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ON A SPECIAL METRIC

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Abstract. In this note, we prove that whenever d is a compatible metric for a hedgehog space J having more than $2^{\mathbf{c}}$ spines, there exists $\epsilon > 0$ and $x \in J$ such that the family $\{B_d(y,\epsilon) : y \in B_d(x,\epsilon)\}$ contains more than $\mathbf{c}$ distinct sets. This result provides a negative answer to a question raised by Nagata in [6]. We also give positive answers to the same question under some extra conditions.

1. Introduction

J. Nagata proved in [5] that each metrizable space X has a compatible metric d such that for every $\epsilon > 0$, the family $\{B_d(x,\epsilon) : x \in X\}$ of all ϵ-balls of (X,d) is closure-preserving, and in [6] he raised the following question:

Does there exist, on each metrizable space X, a compatible metric d such that for every $\epsilon > 0$, the family $\{B_d(x,\epsilon) : x \in X\}$ of all ϵ-balls of (X,d) is hereditarily closure-preserving (as an unindexed family)?

In this note, we answer the above question negatively by proving that there exists no compatible metric d on the hedgehog space $H = J((2^{\mathbf{c}})^+)$ such that for every $\epsilon > 0$, the family $\{B_d(x,\epsilon) : x \in H\}$ of all ϵ-balls is point finite or even point countable (as an unindexed family). We also show that the answer to the above question is positive under some added restrictions.

Notation and terminology. A metric d of a (metrizable) topological space X is a *compatible metric* if the family $\{B_d(x,\epsilon) : x \in X, \epsilon > 0\}$ of all open balls is a base for the topology of X, where $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$. Denote by $\mathbb{I}$ the

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closed unit interval with the ordinary euclidean topology, and for every ordinal number α, let $\mathbb{I}_\alpha$ be the copy of $\mathbb{I}$ obtained by setting $\mathbb{I}_\alpha = \mathbb{I} \times \{\alpha\}$. We denote by H the Hilbert cube $\Pi_{\alpha<\omega}\mathbb{I}_\alpha$ – a Cartesian product of countably infinitely many copies of $\mathbb{I}$. Let κ be an infinite cardinal number. The *hedgehog* $J(\kappa)$ with κ *spines* ([9] and [3]) is the (metrizable) space whose underlying set is obtained from the union $\bigcup\{\mathbb{I}_\alpha : \alpha < \kappa\}$ by identifying all the zero points $(0, \alpha)$ into one point, which we denote by $\mathbf{0}$, and whose topology is defined as follows: for every $\alpha < \kappa$, the set $\mathbb{I}_\alpha \setminus \{(0, \alpha)\}$ is open in $J(\kappa)$ and has the same relative topology in $J(\kappa)$ as in $\mathbb{I}_\alpha$; the point $\mathbf{0}$ has the sets $\{\mathbf{0}\} \cup \{(r, \alpha) \in J(\kappa) : 0 < r < \epsilon\}$, for $\epsilon > 0$, as a neighborhood base. A family $\mathcal{U}$ of subsets of a topological space X is *closure-preserving* if for each subfamily $\mathcal{U}'$ of $\mathcal{U}$, $\mathrm{Cl}\bigcup\{U : U \in \mathcal{U}'\} = \bigcup\{\mathrm{Cl}U : U \in \mathcal{U}'\}$. The family $\mathcal{U}$ is *hereditarily closure-preserving*, if for any choice of sets $L_U \subseteq U$, for $U \in \mathcal{U}$, the family $\{L_U : U \in \mathcal{U}\}$ is closure-preserving. It is well known and easy to prove that every locally finite family is hereditarily closure-preserving.

The symbol $\mathbf{c}$ denotes the cardinality of the continuum. For a set S, the symbol $[S]^2$ is used to denote the family consisting of all two-element subsets of a set S.

For terms not defined here, refer to [2].

2. The Results

The following lemma contains just the special case "$\alpha = \mathbf{c}$" of the Erdös-Rado Theorem $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$ (see e.g., [8], p.9).

Lemma 2.1. (Erdös-Rado) *Let S be a set of cardinality $|S| = (2^{\mathbf{c}})^+$. If $\mathcal{P}$ is a partition of $[S]^2$ with $|\mathcal{P}| \leq \mathbf{c}$, then there exist $A \subseteq S$ and $P \in \mathcal{P}$ such that $|A| > \mathbf{c}$ and $[A]^2 \subseteq P$.*

Proposition 2.2. *Let $\kappa = (2^{\mathbf{c}})^+$, and let d be a compatible metric for the hedgehog space $J(\kappa)$. Then there exists $\epsilon > 0$ and $x \in J(\kappa)$ such that the family $\{B_d(y, \epsilon) : y \in B_d(x, \epsilon)\}$ contains more than $\mathbf{c}$ distinct sets.*

Proof. For every $(r, \alpha) \in J(\kappa) \setminus \{\mathbf{0}\}$, denote the point (r, α) by r_α. For each $\alpha < \kappa$, let $0_\alpha = \mathbf{0}$. Denote by Q the set of all rational numbers in $\mathbb{I}$.

To prove the proposition, assume on the contrary that there exists a compatible metric d on $J(\kappa)$ such that for all $\epsilon > 0$ and $x \in J(\kappa)$, the family $\{B_d(y, \epsilon) : y \in B_d(x, \epsilon)\}$ contains at most $\mathbf{c}$ distinct sets. For each $\alpha < \kappa$, define a function $f_\alpha : Q \times Q \to [0, \infty)$ by setting $f_\alpha(y, z) = d(y_\alpha, z_\alpha)$ for $y_\alpha, z_\alpha \in \mathbb{I}_\alpha$ with $y, z \in Q$. Since there are only $\mathbf{c}^\omega = \mathbf{c}$ different functions $Q \times Q \to [0, \infty)$, we can find a set $K \subseteq \kappa$ with cardinality κ such that for all $\alpha, \beta \in K$, $f_\alpha = f_\beta$. Without loss of generality, we may assume that $K = \kappa$. We then have, for all $y, z \in Q$, that

$d(y_\alpha, z_\alpha) = d(y_\beta, z_\beta)$ for any pair $\alpha, \beta \in \kappa$. Since d is continuous, the following stronger condition holds:

$$d(y_\alpha, z_\alpha) = d(y_\beta, z_\beta) \text{ for all } \alpha, \beta \in \kappa \text{ and } y, z \in \mathbb{I} . \tag{1}$$

For all $\alpha, \beta \in \kappa$ with $\alpha \neq \beta$, define $g_{\alpha,\beta} : Q \times Q \to [0, \infty)$ by setting $g_{\alpha,\beta}(y, z) = d(y_\alpha, z_\beta)$ for all $y, z \in Q$. Since there are only $\mathbf{c}^\omega = \mathbf{c}$ functions $Q \times Q \to [0, \infty)$, we can use Lemma 1 to find a set $L \subseteq \kappa$ with cardinality bigger than $\mathbf{c}$ such that for all $\alpha, \beta, \alpha', \beta' \in L$ with $\alpha \neq \beta$ and $\alpha' \neq \beta'$ we have that $g_{\alpha,\beta} = g_{\alpha',\beta'}$. We then have, for all $y, z \in Q$, that $d(y_\alpha, z_\beta) = d(y_{\alpha'}, z_{\beta'})$ for all $\alpha, \beta, \alpha', \beta' \in L$ with $\alpha \neq \beta$ and $\alpha' \neq \beta'$. By continuity of d, the following holds:

$$d(y_\alpha, z_\beta) = d(y_{\alpha'}, z_{\beta'}) \text{ if } y, z \in \mathbb{I}, \alpha, \beta, \alpha', \beta' \in L, \alpha \neq \beta \text{ and } \alpha' \neq \beta' . \tag{2}$$

Let $y \in \mathbb{I}$. Note that it follows from Condition (2) that, whenever $\alpha \neq \beta$, the distance $d(y_\alpha, \mathbb{I}_\beta) = \min\{d(y_\alpha, z_\beta) : z \in \mathbb{I}\}$ is independent of $\alpha, \beta \in L$. Hence we can denote by $D(y)$ the number $d(y_\alpha, \mathbb{I}_\beta)$, for $\alpha, \beta \in L$, $\alpha \neq \beta$. Note that $D(y) > 0$ whenever $y > 0$.

Claim. For each $y \in (0, 1]$, there is $\epsilon > 0$ such that $z \in (y - \epsilon, y + \epsilon)$ implies $D(z) \leq D(y)$. To prove the Claim, note that $B_d(y_\omega, D(y))$ is a neighborhood of y_ω and that there thus exists $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \times \{\omega\} \subseteq B_d(y_\omega, D(y))$. By Condition (1), we have that $(y - \epsilon, y + \epsilon) \times \{\alpha\} \subseteq B_d(y_\alpha, D(y))$ for every $\alpha \in L$. We show that if $z \in (y - \epsilon, y + \epsilon)$, then $D(z) \leq D(y)$. Assume that this is not true, and let $z \in (y - \epsilon, y + \epsilon)$ be such that $D(z) > D(y)$. Let $\alpha \in L$. Note that there exists $w \in [0, 1]$ such that $d(y_\alpha, w_\beta) = D(y)$ for each $\beta \in L \setminus \{\alpha\}$. Since $D(z) > D(y)$, we have that $w_\beta \in B_d(y_\alpha, D(z))$; by condition (2) it follows that $w_\alpha \in B_d(y_\beta, D(z))$ for each $\beta \in L \setminus \{\alpha\}$. Note that, for all $\beta, \beta' \in L$, $\beta' \neq \beta$, we have that $d(z_\beta, y_{\beta'}) \geq D(z)$, and hence that $z_\beta \notin B_d(y_{\beta'}, D(z))$; on the other hand, we have that $z_\beta \in (y - \epsilon, y + \epsilon) \times \{\beta\} \subseteq B_d(y_\beta, D(y)) \subseteq B_d(y_\beta, D(z))$. The foregoing shows that $B_d(y_\beta, D(z)) \neq B_d(y_{\beta'}, D(z))$ whenever $\beta, \beta' \in L$, $\beta \neq \beta'$. As a consequence, the subfamily $\{B_d(y_\beta, D(z)) : \beta \in L\}$ of $\{B_d(y_\beta, D(z)) : y_\beta \in B_d(w_\alpha, D(z))\}$ contains $|L| > \mathbf{c}$ distinct sets in contradiction to our assumption. Hence the Claim is proved.

Note that D is a continuous function $\mathbb{I} \to \mathbb{R}$. The Claim shows that, for each $y \in (0, 1]$, there is an $\epsilon > 0$, such that $D(y)$ is the maximum value in $(y - \epsilon, y + \epsilon)$. By an elementary argument of mathematical analysis, we can prove that $D(y)$ is constant on $(0, 1]$. It follows that $D(y) = D(0) = 0$ for every $y \in \mathbb{I}$, and this is clearly impossible. □

Every hedgehog space is a first countable T_1-space without isolated points, and hence Corollary 3 of [1] shows that every hereditarily closure-preserving open family in a hedgehog space is locally finite; as a consequence, we have the following solution to Nagata's problem.

Corollary 2.3. *For $\kappa = (2^{\mathfrak{c}})^+$, the hedgehog $J(\kappa)$ has no compatible metric d such that for every $\epsilon > 0$, the family of ϵ-balls $\{B_d(x, \epsilon) : x \in J(\kappa)\}$ is hereditarily closure-preserving (as an unindexed family).*

In the following, we show that under certain added conditions the answer to Nagata's question is positive.

The following result is due to Nagata (see [5, proofs of Theorems V.3 and V.4]).

Proposition 2.4. ([5]) *Every metrizable space X admits a compatible metric d such that, for each $\epsilon > 0$, there is a locally finite open cover $\mathcal{U}$ of X such that, for every $x \in X$, the set $B_d(x, \epsilon)$ is the union of finitely many members of $\mathcal{U}$.*

The following is an immediate consequence of the above result.

Corollary 2.5. *Every compact metrizable space X admits a compatible metric d such that for every $\epsilon > 0$ there are only finitely many distinct ϵ-balls $B_d(x, \epsilon)$, $x \in X$.*

Proposition 2.6. *A metrizable space X is separable if and only if X has a compatible metric d on X such that for every $\epsilon > 0$, there are only finitely many distinct ϵ-balls $B_d(x, \epsilon)$, $x \in X$.*

PROOF. Sufficiency of the condition follows from total boundedness of the metric d.
Necessity follows by Corollary 2 since every separable metrizable space has a metrizable compactification. □

In [7, Section 2.3], a metrizable space is called *strongly metrizable* provided the space has a base which is the union of countably many star-finite coverings. By a result of K. Morita (see [4], or [7, Proposition 2.3.27]), a strongly metrizable space of weight κ embeds in the space $H \times B(\kappa)$, where H is the Hilbert cube and $B(\kappa) = D(\kappa)^\omega$ with $D(\kappa)$ the discrete space on κ. Since $H \times B(\kappa)$ is strongly paracompact, we see that strongly metrizable spaces coincide with subspaces of strongly paracompact metrizable spaces.

Proposition 2.7. *A topological space X is strongly metrizable if, and only if, X has a compatible metric d such that for every $\epsilon > 0$, the family $\{B_d(x, \epsilon) : x \in X\}$ of all ϵ-balls is star-finite (as an unindexed family).*

PROOF. *Sufficiency* of the condition follows directly from the definition of a strongly metrizable space.
Necessity. The Baire space $B(\kappa)$ is a strongly zero-dimensional metrizable space and therefore it has a compatible non-archimedean metric d. For d, every family $\{B_d(a, \epsilon) : a \in B(\kappa)\}$ is disjoint and hence star-finite. By Corollary 2.5, the Hilbert cube H has a compatible metric ρ with every family $\{B_\rho(x, \epsilon) : x \in H\}$ finite. It is easy to see that the (compatible) metric δ of $H \times B(\kappa)$, defined by setting $\delta((x, a), (y, b)) = \max(\rho(x, y), d(a, b))$, has the property that every family $\{B_\delta(p, \epsilon) : p \in H \times B(\kappa)\}$, where $\epsilon > 0$, is star-finite; it follows that also every restriction of δ to a subspace of $H \times B(\kappa)$ has the same property. The conclusion now follows by the result of Morita mentioned above. □

Problem *Is it possible to find a metrizable space X which is "smaller" than $J((2^{\mathbf{c}})^+)$ (e.g., with respect to weight or cardinality) and which admits no compatible metric d with all the families $\{B_d(x, \epsilon) : x \in X\}$, $\epsilon > 0$, locally finite? In particular, does $J(\omega_1)$ have a compatible metric d with all the families $\{B_d(x, \epsilon) : x \in X\}$, $\epsilon > 0$, locally finite?*

Note added on Feb. 29th, 2000: The authors have been informed that G. Gruenhage has solved the above problem by showing that $J(\omega_1)$ has no compatible metric d such that $\{B_d(x, \epsilon) : x \in J(\omega_1)\}$ is locally finite for every $\epsilon > 0$.

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References

[1] D. Burke, R. Engelking and D. Lutzer, Hereditarily closure-preserving collections and metrization, Proc. Amer. Math. Soc. *51*(1975)483-488.

[2] R. Engelking, General Topology (PWN, Warszawa, 1977).

[3] H. Kowalsky, Einbettung metrischer Räume, Arch. Math. *8* (1957) 336-339

[4] K. Morita, Normal families and dimension theory for metric spaces, Math. Ann. *128* (1954) 350-362

[5] J. Nagata, Modern Dimension Theory (North-Holland Publ. Co., Amsterdam, 1965)

[6] J. Nagata, Remarks on metrizability and generalized metric space, Topology and its Appl. *91* (1999) 71-77

[7] A.R. Pears, Dimension Theory of General Spaces (Cambridge Univ. Press, Cambridge, 1975)

[8] M.E. Rudin, Lectures on Set Theoretic Topology, Regional Confer. Series in Math. *23* (1975)

[9] P. Urysohn, Sur un espace mtrique universel, Bull. Sci. Math. *51* (1927) 43-64 and 74-90

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