the Lemma. The second line follows from two applications of the first, one in each direction of containment. The third line follows from the second and the fact that $\Omega_1^* \cup \ldots \cup \Omega_w^* = (\Omega_1 \cup \ldots \cup \Omega_w)^*$. □

2. Pseudogrilles and $\mathcal{P}$-decompositions

Definition 2.1. $\mathcal{P}$ is a *pseudogrille* for P provided that

- **(a):** $\mathcal{P} \subseteq C(P)$,
- **(b):** $\mathcal{P}$ covers P irreducibly, and
- **(c):** for each $M \in C(P)$, the set $\mathcal{P}(M)^*$ is closed in P(or, equivalently, by (a) and (b), $\mathcal{P}(M)^*$ is a subcontinuum of P containing M).

Example 2.2. $\mathcal{P} = \{\{x\} : x \in P\}$, $\mathcal{P} = \{P\}$, *or $\mathcal{P}$ = any upper semicontinuous decomposition of P into continua is a pseudogrille for P. (One can apply complementation in Theorem 3–32 of* [Ho]*.)*

Example 2.3. *Let $P = [0,1] \times [0,1]$ and, for any fixed positive integer n, let $\mathcal{P}_n = \{[\frac{j-1}{n}, \frac{j}{n}] \times [\frac{k-1}{n}, \frac{k}{n}] : 1 \leq j, k \leq n\}$. In particular, for later comment, we express $\mathcal{P}_4$ as $\{G_1, G_2, ..., G_{16}\}$, where $G_1, G_2, ..., G_{16}$ describes a spiral pattern with $G_1 = [0, 1/4] \times [3/4, 1]$, $G_2 = [0, 1/4] \times [1/2, 3/4]$, and so forth.*

The term *pseudogrille* was chosen because of the similarity such collections bear to the grille decompositions defined in [Mo]. Clearly, every finite irreducible covering of the Hausdorff continuum P by subcontinua is a pseudogrille for P. In the next Proposition 2^P denotes the topological space of all nonempty closed subsets of P with the Vietoris topology (see [Mi]), and $C(P)$ is given the relative topology from 2^P.

Proposition 2.4. *Suppose $\mathcal{P}$ is a closed subspace of $C(P)$, and $\mathcal{P}$ covers P irreducibly. Then $\mathcal{P}$ is a pseudogrille for P.*

Proof. By hypothesis, (a) and (b) of Definition 2.1 hold. To see that (c) also holds, suppose $M \in C(P)$. Then $M \in 2^P$, and $\{A \in 2^P : A \cap M \neq \emptyset\}$ is closed in 2^P by 2.2.2 of [Mi]. Now, $\mathcal{P}$ is closed in $C(P)$ and $C(P)$ is closed in 2^P by 4.13.5 of [Mi]. Thus $\mathcal{P}$ is closed in 2^P. Therefore, the collection $\mathcal{P}(M) = \mathcal{P} \cap \{A \in 2^P : A \cap M \neq \emptyset\}$ is closed in 2^P. Then, by 2.5.2 of [Mi], $\mathcal{P}(M)^*$ is closed in P. Since M was any member of $C(P)$, we conclude that (c) of Definition 2.1 does in fact hold, along with (a) and (b). Thus $\mathcal{P}$ is a pseudogrille for P. □

For the balance of this paper, $\mathcal{P}$ denotes an arbitrary pseudogrille for an arbitrary Hausdorff continuum P. We now give an alternate form of the $*$ notation for subcollections of $\mathcal{P}$.

Definition 2.5. For any $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}^\star$ is defined to be $\mathcal{A}^*$.

Our use of the $\star$ notation in place of the $*$ notation is intended to emphasize when the collection $\mathcal{A}$ *is known to be* a subcollection of $\mathcal{P}$. To clarify matters further, each of the script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... , $\mathcal{Z}$ will be used only to denote a subcollection of $\mathcal{P}$.

Definition 2.6. If $H \in C(P)$ and $H = \mathcal{H}^\star$ for some $\mathcal{H} \subseteq \mathcal{P}$, we say that H is a *$\mathcal{P}$-continuum*. More specifically, if $\mathcal{Z}$ is any subcollection of $\mathcal{P}$, $H \in C(P)$, and $H = \mathcal{H}^\star$ for some $\mathcal{H} \subseteq \mathcal{Z}$, we say that H is a $\mathcal{Z}$-continuum.

Example 2.7. *In Example 2.3 with $\mathcal{P} = \mathcal{P}_4$, the set $G_2 \cup G_3 \cup G_5$ is a $\mathcal{P}$-continuum.*

Some relevant notes follow. (1) For each $\mathcal{P}$-continuum H there is, by Lemma 1.1 and (b) of Definition 2.1, exactly one collection $\mathcal{H} \subseteq \mathcal{P}$ such that $H = \mathcal{H}^\star$. (2) Ordinarily, the form $\mathcal{H}^\star$ will be used in place of H when we refer to a $\mathcal{P}$-continuum. (3) Since $\mathcal{P} \subseteq C(P)$, each $G \in \mathcal{P}$ is a $\mathcal{P}$-continuum equal to $\{G\}^\star$. (4) If $M \in C(P)$ then $\mathcal{P}(M)^\star$ is a $\mathcal{P}$-continuum containing M, by (c) in the definition of a pseudogrille. (5) The letter G is customarily used to denote an element of $\mathcal{P}$.

Definition 2.8. A compactum $Z \subseteq P$ such that $Z = \mathcal{Z}^\star$ for some $\mathcal{Z} \subseteq \mathcal{P}$ is called a *$\mathcal{P}$-compactum*.

Again by Lemma 1.1, $\mathcal{Z}$ is uniquely determined by Z. Note also that each $\mathcal{P}$-continuum is a connected $\mathcal{P}$-compactum, and vice versa.

Lemma 2.9. *If $\mathcal{Z}^\star$ is a $\mathcal{P}$-compactum and $G \in \mathcal{P}$, then $G \subseteq \mathcal{Z}^\star \Longleftrightarrow G \in \mathcal{Z}$.*

PROOF. Apply the first line of Lemma 1.1 with $\Omega = \mathcal{P}$, $\Omega_1 = \{G\}$ and $\Omega_2 = \mathcal{Z}$. $\square$

Lemma 2.10. *If $\mathcal{Z}^\star$ is a $\mathcal{P}$-compactum and C is a component of $\mathcal{Z}^\star$ then C is the $\mathcal{Z}$-continuum $\mathcal{Z}(C)^\star$.*

PROOF. C is a continuum, being a component of a compactum, so it suffices to show that $C = \mathcal{Z}(C)^\star$. First, suppose $p \in \mathcal{Z}(C)^\star$. Then there exists $G \in \mathcal{Z}(C)$ with $p \in G$. Since $G \in \mathcal{Z}(C)$ we have $G \cap C \neq \emptyset$. Therefore, $G \cup C$ is a subcontinuum of $\mathcal{Z}^\star$. Then, since C is a component of $\mathcal{Z}^\star$, we have $G \cup C = C$. Thus $p \in G \subseteq C$. Hence, as p was an arbitrary point of $\mathcal{Z}(C)$, we have $\mathcal{Z}(C)^\star \subseteq C$. For the reverse inclusion, suppose that $p \in C$. Since $C \subseteq \mathcal{Z}^\star$ we have $p \in \mathcal{Z}^\star$. So there exists $G \in \mathcal{Z}$ with $p \in G$. Thus, as $p \in G \cap C$, $G \cap C \neq \emptyset$. Hence $G \in \mathcal{Z}(C)$. Therefore, as $p \in G$ and $G \subseteq \mathcal{Z}(C)^\star$, it follows that $p \in \mathcal{Z}(C)^\star$. This proves that $C \subseteq \mathcal{Z}(C)^\star$. Thus $C = \mathcal{Z}(C)^\star$, as required. $\square$

Of course, the $\mathcal{Z}$-continuum $\mathcal{Z}(C)^\star$ is also a $\mathcal{P}$-continuum, since $\mathcal{Z} \subseteq \mathcal{P}$.

Definition 2.11. A quasichain (or a quasichain of $\mathcal{P}$-continua) is a finite nonvoid collection of $\mathcal{P}$-continua, $\Omega = \{\mathcal{H}_1^\star, ..., \mathcal{H}_w^\star\}$, such that

- **(a):** Ω^* *is connected (or, equivalently, is a $\mathcal{P}$-continuum), and*
- **(b):** Ω *covers Ω^* irreducibly (i.e., $\mathcal{H}_i^\star \nsubseteq \bigcup_{j \neq i} \mathcal{H}_j^\star$ for each i, $1 \leq i \leq w$).*

Example 2.12. *In Example 2.3 with $\mathcal{P} = \mathcal{P}_4$, let $\mathcal{H}_0 = \{G_i : 1 \leq i \leq 8\}$, $\mathcal{H}_1 = \{G_7, G_8, G_9\}$, $\mathcal{H}_2 = \{G_8, G_9, G_{10}\}$, $\mathcal{H}_3 = \{G_8, G_9, G_{10}, G_{11}, G_{12}, G_1, G_2\}$, $\mathcal{H}_4 = \{G_{11}, G_{12}, G_1, G_2\}$, and $\mathcal{H}_5 = \{G_2, G_3, G_4, G_5, G_6, G_8\}$. Then the following are quasichains: $\Omega_1 = \{\mathcal{H}_0^\star, \mathcal{H}_3^\star\}$, $\Omega_2 = \{\mathcal{H}_0^\star, \mathcal{H}_2^\star, \mathcal{H}_4^\star\}$, $\Omega_3 = \{\mathcal{H}_0^\star, \mathcal{H}_1^\star, \mathcal{H}_4^\star\}$, $\Omega_4 = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, \mathcal{H}_4^\star, \mathcal{H}_5^\star\}$, and $\Omega_5 = \{\mathcal{H}_1^\star, \mathcal{H}_3^\star, \mathcal{H}_5^\star\}$. However, $\{\mathcal{H}_0^\star, \mathcal{H}_1^\star, \mathcal{H}_3^\star\}$ is not a quasichain.*

The equivalence mentioned in the first part of Definition 2.11 follows from the equality $(\mathcal{H}_1 \cup \mathcal{H}_2 \cup ... \cup \mathcal{H}_w)^\star = \mathcal{H}_1^\star \cup \mathcal{H}_2^\star \cup ... \cup \mathcal{H}_w^\star$. We note that a quasichain is a kind of *weak chain* or *pseudochain*, as these are defined in [Na] and [Le]. See Lemma 8.13 in [Na]**.**

Definition 2.13. A quasichain $\Omega = \{\mathcal{H}_1^*, ..., \mathcal{H}_w^*\}$ is said to have *order one (or to be of order one)* if $\mathcal{H}_i^* \cap \mathcal{H}_j^* \cap \mathcal{H}_k^* = \emptyset$ whenever $1 \leq i < j < k \leq w$.

Definition 2.14. A $\mathcal{P}$-decomposition of a $\mathcal{P}$-continuum $\mathcal{H}^\star$ is a *quasichain of order one*, $\Upsilon = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$, such that (a) $w \geq 2$, (b) $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup ... \cup \mathcal{H}_w$, and (c) if $G^{'} \in \mathcal{H}_i$ and $G^{''} \in \mathcal{H}_j$, $1 \leq i, j \leq w$, with $G^{'} \cap G^{''} \neq \emptyset$, then either $G^{'} \in \mathcal{H}_j$ or $G^{''} \in \mathcal{H}_i$.

Example 2.15. *In Example 2.12, if we let* $\mathcal{H} = \{G_1, G_2, ..., G_{12}\}$*, then* $\Upsilon_1 = \{\mathcal{H}_0^\star, \mathcal{H}_3^\star\}$, $\Upsilon_2 = \{\mathcal{H}_0^\star, \mathcal{H}_2^\star, \mathcal{H}_4^\star\}$*, and* $\Upsilon_3 = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, \mathcal{H}_4^\star, \mathcal{H}_5^\star\}$ *are* $\mathcal{P}$*-decompositions of the* $\mathcal{P}$*-continuum* $\mathcal{H}^\star$*. But* $\{\mathcal{H}_0^\star, \mathcal{H}_1^\star, \mathcal{H}_4^\star\}$ *is not a* $\mathcal{P}$*-decomposition of* $\mathcal{H}^\star$ *(since* $G_9 \in \mathcal{H}_1 \backslash \mathcal{H}_4$ *and* $G_{11} \in \mathcal{H}_4 \backslash \mathcal{H}_1$ *and* $G_9 \cap G_{11} \neq \emptyset$*), and neither is* $\{\mathcal{H}_1^\star, \mathcal{H}_3^\star, \mathcal{H}_5^\star\}$*, since this quasichain does not have order one.*

Example 2.16. *Let* M *and* N *be proper subcontinua of a continuum* P*. Let* $\mathcal{P} = \{\{x\} : x \in P\}$, $\mathcal{H}_1 = \{\{x\} : x \in M\}$*, and* $\mathcal{H}_2 = \{\{x\} : x \in N\}$*. Then* $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\} = \{M, N\}$ *is a* $\mathcal{P}$*-decomposition of* P*.*

The condition that a $\mathcal{P}$-decomposition have order one implies that a metrizable continuum P cannot have $\mathcal{P}$-decompositions of arbitrarily small mesh unless P is one-dimensional and locally connected. This fact, which we will not use, follows from Theorem 1.6.12 in [En].

Lemma 2.17. *Condition (b) in Definition 2.14 is equivalent to the condition (b′)* $\Upsilon^* = \mathcal{H}^\star$*.*

PROOF. If $\Upsilon^* = \mathcal{H}^\star$ then $\Upsilon^* = \mathcal{H}^\star = \mathcal{H}_1^\star \cup \mathcal{H}_2^\star \cup ... \cup \mathcal{H}_w^\star = (\mathcal{H}_1 \cup \mathcal{H}_2 \cup ... \cup \mathcal{H}_w)^\star$, and we can apply the last part of Lemma 1.1 to conclude that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup ... \cup \mathcal{H}_w$. These equations also guarantee the other half of the equivalence. □

Lemma 2.18. *Let* $\Upsilon = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$ *be a* $\mathcal{P}$*-decomposition of the* $\mathcal{P}$*-continuum* $\mathcal{H}^\star$*. Then* $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star = (\mathcal{H}_i \cap \mathcal{H}_j)^\star$ *whenever* $1 \leq i, j \leq w$*.*

PROOF. The inclusion $(\mathcal{H}_i \cap \mathcal{H}_j)^\star \subseteq \mathcal{H}_i^\star \cap \mathcal{H}_j^\star$ follows from $\mathcal{H}_i \cap \mathcal{H}_j \subseteq \mathcal{H}_i$ and $\mathcal{H}_i \cap \mathcal{H}_j \subseteq \mathcal{H}_j$. For the reverse inclusion, suppose $x \in \mathcal{H}_i^\star \cap \mathcal{H}_j^\star$. Then there exist $G^{'} \in \mathcal{H}_i$ and $G^{''} \in \mathcal{H}_j$ so that $x \in G^{'} \cap G^{''}$. Thus $G^{'} \cap G^{''} \neq \emptyset$, and by condition (c) in Definition 2.14 we have either $G^{'} \in \mathcal{H}_j$ or $G^{''} \in \mathcal{H}_i$. In the first case, $x \in G^{'} \in \mathcal{H}_i \cap \mathcal{H}_j$ and hence $x \in (\mathcal{H}_i \cap \mathcal{H}_j)^\star$, while in the second case we have $x \in G^{''} \in \mathcal{H}_i \cap \mathcal{H}_j$, and so $x \in (\mathcal{H}_i \cap \mathcal{H}_j)^\star$ again. Thus $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star \subseteq (\mathcal{H}_i \cap \mathcal{H}_j)^\star$. Hence $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star = (\mathcal{H}_i \cap \mathcal{H}_j)^\star$. □

Lemma 2.19. *Suppose* $\Upsilon = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$ *is a* $\mathcal{P}$*-decomposition of the* $\mathcal{P}$*-continuum* $\mathcal{H}^\star$*. Let* $W_1, \ldots, W_m$ *be* $m \geq 2$ *disjoint nonempty sets whose union is* $\{1, 2, \ldots, w\}$*. For* $1 \leq i \leq m$ *let* $\mathcal{H}_{W_i} = \bigcup_{r \in W_i} \mathcal{H}_r$*, and assume that every* $\mathcal{H}_{W_i}^\star$ *is connected. Then the collection* $\Psi = \{\mathcal{H}_{W_1}^\star, \mathcal{H}_{W_2}^\star, \ldots, \mathcal{H}_{W_m}^\star\}$ *is also a* $\mathcal{P}$*-decomposition of* $\mathcal{H}^\star$*.*

PROOF. $\mathcal{H}^\star_{W_1}, \ldots, \mathcal{H}^\star_{W_m}$ are $\mathcal{P}$-continua whose union is $\mathcal{H}^\star_1 \cup \mathcal{H}^\star_2 \cup ... \cup \mathcal{H}^\star_w = \Upsilon^* = \mathcal{H}^\star$. Let $M = \{1, 2, \ldots, m\}$ and $W = \{1, 2, \ldots, w\}$. Note that for each set $M' \subseteq M$ we have

$$\bigcup_{i \in M'} \mathcal{H}^\star_{W_i} = \bigcup_{i \in M'} \bigcup_{r \in W_i} \mathcal{H}^\star_r = \bigcup \{\mathcal{H}^\star_r : r \in \bigcup_{i \in M'} W_i\}.$$

Therefore, as W is the disjoint union of the nonempty sets $W_1, \ldots, W_m$ and as Υ covers $\mathcal{H}^\star$ irreducibly (Υ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$), for sets $M' \subseteq M$ one has

$$M' = M \Longleftrightarrow \bigcup_{i \in M'} W_i = W \Longleftrightarrow \bigcup_{i \in M'} \mathcal{H}^\star_{W_i} = \mathcal{H}^\star.$$

Thus Ψ also covers $\mathcal{H}^\star$ irreducibly. So Ψ is a quasichain, and $|\Psi| = m \geq 2$. Now suppose that

$$\mathcal{H}^\star_{W_i} \cap \mathcal{H}^\star_{W_j} \cap \mathcal{H}^\star_{W_k} \neq \emptyset \text{ for some } i,\, j,\, k \in M.$$

Then there exist $r_i \in W_i$, $r_j \in W_j$, and $r_k \in W_k$ with $\mathcal{H}_{r_i} \subseteq \mathcal{H}_{W_i}, \mathcal{H}_{r_j} \subseteq \mathcal{H}_{W_j}, \mathcal{H}_{r_k} \subseteq \mathcal{H}_{W_k}$, and

$$\mathcal{H}^\star_{r_i} \cap \mathcal{H}^\star_{r_j} \cap \mathcal{H}^\star_{r_k} \neq \emptyset.$$

Since Υ has order one (Υ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$), $|\{r_i\ , r_j\ , r_k\}| \leq 2$. Without loss of generality we can assume $r_i = r_j$. Hence, $W_i\ \cap W_j \neq \emptyset$. Then, since the sets $W_1, \ldots, W_m$ are disjoint, $i = j$. Thus, the collection Ψ also has order one. Now, to conclude that Ψ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ it only remains to establish that property (c) of Definition 2.14 holds for Ψ. So, suppose that $G^{'} \in \mathcal{H}_{W_i}$ and $G^{''} \in \mathcal{H}_{W_j}$ with $G^{'} \cap G^{''} \neq \emptyset$. We must show that either $G^{'} \in \mathcal{H}_{W_j}$ or $G^{''} \in \mathcal{H}_{W_i}$. There exist $r \in W_i$ and $s \in W_j$ such that $G' \in \mathcal{H}_r \subseteq \mathcal{H}_{W_i}$ and $G^{''} \in \mathcal{H}_s \subseteq \mathcal{H}_{W_j}$. Then, since Υ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$, it follows that either $G^{'} \in \mathcal{H}_s$ or $G^{''} \in \mathcal{H}_r$. Hence, either $G^{'} \in \mathcal{H}_{W_j}$ or $G^{''} \in \mathcal{H}_{W_i}$, as required. This completes the proof. $\square$

3. $\mathcal{P}$-UNICOHERENCE AND CHAINS

Definition 3.1. A $\mathcal{P}$-continuum $\mathcal{H}^\star$ is said to be $\mathcal{P}$-*unicoherent* if $\mathcal{H}^\star_1 \cap \mathcal{H}^\star_2$ is connected whenever $\{\mathcal{H}^\star_1, \mathcal{H}^\star_2\}$ is a two-element $\mathcal{P}$-decomposition of $\mathcal{H}^\star$.

Considering Examples 2.12 and 2.15, we find that the $\mathcal{P}$-continuum $\mathcal{H}^\star$ is not $\mathcal{P}$-unicoherent since, e. g., $\Upsilon_1 = \{\mathcal{H}^\star_0, \mathcal{H}^\star_3\}$ is a two-element $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ and $\mathcal{H}^\star_0 \cap \mathcal{H}^\star_3 = (G_1 \cup G_2) \mid G_8$.

Proposition 3.2. *If $\mathcal{P}$ is a pseudogrille for P and $\mathcal{H}^\star$ is a unicoherent $\mathcal{P}$-continuum, then $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent.*

Proof. Let $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\}$ be any two-element $\mathcal{P}$-decomposition of $\mathcal{H}^\star$. Then $\mathcal{H}_1^\star$ and $\mathcal{H}_2^\star$ are $\mathcal{P}$-continua, and hence are continua, whose union is $\mathcal{H}^\star$. Thus, as $\mathcal{H}^\star$ is unicoherent, $\mathcal{H}_1^\star \cap \mathcal{H}_2^\star$ is connected □

Proposition 3.3. *Let H be a subcontinuum of P and $\mathcal{P} = \{\{x\} : x \in P\}$. Then H is unicoherent if and only if H is $\mathcal{P}$-unicoherent.*

Proof. For the sufficiency of the condition apply Proposition 3.2. For the necessity, suppose H is $\mathcal{P}$-unicoherent and $H = M \cup N$, where M and N are subcontinua of P. If $M = H$ or $N = H$ then $M \cap N = H$ is indeed connected. On the other hand, if M and N are proper subcontinua of H let $\mathcal{H}_1 = \{\{x\} : x \in M\}$ and $\mathcal{H}_2 = \{\{x\} : x \in N\}$. Then $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\} = \{M, N\}$ is a two-element $\mathcal{P}$-decomposition of H, so that $\mathcal{H}_1^\star \cap \mathcal{H}_2^\star = M \cap N$ is connected by the $\mathcal{P}$-unicoherence of H. □

Proposition 3.4. *A $\mathcal{P}$-continuum $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent if and only if $\mathcal{H}_1^\star \cap \mathcal{H}_2^\star$ is a $\mathcal{P}$-continuum equal to $(\mathcal{H}_1 \cap \mathcal{H}_2)^\star$ whenever $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\}$ is a two-element $\mathcal{P}$-decomposition of $\mathcal{H}^\star$.*

Proof. Apply Lemma 2.18 and the definition of $\mathcal{P}$-unicoherence. □

Types of chains defined in [Bi], [Le] and [Di] are now introduced. Use is also made of the terminology introduced in the last part of Definition 2.11.

Definition 3.5. A circular chain (or a circular chain of $\mathcal{P}$-continua) is a quasichain Θ which can be expressed as $\Theta = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$, with $w \geq 3$ and with $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star \neq \emptyset \iff |i - j| \leq 1$ or $\{i, j\} = \{1, w\}$.

Definition 3.6. A treechain (or a treechain of $\mathcal{P}$-continua) is a quasichain that contains no circular chain.

Example 3.7. *With respect to Example 2.15, Υ_1 is a treechain, while Υ_2 and Υ_3 are not.*

Lemma 3.8. *Let Υ be a treechain of $\mathcal{P}$-continua. Then there does not exist a finite sequence $H_1, H_2, H_3, \ldots, H_m$, of distinct elements of Υ such that $m \geq 3$, $H_1 \cap H_m \neq \emptyset$, and $H_i \cap H_{i+1} \neq \emptyset$ for $1 \leq i < m$.*

PROOF. Suppose that such a finite sequence does exist. Then there exist $i,\, j \in \{1, 2, 3, \ldots, m\}$ such that (i) $j - i \geq 2$, and (ii) $H_i \cap H_j \neq \emptyset$. (We could choose $i = 1$ and $j = m$.) Hence there exist $i,\, j \in \{1, 2, 3, \ldots, m\}$ satisfying (i) and (ii) where $j - i$ is minimal. For this i and j, let $\Theta = \{H_i, H_{i+1}, H_{i+2}, \ldots, H_j\}$. Then, by the minimality of the difference $j - i$, for all $r, s \in \{i,\, i+1,\, i+2, \ldots, j\}$ we have

$$H_r \cap H_s \neq \emptyset \Longleftrightarrow |r - s| \leq 1 \text{ or } \{r, s\} = \{i,\, j\}.$$

Hence Θ is a circular chain. But, as $\Theta \subseteq \Upsilon$, this contradicts the fact that Υ is a treechain. $\square$

The next two Lemmas are key to proving Lemma 6.1, which shows that the nerves of the coverings ΣJ and ΣK, described in the Introduction and in Theorem 6.3, are trees.

Lemma 3.9. *A $\mathcal{P}$-continuum $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent if and only if every $\mathcal{P}$-decomposition $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$ of $\mathcal{H}^\star$ is a treechain where, for $1 \leq i, j \leq w$, $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star$ is either empty or a $\mathcal{P}$-continuum equal to $(\mathcal{H}_i \cap \mathcal{H}_j)^\star$.*

PROOF. Let the given condition hold, and suppose that $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\}$ is any two-element $\mathcal{P}$-decomposition of $\mathcal{H}^\star$. Since $\mathcal{H}_1^\star$ and $\mathcal{H}_2^\star$ are continua whose union is the continuum $\mathcal{H}^\star$, the set $\mathcal{H}_1^\star \cap \mathcal{H}_2^\star$ is nonempty, and hence is a $\mathcal{P}$-continuum by the given condition. Thus $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent.

For the converse, let $\mathcal{H}^\star$ be a $\mathcal{P}$-unicoherent $\mathcal{P}$-continuum. The argument is by induction on w, the cardinality of the $\mathcal{P}$-decomposition $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star\}$. If $w = 2$ then $\{\mathcal{H}_1^\star, \mathcal{H}_2^\star\}$ is indeed a treechain and $\mathcal{H}_1^\star \cap \mathcal{H}_2^\star$ is a $\mathcal{P}$-continuum equal to $(\mathcal{H}_1 \cap \mathcal{H}_2)^\star$ by Proposition 3.4. So assume the conclusions hold for each $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ of cardinality at most w, and let $\Upsilon = \{\mathcal{H}_1^\star, \mathcal{H}_2^\star, ..., \mathcal{H}_w^\star, \mathcal{H}_{w+1}^\star\}$ be a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ of cardinality $w + 1 \geq 3$. By Lemma 2.18 we have $\mathcal{H}_i^\star \cap \mathcal{H}_j^\star = (\mathcal{H}_i \cap \mathcal{H}_j)^\star$ for $1 \leq i,\, j \leq w + 1$.

Now, suppose there existed $m,\, n \in \{1, \ldots, w, w+1\}$ where $\mathcal{H}_m^\star \cap \mathcal{H}_n^\star$ is not connected. Then $\mathcal{H}_m^\star \cup \mathcal{H}_n^\star$ is connected. Also, since $w + 1 \geq 3$ and Υ covers $\Upsilon^* = \mathcal{H}^\star$ irreducibly, $\mathcal{H}_m^\star \cup \mathcal{H}_n^\star$ is a proper subcontinuum of the continuum $\mathcal{H}^\star = \bigcup_{1 \leq j \leq w+1} \mathcal{H}_j^\star$. Then there exists $k \in \{1, \ldots, w, w+1\} \backslash \{m, n\}$ such that $\mathcal{H}_k^\star \cap (\mathcal{H}_m^\star \cup \mathcal{H}_n^\star) \neq \emptyset$. Furthermore, we can assume $\mathcal{H}_k^\star \cap \mathcal{H}_n^\star \neq \emptyset,\ k = w,\ n = w + 1$, and $m = 1$. Thus,

$$\mathcal{H}_w^\star \cap \mathcal{H}_{w+1}^\star \neq \emptyset, \text{ and } \mathcal{H}_1^\star \cap \mathcal{H}_{w+1}^\star \text{ is not connected.}$$

We have $\mathcal{H}_1^\star \cap \mathcal{H}_{w+1}^\star = A\,|B$, where sets A and B are compacta. For each $i \in \{1,\dots,w-1\}$ let $W_i = \{i\}$, and let $W_w = \{w, w+1\}$. Define $\mathcal{H}_{W_i} = \bigcup_{r\in W_i}\mathcal{H}_r$ for $1 \le i \le w$. Then, by Lemma 2.19, $\{\mathcal{H}_{W_1}^\star, \mathcal{H}_{W_2}^\star, ..., \mathcal{H}_{W_w}^\star\}$ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ of cardinality w. By the induction hypothesis, every $\mathcal{H}_{W_i}^\star \cap \mathcal{H}_{W_j}^\star$ is connected. Thus

$$\mathcal{H}_{W_1}^\star \cap \mathcal{H}_{W_w}^\star \text{ is connected.}$$

Also,

$$\mathcal{H}_{W_1}^\star \cap \mathcal{H}_{W_w}^\star = \mathcal{H}_1^\star \cap (\mathcal{H}_w^\star \cup \mathcal{H}_{w+1}^\star) = (\mathcal{H}_1^\star \cap \mathcal{H}_w^\star) \cup (\mathcal{H}_1^\star \cap \mathcal{H}_{w+1}^\star). \tag{2}$$

Since Υ (being a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$) has order one, the closed sets $(\mathcal{H}_1^\star \cap \mathcal{H}_w^\star)$ and $(\mathcal{H}_1^\star \cap \mathcal{H}_{w+1}^\star)$ are disjoint. Therefore, as $\mathcal{H}_1^\star \cap \mathcal{H}_{w+1}^\star = A\,|B$, the closed sets $\mathcal{H}_1^\star \cap \mathcal{H}_w^\star$ and B are disjoint. Moreover, by (2), $\mathcal{H}_{W_1}^\star \cap \mathcal{H}_{W_w}^\star = ((\mathcal{H}_1^\star \cap \mathcal{H}_w^\star) \cup A)\,|B$. But this contradicts the connectedness of $\mathcal{H}_{W_1}^\star \cap \mathcal{H}_{W_w}^\star$. The contradiction shows that if $m, n \in \{1,\dots,w,w+1\}$ then $\mathcal{H}_m^\star \cap \mathcal{H}_n^\star$ is connected.

Finally, suppose that Υ is not a treechain. Then there is a circular chain $\Theta \subseteq \Upsilon$. Since $|\Theta| \ge 3$, one can assume $\Theta = \{\mathcal{H}_m^\star, \mathcal{H}_{m+1}^\star, \mathcal{H}_{m+2}^\star, ..., \mathcal{H}_{w+1}^\star\}$ for some $m \in \{1,\dots,w-1\}$. One can further assume (see Definition 3.5) that for $m \le j,\, k \le w+1$ we have $\mathcal{H}_j^\star \cap \mathcal{H}_k^\star \ne \emptyset \Leftrightarrow |j-k| \le 1$ or $\{j,k\} = \{m, w+1\}$. Hence,

$$\Theta(\mathcal{H}_m^\star) = \{\mathcal{H}_m^\star, \mathcal{H}_{m+1}^\star, \mathcal{H}_{w+1}^\star\}. \tag{3}$$

For each $i \in \{1,\dots,m\}$ let $W_i = \{i\}$. Let $W_{m+1} = \{m+1, m+2, \dots, w+1\}$ and define $\mathcal{H}_{W_i} = \bigcup_{r\in W_i}\mathcal{H}_r$ for $1 \le i \le m+1$. $\mathcal{H}_{W_{m+1}}^\star$ is connected since $\mathcal{H}_j^\star \cap \mathcal{H}_{j+1}^\star \ne \emptyset$ for $m \le j \le w$. By Lemma 2.19, $\{\mathcal{H}_{W_1}^\star, ..., \mathcal{H}_{W_m}^\star, \mathcal{H}_{W_{m+1}}^\star\}$ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ of cardinality $m+1 \le w$. Hence, the induction hypothesis gives that every intersection $\mathcal{H}_{W_i}^\star \cap \mathcal{H}_{W_j}^\star$ is connected. Thus

$$\mathcal{H}_{W_m}^\star \cap \mathcal{H}_{W_{m+1}}^\star \text{ is connected.}$$

Also, by (3),

$$\mathcal{H}_{W_m}^\star \cap \mathcal{H}_{W_{m+1}}^\star = \mathcal{H}_m^\star \cap \bigcup_{m+1\le r\le w+1}\mathcal{H}_r^\star = (\mathcal{H}_m^\star \cap \mathcal{H}_{m+1}^\star) \cup (\mathcal{H}_m^\star \cap \mathcal{H}_{w+1}^\star). \tag{4}$$

Since Υ has order one, we have $(\mathcal{H}_m^\star \cap \mathcal{H}_{m+1}^\star) \cap (\mathcal{H}_m^\star \cap \mathcal{H}_{w+1}^\star) = \emptyset$. Then, by (4) and (3), $\mathcal{H}_{W_m}^\star \cap \mathcal{H}_{W_{m+1}}^\star = (\mathcal{H}_m^\star \cap \mathcal{H}_{m+1}^\star)\,|\,(\mathcal{H}_m^\star \cap \mathcal{H}_{w+1}^\star)$. This contradicts the connectedness of $\mathcal{H}_{W_m}^\star \cap \mathcal{H}_{W_{m+1}}^\star$. The contradiction shows Υ to be a treechain. $\square$

Lemma 3.10. *Let $\mathcal{H}^\star$ be a $\mathcal{P}$-continuum and $\{\mathcal{P}_{-1}^\star, \mathcal{P}_1^\star\}$ be a $\mathcal{P}$-decomposition of P with $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star$ and $\mathcal{H}^\star \nsubseteq \mathcal{P}_{-1}^\star$. Let $\mathcal{H}_i = \mathcal{H} \cap \mathcal{P}_i$ for each $i = -1,\, 1$. Assume* *that $\mathcal{H}^\star_{-1}$ is a compactum with finitely many components, $\mathcal{H}^\star_{-1,1},\dots,\mathcal{H}^\star_{-1,m}$, and that $\mathcal{H}^\star_1$ is a compactum with finitely many components, $\mathcal{H}^\star_{1,1},\dots,\mathcal{H}^\star_{1,n}$. Then $\Upsilon=\Upsilon_{-1}\cup\Upsilon_1$ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$, where*

$$\Upsilon_{-1}=\{\mathcal{H}^\star_{-1,j}:1\le j\le m \text{ and } \mathcal{H}^\star_{-1,j}\nsubseteq\mathcal{H}^\star_{1,k}\ \text{ for } 1\le k\le n\}$$
$$\Upsilon_{1}=\{\mathcal{H}^\star_{1,k}:1\le k\le n \text{ and } \mathcal{H}^\star_{1,k}\nsubseteq\mathcal{H}^\star_{-1,j}\ \text{ for } 1\le j\le m\}.$$

Proof. Since $P=\mathcal{P}^\star$, (b) of Definition 2.14 gives $\mathcal{P}_{-1}\cup\mathcal{P}_1=\mathcal{P}$. Thus,

$$\begin{aligned}\mathcal{H}^\star_{-1}\cup\mathcal{H}^\star_1&=(\mathcal{H}_{-1}\cup\mathcal{H}_1)^\star=((\mathcal{H}\cap\mathcal{P}_{-1})\cup(\mathcal{H}\cap\mathcal{P}_1))^\star=\\ &(\mathcal{H}\cap(\mathcal{P}_{-1}\cup\mathcal{P}_1))^\star=(\mathcal{H}\cap\mathcal{P})^\star=\mathcal{H}^\star,\ \text{ a } \mathcal{P}\text{-continuum.}\end{aligned}\tag{5}$$

By Lemma 2.10, each $\mathcal{H}^\star_{-1,j}$ is an $\mathcal{H}_{-1}$-continuum and each $\mathcal{H}^\star_{1,k}$ is an $\mathcal{H}_1$-continuum. We claim that

$$\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1=(\mathcal{H}_{-1}\cap\mathcal{H}_1)^\star \text{ is a } \mathcal{P}\text{-compactum.}\tag{6}$$

For, as $\mathcal{H}^\star_{-1}$ and $\mathcal{H}^\star_1$ are $\mathcal{P}$-compacta whose union is a $\mathcal{P}$-continuum, $\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1$ is nonempty and compact. Thus $\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1$ is a compactum. Since $\mathcal{H}_{-1}\cap\mathcal{H}_1\subseteq\mathcal{H}_{-1}$,$\mathcal{H}_1$ we have $(\mathcal{H}_{-1}\cap\mathcal{H}_1)^\star\subseteq\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1$. For the reverse inclusion, suppose that $p\in\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1$. Choose $G'\in\mathcal{H}_{-1}$ and $G''\in\mathcal{H}_1$ with $p\in G'\cap G''$. Then since $G'\cap G''\neq\emptyset$, $G'\in\mathcal{H}_{-1}\subseteq\mathcal{P}_{-1}$, $G''\in\mathcal{H}_1\subseteq\mathcal{P}_1$, and $\{\mathcal{P}^\star_{-1},\mathcal{P}^\star_1\}$ is a $\mathcal{P}$-decomposition of P, one has either $G'\in\mathcal{P}_1$ or $G''\in\mathcal{P}_{-1}$ (by (c) in Definition 2.14). Thus, either $G'\in\mathcal{H}_{-1}\cap\mathcal{P}_1=(\mathcal{P}_{-1}\cap\mathcal{H})\cap\mathcal{P}_1=\mathcal{H}_{-1}\cap\mathcal{H}_1$ or else $G''\in\mathcal{H}_1\cap\mathcal{P}_{-1}=(\mathcal{P}_1\cap\mathcal{H})\cap\mathcal{P}_{-1}=\mathcal{H}_1\cap\mathcal{H}_{-1}$. Consequently, either $G'\subseteq(\mathcal{H}_{-1}\cap\mathcal{H}_1)^\star$ or $G''\subseteq(\mathcal{H}_1\cap\mathcal{H}_{-1})^\star$. In either case, since $p\in G'\cap G''$, we have $p\in(\mathcal{H}_1\cap\mathcal{H}_{-1})^\star$. Thus $\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1\subseteq(\mathcal{H}_1\cap\mathcal{H}_{-1})^\star$. Therefore, $\mathcal{H}^\star_{-1}\cap\mathcal{H}^\star_1=(\mathcal{H}_{-1}\cap\mathcal{H}_1)^\star$, and (6) holds. The following will also be proved.

$$\text{If } 1\le j\le m,\ 1\le k\le n, \text{ and } \mathcal{H}^\star_{-1,j}\subseteq\mathcal{H}^\star_{1,k},\ \text{then } \mathcal{H}^\star_{1,k}\in\Upsilon_1.\tag{7}$$

For otherwise there exists l, $1\le l\le m$, with $\mathcal{H}^\star_{-1,j}\subseteq\mathcal{H}^\star_{1,k}\subseteq\mathcal{H}^\star_{-1,l}$. Then the components $\mathcal{H}^\star_{-1,j}$ and $\mathcal{H}^\star_{-1,l}$ of $\mathcal{H}^\star_{-1}$ are equal to each other and to $\mathcal{H}^\star_{1,k}$. Furthermore, as $\mathcal{H}^\star=\mathcal{H}^\star_{-1}\cup\mathcal{H}^\star_1$ by (5), we have

$$\mathcal{H}^\star=\mathcal{H}^\star_{1,k}\cup\bigcup_{k'\neq k}\mathcal{H}^\star_{1,k'}\cup\bigcup_{j'\neq j}\mathcal{H}^\star_{-1,j'}.$$

Also, $\mathcal{H}^\star_{1,k'}\cap\mathcal{H}^\star_{1,k}=\emptyset$ whenever $k'\neq k$, and $\mathcal{H}^\star_{-1,j'}\cap\mathcal{H}^\star_{1,k}=\mathcal{H}^\star_{-1,j'}\cap\mathcal{H}^\star_{-1,j}=\emptyset$ whenever $j'\neq j$. Then, since $\mathcal{H}^\star_{-1,j}=\mathcal{H}^\star_{1,k}$, $\mathcal{H}^\star_{1,k}$ and $\bigcup_{k'\neq k}\mathcal{H}^\star_{1,k'}\cup\bigcup_{j'\neq j}\mathcal{H}^\star_{-1,j'}$ are disjoint compact sets whose union is the continuum $\mathcal{H}^\star$. This implies that $\bigcup_{k'\neq k}\mathcal{H}^\star_{1,k'}\cup\bigcup_{j'\neq j}\mathcal{H}^\star_{-1,j'}=\emptyset$, and hence that $\mathcal{H}^\star=\mathcal{H}^\star_{1,k}=\mathcal{H}^\star_1$. Thus, by Lemma 1.1, $\mathcal{H}=\mathcal{H}_1=\mathcal{P}_1\cap\mathcal{H}$. Hence $\mathcal{H}\subseteq\mathcal{P}_1$ and $\mathcal{H}^\star\subseteq\mathcal{P}^\star_1$. But this contradicts the hypothesis $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star$. We conclude that (7) must hold. Analogous to (7), one has

(8) If $1 \le j \le m$, $1 \le k \le n$, and $\mathcal{H}_{1,k}^\star \subseteq \mathcal{H}_{-1,j}^\star$, then $\mathcal{H}_{-1,j}^\star \in \Upsilon_{-1}$.

From (7) and (8) there follows $\Upsilon_{-1}^* \cup \Upsilon_1^* = \bigcup_{1\le j\le m} \mathcal{H}_{-1,j}^\star \cup \bigcup_{1\le k\le n} \mathcal{H}_{1,k}^\star$. Thus, as $\Upsilon = \Upsilon_{-1} \cup \Upsilon_1$,

$$\Upsilon^* = \Upsilon_{-1}^* \cup \Upsilon_1^* = \bigcup_{1\le j\le m} \mathcal{H}_{-1,j}^\star \cup \bigcup_{1\le k\le n} \mathcal{H}_{1,k}^\star = \mathcal{H}_{-1}^\star \cup \mathcal{H}_1^\star = \mathcal{H}^\star.$$

That is, Υ covers $\mathcal{H}^\star$. Moreover, we claim that

$$\Upsilon \text{ covers } \mathcal{H}^\star \text{ irreducibly.}$$

For if, say, $\mathcal{H}_{1,k}^\star \in \Upsilon_1$, then for all j, $1 \le j \le m$, we have $\mathcal{H}_{1,k}^\star \nsubseteq \mathcal{H}_{-1,j}^\star$. Then $\mathcal{H}_{1,k}^\star \nsubseteq \Upsilon_{-1}^*$, since Υ_{-1}^* is a disjoint union of some of the finitely many closed components $\mathcal{H}_{-1,j}^\star$. Thus, as it is also true that $\mathcal{H}_{1,k}^\star$ intersects no member of Υ_1 apart from itself, we have $\mathcal{H}_{1,k}^\star \nsubseteq (\Upsilon_1 \backslash \{\mathcal{H}_{1,k}^\star\})^* \cup \Upsilon_{-1}^* = (\Upsilon \backslash \{\mathcal{H}_{1,k}^\star\})^*$. By a symmetric argument, if $\mathcal{H}_{-1,,j}^\star \in \Upsilon_{-1}$ then $\mathcal{H}_{-1,j}^\star \nsubseteq (\Upsilon \backslash \{\mathcal{H}_{-1,j}^\star\})^*$. We have thus shown that Υ covers $\mathcal{H}^\star$ ($= \Upsilon^*$) irreducibly. Hence Υ is a quasichain. Also, as $\Upsilon = \Upsilon_{-1} \cup \Upsilon_1$ and each Υ_i is a collection of disjoint sets, Υ has order one. Furthermore, $|\Upsilon| > 1$, as otherwise $\Upsilon = \Upsilon_i$ for some $i \in \{-1, 1\}$ and then $\mathcal{H}^\star = \Upsilon^* = \Upsilon_i^* \subseteq \mathcal{H}_i^\star \subseteq \mathcal{P}_i^\star$, again contradicting the hypothesis $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star, \mathcal{P}_{-1}^\star$. Thus, (a) of Definition 2.14 holds. Since $\mathcal{H}^\star = \Upsilon^*$, Lemma 2.17 shows that (b) of Definition 2.14 holds. To show that Υ is a $\mathcal{P}$-decomposition of $\mathcal{H}^\star$ it only remains to show that (c) of Definition 2.14 holds. So suppose there exist sets $\mathcal{M}^\star, \mathcal{N}^\star \in \Upsilon = \Upsilon_{-1} \cup \Upsilon_1$ and $G' \in \mathcal{M}$ and $G'' \in \mathcal{N}$ with $G' \cap G'' \neq \emptyset$. Notice that $\mathcal{M}^\star, \mathcal{N}^\star$, G' and G'' are all continua. We need to prove that

$$G' \in \mathcal{N} \text{ or } G'' \in \mathcal{M}. \tag{9}$$

Now if $\mathcal{M}^\star = \mathcal{N}^\star$ then $\mathcal{M} = \mathcal{N}$ (by Lemma 1.1) and $G' \in \mathcal{M} = \mathcal{N}$, so (9) holds. On the other hand, suppose $\mathcal{M}^\star \neq \mathcal{N}^\star$. Then since $\mathcal{M}^\star, \mathcal{N}^\star \in \Upsilon_{-1} \cup \Upsilon_1$ and $\emptyset \neq G' \cap G'' \subseteq \mathcal{M}^\star \cap \mathcal{N}^\star$, and since each Υ_i is a collection of disjoint sets, there exist j, k so that $\{\mathcal{M}^\star, \mathcal{N}^\star\} = \{\mathcal{H}_{-1,j}^\star, \mathcal{H}_{1,k}^\star\}$. Without loss of generality we can assume

$$\mathcal{M}^\star = \mathcal{H}_{-1,j}^\star \text{ and } \mathcal{N}^\star = \mathcal{H}_{1,k}^\star. \tag{10}$$

Then, by Lemma 1.1,

$$G' \in \mathcal{M} = \mathcal{H}_{-1,j} \subseteq \mathcal{H}_{-1} \subseteq \mathcal{P}_{-1} \text{ and } G'' \in \mathcal{N} = \mathcal{H}_{1,k} \subseteq \mathcal{H}_1 \subseteq \mathcal{P}_1. \tag{11}$$

Moreover, as $G' \cap G'' \neq \emptyset$ and (by hypothesis) $\{\mathcal{P}_{-1}^{\star}, \mathcal{P}_1^{\star}\}$ is a $\mathcal{P}$-decomposition of P, it follows from (c) of Definition 2.14 for this $\mathcal{P}$-decomposition that either $G' \in \mathcal{P}_1$ or $G'' \in \mathcal{P}_{-1}$. Suppose $G' \in \mathcal{P}_1$. Then $G' \in \mathcal{H}_{-1} \cap \mathcal{P}_1 \subseteq \mathcal{H} \cap \mathcal{P}_1 = \mathcal{H}_1$. Hence $G' \subseteq \mathcal{H}_1^{\star}$. Since G' is connected, there is one component $\mathcal{H}_{1,k'}^{\star}$ of $\mathcal{H}_1^{\star}$ with $G' \subseteq \mathcal{H}_{1,k'}^{\star}$. Thus, as $G'' \subseteq \mathcal{H}_{1,k}^{\star}$ (by (11)), we have $\emptyset \neq G' \cap G'' \subseteq \mathcal{H}_{1,k'}^{\star} \cap \mathcal{H}_{1,k}^{\star}$. Thus the components $\mathcal{H}_{1,k'}^{\star}$ and $\mathcal{H}_{1,k}^{\star}$ are identical, and hence $G' \subseteq \mathcal{H}_{1,k}^{\star}$. Then, by Lemma 2.9 and (11), $G' \in \mathcal{H}_{1,k} = \mathcal{N}$, and so (9) holds if $G' \in \mathcal{P}_1$. If $G'' \in \mathcal{P}_{-1}$ a dual argument gives that $G'' \in \mathcal{M}$, and hence (9) holds again. This completes the proof that (c) of Definition 2.14 holds for Υ, so that, as noted above, Υ is a $\mathcal{P}$-decomposition of $\mathcal{H}^{\star}$. □

4. An Approximation Technique

In this section it is shown that the unicoherence of a continuum P is equivalent to the $\mathcal{P}$-unicoherence of P for arbitrarily fine pseudogrilles $\mathcal{P}$.

Proposition 4.1. *Let a Hausdorff continuum P be the union of proper subcontinua Y and Z where $Y \cap Z$ is not connected. Then there are open neighborhoods O_Y of Y and O_Z of Z such that $Y \nsubseteq O_Z$, $Z \nsubseteq O_Y$, and $R \cap S$ is not connected for all sets R and S satisfying $Y \subseteq R \subseteq O_Y$ and $Z \subseteq S \subseteq O_Z$.*

Proof. We have $Y \cap Z = E\,|F$. Here, E and F are disjoint compacta whose union is $Y \cap Z$. By the normality of P (Theorem 2-3 in [Ho]), there are disjoint open neighborhoods U_E of E and U_F of F in P. Since Y is a continuum meeting both of the disjoint open sets U_E and U_F, $Y\backslash(U_E \cup U_F)$ is a compactum. Similarly $Z\backslash(U_E \cup U_F)$ is a compactum. Also, the compacta $Y\backslash(U_E \cup U_F)$ and $Z\backslash(U_E \cup U_F)$ are disjoint, because $Y \cap Z = E\,|F \subseteq U_E \cup U_F$. Again, as P is normal, there are disjoint open neighborhoods W_Y of $Y\backslash(U_E \cup U_F)$ and W_Z of $Z\backslash(U_E \cup U_F)$. Define

$$O_Y = W_Y \cup U_E \cup U_F \text{ and } O_Z = W_Z \cup U_E \cup U_F.$$

Note that $Y \subseteq O_Y$ and $Z \subseteq O_Z$. Also, if R and S are any sets with $Y \subseteq R \subseteq O_Y$ and $Z \subseteq S \subseteq O_Z$, we have

$$E \cup F = Y \cap Z \subseteq R \cap S \subseteq O_Y \cap O_Z = U_E \cup U_F = U_E\,|U_F\ .$$

Hence,

$$R \cap S \subseteq U_E\,|U_F\ .$$

Then, since $\emptyset \neq E \subseteq R \cap S \cap U_E$ and $\emptyset \neq F \subseteq R \cap S \cap U_F$, the set $R \cap S$ is necessarily disconnected. That is,

(12) $\quad R \cap S$ is not connected whenever $Y \subseteq R \subseteq O_Y$ and $Z \subseteq S \subseteq O_Z$.

To complete the proof it suffices to show that $Y \nsubseteq O_Z$ and $Z \nsubseteq O_Y$. We suppose $Z \subseteq O_Y$. Let $R = P$ and $S = Z$. Then $Y \subseteq R = P = Y \cup Z \subseteq O_Y$ and $Z = S \subseteq O_Z$, so that $R \cap S$ is not connected, by (12). But in this case $R \cap S$ is just the continuum Z. The contradiction shows that $Z \nsubseteq O_Y$. A similar argument gives that $Y \nsubseteq O_Z$. $\square$

Proposition 4.2. *A Hausdorff continuum P is unicoherent if and only if (*) every three-element open cover $\Im$ of P has a refinement $\mathcal{P}$ such that $\mathcal{P}$ is a pseudogrille for P and P is $\mathcal{P}$-unicoherent.*

Proof. If P is unicoherent and $\Im$ is any three-element open cover of P then the discrete pseudogrille $\mathcal{P} = \{\{x\} : x \in P\}$ has the desired properties. Conversely, suppose that P satisfies (*) but is not unicoherent. We will derive a contradiction to Proposition 4.1. Since P is not unicoherent there are proper subcontinua Y and Z of P such that $P = Y \cup Z$ and $Y \cap Z$ is not connected. Let O_Y and O_Z be as guaranteed by Proposition 4.1. Thus,

(13) $\quad R \cap S$ is not connected whenever $Y \subseteq R \subseteq O_Y$ and $Z \subseteq S \subseteq O_Z$.

Note also that $Y \backslash (O_Y \cap O_Z)$ is nonvoid since $Y \nsubseteq O_Z$ (Proposition 4.1). Hence, as $Y \cap Z \subseteq O_Y \cap O_Z$, the sets $Y \backslash (O_Y \cap O_Z)$ and Z are disjoint compacta. By the normality of P there exists an open set U_Y such that $Y \backslash (O_Y \cap O_Z) \subseteq U_Y$ and $U_Y \cap Z = \emptyset$. Let $V_Y = U_Y \cap O_Y$. Since $Y \subseteq O_Y$ (Proposition 4.1) we have

(14) $$\emptyset \neq Y \backslash (O_Y \cap O_Z) \subseteq V_Y \subseteq O_Y \text{ and } V_Y \cap Z = \emptyset.$$

Similarly, there is an open set V_Z with

(15) $$\emptyset \neq Z \backslash (O_Y \cap O_Z) \subseteq V_Z \subseteq O_Z \text{ and } V_Z \cap Y = \emptyset.$$

Let $\Im = \{V_Y, O_Y \cap O_Z, V_Z\}$. From (14) and (15) and $P = Y \cup Z$, we find that $\Im$ is a three-element open cover of P. Thus, by hypothesis, there exists a refinement $\mathcal{P}$ of $\Im$ such that

$\mathcal{P}$ is a pseudogrille for P and P is $\mathcal{P}$-unicoherent.

We will show that $\{\mathcal{P}(Y)^\star, \mathcal{P}(Z)^\star\}$ is a $\mathcal{P}$-decomposition of P by proving (16)-(19) below. Note that if $G \in \mathcal{P}$ and $G \cap Y \neq \emptyset$ then, since $V_Z \cap Y = \emptyset$ by (15), we have $G \nsubseteq V_Z$. Hence, as $\mathcal{P}$ is a refinement of $\Im$, if $G \in \mathcal{P}$ and $G \cap Y \neq \emptyset$ then either $G \subseteq V_Y \subseteq O_Y$ or $G \subseteq O_Y \cap O_Z \subseteq O_Y$. Thus,

$$Y \subseteq \mathcal{P}(Y)^\star \subseteq O_Y, \text{ and } \mathcal{P}(Y)^\star \neq P. \tag{16}$$

The second part of (16) follows from $Z \nsubseteq O_Y$ (Proposition 4.1). Likewise,

$$Z \subseteq \mathcal{P}(Z)^\star \subseteq O_Z, \text{ and } \mathcal{P}(Z)^\star \neq P. \tag{17}$$

Furthermore, as $\mathcal{P}$ is a pseudogrille for $P = Y \cup Z$ and $Y, Z \in C(P)$,

$$\mathcal{P}(Y)^\star \text{ and } \mathcal{P}(Z)^\star \text{ are } \mathcal{P}\text{-continua whose union is } P. \tag{18}$$

Now suppose there exists $G_Y \in \mathcal{P}(Y)$ and $G_Z \in \mathcal{P}(Z)$ with $G_Y \cap G_Z \neq \emptyset$. Then $G_Y \cup G_Z$ is a continuum intersecting both Y and Z. Then, as $P = Y \cup Z$, it follows that the continuum $G_Y \cup G_Z$ is the union of the compacta $(G_Y \cup G_Z) \cap Y$ and $(G_Y \cup G_Z) \cap Z$. Hence,

$$\emptyset \neq ((G_Y \cup G_Z) \cap Y) \cap ((G_Y \cup G_Z) \cap Z) = (G_Y \cup G_Z) \cap Y \cap Z.$$

Choose $p \in (G_Y \cup G_Z) \cap Y \cap Z$. If $p \in G_Y$ then $G_Y \in \mathcal{P}(Y) \cap \mathcal{P}(Z)$, whereas if $p \in G_Z$ then $G_Z \in \mathcal{P}(Y) \cap \mathcal{P}(Z)$. Hence, either $G_Y \in \mathcal{P}(Z)$ or $G_Z \in \mathcal{P}(Y)$. We have shown:

$$\begin{gathered} \text{If } G_Y \in \mathcal{P}(Y),\ G_Z \in \mathcal{P}(Z), \text{ and } G_Y \cap G_Z \neq \emptyset, \\ \text{then } G_Y \in \mathcal{P}(Z) \text{ or } G_Z \in \mathcal{P}(Y). \end{gathered} \tag{19}$$

From (16), (17), (18) and (19) we conclude that $\{\mathcal{P}(Y)^\star, \mathcal{P}(Z)^\star\}$ is a $\mathcal{P}$-decomposition of $\mathcal{P}^\star = P$. Thus, as P is $\mathcal{P}$-unicoherent, $\mathcal{P}(Y)^\star \cap \mathcal{P}(Z)^\star$ is connected. However, in view of (16) and (17), this is a contradiction of (13). The contradiction shows that P must in fact be unicoherent if it has the property (*) given in the statement. □

A reading of the last Proposition's statement and the first sentence of its proof indicates that the following corollary has been proved.

Corollary 4.3. *A Hausdorff continuum P is unicoherent if and only if (*) every open cover $\Im$ of P has a refinement $\mathcal{P}$ such that $\mathcal{P}$ is a pseudogrille for P and P is $\mathcal{P}$-unicoherent.*

Proposition 4.4. *A metric continuum (P, ρ) is unicoherent if and only if (**) for every $\varepsilon > 0$ there exists a pseudogrille $\mathcal{P}$ for P such that P is $\mathcal{P}$-unicoherent and mesh$(\mathcal{P}) < \varepsilon$. (mesh$(\mathcal{P}) \equiv \sup\{\rho$-diameter$(G) : G \in \mathcal{P}\}$.)*

Proof. If P is unicoherent then the pseudogrille $\mathcal{P} = \{\{x\} : x \in P\}$ can always be chosen in (**). That is, P is $\mathcal{P}$-unicoherent (Proposition 3.3) and $mesh(\mathcal{P}) < \varepsilon$. For the converse we suppose that (**) holds and apply Proposition 4.2. Let $\Im$ be an arbitrary three-element open cover of P. We want to produce a pseudogrille $\mathcal{P}$ for P such that P is $\mathcal{P}$-unicoherent and $\mathcal{P}$ is a refinement of $\Im$. By Theorem 1−32 in [Ho], the cover $\Im$ has a Lebesgue number $\varepsilon > 0$, i. e., if S is any subset of P with ρ-$diameter(S) < \varepsilon$ then S is contained in some element of $\Im$. By (**) there exists a pseudogrille $\mathcal{P}$ for P such that P is $\mathcal{P}$-unicoherent and $mesh(\mathcal{P}) < \varepsilon$. Each member of $\mathcal{P}$ is a subset of P with ρ-$diameter < \varepsilon$, and hence is contained in some member of $\Im$. Thus, $\mathcal{P}$ is a refinement of $\Im$. Since $\Im$ was an arbitrary three-element open cover of P we conclude from Proposition 4.2 that P is unicoherent. □

5. Graph-theoretic Tools

A graph-theoretic result is now proven to prepare the main result of the paper. Those acquainted with Graph Theory will recognize in the following definition a type of *bipartite* graph, i.e., a graph whose vertices admit a *two-coloring*. In the next section such a graph will be the *nerve* of a finite covering, ΣJ, of a $\mathcal{P}$-continuum $J = \mathcal{J}^\star$. (The terms *bipartite*, *two-coloring* and *nerve* are defined in [Wi] and [En].) Each element of ΣJ will be a component of $(\mathcal{J} \cap \mathcal{P}_{-1})^\star$ or of $(\mathcal{J} \cap \mathcal{P}_1)^\star$, where $\{\mathcal{P}_{-1}^\star, \mathcal{P}_1^\star\}$ is a two-element $\mathcal{P}$-decomposition of P.

Recall that a *partition* of a nonempty set T is a collection of disjoint nonempty sets whose union is T. The partition member that contains the element m of T could be denoted by (m). The members of the partition are simply the equivalence classes of an equivalence relation associated with the partition. (Two elements of T are equivalent if and only if they belong to the same member of the partition.) $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, of a finite graph G. Throughout we let T^{-1} and T^1 be nonempty finite sets of negative and positive integers, respectively, and let $T = T^{-1} \cup T^1$. A partition of T will be said to be *sign-preserving* if each partition member (m) is a subset either of T^{-1} or of T^1. By a *signed graph* $T_{(\ )}$ *defined on* T we mean a finite graph whose vertices are the members (m), $m \in T$, of a sign-preserving partition of T and whose edges never join vertices of the same sign, i. e., if $(m)(n) \in E(T_{(\ )})$ then either $(m) \subseteq T^{-1}$ and $(n) \subseteq T^1$ or $(m) \subseteq T^1$ and $(n) \subseteq T^{-1}$. The phrase *defined on* T is usually omitted.

Note that $(m)(n) = (n)(m)$. That is to say, we are not discussing directed graphs. No loop can be an edge of a signed graph, i. e., $(m)(m)$ is never an edge of $T_{(\ )}$. Neither are multiple edges joining the same pair of vertices allowed. It is thus the case that edges $(m)(n)$ and $(m')(n')$ of $T_{(\ )}$ are equal if and only if $\{(m),(n)\} = \{(m'),(n')\}$.

A *walk in* $T_{(\ )}$ is a finite sequence of vertices of $T_{(\ )}$,

$$(m_1),\, (m_2),\, \dots\, ,\, (m_t)$$

such that $t \geq 2$ and $(m_r)(m_{r+1}) \in E(T_{(\ )})$ for $1 \leq r < t$. The vertices (m_1) and (m_t) are called the *end-vertices* of the walk, and the integer $t-1$ is the walk's *length*. If the vertices in the sequence are distinct, the walk is called a *path*. $T_{(\ )}$ is said to be *connected* if every two of its vertices are the end-vertices of some walk in $T_{(\ )}$. Given distinct vertices in a connected signed graph $T_{(\ )}$, consider a walk (m_1), (m_2), $\dots$, (m_t) in $T_{(\ )}$ of minimal length such that $(m_1) = (m)$ and $(m_t) = (m')$. This walk is necessarily a path with end-vertices (m) and (m'), since if $(m_r) = (m_s)$ for some $r < s$ then by deleting $(m_r), \dots , (m_{s-1})$ one would obtain a walk of smaller length with the same end-vertices, (m) and (m'). Thus:

If (m) and (m') are distinct vertices in the connected signed graph $T_{(\ )}$, then there is a path in $T_{(\ )}$ with end-vertices (m) and (m').

A *circuit* in $T_{(\ )}$ is a walk (m_1), (m_2), $\dots$, (m_t) such that $t \geq 5$, $(m_1) = (m_t)$, and (m_1), $(m_2), \dots$, (m_{t-1}) is a path. A connected signed graph having no circuits is called a *signed tree.* The idea of the next Lemma is simple.

Lemma 5.1. *If* (m_1), (m_2), $\dots$, (m_t) *is a walk in the signed tree* $T_{(\ )}$ *such that* $(m_1) = (m_r) \Longleftrightarrow r \in \{1,t\}$, *then* $(m_2) = (m_{t-1})$.

Proof. Otherwise there exists a walk (m_1), (m_2), $\dots$, (m_t) in $T_{(\ )}$ of minimal length with $(m_2) \neq (m_{t-1})$ and with $(m_1) = (m_r) \Longleftrightarrow r \in \{1,t\}$. Then $t \neq 1, 3$. Also, t is odd, since $T_{(\ )}$ is a signed graph and $(m_1) = (m_t)$. Thus $t \geq 5$ and, by the minimality of length of the walk (m_1), (m_2), $\dots$, (m_t), the vertices (m_2), $\dots$, (m_{t-1}) are distinct. Also, as $(m_1) = (m_r) \Longleftrightarrow r \in \{1,t\}$, the vertices (m_1), $\dots$, (m_{t-1}) are distinct. Hence (m_1), (m_2), $\dots$, (m_t) is a circuit in the signed tree $T_{(\ )}$, which is impossible. $\square$

In the following Lemma a connected signed graph $T_{(\ )}$ and signed trees $T_{[\ ]}$ and $T_{\langle\ \rangle}$, all defined on the same underlying set T, are combined with an equivalence relation $\sim$ on $E(T_{(\ )})$. For any subset S of T, we let $[S]$ denote the set $\bigcup_{m\in S}[m]$ and $\langle S\rangle$ denote the set $\bigcup_{m\in S}\langle m\rangle$.

Lemma 5.2. *(Lemma of Folding-knives) Let $T_{[\,]}$ and $T_{\langle\,\rangle}$ be signed trees and $T_{(\,)}$ be a connected signed graph on $T = T^{-1} \cup T^{1}$. Suppose that $(m) = \{m\}$ for each $m \in T$. Let $\sim$ be an equivalence relation on $E(T_{(\,)})$. Then $\sim$ has just one equivalence class if all the following hold.*

(1) *$[m][n] \in E(T_{[\,]})$ and $\langle m\rangle\langle n\rangle \in E(T_{\langle\,\rangle})$ whenever $m, n \in T$ and $(m)(n) \in E(T_{(\,)})$.*

(2) *If $m, n, m', n' \in T$, if $(m)(n), (m')(n') \in E(T_{(\,)})$, and if either $[m][n] = [m'][n']$ or $\langle m\rangle\langle n\rangle = \langle m'\rangle\langle n'\rangle$, then $(m)(n) \sim (m')(n')$.*

($3a$) *If $i \in \{-1, 1\}$ and $\emptyset \neq S \subset T^{i}$ with $[S] \neq T^{i}$, then there exists $k \in [S]$ with $\langle k\rangle \nsubseteq [S]$.*

($3b$) *If $i \in \{-1, 1\}$ and $\emptyset \neq S \subset T^{i}$ with $\langle S\rangle \neq T^{i}$, then there exists $k \in \langle S\rangle$ with $[k] \nsubseteq \langle S\rangle$.*

Proof. The proof proceeds by induction on $\left|V(T_{[\,]})\right| + \left|V(T_{\langle\,\rangle})\right|$. Choose $m_0 \in T^1$and $n_0 \in T^{-1}$. Since $[m] \neq [n]$ and $\langle m\rangle \neq \langle n\rangle$ whenever $m \in T^{i}$, $n \in T^{-i}$, the first case to consider is when $\left|V(T_{[\,]})\right| + \left|V(T_{\langle\,\rangle})\right| = 4$, i. e., when $T^1 = [m_0] = \langle m_0\rangle$, $T^{-1} = [n_0] = \langle n_0\rangle$, and

$$V(T_{[\,]}) = \{[m_0], [n_0]\},\ E(T_{[\,]}) = \{[m_0][n_0]\}$$
$$V(T_{\langle\,\rangle}) = \{\langle m_0\rangle, \langle n_0\rangle\},\ E(T_{\langle\,\rangle}) = \{\langle m_0\rangle\langle n_0\rangle\}$$

The connected signed graph $T_{(\,)}$ has at least the two vertices (m_0) and (n_0), and hence has at least one edge. Thus, there exist $m' \in T^1$ and $n' \in T^{-1}$ with $(m')(n') \in E(T_{(\,)})$. By (1), $[m'][n'] \in E(T_{[\,]})$. Then, since $m' \in T^1 = [m_0]$ and $n' \in T^{-1} = [n_0]$, it follows that $[m'][n'] = [m_0][n_0]$. Moreover, whenever $m \in T^1$ and $n \in T^{-1}$, we have $[m][n] \in E(T_{[\,]})$, and therefore $[m][n] = [m_0][n_0]$. Thus, if $m, n, m', n' \in T$ and $(m)(n)$, $(m')(n') \in E(T_{(\,)})$ then $[m][n] \in E(T_{[\,]})$, $[m'][n'] \in E(T_{[\,]})$, and $[m][n] = [m_0][n_0] = [m'][n']$. Hence, by (2), $(m)(n) \sim (m')(n')$. Therefore, $\sim$ has just one equivalence class in this case.

Now suppose $\sim$ has only one equivalence class when all the hypotheses of Lemma 5.2 hold and $\left|V(T_{[\,]})\right| + \left|V(T_{\langle\,\rangle})\right| \leq w$, where w is some integer ≥ 4. Assume, then, that $T_{(\,)}$ is a connected signed graph on a set $T = T^{-1} \cup T^{1}$ with $(m) = \{m\}$ for each $m \in T$, that

$$T_{[\,]} \text{ and } T_{\langle\,\rangle} \text{ are signed trees on } T \text{ with } \left|V(T_{[\,]})\right| + \left|V(T_{\langle\,\rangle})\right| = w + 1,$$

that $\sim$ is an equivalence relation on $E(T_{(\,)})$, and that (1), (2), ($3a$) and ($3b$) hold. To show $\sim$ has only one equivalence class we first establish the following.

(20) There exist $n_1, n_2, n_2', n_3 \in T$ such that $(n_1)(n_2) \sim (n_2')(n_3)$ and either $\langle n_2\rangle = \langle n_2'\rangle$ and $\langle n_1\rangle \neq \langle n_3\rangle$ or $[n_2] = [n_2']$ and $[n_1] \neq [n_3]$.

To prove (20), let $\Pi_{(\,)}$ denote the collection of all paths $(m_1), (m_2), \dots, (m_t)$ in $T_{(\,)}$ such that *either* $\langle m_1\rangle = \langle m_t\rangle$ and $[m_1] \neq [m_t]$ *or* $[m_1] = [m_t]$ and $\langle m_1\rangle \neq \langle m_t\rangle$. We claim $\Pi_{(\,)} \neq \emptyset$. For, since $\left|V(T_{[\,]})\right| + \left|V(T_{\langle\,\rangle})\right| > 4$, there exists $i \in \{-1, 1\}$ and $j \in T^i$ such that either $[j] \neq T^i$ or $\langle j\rangle \neq T^i$. Suppose first that $[j] \neq T^i$ for this i. By ($3a$), with $S = \{j\}$, there exists $k \in [j]$ with $\langle k\rangle \nsubseteq [j]$. One can then find $k' \in \langle k\rangle$ with $k' \notin [j]$. Thus $[k'] \neq [j] = [k]$. Hence

$$\langle k'\rangle = \langle k\rangle \text{ and } [k'] \neq [k].$$

Then $k' \neq k$, and $(k') = \{k'\} \neq \{k\} = (k)$. Let $R = (m_1), (m_2), \dots, (m_t)$ be a path in the connected signed graph $T_{(\,)}$ with end-vertices $(m_1) = (k')$ and $(m_t) = (k)$. Thus, $m_1 = k'$, $m_t = k$, and we have $R \in \Pi_{(\,)}$. So $\Pi_{(\,)} \neq \emptyset$ if $[j] \neq T^i$. Similarly, in case that $\langle j\rangle \neq T^i$, one can use ($3b$) to show $\Pi_{(\,)} \neq \emptyset$. Since $\Pi_{(\,)} \neq \emptyset$, we can choose a path

$$(m_1), (m_2), \dots, (m_t) = Q \in \Pi_{(\,)}$$

of minimal length. Also, we assume, without loss of generality, that

$$\langle m_1\rangle = \langle m_t\rangle \text{ and } [m_1] \neq [m_t]. \tag{21}$$

Now $m_1 \neq m_t$, so $(m_1) \neq (m_t)$. Also, since $T_{(\,)}$ is a signed graph and $(m_1)(m_2) \in E(T_{(\,)})$, m_1 and m_2 must have opposite signs. Thus, as $T_{\langle\,\rangle}$ is also a signed graph, $\langle m_1\rangle \neq \langle m_2\rangle$. Therefore, since $\langle m_1\rangle = \langle m_t\rangle$ and $t \geq 2$, one has $t \geq 3$. We now aim to use Lemma 5.1, and claim for this purpose that

$$\langle m_r\rangle \neq \langle m_1\rangle \text{ for } 1 < r < t. \tag{22}$$

To see this, fix $r \in \{2, \dots, t-1\}$, let $R' = (m_r), (m_{r+1}), \dots, (m_t)$ and let $R'' = (m_1), (m_2), \dots, (m_r)$. Then $R', R'' \notin \Pi_{(\,)}$, since the lengths of these paths are less than the length of Q. Now, as $R'' \notin \Pi_{(\,)}$, if $\langle m_r\rangle = \langle m_1\rangle$ then $[m_r] = [m_1]$. But then, by (21), $[m_r] = [m_1] \neq [m_t]$ and $\langle m_r\rangle = \langle m_1\rangle = \langle m_t\rangle$, which says that $R' \in \Pi_{(\,)}$. Since this is not the case, we must have $\langle m_r\rangle \neq \langle m_1\rangle$. So (22) holds. By (22) and (21), $W = \langle m_1\rangle, \langle m_2\rangle, \langle m_3\rangle, \dots, \langle m_t\rangle$ is a walk in the signed tree $T_{\langle\,\rangle}$ such that $\langle m_1\rangle = \langle m_r\rangle \Leftrightarrow r \in \{1, t\}$. Thus, as a signed tree is connected by definition, it follows from Lemma 5.1 that

$$\langle m_2\rangle = \langle m_{t-1}\rangle.$$

Define

$$n_1 = m_1, \qquad n_2 = m_2, \qquad n_2' = m_{t-1}, \qquad n_3 = m_t.$$

Then $(n_1)(n_2) = (m_1)(m_2) \in E(T_{(\,)})$ and $(n_2')(n_3) = (m_{t-1})(m_t) \in E(T_{(\,)})$. Furthermore, $\langle n_1\rangle\langle n_2\rangle = \langle m_1\rangle\langle m_2\rangle = \langle m_t\rangle\langle m_{t-1}\rangle = \langle n_3\rangle\langle n_2'\rangle = \langle n_2'\rangle\langle n_3\rangle$. Then by (2) we have $(n_1)(n_2) \sim (n_2')(n_3)$, which is the first part of (20). Now, as noted earlier, $t \geq 3$. If $t = 3$ then $n_2 = n_2'$, $[n_2] = [n_2']$ and, by (21), $[n_1] \neq [n_3]$. Thus (20) holds in its entirety should t equal 3. So suppose $t > 3$. Consider the path $Z = (m_2), (m_3), (m_4), \ldots, (m_{t-1})$. Since Z has length less than that of Q, Z is not an element of $\Pi_{(\,)}$. Thus, as $\langle m_2\rangle = \langle m_{t-1}\rangle$ and $Z \notin \Pi_{(\,)}$, one has $[m_2] = [m_{t-1}]$. Hence $[n_2] = [n_2']$ again, and $[n_1] \neq [n_3]$ again by (21). This completes the proof of (20).

We now apply (20) together with the inductive hypothesis to show that $\sim$ has exactly one equivalence class. By the symmetry of hypotheses, we can and shall assume:

(23) There exist $n_1, n_2, n_2', n_3 \in T$ such that $(n_1)(n_2) \sim (n_2')(n_3)$, $[n_2] = [n_2']$ and $[n_1] \neq [n_3]$.

Thus, by (1), $[n_1][n_2]$ and $[n_2'][n_3]$ are distinct edges of $T_{[\,]}$ sharing the common vertex $[n_2] = [n_2']$. Hence

$$[n_1], [n_3] \subseteq T^i \text{ for some } i \in \{-1, 1\}. \tag{24}$$

Consider the partition of T whose members, denoted by $[[m]]$ for $m \in T$, are defined by

$$[[m]] = \begin{cases} [n_1] \cup [n_3] & \text{if } m \in [n_1] \cup [n_3] \\ [m] & \text{otherwise} \end{cases}$$

We define a quotient graph $T_{[[\,]]}$ of $T_{[\,]}$ as follows:

$$V(T_{[[\,]]}) = \{[[m]] : m \in T\}$$

$$E(T_{[[\,]]}) = \{[[m]][[n]] : m, n \in T \text{ and } [m][n] \in E(T_{[\,]})\}$$

To understand $T_{[[\,]]}$ note first that the natural map given by $[m] \mapsto [[m]]$ maps only the distinct vertices $[n_1]$ and $[n_3]$ of $T_{[\,]}$ to $[[n_1]] = [[n_3]]$, and is one-to-one on $V(T_{[\,]}) \backslash \{[n_1], [n_3]\}$. Likewise, the canonical map

(25)

$[m][n] \mapsto [[m]][[n]]$ maps only the distinct edges $[n_1][n_2]$ and $[n_3][n_2]$ to $[[n_1]][[n_2]] = [[n_3]][[n_2]]$, and is one-to-one on $E(T_{[\,]}) \backslash \{[n_1][n_2], [n_3][n_2]\}$.

The identification represents the folding together of the incident edges $[n_1][n_2]$ and $[n_3][n_2]$ of the tree $T_{[\,]}$. From these considerations, including (24), we conclude that $T_{[[\,]]}$ is a connected signed graph with

$$\left|V(T_{[[\,]]})\right| = \left|V(T_{[\,]})\right| - 1 \text{ and } \left|E(T_{[[\,]]})\right| = \left|E(T_{[\,]})\right| - 1.$$

Since $T_{[\,]}$ is a tree, Theorem 9A of [Wi] tells us that

$$\left|E(T_{[\,]})\right| = \left|V(T_{[\,]})\right| - 1.$$

Then $\left|V(T_{[[\,]]})\right| + \left|V(T_{\langle\,\rangle})\right| = \left|V(T_{[\,]})\right| - 1 + \left|V(T_{\langle\,\rangle})\right| = w$, and $\left|E(T_{[[\,]]})\right| = \left|E(T_{[\,]})\right| - 1 = (\left|V(T_{[\,]})\right| - 1) - 1 = \left|V(T_{[\,]})\right| - 2 = \left|V(T_{[[\,]]})\right| - 1$. Thus,

$$\left|V(T_{[[\,]]})\right| + \left|V(T_{\langle\,\rangle})\right| = w \text{ and } T_{[[\,]]} \text{ is a signed tree.} \tag{26}$$

We claim that the following hold.

(1′) *$[[m]][[n]] \in E(T_{[[\,]]})$ and $\langle m\rangle\langle n\rangle \in E(T_{\langle\,\rangle})$ whenever $m, n \in T$ and $(m)(n) \in E(T_{(\,)})$.*

(2′) *If $m, n, m', n' \in T$, $(m)(n), (m')(n') \in E(T_{(\,)})$, and if either $[[m]][[n]] = [[m']][[n']]$ or $\langle m\rangle\langle n\rangle = \langle m'\rangle\langle n'\rangle$, then $(m)(n) \sim (m')(n')$.*

(3a′) *If $i \in \{-1, 1\}$ and $\emptyset \neq S \subset T^i$ with $[[S]] \neq T^i$, then there exists $k \in [[S]]$ with $\langle k\rangle \not\subseteq [[S]]$.*

(3b′) *If $i \in \{-1, 1\}$ and $\emptyset \neq S \subset T^i$ with $\langle S\rangle \neq T^i$, then there exists $k \in \langle S\rangle$ with $[[k]] \not\subseteq \langle S\rangle$.*

To prove (1′), suppose that $m, n \in T$ and $(m)(n) \in E(T_{(\,)})$. Then, by (1), $[m][n] \in E(T_{[\,]})$ and $\langle m\rangle\langle n\rangle \in E(T_{\langle\,\rangle})$. Thus, by the definition of $E(T_{[[\,]]})$, $[[m]][[n]] \in E(T_{[[\,]]})$.

To prove (2′), suppose $m, n, m', n' \in T$ and $(m)(n)$, $(m')(n') \in E(T_{(\,)})$. If $\langle m\rangle\langle n\rangle = \langle m'\rangle\langle n'\rangle$ then $(m)(n) \sim (m')(n')$ by (2). So assume $[[m]][[n]] = [[m']][[n']]$. If $[m][n] = [m'][n']$ then $(m)(n) \sim (m')(n')$ by (2). On the other hand, if $[m][n] \neq [m'][n']$ then one has $\{[m][n], [m'][n']\} = \{[n_1][n_2], [n_3][n_2]\} = \{[n_1][n_2], [n_2'][n_3]\}$, by (25) and (23). Here, it is no loss of generality to assume that

$$m \in [n_1], \qquad n \in [n_2], \qquad m' \in [n_2'], \qquad n' \in [n_3].$$

Then $[m][n] = [n_1][n_2]$ and $[m'][n'] = [n_2'][n_3]$, so that, by (2), $(m)(n) \sim (n_1)(n_2)$ and $(m')(n') \sim (n_2')(n_3)$. Also, by (23), $(n_1)(n_2) \sim (n_2')(n_3)$. Hence, by the transitivity and symmetry of $\sim$, we have $(m)(n) \sim (m')(n')$.

To prove $(3a')$, suppose that $i \in \{-1,1\}$, $\emptyset \neq S \subset T^i$ and $[[S]] \neq T^i$. Let $S' = [[S]] = \bigcup_{m\in S}[[m]]$. Then $\emptyset \neq S' \subset T^i$. Also, each partition member $[[m]]$ is a union of sets of the form $[n]$ (some $n \in T^i$). Hence, $S' = [[S]]$ is a union of partition elements $[n]$. Thus, $[S'] = S' = [[S]]$. We have $\emptyset \neq S' \subset T^i$ and $[S'] = [[S]] \neq T^i$. By $(3a)$, there exists $k \in [S'] = [[S]]$ with $\langle k\rangle \nsubseteq [S'] = [[S]]$. This is $(3a')$.

Next, suppose $i \in \{-1,1\}$, $\emptyset \neq S \subset T^i$ and $\langle S\rangle \neq T^i$. By $(3b)$ there exists $k \in \langle S\rangle$ with $[k] \nsubseteq \langle S\rangle$. Then, as $[k] \subseteq [[k]]$, we have $[[k]] \nsubseteq \langle S\rangle$. This is $(3b')$.

Now, in view of $(1')$, $(2')$, $(3a')$, $(3b')$ and (26), the induction hypothesis guarantees that $\sim$ has exactly one equivalence class, as required. $\square$

6. Special Unions of Unicoherent Continua

Lemma 6.1. *Suppose $\{\mathcal{P}^\star_{-1}, \mathcal{P}^\star_1\}$ is a $\mathcal{P}$-decomposition of P. Let $\mathcal{H}^\star$ be a $\mathcal{P}$-continuum such that $(\mathcal{H}\cap\mathcal{P}_i)^\star = \mathcal{H}^\star \cap \mathcal{P}_i{}^\star \neq \emptyset$ for each $i \in \{-1,1\}$. Let $\Sigma\mathcal{H}^\star = \{\mathcal{A}^\star : \mathcal{A}^\star$ is a component of $(\mathcal{H}\cap\mathcal{P}_{-1})^\star$ or of $(\mathcal{H}\cap\mathcal{P}_1)^\star\}$, and suppose $T = T^{-1}\cup T^1$, where T^{-1} and T^1 are finite nonempty sets of negative and positive integers, respectively. Let $\sigma : T \longrightarrow \Sigma\mathcal{H}^\star$ be a surjection such that $\sigma(m)$ is a component of $(\mathcal{H}\cap\mathcal{P}_i)^\star$ if and only if $m \in T^i$. Define $\lceil m\rceil = \sigma^{-1}(\sigma(m))$ for each $m \in T$. Define*

$$V(T_{\lceil\rceil}) = \{\lceil m\rceil : m \in T\}$$
$$E(T_{\lceil\rceil}) = \{\lceil m\rceil\lceil n\rceil : \sigma(m) \neq \sigma(n) \text{ and } \sigma(m)\cap\sigma(n) \neq \emptyset\}$$

(No distinction is made between $\lceil m\rceil\lceil n\rceil$ and $\lceil n\rceil\lceil m\rceil$.) Then $T_{\lceil\rceil}$ is a connected signed graph. Also, if $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent then $T_{\lceil\rceil}$ is a signed tree and $\sigma(m)\cap\sigma(n)$ is a connected subset of $\mathcal{P}^\star_{-1}\cap\mathcal{P}^\star_1$ for all $m \in T^{-1}$, $n \in T^1$.

Proof. The hypothesis that $\sigma(m)$ is a component of $(\mathcal{H}\cap\mathcal{P}_i)^\star$ if and only if $m \in T^i$, for each $i \in \{-1,1\}$, implies that $\lceil m\rceil = \sigma^{-1}(\sigma(m)) \subseteq T^i$ whenever $i \in \{-1,1\}$ and $m \in T^i$. Therefore, $V(T_{\lceil\rceil})$ is a sign-preserving partition of T. Moreover, as the components of $(\mathcal{H}\cap\mathcal{P}_{-1})^\star$ are pairwise disjoint and the components of $(\mathcal{H}\cap\mathcal{P}_1)^\star$ are pairwise disjoint, $\sigma(m)\cap\sigma(n) = \emptyset$ whenever m and n have the same sign and $\sigma(m) \neq \sigma(n)$. Thus $\lceil m\rceil\lceil n\rceil \notin E(T_{\lceil\rceil})$ if m and n have the same sign. Therefore, $T_{\lceil\rceil}$ is a signed graph defined on T.

Since $(\mathcal{H}\cap\mathcal{P}_i)^\star \equiv \mathcal{H}^\star\cap\mathcal{P}_i{}^\star$ is a $\mathcal{P}$-compactum, Lemma 2.10 guarantees that each element of $\Sigma\mathcal{H}^\star$ is a subcontinuum of $\mathcal{H}^\star$. Also, the collection $\Sigma\mathcal{H}^\star$ is finite since σ is a function from the finite set T onto $\Sigma\mathcal{H}^\star$. Now suppose m, $n \in T$ with $\lceil m\rceil$, $\lceil n\rceil \in V(T_{\lceil\rceil})$ and $\lceil m\rceil \neq \lceil n\rceil$. Then $\sigma(m)$ and $\sigma(n)$ are distinct elements of $\Sigma\mathcal{H}^\star$. Moreover, since $\mathcal{H}^\star$ is connected and $\mathcal{H}^\star = \bigcup\{\mathcal{A}^\star : \mathcal{A}^\star \in \Sigma\mathcal{H}^\star\}$, there exist (see 8.12 in [Na]) elements $\mathcal{A}^\star_1, \mathcal{A}^\star_2, \ldots, \mathcal{A}^\star_d$ of $\Sigma\mathcal{H}^\star$ such that $\sigma(m) = \mathcal{A}^\star_1 \neq$

$\mathcal{A}_d^\star = \sigma(n)$, and $\mathcal{A}_c^\star \cap \mathcal{A}_{c+1}^\star \neq \emptyset$ for $1 \leq c < d$. Furthermore, we can assume that $\mathcal{A}_c^\star \neq \mathcal{A}_{c+1}^\star$ for $1 \leq c < d$. By the surjectivity of σ, T contains integers $m = l_1, l_2, \ldots, l_d = n$ such that $\sigma(l_c) = \mathcal{A}_c^\star$ for $1 \leq c \leq d$. Then,

$$\sigma(l_c) \neq \sigma(l_{c+1}) \text{ and } \sigma(l_c) \cap \sigma(l_{c+1}) \neq \emptyset \text{ for } 1 \leq c < d.$$

Thus, $\lceil l_1 \rceil, \lceil l_2 \rceil, \ldots, \lceil l_d \rceil$ is a walk in $T_{\lceil\rceil}$ with end-vertices $\lceil l_1 \rceil = \lceil m \rceil$ and $\lceil l_d \rceil = \lceil n \rceil$. Hence, as $\lceil m \rceil, \lceil n \rceil$ were arbitrary distinct vertices of $T_{\lceil\rceil}$, $T_{\lceil\rceil}$ is a connected signed graph.

For the rest of the proof assume that $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent. Let

$$\begin{aligned}
&\Upsilon_{-1} = \{\sigma(n) : n \in T^1 \text{ and } \sigma(n) \nsubseteq \sigma(m) \text{ for all } m \in T^{-1}\},\\
&\Upsilon_1 = \{\sigma(m) : m \in T^{-1} \text{ and } \sigma(m) \nsubseteq \sigma(n) \text{ for all } n \in T^1\}, \text{ and}\\
&\Upsilon = \Upsilon_{-1} \cup \Upsilon_1.
\end{aligned}$$

To show that $T_{\lceil\rceil}$ is a treechain we argue by contradiction, as follows. Suppose $T_{\lceil\rceil}$ had a circuit $\lceil m_1 \rceil, \lceil m_2 \rceil, \lceil m_3 \rceil, \lceil m_4 \rceil, \ldots, \lceil m_w \rceil$. Thus, $w \geq 5$, $\lceil m_1 \rceil = \lceil m_w \rceil$, and $\lceil m_r \rceil \neq \lceil m_s \rceil$ for $1 \leq r < s < w$. Hence,

(27) the continua $\sigma(m_1), \sigma(m_2), \sigma(m_3), \sigma(m_4), \ldots, \sigma(m_{w-1})$ are distinct.

Now, for $1 \leq r < w$, $\lceil m_r \rceil = \sigma^{-1}(\sigma(m_r))$ and, by the definition of a circuit in $T_{\lceil\rceil}$, $\lceil m_r \rceil \lceil m_{r+1} \rceil \in E(T_{\lceil\rceil})$. Thus,

(28) $$\sigma(m_r) \neq \sigma(m_{r+1}) \text{ and } \sigma(m_r) \cap \sigma(m_{r+1}) \neq \emptyset \text{ for } 1 \leq r < w.$$

Note that $\sigma(m_1) = \sigma(m_w)$, since $\sigma^{-1}(\sigma(m_1)) = \lceil m_1 \rceil = \lceil m_w \rceil = \sigma^{-1}(\sigma(m_w))$. Then, by the second part of (28) with $r = w - 1$, we have

(29) $$\sigma(m_1) = \sigma(m_w) \text{ and } \sigma(m_{w-1}) \cap \sigma(m_1) \neq \emptyset.$$

Without loss of generality it can be assumed that

$$\lceil m_r \rceil \subseteq T^{-1} \iff r = 1, 3, 5, \ \ldots, w$$

$$\lceil m_r \rceil \subseteq T^1 \iff r = 2, 4, \ \ldots, w-1.$$

So if r is even then m_1 and m_r have opposite signs, and hence $\sigma(m_1) \neq \sigma(m_r)$. Thus, as $\sigma(m_1) = \sigma(m_w)$,

$$w \text{ is odd and } w \geq 5.$$

Since $\mathcal{H}^\star$ is a $\mathcal{P}$-continuum and $(\mathcal{H} \cap \mathcal{P}_{-1})^\star$ has distinct components $\sigma(m_1)$ and $\sigma(m_3)$, $\mathcal{H}^\star \neq (\mathcal{H} \cap \mathcal{P}_{-1})^\star$. Hence $\mathcal{H} \neq \mathcal{H} \cap \mathcal{P}_{-1}$. Thus $\mathcal{H} \nsubseteq \mathcal{P}_{-1}$, and, by Lemma 1.1, $\mathcal{H}^\star \nsubseteq \mathcal{P}_{-1}^\star$. Similarly, as $\sigma(m_2)$ and $\sigma(m_4)$ are distinct components of $(\mathcal{H} \cap \mathcal{P}_1)^\star$, we have $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star$. Since σ is a surjection, $\mathcal{H}^\star \nsubseteq \mathcal{P}_{-1}^\star$, $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star$, and $\mathcal{H}^\star$ is $\mathcal{P}$-unicoherent, Lemmas 3.10 and 3.9 guarantee that

$$\Upsilon \text{ is a } \mathcal{P}\text{-decomposition of } \mathcal{H}^\star \text{ and a treechain.}$$

The components $\sigma(m_1)$, $\sigma(m_3), \ldots , \sigma(m_{w-2})$ of $(\mathcal{H}\cap\mathcal{P}_{-1})^\star$ all belong to Υ_1 since, by (28), (29) and (27), each one intersects at least two components of $(\mathcal{H} \cap \mathcal{P}_1)^\star$. Likewise, the components $\sigma(m_2)$, $\sigma(m_4), \ldots , \sigma(m_{w-1})$ of $(\mathcal{H} \cap \mathcal{P}_1)^\star$ all belong to Υ_{-1}. Then,

$$\{\sigma(m_1), \sigma(m_2), \ldots , \sigma(m_{w-1})\} \subseteq \Upsilon_{-1} \cup \Upsilon_1 = \Upsilon.$$

We have seen that $\sigma(m_1), \sigma(m_2), \ldots , \sigma(m_{w-1})$ are distinct members of the treechain Υ with $w \geq 5$, that $\sigma(m_1)\cap\sigma(m_{w-1}) \neq \emptyset$, and that $\sigma(m_r)\cap\sigma(m_{r+1}) \neq \emptyset$ for $1 \leq r < w$. However, this contradicts Lemma 3.8. The contradiction shows that $T_{\lceil\rceil}$ in fact has no circuit. Thus, as $T_{\lceil\rceil}$ has already been seen to be a connected signed graph, $T_{\lceil\rceil}$ is a signed tree.

Now suppose $m \in T^{-1}$ and $n \in T^1$. Then $\sigma(m) \cap \sigma(n) \subseteq (\mathcal{H} \cap \mathcal{P}_{-1})^\star \cap (\mathcal{H} \cap \mathcal{P}_1)^\star \subseteq \mathcal{P}_{-1}^\star \cap \mathcal{P}_1^\star$. We must show that $\sigma(m) \cap \sigma(n)$ is connected, and to do so it can be assumed that $\emptyset \neq \sigma(m) \cap \sigma(n)$, $\sigma(m) \nsubseteq \sigma(n)$, and $\sigma(n) \nsubseteq \sigma(m)$. Hence, $\sigma(n) \neq \mathcal{H}^\star \neq \sigma(m)$. Since $\sigma(m)$ is a component of $(\mathcal{H} \cap \mathcal{P}_{-1})^\star$ distinct from $\mathcal{H}^\star$, one has $(\mathcal{H} \cap \mathcal{P}_{-1})^\star \neq \mathcal{H}^\star$. So $\mathcal{H} \nsubseteq \mathcal{P}_{-1}$ and, by Lemma 1.1, $\mathcal{H}^\star \nsubseteq \mathcal{P}_{-1}^\star$. Similarly, as $\sigma(n)$ is a component of $(\mathcal{H} \cap \mathcal{P}_1)^\star$ distinct from $\mathcal{H}^\star$, one obtains $\mathcal{H}^\star \nsubseteq \mathcal{P}_1^\star$. Then Lemmas 3.10 and 3.9 again give

$$\Upsilon \text{ is a } \mathcal{P}\text{-decomposition of } \mathcal{H}^\star \text{ and a treechain.}$$

Since $\sigma(n)$ is a component of $(\mathcal{H} \cap \mathcal{P}_1)^\star$ that intersects but fails to contain $\sigma(m)$, $\sigma(m)$ is not a subset of any component of $(\mathcal{H} \cap \mathcal{P}_1)^\star$. Consequently, $\sigma(m) \in \Upsilon_1$. Similarly, one obtains $\sigma(n) \in \Upsilon_{-1}$. Hence $\sigma(m)$, $\sigma(n) \in \Upsilon$ and, by Lemma 3.9, $\sigma(m) \cap \sigma(n)$ is connected. $\square$

Lemma 6.2. *Let K be a closed subset of a Hausdorff compactum Q and let L be a continuum in Q that intersects both K and $Q\backslash K$. Then each component of $L \cap K$ intersects $Bd_Q(K)$.*

Proof. Let C be any component of $L \cap K$. $L \cap K$ is a nonempty proper closed subset of the continuum L. Note that $Bd_L(L \cap K) \subseteq Bd_Q(L \cap K)$. Also, $C \cap Bd_L(L \cap K) \neq \emptyset$ by Theorem 3, page 173, of [Ku]. Choose $x \in C \cap Bd_L(L \cap K)$.

Then $x \notin Int_L(L \cap K)$ and

$$x \in C \cap Bd_Q(L \cap K) = C \cap (L \cap K) \backslash Int_Q(L \cap K).$$

We have $x \notin Int_Q(K)$, as otherwise $L \cap Int_Q(K)$ is a subset of $L \cap K$ that contains x and is open in L, contrary to $x \notin Int_L(L \cap K)$. Thus, $x \in K \backslash Int_Q(K) = Bd_Q(K)$. Therefore, $x \in C \cap Bd_Q(K)$. So C intersects $Bd_Q(K)$. □

Theorem 6.3. *Suppose the Hausdorff continuum P is the union of unicoherent continua J and K. If $J \cap K$ is connected and locally connected, then P is unicoherent.*

Proof. We can assume without loss of generality that J and K are proper subcontinua of P, so

$$K \backslash J = P \backslash J \neq \emptyset \neq P \backslash K = J \backslash K. \tag{30}$$

Since $J \cup K = P$ and J and K are closed, $Bd(J) = J \backslash Int(J) \subseteq J \cap K$ and $Bd(K) = K \backslash Int(K) \subseteq J \cap K$. Let $V = J \cap K$. Then

$J \cap K = V$ is a locally connected continuum with $Bd_P(J) \cup Bd_P(K) \subseteq V$.

We apply Proposition 4.2. Let $\Im$ be any three-element open cover of P. For each $x \in V$ there exists $O(x) \in \Im$ with $x \in O(x)$. Then $V \cap O(x)$ is an open neighborhood of x in V. Since V is a locally connected Hausdorff compactum, one can find a connected relatively open set $V(x) \subseteq V$ with $x \in V(x) \subseteq Cl(V(x)) \subseteq V \cap O(x)$. Since V is compact there exist finitely many elements of V, $x_1, \ldots, x_s$, such that the collection of continua

$$\mathcal{V} = \{Cl(V(x_1)), \ldots, Cl(V(x_s))\}$$

covers V irreducibly. Define

$$\begin{gathered}\mathcal{P}(J \backslash K) = \{\{x\} : x \in J \backslash K\}\ ,\ \mathcal{P}(K \backslash J) = \{\{x\} : x \in K \backslash J\}\ ,\\ \mathcal{J} = \mathcal{V} \cup \mathcal{P}(J \backslash K)\ ,\ \mathcal{K} = \mathcal{V} \cup \mathcal{P}(K \backslash J),\\ \text{and } \mathcal{P} = \mathcal{J} \cup \mathcal{K}.\end{gathered}$$

Then

$$\begin{gathered}\mathcal{P}(J \backslash K)^\star = J \backslash K\ ,\ \mathcal{V}^\star = V\ ,\ \mathcal{P}(K \backslash J)^\star = K \backslash J\ ,\\ \mathcal{J}^\star = J\ ,\ \mathcal{K}^\star = K, \text{ and } \mathcal{P}^\star = P.\end{gathered}$$

Note that the nonempty sets V, $J \backslash K$ and $K \backslash J$ are pairwise disjoint, and $\mathcal{P}$ is the union of the nonempty disjoint collections $\mathcal{V}, \mathcal{P}(J \backslash K)$ and $\mathcal{P}(K \backslash J)$. By construction, $\mathcal{P}$ is a refinement of $\Im$. Moreover, $\mathcal{P} \subset C(P)$ and $\mathcal{P}$ covers P irreducibly. Also, for each $M \in C(P)$ the set $\mathcal{P}(M)^\star$ is the union of M and all those elements of $\mathcal{V}$ that intersect M, and hence is a closed set. Thus, $\mathcal{P}$ is a pseudogrille for P. Now, to complete the proof of Theorem 6.3 (by Proposition 4.2) it suffices to show that P is $\mathcal{P}$-unicoherent.

Let $\{\mathcal{P}^\star_{-1}, \mathcal{P}^\star_1\}$ be a two-element $\mathcal{P}$-decomposition of P. It must be proven that $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ is connected. (We note that, although the proof given here is a direct argument based upon Proposition 4.2, $\mathcal{P}^\star_{-1}$ and $\mathcal{P}^\star_1$ portray the sets P'_{-1} and P'_1 described in the first part of the Introduction. Also, the roles of J_i and K_i are played by $\mathcal{J}^\star_i$ and $\mathcal{K}^\star_i$, while $\mathcal{V}(P_{-1})$ and $\mathcal{V}(P_1)$ are the collections $\mathcal{V}_{-1} \cup \ldots \cup \mathcal{V}_{-u}$ and $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_v$, respectively.)

Since $\{\mathcal{P}^\star_{-1}, \mathcal{P}^\star_1\}$ is a $\mathcal{P}$-decomposition of $P = \mathcal{P}^\star$, one has $\mathcal{P}_{-1} \cup \mathcal{P}_1 = \mathcal{P}$. Thus, $\mathcal{P}_{-1} \cup \mathcal{P}_1$ is the union of the nonempty disjoint collections $\mathcal{V}$, $\mathcal{P}(J \backslash K)$ and $\mathcal{P}(K \backslash J)$. For each $i \in \{-1, 1\}$, let

$$\mathcal{J}_i = \mathcal{J} \cap \mathcal{P}_i \ \text{ and } \mathcal{K}_i = \mathcal{K} \cap \mathcal{P}_i.$$

Note that for each $x \in J \cap \mathcal{P}^\star_1$ there exists $G_x \in \mathcal{P}_1 \backslash \mathcal{P}(K \backslash J)$ with $x \in G_x$. Thus $x \in G_x \in \mathcal{P}_1 \cap (\mathcal{V} \cup \mathcal{P}(J \backslash K)) = \mathcal{P}_1 \cap \mathcal{J}$. Consequently, $J \cap \mathcal{P}^\star_1 \subseteq (\mathcal{P}_1 \cap \mathcal{J})^\star = \mathcal{J}^\star_1$. Also, $\mathcal{J}^\star_1 = (\mathcal{J} \cap \mathcal{P}_1)^\star \subseteq \mathcal{J}^\star \cap \mathcal{P}^\star_1 = J \cap \mathcal{P}^\star_1$. So $\mathcal{J}^\star_1 = J \cap \mathcal{P}^\star_1$. By repeating this reasoning for $\mathcal{J}^\star_{-1}$, $\mathcal{K}^\star_{-1}$ and $\mathcal{K}^\star_1$, one obtains $\mathcal{J}^\star_i = J \cap \mathcal{P}^\star_i$ and $\mathcal{K}^\star_i = K \cap \mathcal{P}^\star_i$ for each $i \in \{-1, 1\}$.

$$\mathcal{J}^\star_i = J \cap \mathcal{P}^\star_i \text{ and } \mathcal{K}^\star_i = K \cap \mathcal{P}^\star_i \text{ for each } i \in \{-1, 1\}. \tag{31}$$

These equalities can be written as $(\mathcal{J} \cap \mathcal{P}_i)^\star = \mathcal{J}^\star \cap \mathcal{P}^\star_i$ and $(\mathcal{K} \cap \mathcal{P}_i)^\star = \mathcal{K}^\star \cap \mathcal{P}^\star_i$, and hence represent hypotheses to be used in two settings of Lemma 6.1 in Case 2 below. It also follows from the definitions of $\mathcal{V}$ and $\mathcal{P}$ that $(\mathcal{V} \cap \mathcal{P}_i)^\star = \mathcal{V}^\star \cap \mathcal{P}^\star_i$ for each i, which will allow a third application of Lemma 6.1. The proof of the connectedness of $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ breaks into two cases.

Case 1 $\qquad \mathcal{J}_{-1} = \emptyset$ or $\mathcal{J}_1 = \emptyset$ or $\mathcal{K}_{-1} = \emptyset$ or $\mathcal{K}_1 = \emptyset$.

We prove $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ is connected when $\mathcal{J}_1 = \emptyset$, the proof in each of the other three subcases being analogous.

Since $\mathcal{J} \subseteq \mathcal{P} = \mathcal{P}_{-1} \cup \mathcal{P}_1$ and $\mathcal{J} \cap \mathcal{P}_1 = \mathcal{J}_1 = \emptyset$, one has $\mathcal{J} \subseteq \mathcal{P}_{-1}$. Moreover, $\mathcal{V} \subseteq \mathcal{J} \subseteq \mathcal{P}_{-1}$. Therefore, as $V = J \cap K \subseteq K$,

$$V = \mathcal{V}^\star \subseteq \mathcal{P}^\star_{-1} \cap K.$$

Since $\mathcal{J} \cap \mathcal{P}_1 = \emptyset$, we have $\mathcal{P}_1 \subseteq \mathcal{P} \backslash \mathcal{J} = \mathcal{P} \backslash (\mathcal{V} \cup \mathcal{P}(J \backslash K)) = \mathcal{P}(K \backslash J)$. Thus $\mathcal{P}_1^\star \subseteq \mathcal{P}(K \backslash J)^\star \subseteq K$, and

$$\mathcal{P}_1^\star \cap \mathcal{P}_{-1}^\star = (\mathcal{P}_1^\star \cap K) \cap \mathcal{P}_{-1}^\star = \mathcal{P}_1^\star \cap (\mathcal{P}_{-1}^\star \cap K). \tag{32}$$

Also,

$$K = P \cap K = (\mathcal{P}_1^\star \cup \mathcal{P}_{-1}^\star) \cap K = \mathcal{P}_1^\star \cup (\mathcal{P}_{-1}^\star \cap K). \tag{33}$$

By (30) there exists $x \in J \backslash K$. Since $\mathcal{P}(J \backslash K) \cap \mathcal{P}_1 \subseteq \mathcal{J} \cap \mathcal{P}_1 = \emptyset$, we have $\mathcal{P}(J \backslash K) \cap \mathcal{P}_1 = \emptyset$. Thus $\{x\} \in \mathcal{P}(J \backslash K) \backslash \mathcal{P}_1 \subseteq \mathcal{P}(J \backslash K) \cap \mathcal{P}_{-1}$. Hence $x \in (J \backslash K) \cap \mathcal{P}_{-1}^\star$. We have $x \in (P \backslash K) \cap \mathcal{P}_{-1}^\star$ and $V \subseteq \mathcal{P}_{-1}^\star \cap K$. So the subcontinuum $\mathcal{P}_{-1}^\star$ of P intersects both $P \backslash K$ and K. By Lemma 6.2, each component of $\mathcal{P}_{-1}^\star \cap K$ intersects $Bd_P(K)$. Therefore, as $Bd_P(K) \subseteq V$, each component of $\mathcal{P}_{-1}^\star \cap K$ intersects V. Hence, as the continuum V is a subset of $\mathcal{P}_{-1}^\star \cap K$, $\mathcal{P}_{-1}^\star \cap K$ is a subcontinuum of K. Then by (33) K is the union of its subcontinua $\mathcal{P}_1^\star$ and $\mathcal{P}_{-1}^\star \cap K$. Thus, by the assumed unicoherence of K, the set $\mathcal{P}_1^\star \cap (\mathcal{P}_{-1}^\star \cap K)$ is connected. Therefore, by (32), $\mathcal{P}_{-1}^\star \cap \mathcal{P}_1^\star$ is connected.

Case 2 $\quad \mathcal{J}_{-1} \neq \emptyset$ and $\mathcal{J}_1 \neq \emptyset$ and $\mathcal{K}_{-1} \neq \emptyset$ and $\mathcal{K}_1 \neq \emptyset$.

In Case 2 , boundary-bumping gives the following.

$$\begin{aligned}&\text{Suppose } i \in \{-1, 1\} \text{ and } A \text{ is any component of } \mathcal{J}_i^\star \text{ or } \mathcal{K}_i^\star.\\ &\text{Then } A \cap V \neq \emptyset \text{ and } A \text{ is a } \mathcal{P}_i\text{-continuum, } \mathcal{A}^\star, \text{ with } \mathcal{A} \cap \mathcal{V} \neq \emptyset.\end{aligned} \tag{34}$$

For example, let A be any component of $\mathcal{J}_1^\star$. A is nonempty and, by (31), A is a component of $J \cap \mathcal{P}_1^\star$. To see that $A \cap V$ is nonempty we first suppose that $\mathcal{P}_1^\star \subseteq J$. Then $J \cap \mathcal{P}_1^\star = \mathcal{P}_1^\star$ is connected, and hence $A = \mathcal{P}_1^\star$. Moreover, $\mathcal{K}_1 \neq \emptyset$, so $\emptyset \neq \mathcal{K}_1^\star = K \cap \mathcal{P}_1^\star = K \cap A = K \cap (A \cap J) = A \cap V$. Suppose next that $\mathcal{P}_1^\star \nsubseteq J$. Then the subcontinuum $\mathcal{P}_1^\star$ of P intersects both $P \backslash J$ and J. By Lemma 6.2, each component of $J \cap \mathcal{P}_1^\star$ intersects $Bd_P(J)$. Therefore, as $Bd_P(J) \cup Bd_P(K) \subseteq V$, each component of $J \cap \mathcal{P}_1^\star$ intersects V. Hence, $A \cap V \neq \emptyset$. We have shown that if A is any component of $\mathcal{J}_1^\star$ then $A \cap V \neq \emptyset$. By Lemma 2.10, $A = \mathcal{A}^\star$ for some $\mathcal{A} \subseteq \mathcal{J}_1 = \mathcal{P}_1 \cap \mathcal{J}$. That is, A is the $\mathcal{P}_1$-continuum $\mathcal{A}^\star$. Then, since $A \cap V \neq \emptyset$ there exists $G \in \mathcal{A}$ with $G \cap V \neq \emptyset$. We have

$$G \in \mathcal{A} \subseteq \mathcal{J}_1 = \mathcal{P}_1 \cap \mathcal{J} = \mathcal{P}_1 \cap (\mathcal{V} \cup \mathcal{P}(J \backslash K)).$$

But $G \notin \mathcal{P}(J\backslash K)$ since $G \cap V \neq \emptyset$. Thus, $G \in \mathcal{A} \cap \mathcal{V}$. Therefore $\mathcal{A} \cap \mathcal{V} \neq \emptyset$, as required. Analogous reasoning for components of $\mathcal{J}^\star_{-1}$, $\mathcal{K}^\star_1$, and $\mathcal{K}^\star_{-1}$ establishes (34).

Define $\mathcal{Z} = \mathcal{P}_{-1} \cap \mathcal{P}_1$. By Lemma 2.18,

$$\mathcal{Z}^\star = (\mathcal{P}_{-1} \cap \mathcal{P}_1)^\star = \mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1.$$

By Lemma 2.10,

$$C = \mathcal{Z}(C)^\star \text{ (a } \mathcal{P}\text{-continuum) for each component } C \text{ of } \mathcal{Z}^\star.$$

Moreover, we claim that

$$\mathcal{Z}(C) \cap \mathcal{V} \neq \emptyset \text{ for each component } C \text{ of } \mathcal{Z}^\star. \tag{35}$$

For if not then there is a component $C = \mathcal{Z}(C)^\star$ of $\mathcal{Z}^\star$ with $\mathcal{Z}(C) \subseteq \mathcal{P}\backslash\mathcal{V} = \mathcal{P}(J\backslash K) \cup \mathcal{P}(K\backslash J)$. Then the $\mathcal{P}$-continuum $\mathcal{Z}(C)^\star$ is contained in the union of the disjoint open sets $\mathcal{P}(J\backslash K)^\star = J\backslash K = P\backslash K$ and $\mathcal{P}(K\backslash J)^\star = K\backslash J = P\backslash J$. Hence, either $\mathcal{Z}(C)^\star \subseteq J\backslash K$ or $\mathcal{Z}(C)^\star \subseteq K\backslash J$. Without loss of generality we assume $\mathcal{Z}(C)^\star \subseteq J\backslash K$. Thus $\mathcal{Z}(C)^\star$ does not intersect the continuum $V = J \cap K$, and $\mathcal{Z}(C)^\star$ is a component of the Hausdorff compactum $\mathcal{Z}^\star \cap J$. Then, by Theorem 2$-$15 in [Ho], $(\mathcal{Z}^\star \cap J)\backslash V$ contains a set Y that is open in $\mathcal{Z}^\star \cap J$, has empty boundary in $\mathcal{Z}^\star \cap J$, and contains $\mathcal{Z}(C)^\star$. Since Y has empty boundary in $\mathcal{Z}^\star \cap J$, Y is closed as well as open in $\mathcal{Z}^\star \cap J$. Note that

$$Y = ((\mathcal{Z}^\star \cap J) \cup V)\backslash((\mathcal{Z}^\star \cap J)\backslash Y) \cup V).$$

Now Y and $(\mathcal{Z}^\star \cap J)\backslash Y$ are closed in the closed subset $\mathcal{Z}^\star \cap J$ of $(\mathcal{Z}^\star \cap J) \cup V$, so Y and $(\mathcal{Z}^\star \cap J)\backslash Y$ are closed subsets of $(\mathcal{Z}^\star \cap J) \cup V$. Also, V is closed in $(\mathcal{Z}^\star \cap J) \cup V$. So $((\mathcal{Z}^\star \cap J)\backslash Y) \cup V$ is a closed subset of $(\mathcal{Z}^\star \cap J) \cup V$. Therefore, $Y = ((\mathcal{Z}^\star \cap J) \cup V)\backslash((\mathcal{Z}^\star \cap J)\backslash Y) \cup V)$ is open in $(\mathcal{Z}^\star \cap J) \cup V$. To sum up, $\emptyset \neq \mathcal{Z}(C)^\star \subseteq Y$, $Y \cap V = \emptyset$, and Y is closed and open in $(\mathcal{Z}^\star \cap J) \cup V$. Thus,

$$(\mathcal{Z}^\star \cap J) \cup V \text{ is not connected.}$$

Now let R be the compactum $(J \cap \mathcal{P}^\star_{-1}) \cup V$. By (31) and (34), each component of $J \cap \mathcal{P}^\star_{-1}$ is a continuum that intersects the continuum V. Thus, R is connected. Hence R is a continuum. Similarly, the compactum $S = (J \cap \mathcal{P}^\star_1) \cup V$ is a continuum. Note that $R \cup S = V \cup (J \cap (\mathcal{P}^\star_{-1} \cup \mathcal{P}^\star_1)) = J$. Then, by the assumed unicoherence of J, $R \cap S$ is connected. But $R \cap S = (\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1 \cap J) \cup V = (\mathcal{Z}^\star \cap J) \cup V$, which is not connected. This contradiction establishes (35).

Let C be a component of $\mathcal{Z}^\star$. By (35), $\mathcal{Z}(C)\cap\mathcal{V}$ is nonempty, so $\mathcal{Z}\cap\mathcal{V}=\mathcal{P}_{-1}\cap\mathcal{P}_1\cap\mathcal{V}$ is nonempty. Observe then that for each $i\in\{-1,1\}$, since $0<|\mathcal{V}\cap\mathcal{P}_i|\leq|\mathcal{V}|<\infty$, the $\mathcal{P}$-compactum $(\mathcal{V}\cap\mathcal{P}_i)^\star$ has finitely many components, each one a union of elements of $\mathcal{V}$. Let $\mathcal{V}^\star_{-1},\dots,\mathcal{V}^\star_{-u}$ be the components of $(\mathcal{V}\cap\mathcal{P}_{-1})^\star$ and $\mathcal{V}^\star_1,\dots,\mathcal{V}^\star_v$ be the components of $(\mathcal{V}\cap\mathcal{P}_1)^\star$. Let $T^{-1}=\{-1,\ \dots,-u\}$, $T^1=\{1,\dots,v\}$, and $T=T^{-1}\cup T^1$. By Lemma 2.10,

$$\mathcal{V}_m\subseteq\mathcal{V}\cap\mathcal{P}_1 \text{ for each } m\in T^1 \text{ and } \mathcal{V}_n\subseteq\mathcal{V}\cap\mathcal{P}_{-1} \text{ for each } n\in T^{-1}.$$

Now suppose $\mathcal{V}^\star_m=\mathcal{V}^\star_n$ for some $m\in T^1$ and $n\in T^{-1}$. Then $\mathcal{V}_m=\mathcal{V}_n\subseteq\mathcal{V}$ by Lemma 1.1, and we claim that $\mathcal{V}_m=\mathcal{V}_n=\mathcal{V}$. For if $\mathcal{V}_m=\mathcal{V}_n\neq\mathcal{V}$ then, since $\mathcal{V}^\star$ is a continuum and $|\mathcal{V}|<\infty$, there exists $G\in\mathcal{V}\backslash\mathcal{V}_m=\mathcal{V}\backslash\mathcal{V}_n$ such that $\emptyset\neq G\cap\mathcal{V}^\star_m=G\cap\mathcal{V}^\star_n$. Also, by Lemma 2.9, $G\nsubseteq\mathcal{V}^\star_m=\mathcal{V}^\star_n$. Moreover, $G\notin\mathcal{P}_{-1}$, as otherwise $\mathcal{V}^\star_n\cup G$ is a connected subset of $(\mathcal{V}\cap\mathcal{P}_{-1})^\star$ properly containing the component $\mathcal{V}^\star_n$ of $(\mathcal{V}\cap\mathcal{P}_{-1})^\star$. Likewise, $G\notin\mathcal{P}_1$, else $\mathcal{V}^\star_m\cup G$ is a connected subset of $(\mathcal{V}\cap\mathcal{P}_1)^\star$ properly containing the component $\mathcal{V}^\star_m$ of $(\mathcal{V}\cap\mathcal{P}_1)^\star$. But $G\in\mathcal{P}=\mathcal{P}_{-1}\cup\mathcal{P}_1$. This contradiction shows that $\mathcal{V}_m=\mathcal{V}_n=\mathcal{V}$. Thus, $\mathcal{V}\subseteq\mathcal{P}_{-1}\cap\mathcal{P}_1=\mathcal{Z}$. Hence, by (35), each component $C=\mathcal{Z}(C)^\star$ of $\mathcal{Z}^\star$ contains the continuum $\mathcal{V}^\star=V$. Thus $\mathcal{Z}^\star=\mathcal{P}^\star_{-1}\cap\mathcal{P}^\star_1$ is connected if $\mathcal{V}^\star_m=\mathcal{V}^\star_n$ for some $m\in T^1$ and $n\in T^{-1}$. Therefore, for the balance of the proof we can and will assume that

$$\mathcal{V}^\star_m\neq\mathcal{V}^\star_n \text{ whenever } m\in T^1 \text{ and } n\in T^{-1}. \tag{36}$$

Recalling (34) and (31), let $\Sigma\mathcal{J}^\star=\{\mathcal{A}^\star:\mathcal{A}^\star \text{ is a component of } \mathcal{J}^\star_{-1} \text{ or of } \mathcal{J}^\star_1\}$. For each $i\in\{-1,1\}$ and $m\in T^i$, the set $\mathcal{V}^\star_m$ is a component of $(\mathcal{V}\cap\mathcal{P}_i)^\star$, and since $\mathcal{V}\subseteq\mathcal{J}$, $\mathcal{V}^\star_m$ is contained in some component of $(\mathcal{J}\cap\mathcal{P}_i)^\star=\mathcal{J}^\star_i$. For each $i\in\{-1,1\}$ and $m\in T^i$ let $\sigma_J(m)$ be the component of $\mathcal{J}^\star_i$ that contains $\mathcal{V}^\star_m$. We claim the map $\sigma_J:T\longrightarrow\Sigma\mathcal{J}^\star$ is a surjection. For suppose $i\in\{-1,1\}$ and $\mathcal{A}^\star$ is a component of $\mathcal{J}^\star_i$. By (34) $\mathcal{A}^\star$ is a $\mathcal{P}_i$-continuum, so $\mathcal{A}\subseteq\mathcal{P}_i$, and there exists $G\in\mathcal{V}\cap\mathcal{A}$. Thus, $G\in\mathcal{V}\cap\mathcal{P}_i$. Let $\mathcal{V}^\star_m$ be the component of $(\mathcal{V}\cap\mathcal{P}_i)^\star$ that contains G. Then $\sigma_J(m)=\mathcal{A}^\star$. Hence σ_J is a surjective map. It will next be shown that

$$\sigma_J(r) \text{ is a component of } \mathcal{J}^\star_i \text{ if and only if } r\in T^i,\ \text{ for } i\in\{-1,1\}. \tag{37}$$

For otherwise there exist $i\in\{-1,1\}$ and $r\in T^{-i}$ so that $\sigma_J(r)$ is a component of $\mathcal{J}^\star_i=(\mathcal{J}\cap\mathcal{P}_i)^\star$. By definition of σ_J, and as $r\in T^{-i}$, $\sigma_J(r)$ is also the component $\mathcal{A}^\star$ of $\mathcal{J}^\star_{-i}=(\mathcal{J}\cap\mathcal{P}_{-i})^\star$ that contains $\mathcal{V}^\star_r$. Thus $\mathcal{A}^\star$ is a component of $(\mathcal{J}\cap\mathcal{P}_i)^\star$

and a component of $(\mathcal{J} \cap \mathcal{P}_{-i})^\star$. Then Lemma 2.10 gives

$$\mathcal{V}_r \subseteq \mathcal{A} \subseteq (\mathcal{J} \cap \mathcal{P}_{-i}) \cap (\mathcal{J} \cap \mathcal{P}_i) \subseteq \mathcal{P}_{-1} \cap \mathcal{P}_1.$$

Now, by (36), either $\mathcal{V}_1 \neq \mathcal{V}$ or $\mathcal{V}_{-1} \neq \mathcal{V}$. Assume first that $\mathcal{V}_1 \neq \mathcal{V}$. By Lemma 1.1, $\mathcal{V}_1^\star \neq \mathcal{V}^\star$. Then, since $\mathcal{V}_1^\star$ is a component of $(\mathcal{V} \cap \mathcal{P}_1)^\star$ not equal to the connected set $\mathcal{V}^\star$, we have $\mathcal{V} \neq \mathcal{V} \cap \mathcal{P}_1$. So $\mathcal{V} \nsubseteq \mathcal{P}_1$. Thus $\mathcal{V}_r \neq \mathcal{V}$. Similarly, if $\mathcal{V}_{-1} \neq \mathcal{V}$ a symmetric argument gives $\mathcal{V} \nsubseteq \mathcal{P}_{-1}$ and, again, $\mathcal{V}_r \neq \mathcal{V}$. So $\mathcal{V}_r \subset \mathcal{V}$ and $0 < |\mathcal{V}_r| < |\mathcal{V}| < \infty$. Moreover, $\mathcal{V}^\star$ and each $G \in \mathcal{V}$ are continua. Hence there exists $G \in \mathcal{V} \backslash \mathcal{V}_r$ with $\emptyset \neq G \cap \mathcal{V}_r^\star$. By Lemma 2.9, $G \nsubseteq \mathcal{V}_r^\star$. Then, as $r \in T^{-i}$ and $\mathcal{V}_r^\star$ is a component of $(\mathcal{P}_{-i} \cap \mathcal{V})^\star$ that intersects but does not contain the element G of $\mathcal{V}$, we have $G \in \mathcal{P} \backslash \mathcal{P}_{-i}$. Thus, as $\mathcal{A} \subseteq \mathcal{P}_{-1} \cap \mathcal{P}_1$, we have $G \notin \mathcal{A}$. Therefore, by Lemma 2.9, $G \nsubseteq \mathcal{A}^\star$. But $\emptyset \neq G \cap \mathcal{V}_r^\star \subseteq G \cap \mathcal{A}^\star$ and $G \in \mathcal{V} \backslash \mathcal{P}_{-i} \subseteq \mathcal{J} \cap \mathcal{P}_i$. Then, as $\mathcal{A}^\star$ is a component of $(\mathcal{J} \cap \mathcal{P}_i)^\star$ that intersects G and $G \in \mathcal{J} \cap \mathcal{P}_i$, one has $G \subseteq \mathcal{A}^\star$. This is a contradiction. The contradiction shows that (37) holds. Next, let $[m] = \sigma_J^{-1}(\sigma_J(m))$ for each $m \in T$. Define

$$V(T_{[\,]}) = \{[m] : m \in T\}$$
$$E(T_{[\,]}) = \{[m]\,[n] : \sigma_J(m) \neq \sigma_J(n) \text{ and } \sigma_J(m) \cap \sigma_J(n) \neq \emptyset\}$$

No distinction is made between $[m]\,[n]$ and $[n]\,[m]$ .Then, since $J = \mathcal{J}^\star$ is unicoherent, Lemma 6.1 says that $T_{[\,]}$ is a signed tree on T and

(38) $\quad \sigma_J(m) \cap \sigma_J(n)$ is a connected subset of $\mathcal{Z}^\star$ for all $m \in T^1,\ n \in T^{-1}$.

Similarly, we let $\Sigma\mathcal{K}^\star = \{\mathcal{B}^\star : \mathcal{B}^\star$ is a component of $\mathcal{K}_{-1}^\star$ or of $\mathcal{K}_1^\star\}$. For each $i \in \{-1, 1\}$ and $m \in T^i$ let $\sigma_K(m)$ be the component of $\mathcal{K}_i^\star$ that contains $\mathcal{V}_m^\star$. Then $\sigma_K : T \longrightarrow \Sigma\mathcal{K}^\star$ is a surjection by (34), and by a proof symmetric to that of (37) we have

(39) $\quad \sigma_K(r)$ is a component of $\mathcal{K}_i^\star$ if and only if $r \in T^i$, for $i \in \{-1, 1\}$.

Let $\langle m \rangle = \sigma_K^{-1}(\sigma_K(m))$ for each $m \in T$, and define

$$V(T_{\langle\,\rangle}) = \{\langle m \rangle : m \in T\}$$
$$E(T_{\langle\,\rangle}) = \{\langle m \rangle \langle n \rangle : \sigma_K(m) \neq \sigma_K(n) \text{ and } \sigma_K(m) \cap \sigma_K(n) \neq \emptyset\}$$

No distinction is made between $\langle m \rangle \langle n \rangle$ and $\langle n \rangle \langle m \rangle$. Again by Lemma 6.1, $T_{\langle\,\rangle}$ is a signed tree on T and

(40) $\quad \sigma_K(m) \cap \sigma_K(n)$ is a connected subset of $\mathcal{Z}^\star$ for all $m \in T^1,\ n \in T^{-1}$.

A third connected signed graph, $T_{(\,)}$, is defined on T as follows. Let $(m) = \{m\}$ for each $m \in T$. Let $\Sigma\mathcal{V}^\star = \{\mathcal{V}_m^\star : -u \leq m \leq -1\} \cup \{\mathcal{V}_n^\star : 1 \leq n \leq v\}$. Define

$$V(T_{(\,)}) = \{(m) : m \in T\}$$
$$E(T_{(\,)}) = \{(m)(n) : m \in T^i,\ n \in T^{-i} \text{ for } i = -1 \text{ or } 1,\ \text{and } \mathcal{V}_m^\star \cap \mathcal{V}_n^\star \neq \emptyset\}$$

No distinction is made between $(m)(n)$ and $(n)(m)$. For each $m \in T$ let $\sigma_V(m) = \mathcal{V}_m^\star$. By (36) and the definition of the components $\mathcal{V}_m^\star$, the map $\sigma_V : T \longrightarrow \Sigma\mathcal{V}^\star$ is a bijection, and $\sigma_V(m)$ is a component of $(\mathcal{P}_i \cap \mathcal{V})^\star$ if and only if $m \in T^i$. Thus $\sigma_V^{-1}(\sigma_V(m)) = \{m\} = (m)$ for each $m \in T$. Also, $\mathcal{V}^\star = V$ is a $\mathcal{P}$-continuum, and by the various definitions we have $(\mathcal{V} \cap \mathcal{P}_i)^\star = \mathcal{V}^\star \cap \mathcal{P}_i^\star$ for each i. Therefore, by Lemma 6.1, $T_{(\,)}$ is a connected signed graph on T. We prove the following assertion.

(41) If $\mathcal{A}^\star \in \Sigma\mathcal{J}^\star$ and $\mathcal{B}^\star \in \Sigma\mathcal{K}^\star$ then $\mathcal{A}^\star \cap \mathcal{B}^\star = (\mathcal{A} \cap \mathcal{B} \cap \mathcal{V})^\star$.

Clearly $(\mathcal{A}\cap\mathcal{B}\cap\mathcal{V})^\star \subseteq \mathcal{A}^\star\cap\mathcal{B}^\star$. For the reverse inclusion assume $i, j \in \{-1, 1\}$ and $x \in \mathcal{A}^\star \cap \mathcal{B}^\star$, where $\mathcal{A}^\star$ is a component of $\mathcal{J}_i^\star = (\mathcal{J} \cap \mathcal{P}_i)^\star$ and $\mathcal{B}^\star$ is a component of $\mathcal{K}_j^\star = (\mathcal{K} \cap \mathcal{P}_j)^\star$. We want to show that $x \in (\mathcal{A} \cap \mathcal{B} \cap \mathcal{V})^\star$. By Lemma 2.10, $\mathcal{A} \subseteq \mathcal{J} \cap \mathcal{P}_i$ and $\mathcal{B} \subseteq \mathcal{K} \cap \mathcal{P}_j$. Select $G' \in \mathcal{A}$ and $G'' \in \mathcal{B}$ with $x \in G' \cap G''$. Then we have $x \in \mathcal{A}^\star \cap \mathcal{B}^\star \subseteq \mathcal{J}^\star \cap \mathcal{K}^\star = J \cap K = V$. Hence

$$G', G'' \in \mathcal{P}\backslash(\mathcal{P}(J\backslash K) \cup \mathcal{P}(K\backslash J)) = \mathcal{V} = \mathcal{J} \cap \mathcal{K}.$$

Then, since $x \in G'\cap G''$, $G' \in \mathcal{A} \subseteq \mathcal{P}_i$ and $G'' \in \mathcal{B} \subseteq \mathcal{P}_j$, it follows from condition (c) of Definition 2.14 (as applied to the $\mathcal{P}$-decomposition $\{\mathcal{P}_{-1}^\star, \mathcal{P}_1^\star\}$ of P) that either $G'' \in \mathcal{P}_i$ or $G' \in \mathcal{P}_j$. If $G'' \in \mathcal{P}_i$ then $x \in G'' \in \mathcal{J} \cap \mathcal{P}_i$. Then, as x lies in the component $\mathcal{A}^\star$ of $(\mathcal{J} \cap \mathcal{P}_i)^\star$, we have $G'' \subseteq \mathcal{A}^\star$. Hence, by Lemma 2.9, $G'' \in \mathcal{A}$. This yields $x \in G'' \in \mathcal{A}\cap\mathcal{B}\cap\mathcal{V}$, and $x \in (\mathcal{A}\cap\mathcal{B}\cap\mathcal{V})^\star$, as desired. On the other hand, if $G' \in \mathcal{P}_j$ then a symmetric argument yields $x \in G' \in \mathcal{A}\cap\mathcal{B}\cap\mathcal{V}$, and again $x \in (\mathcal{A} \cap \mathcal{B} \cap \mathcal{V})^\star$. This shows that $\mathcal{A}^\star \cap \mathcal{B}^\star \subseteq (\mathcal{A} \cap \mathcal{B} \cap \mathcal{V})^\star$, and completes the proof of (41). We now claim that

(42) If $m \in T^{-1}$ and $n \in T^1$ then $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star = (\mathcal{V}_m \cap \mathcal{V}_n)^\star$, and this set is contained in some component of $\mathcal{Z}^\star$.

Clearly, $(\mathcal{V}_m \cap \mathcal{V}_n)^\star \subseteq \mathcal{V}_m^\star \cap \mathcal{V}_n^\star$. For the opposite inclusion assume $p \in \mathcal{V}_m^\star \cap \mathcal{V}_n^\star$. Then there exist $G' \in \mathcal{V}_m \subseteq \mathcal{P}_{-1} \cap \mathcal{V}$ and $G'' \in \mathcal{V}_n \subseteq \mathcal{P}_1 \cap \mathcal{V}$ with $p \in G' \cap G''$. Moreover, as $\{\mathcal{P}_{-1}^\star, \mathcal{P}_1^\star\}$ is a $\mathcal{P}$-decomposition of $\mathcal{P}^\star = P$, we have (by (c) of Definition 2.14) either $G' \in \mathcal{P}_1$ or $G'' \in \mathcal{P}_{-1}$. Suppose that $G' \in \mathcal{P}_1$. Thus, $p \in G' \in \mathcal{P}_1 \cap \mathcal{P}_{-1} \cap \mathcal{V}$. Then, because $\mathcal{V}_m^\star$ is the component of $(\mathcal{P}_{-1} \cap \mathcal{V})^\star$ containing p and $\mathcal{V}_n^\star$ is the component of $(\mathcal{P}_1 \cap \mathcal{V})^\star$ containing p, there follows $G' \in \mathcal{V}_m$ and $G' \in \mathcal{V}_n$. Therefore, $p \in G' \subseteq (\mathcal{V}_m \cap \mathcal{V}_n)^\star$ if $G' \in \mathcal{P}_1$. Similarly, $p \in$

$G'' \subseteq (\mathcal{V}_m \cap \mathcal{V}_n)^\star$ if $G'' \in \mathcal{P}_{-1}$. Hence, as p was an arbitrary element of $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star$, we have $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star \subseteq (\mathcal{V}_m \cap \mathcal{V}_n)^\star$. Thus $(\mathcal{V}_m \cap \mathcal{V}_n)^\star = \mathcal{V}_m^\star \cap \mathcal{V}_n^\star$. For the second conclusion of (42), note that $\sigma_J(m) \cap \sigma_J(n)$ is a connected subset of $\mathcal{Z}^\star$ by (38). Then, as $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star \subseteq \sigma_J(m) \cap \sigma_J(n) \subseteq \mathcal{Z}^\star$, $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star$ is contained in some component of $\mathcal{Z}^\star$.

Define a binary relation $\sim$ on $E(T_{(\,)})$ by

$$(m)(n) \sim (m')(n') \iff \text{there is a component } C \text{ of } \mathcal{Z}^\star \text{ with } (\mathcal{V}_m^\star \cap \mathcal{V}_n^\star) \cup (\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star) \subseteq C.$$

Clearly, $\sim$ is symmetric. By (42), $\sim$ is reflexive. Suppose $(m)(n) \sim (m')(n')$ and $(m')(n') \sim (m'')(n'')$. Since $(m)(n)$, $(m')(n')$, and $(m'')(n'')$ all belong to $E(T_{(\,)})$, we have $m \neq n$ and $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star \neq \emptyset$, $m' \neq n'$ and $\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star \neq \emptyset$, and $m'' \neq n''$ and $\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star \neq \emptyset$. There are components C and C' of $\mathcal{Z}^\star$ with $(\mathcal{V}_m^\star \cap \mathcal{V}_n^\star) \cup (\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star) \subseteq C$ and $(\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star) \cup (\mathcal{V}_{m''}^\star \cap \mathcal{V}_{n''}^\star) \subseteq C'$. Since the nonempty set $\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star$ is contained in $C \cap C'$, we have $C = C'$. Therefore, $(\mathcal{V}_m^\star \cap \mathcal{V}_n^\star) \cup (\mathcal{V}_{m''}^\star \cap \mathcal{V}_{n''}^\star) \subseteq C$. Thus $(m)(n) \sim (m'')(n'')$. Hence $\sim$ is transitive as well as symmetric and reflexive, and so $\sim$ is an equivalence relation on $E(T_{(\,)})$. It will follow from the Folding-knives Lemma (Lemma 5.2) that $\sim$ has just one equivalence class if it can be shown that hypotheses $(1), (2), (3a)$ and $(3b)$ of that Lemma hold.

To prove that (1) of Lemma 5.2 holds, suppose $(m)(n) \in E(T_{(\,)})$. Then $m \in T^i$, $n \in T^{-i}$ for some $i \in \{-1, 1\}$, and $\mathcal{V}_m^\star \cap \mathcal{V}_n^\star \neq \emptyset$. By (37) we have $\sigma_J(m) \neq \sigma_J(n)$, and by (39) we have $\sigma_K(m) \neq \sigma_K(n)$. Also, $\mathcal{V}_m^\star \subseteq \sigma_J(m) \cap \sigma_K(m)$ and $\mathcal{V}_n^\star \subseteq \sigma_J(n) \cap \sigma_K(n)$. Thus, $\emptyset \neq \mathcal{V}_m^\star \cap \mathcal{V}_n^\star \subseteq \sigma_J(m) \cap \sigma_J(n)$, and $\emptyset \neq \mathcal{V}_m^\star \cap \mathcal{V}_n^\star \subseteq \sigma_K(m) \cap \sigma_K(n)$. Hence, $[m][n] \in E(T_{[\,]})$ and $\langle m\rangle\langle n\rangle \in E(T_{\langle\,\rangle})$.

To prove (2) holds, let $m, n, m', n' \in T$ and $(m)(n)$, $(m')(n') \in E(T_{(\,)})$. Assume first that $[m][n] = [m'][n']$. Without loss of generality suppose that $m, m' \in T^1$ and $n, n' \in T^{-1}$, so $[m] = [m']$ and $[n] = [n']$. We have $\sigma_J(m) = \sigma_J(m')$ since $m \in [m] = [m'] = \sigma_J^{-1}(\sigma_J(m'))$, and $\sigma_J(n) = \sigma_J(n')$ since $n \in [n] = [n'] = \sigma_J^{-1}(\sigma_J(n'))$. Hence

$$(\mathcal{V}_m^\star \cap \mathcal{V}_n^\star) \cup (\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star) \subseteq (\sigma_J(m) \cap \sigma_J(n)) \cup (\sigma_J(m') \cap \sigma_J(n')\ ) = \sigma_J(m) \cap \sigma_J(n).$$

Also, by (38), $\sigma_J(m) \cap \sigma_J(n)$ is a connected subset of $\mathcal{Z}^\star$. Thus, there is a component C of $\mathcal{Z}^\star$ with $(\mathcal{V}_m^\star \cap \mathcal{V}_n^\star) \cup (\mathcal{V}_{m'}^\star \cap \mathcal{V}_{n'}^\star) \subseteq C$. That is to say, $(m)(n) \sim (m')(n')$. Now, if $\langle m\rangle\langle n\rangle = \langle m'\rangle\langle n'\rangle$ then a symmetric argument using (40) shows again that $(m)(n) \sim (m')(n')$. This completes the proof that (2) holds.

We now prove that $(3a)$ and $(3b)$ hold. Suppose that $i \in \{-1, 1\}$ and $\emptyset \neq S' \subset T^i$ with $[S'] = \bigcup_{s \in S'} [s] \neq T^i$. Assume there does not exist $k \in [S']$ with

$\langle k\rangle \nsubseteq [S']$. (We will derive a contradiction.) Let $S = [S']$. Then, since S is a union of equivalence classes of the form $[m']$ (some $m' \in S'$), we have $[S] = S$. So

$$\emptyset \neq [S] = S = [S'] \subset T^i,$$

and there does not exist $k \in S$ with $\langle k\rangle \nsubseteq S$. Thus, $\langle m\rangle \subseteq S$ for all $m \in S$. Hence,

$$\emptyset \neq \langle S\rangle = \bigcup_{m\in S} \langle m\rangle = S = [S] \subset T^i. \tag{43}$$

Since the maps $\sigma_J : T \longrightarrow \Sigma J$ and $\sigma_K : T \longrightarrow \Sigma K$ are surjections, lines (37) and (39) yield

$$\bigcup_{m\in T^i} \sigma_J(m) = (\mathcal{J} \cap \mathcal{P}_i)^\star \text{ and } \bigcup_{m\in T^i} \sigma_K(m) = (\mathcal{K} \cap \mathcal{P}_i)^\star.$$

The following also holds.

$$\text{If } m \in S \text{ and } m_0 \in T^i\backslash S \text{ then } \sigma_J(m) \cap \sigma_J(m_0) = \emptyset. \tag{44}$$

For, $[m] \subseteq [S] = S$. Consequently, $m_0 \in T^i\backslash S \subseteq T^i\backslash[m] = T^i\backslash\sigma_J^{-1}(\sigma_J(m))$. So $\sigma_J(m_0) \neq \sigma_J(m)$. Moreover, $m \in S \subset T^i$. Since m, $m_0 \in T^i$ and $\sigma_J(m_0)$ and $\sigma_J(m)$ are distinct components of $(\mathcal{J} \cap \mathcal{P}_i)^\star$, we have $\sigma_J(m) \cap \sigma_J(m_0) = \emptyset$. Thus, (44) holds. Now, the same reasoning (and the symmetry in (43)) can be applied to show that

$$\text{If } m \in S \text{ and } m_0 \in T^i\backslash S \text{ then } \sigma_K(m) \cap \sigma_K(m_0) = \emptyset. \tag{45}$$

Define

$$A_i = \bigcup_{m\in S} (\sigma_J(m) \cup \sigma_K(m)) \text{ and } B_i = \bigcup_{m_0\in T^i\backslash S} (\sigma_J(m_0) \cup \sigma_K(m_0)).$$

Notice that $\sigma_J(m) \cup \sigma_K(m)$ is a closed subset of $\mathcal{P}_i^\star$ for every $m \in T^i$. We have $\emptyset \neq S \subset T^i$. Then S and $T^i\backslash S$ are nonempty subsets of the finite set T^i. Therefore A_i and B_i are nonempty closed subsets of $\mathcal{P}_i^\star$. Thus, $A_i \cup B_i = \bigcup_{m\in T^i} \sigma_J(m) \cup \bigcup_{m\in T^i} \sigma_K(m) = (\mathcal{J} \cap \mathcal{P}_i)^\star \cup (\mathcal{K} \cap \mathcal{P}_i)^\star = ((\mathcal{J} \cup \mathcal{K}) \cap \mathcal{P}_i)^\star = (\mathcal{P} \cap \mathcal{P}_i)^\star = \mathcal{P}_i^\star$, which is a $\mathcal{P}$-continuum (as $\{\mathcal{P}_{-1}^\star, \mathcal{P}_{-1}^\star\}$ is a $\mathcal{P}$-decomposition of P). So $A_i \cap B_i \neq \emptyset$. Consequently, one can select $m \in S$ and $m_0 \in T^i\backslash S$ so that the set $(\sigma_J(m) \cap \sigma_J(m_0)) \cup (\sigma_J(m) \cap \sigma_K(m_0)) \cup (\sigma_J(m_0) \cap \sigma_K(m)) \cup (\sigma_K(m) \cap \sigma_K(m_0))$ is nonempty. Then, by (44) and (45),

$$(\sigma_J(m) \cap \sigma_K(m_0)) \cup (\sigma_J(m_0) \cap \sigma_K(m)) \neq \emptyset. \tag{46}$$

Now, suppose $\sigma_J(m) \cap \sigma_K(m_0) \neq \emptyset$. Since m, $m_0 \in T^i$, $\sigma_J(m)$ is some component $\mathcal{A}^\star$ of $(\mathcal{J}\cap\mathcal{P}_i)^\star$ and $\sigma_K(m_0)$ is some component $\mathcal{B}^\star$ of $(\mathcal{K}\cap\mathcal{P}_i)^\star$. Thus, $\emptyset \neq \sigma_J(m) \cap \sigma_K(m_0) = (\mathcal{A}\cap\mathcal{B}\cap\mathcal{V})^\star$, by (41). Select $G' \in \mathcal{V}\cap\mathcal{A}\cap\mathcal{B}$. Since $\mathcal{A} \subseteq \mathcal{J}\cap\mathcal{P}_i$ (by Lemma

2.10), $G' \in \mathcal{V} \cap \mathcal{P}_i$. Let $\mathcal{V}^\star_{m'}$ be the component of $(\mathcal{V} \cap \mathcal{P}_i)^\star$ containing G'. Now m, m_0, $m' \in T^i$ and $G' \subseteq \mathcal{A}^\star \cap \mathcal{B}^\star = \sigma_J(m) \cap \sigma_K(m_0)$. Hence, $\sigma_J(m') = \sigma_J(m)$ and $\sigma_K(m') = \sigma_K(m_0)$. Therefore, $m' \in \sigma_J^{-1}(\sigma_J(m)) \cap \sigma_K^{-1}(\sigma_K(m_0)) = [m] \cap \langle m_0 \rangle$. But $[m] \subseteq [S] = \langle S \rangle$ since $m \in S$, and $\langle m_0 \rangle \subseteq T^i \backslash \langle S \rangle$ since $m_0 \in T^i \backslash \langle S \rangle$. (For if $m_1 \in \langle m_0 \rangle \cap \langle S \rangle = \langle m_0 \rangle \cap S$, then $m_0 \in \langle m_1 \rangle \subseteq \langle S \rangle = S$, contrary to the choice of m_0.) So $m' \in [m] \cap \langle m_0 \rangle \subseteq \langle S \rangle \cap (T^i \backslash \langle S \rangle)$. This contradiction yields $\sigma_J(m) \cap \sigma_K(m_0) = \emptyset$. Similarly, $\sigma_J(m_0) \cap \sigma_K(m) = \emptyset$. This contradiction of (46) shows that (3a) holds. Property (3b) can be established in the same manner as (3a).

We now conclude from Lemma 5.2 that $\sim$ has only one equivalence class. To finish the proof that $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ is connected in this case (Case 2), suppose that C and C' are components of $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$. By Lemma 2.18, C and C' are components of $\mathcal{Z}^\star = (\mathcal{P}_{-1} \cap \mathcal{P}_1)^\star$. Then, from (35), $\mathcal{Z}(C) \cap \mathcal{V} \neq \emptyset$ and $\mathcal{Z}(C') \cap \mathcal{V} \neq \emptyset$. Choose $G \in \mathcal{Z}(C) \cap \mathcal{V}$ and $G' \in \mathcal{Z}(C') \cap \mathcal{V}$. By Lemma 2.10,

$$G \subseteq C \text{ and } G' \subseteq C'.$$

Also, G, $G' \in \mathcal{Z} \cap \mathcal{V} = \mathcal{P}_{-1} \cap \mathcal{P}_1 \cap \mathcal{V}$. Let

$$\begin{gathered}
\mathcal{V}^\star_m \text{ be the component of } (\mathcal{P}_{-1} \cap \mathcal{V})^\star \text{ that contains } G,\\
\mathcal{V}^\star_n \text{ be the component of } (\mathcal{P}_1 \cap \mathcal{V})^\star \text{ that contains } G,\\
\mathcal{V}^\star_{m'} \text{ be the component of } (\mathcal{P}_{-1} \cap \mathcal{V})^\star \text{ that contains } G',\\
\mathcal{V}^\star_{n'} \text{ be the component of } (\mathcal{P}_1 \cap \mathcal{V})^\star \text{ that contains } G'.
\end{gathered}$$

Then $\mathcal{V}^\star_m \cap \mathcal{V}^\star_n \neq \emptyset$, so $(m)(n) \in E(T_{(\,)})$. Similarly, $\mathcal{V}^\star_{m'} \cap \mathcal{V}^\star_{n'} \neq \emptyset$, so that $(m')(n') \in E(T_{(\,)})$. Since $\sim$ has one equivalence class, $(m)(n) \sim (m')(n')$. Thus there is a component C'' of $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ with $(\mathcal{V}^\star_m \cap \mathcal{V}^\star_n) \cup (\mathcal{V}^\star_{m'} \cap \mathcal{V}^\star_{n'}) \subseteq C''$. Therefore, $G \cup G' \subseteq C''$. So $G \subseteq C \cap C''$ and $G' \subseteq C' \cap C''$. Consequently, $C \cap C'' \neq \emptyset \neq C' \cap C''$. Hence $C = C'' = C'$. Thus $\mathcal{P}^\star_{-1} \cap \mathcal{P}^\star_1$ is connected, as required. This completes the proof of Theorem 6.3. □

A natural conjecture related to Theorem 6.3 is the following.

Conjecture 6.4. *There exists a non-unicoherent Hausdorff continuum that is the union of two unicoherent continua having a connected intersection.*

While such a continuum has eluded the grasp of this author, instances have in fact been given by Charles Hagopian, Alejandro Illanes, et al. The referee has kindly provided the following simple variant of their examples. In the complex plane let $T = \{z : |z| = 1\}$, $A' = \{z : 1 < |z| \leq 2\}$, $A = T \cup A'$, and let S be the following union of two disjoint spirals in A':

$$S = \{(1 + e^{-\theta})e^{i\theta} : 0 \leq \theta < \infty\} \cup \{-(1 + e^{-\theta})e^{i\theta} : 0 \leq \theta < \infty\}$$

Denote by U_1 and U_2 the two components of $A' \backslash S$. It can be shown that $A \backslash U_1$ and $A \backslash U_2$ are unicoherent continua whose union is the annulus A and whose intersection is the continuum $T \cup S$.

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