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ABEL'S DIFFERENTIAL EQUATIONS

MARK L. GREEN AND PHILLIP A. GRIFFITHS

This paper is the text of a talk given by the second author at the Chern Conference. Intended for a general audience, it is based on joint work in progress with Mark Green. The purposes of the talk were (i) to discuss Abel's differential equations (DE's) for algebraic curves in both classical and modern form, (ii) to explain in a special case the extension of Abel's DE's to general algebraic varieties that Mark and I have developed, and (iii) to discuss the integration of Abel's DE's in this special case. The emphasis throughout is on the geometric content of the differential equations.

1. Historical perspective; Abel's DE's for algebraic curves
2. Abel's DE's for regular algebraic surfaces defined over $\mathbb{Q}$; discussion of the general case
3. Geometric meaning of Abel's DE's
4. Integration of Abel's DE's; caveat

1. Historical perspective; Abel's DE's for algebraic curves

Our story begins with the classical Abel's theorem. From the earliest days of the calculus mathematicians were interested in integrals

$$I = \int h(x)dx$$

where $h(x)$ is an algebraic function of x. This means that there is an algebraic curve defined by a polynomial equation

$$C = \{f(x, y) = 0\}$$

the roots of which for variable x give $y = y(x)$ as a multi-valued algebraic function of x, and

$$I = \int g(x, y(x))dx$$

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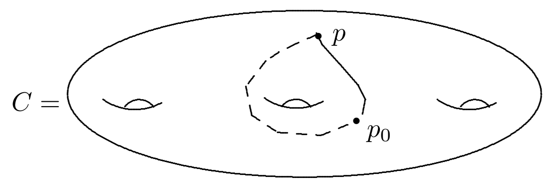
where $g(x,y)$ is a rational function of (x,y). For example, we might have

$$C = \{y^2 = p(x)\}$$

where $p(x)$ is a polynomial in x, and

$$I = \int \frac{dx}{\sqrt{p(x)}} = \int \frac{dx}{y}$$

is called a hyperelliptic integral. In modern terms, we may picture C as a compact Riemann surface



On C we have a meromorphic differential

$$\omega = g(x,y)dx\big|_C$$

and

$$I = \int_{p_0}^{p} \omega \qquad \text{mod periods,}$$

the ambiguity arising from different choices of a path from p_0 to p.

The functions obtained by "inverting" these integrals; i.e., by defining $\varphi(u)$ by

$$u = \int^{\varphi(u)} h(x)dx$$

include some of the most interesting transcendental functions

$$u = \int^{\sin u} \frac{dx}{\sqrt{1-x^2}} \qquad (x^2 + y^2 = 1)$$

$$u = \int^{p(u)} \frac{dx}{\sqrt{x^3 + ax + b}} \qquad (y^2 = x^3 + ax + b).$$

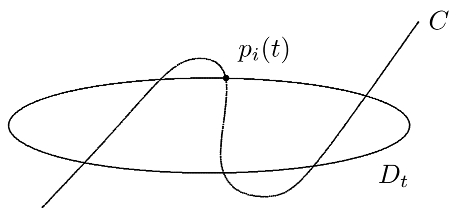
Although the $\varphi(u)$ are transcendental, they satisfy functional equations or "addition theorems", the general form of which was discovered by Abel. For this he considered *abelian sums*, defined as follows: Let

$$D_t = \{h(x,y,t) = 0\}$$

be a family of algebraic curves depending rationally on a parameter t. Write

$$C \cdot D_t = \sum_i (x_i(t), y_i(t))$$

where the $(x_i(t), y_i(t)) = p_i(t)$ are the deg $C \cdot$ deg D_t points of intersection



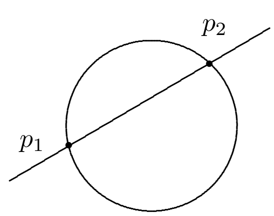
The abelian sum is then

$$I(t) = \sum_i \int^{p_i(t)} g(x, y)dx.$$

Although the individual integrals are very complicated, Abel found that the abelian sum is quite simple:

Theorem. $I(t) = r(t) + \sum_\alpha A_\alpha \log(t - t_\alpha)$ *where the rational function* $r(t)$ *and constants* A_α, t_α *can in practice be determined.*

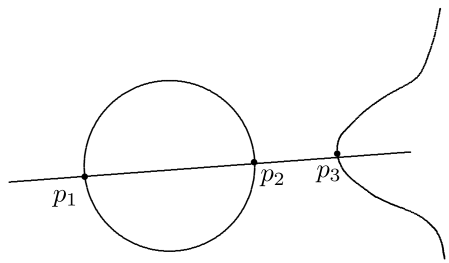
Example. For the circle and a variable line



$$\int^{p_1} \omega + \int^{p_2} \omega = \left\{ \begin{matrix} \text{known} \\ \text{quantity} \end{matrix} \right\}, \quad \omega = \frac{dx}{\sqrt{1 - x^2}}$$

gives the addition theorem for sin u.

Example. For the cubic and a variable line



$$\int^{p_1} \omega + \int^{p_2} \omega + \int^{p_3} \omega = \text{constant}\,, \quad \omega = \frac{dx}{\sqrt{x^3 + ax + b}}$$

gives the addition theorem for $p(u)$.

Concerning the second example, classically one had the

Definition. ω is a *differential of the* 1^{st}*-kind* if $\int \omega$ is locally bounded.

If $f(x, y) = 0$ defines a smooth plane curve of degree n, the differentials of the 1^{st} kind are

$$\omega = \frac{p(x, y)dx}{f_y(x, y)}$$

where $p(x, y)$ is a polynomial of degree $n - 3$.

For differentials of the 1^{st} kind, the A_α are zero and $r(t)$ reduces to a constant. Clearly, ω is a differential of the 1^{st}-kind if, and only if, it has no poles on C. This is in turn equivalent to

$$I(t) = \text{constant}$$

which is the same as

$$I'(t) = 0.$$

Explicitly, this latter equation is

$$\frac{d}{dt}\left(\sum_i \int^{p_i(t)} \omega \right) = \sum\nolimits_i \omega(p_i(t)) = 0$$

where

$$\omega(p_i(t)) = g(x_i(t), y_i(t))x_i'(t)dt.$$

This equation is Abel's DE in classical form; it may be thought of as an infinitesimal functional equation or addition theorem.

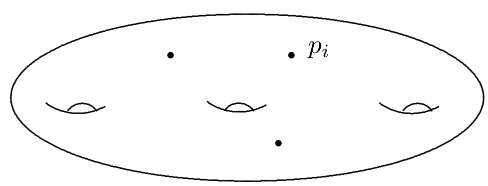
For the modern reformulation, we let X be a smooth algebraic curve and $\omega \in H^0(\Omega^1_{X/\mathbb{C}})$ a regular differential. We denote by

$$X^{(d)} = \underbrace{X \times \cdots \times X}_{d} / \Sigma_d\,, \qquad \Sigma_d = \text{permutation group}$$

the d-fold symmetric product, thought of as the set of configurations

$$z = p_1 + \cdots + p_d$$

of d points on X



It is well known that $X^{(d)}$ is a smooth algebraic variety. We also denote by

$$\mathrm{Tr}\omega(z) = \omega(p_1) + \cdots + \omega(p_d)$$

the regular 1-form on $X^{(d)}$, called the *trace* of ω, induced from the diagonal 1-form $(\omega, \ldots, \omega)$ on X^d, this being obviously invariant under Σ_d.

We will say that two configurations $z_1, z_2 \in X^{(d)}$ are *rationally equivalent* if there is a rational function $f \in \mathbb{C}(X)^*$ with divisor

$$(f) = z_1 - z_2.$$

This is equivalent to there being a regular map

$$F\colon\ \mathbb{P}^1 \to X^{(d)}$$

with $F(0) = z_1$ and $F(\infty) = z_2$; in fact, we may take

$$F(t) = f^{-1}(t)\,, \qquad\qquad t \in \mathbb{P}^1.$$

If $Z \subset Z^{(d)}$ is a subvariety such that any z_1, z_2 in Z are rationally equivalent, then the restriction

$$\mathrm{Tr}\,\omega\big|_Z = 0.$$

By definition, *Abel's differential equations* are

$$(A) \qquad \boxed{\operatorname{Tr}\omega = 0, \qquad \omega \in H^0(\Omega^1_{X/\mathbb{C}})}$$

We think of (A) as the exterior differential system generated by the 1-forms $\operatorname{Tr}\omega$, $\omega \in H^0(\Omega^1_{X/\mathbb{C}})$.

An integral variety of (A) is by definition an algebraic subvariety Z such that all $\operatorname{Tr}\omega\big|_Z = 0$; i.e., such that the equations (A) are satisfied by Z. The classical theorems of Abel and Riemann-Roch may be formulated as stating that:

> (A) *is an involutive exterior differential system with maximal integral variety* $|z|$ *passing through* $z \in X^{(d)}$ *given by*
>
> $$|z| = \{z' \in X^{(d)} \colon (f) = z - z' \text{ for some } f \in \mathbb{C}(X)^*\}.$$

Remark. Even though the 1-forms $\operatorname{Tr}\omega$ are closed, it is by no means automatic that (A) is involutive. The reason is that

$$\delta(z) =: \dim \operatorname{span}\{\{\operatorname{Tr}\omega(z) \colon \omega \in H^0(\Omega^1_{X/\mathbb{C}})\} \subset T^*_z X^{(d)}\}$$

varies with z; in fact, $\delta(z)$ is equal to $g - i(z)$ where g is the genus of X and $i(z)$ is the index of speciality of z. If $\mathcal{I}_r$ is the ideal that defines the condition $\delta(z) \le r$, then for involutivity to hold we must have that

$$d\mathcal{I}_r \subseteq \mathcal{I}_r \cdot \Omega^1_{X^{(d)}/\mathbb{C}} + \{\text{ideal generated by } \operatorname{Tr}\omega\text{'s}\}.$$

This is a non-trivial condition.

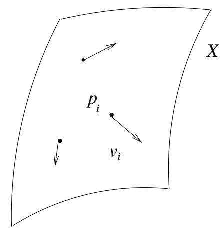
2. Abel's DE's for regular surfaces defined over $\mathbb{Q}$

We were interested in the question: *Is there an analogue of Abel's DE's for general configurations of algebraic cycles on an algebraic variety?* From the work of Bloch and others (cf. [Bloch]), in some formal sense one expects a positive answer. [1] But we were interested in an answer from which one could draw geometric conclusions, at least at the infinitesimal level. For example a specific question is:

[1] Combining his earlier work connecting Chow groups to K-theory with van der Kallen's description of the formal tangent space to $K_2(R)$ for a ring R, Bloch gave an expression for the formal tangent space to the Chow group of 0-cycles on an algebraic surface. Our work identifies Bloch's formal tangent space as the quotient of geometrically defined tangent space to 0-cycles by the tangent space to rational equivalences. This section amounts to making that identification explicit in the simplest special case.

On an algebraic surface X when is

$$\tau = \sum_i (p_i, v_i)\,, \qquad\qquad v_i \in T_{p_i}X$$



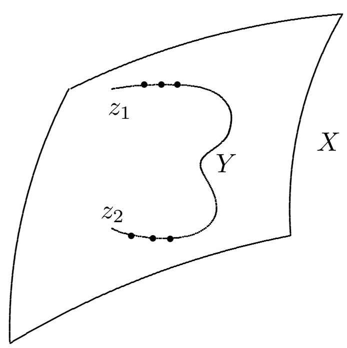
tangent to a rational motion of $\sum_i p_i$*?*

We let $X^{(d)} = \underbrace{X \times \cdots \times X}_{d} / \Sigma_d$ be as before the set of configurations $z = p_1 + \cdots p_d$ of d points on X; as will be of significance below is the fact that the $X^{(d)}$ are singular along the diagonals $p_i = p_j$. From Chow we know that we should consider the equivalence relation generated by: For $z_1, z_2 \in X^{(d)}$ we set

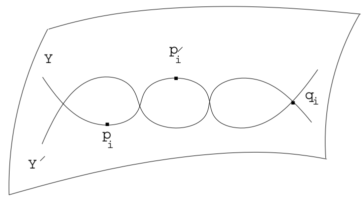
$$z_1 \equiv z_2$$

if there exists an algebraic curve $Y \subset X$ and a rational function $f \in \mathbb{C}(Y)^*$ with

$$(f) = z_1 - z_2$$



From Chow we also know that, in contrast to the case of curves (or divisors on a general variety) we must allow for *cancellations*, such as



$$(f) = \sum_i p_i - \sum_i q_i\,, \qquad f \in \mathbb{C}(Y)^*$$
$$(f') = \sum_i q_i - \sum p'_i\,, \qquad f' \in \mathbb{C}(Y')^*.$$

To move $z = \sum_i p_i$ rationally to $z' = \sum_i p'_i$ we must add $w = \sum_i q_i$ to each. Thus rational equivalence in $X^{(d)}$ requires passing to $X^{(d+d')}$, and consequently we are effectively in $X^{(\infty)}$. Now $X^{(\infty)}$ is a semi-group and we should consider the associated group

$$Z^2(X) = \left\{ \sum_i n_i p_i,\ n_i \in \mathbb{Z} \text{ and } p_i \in X \right\}$$

of 0-cycles on X.[2] We then need to understand the tangent space $TZ^2(X)$ to the space of 0-cycles and the subspace

$$TZ^2_{\mathrm{rat}}(X) \subset TZ^2(X)$$

of tangents to the subgroup $Z^2_{\mathrm{rat}}(X)$ of 0-cycles that are rationally equivalent to zero. Our question then is:

How can we define $TZ^2(X)$ and the space $\Omega^1_{Z^2(X)}$ of 1-forms on $Z^2(X)$, and what are the 1-forms $\alpha \in \Omega^1_{Z^2(X)}$ such that the equations

$$\alpha = 0$$

define $TZ^2_{\mathrm{rat}}(X) \subset TZ^2(X)$?

The definition of $TZ^p(X)$ turns out to have some interesting geometric and algebraic subtleties.

[2]Of course, using the group of divisors on an algebraic curve provides a neat formalism, but its use is not essential in geometrically understanding rational equivalence in this case. For the geometric understanding of configurations of points on algebraic surfaces it is unavoidable that we pass to $X^{(\infty)}$ and then to $Z^2(X)$.

We will illustrate Abel's DE's in a concrete special case. Namely, let $X \subset \mathbb{P}^3$ be a smooth surface given by an affine equation

$$f(x,y,z) = 0\,, \qquad \deg f = n$$

where $f \in \bar{\mathbb{Q}}[x,y,z]$. There are no regular 1-forms on X, and in a similar way as for 1-forms on algebraic curves above the regular 2-forms are spanned by the restrictions to X of

$$\omega = \frac{g(x,y,z)dx\wedge dy}{f_z(x,y,z)}\,, \qquad \deg g \leqq n-4$$

where $g \in \bar{\mathbb{Q}}[x,y,z]$. We consider the earlier question where

$$\begin{cases} p_i = (x_i, y_i, z_i) \\ v_i = \lambda_i \partial/\partial x + \mu_i \partial/\partial y. \end{cases}$$

Setting $h = g/f_z$

$$\omega(p_i)\rfloor v_i = h(p_i)(\mu_i dx - \lambda_i dy) \in T^*_{p_i}X.$$

We then define

$$\alpha_\omega(p_i, v_i) = h(p_i)(\mu_i dx_i - \lambda_i dy_i) \in \Omega^1_{\mathbb{C}/\mathbb{Q}}$$

where $x_i, y_i \in \mathbb{C}$ and $d = d_{\mathbb{C}/\mathbb{Q}}$. The condition that $\tau \in TZ^2_{\mathrm{rat}}(X)$ turns out to be

$$\sum_i \alpha_\omega(p_i, v_i) = 0 \qquad \text{for all } \omega.$$

We write this as

$$(\mathrm{Tr}\,\alpha_\omega)(\tau) = 0$$

where $\tau \in TZ^2(X)$. By definition, *Abel's differential equations* are

$$(A) \qquad \boxed{\mathrm{Tr}\,\alpha_\omega = 0 \qquad \text{for } \omega \text{ as above}}$$

We will discuss later some geometric reasons why the quantities $dx_i, dy_i \in \Omega^1_{\mathbb{C}/\mathbb{Q}}$ appear. As for the definition, for rings R, S with $S \subset R$ the Kähler differentials are defined by

$$\Omega^1_{R/S} = \left\{ \begin{array}{l} R\text{-module linearly generated by expressions } r'dr \\ \text{where } r, r' \in R, \text{ and where the relations} \\ d(rr') = rdr' + r'dr,\ d(r+r') = dr + dr' \\ \text{and } ds = 0 \text{ for } s \in S \text{ are imposed.} \end{array} \right\}$$

If $f(x) \in \mathbb{Q}[x]$ and $f(a) = 0$ is a simple root, then

$$0 = f'(a)da \Longrightarrow da = 0.$$

Additively, if $\alpha_1, \alpha_2, \ldots$ is a transcendence basis for $\mathbb{C}$ over $\mathbb{Q}$

$$\Omega^1_{\mathbb{C}/\mathbb{Q}} \cong \bigoplus_i \mathbb{C}\alpha_i,$$

so that differentials $da, a \in \mathbb{C}$, reflect arithmetic properties of complex numbers.

For algebraic curves the rank of Abel's DE at $z = \sum_i p_i$ has to do with the *geometric* properties of the p_i. For algebraic surfaces it has to do with the *arithmetic* and *geometric* properties of the p_i. More precisely, denoting by Rank $(A)_z$ the dimension of the subspace of $T^*_z Z^n(X)$ given by Abel's DE's for $n = 1, 2$, for for $d \gg 0$ we have

- for algebraic curves, $\mathrm{Rank}(A)_z = g$
- for algebraic surfaces, $\mathrm{Rank}(A)_z \sim p_g(\text{tr deg } z)$

where tr deg $z =$ tr deg $\mathbb{Q}(\ldots, x_i, y_i, \ldots)$. Here $p_g = h^{2,0}(X)$ and the "$\sim$" means that a lower order correction term must be added if X is not regular and/or is not defined over $\mathbb{Q}$.

Corollary (Mumford)**.** *If $p_g \neq 0$ then*

$$\dim CH^2(X(\mathbb{C})) = \infty.$$

Roitman, Voisin and others have extended Mumford's result to state, among other things, that a generic $z \in X^{(d)}$ does not move in a rational equivalence; the above gives a precise meaning to the word "generic".

Corollary. *On a general surface $X \subset \mathbb{P}^3$ of degree $n \geq 5$*

$$(\mathrm{Tr}\, \alpha_\omega)(p, v) = 0$$

has no non-zero solutions for any point $p \in X$.

A corollary, again due to several people, is that X contains no rational curves. The method of proof gives the further result that, for $n \geq 6$, a general X contains no g^1_2's (and hence no rational or hyperelliptic curves) – we suspect that the method may be extended further to show that for $n \geq n_0(d)$ a general X of degree n contains no g^1_d's. In fact, this would all be special cases of the following general principal: On a general surface X in $\mathbb{P}^3$ every curve C is a complete intersection $C = X \cap Y$ where Y is a surface of degree m. Varying Y gives a g^r_{mn} on C which is exceptional in the sense of Brill-Noether theory – i.e., a general curve of genus $g = g(C)$ does not contain a g^r_{mn}. Then *every* curve C on X is *general* among curves containing a g^r_{mn} – i.e., C contains no other exceptional linear series.

At the other extreme, we have the

Corollary. *If X is regular and defined over $\mathbb{Q}$ and the $p_i \in X(\bar{\mathbb{Q}})$, then for any $\tau = \sum_i (p_i, v_i) \in TZ^2(X)$ we have*

$$\tau \in TZ^2_{\text{rat}}(X).$$

This is actually a geometric existence theorem, albeit at the infinitesimal level; namely, given τ there exist curves Y_ν and rational functions $f_\nu \in \mathbb{C}(Y_\nu)^*$ all defined on

$$X_1 = X \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}[\epsilon], \qquad \epsilon^2 = 0$$

such that

$$\tau = \sum_\nu \frac{d}{d\epsilon}(\operatorname{div} f_\nu).$$

This is an infinitesimal version of the well-known conjecture of Bloch-Beilinson (cf. [Ramakrishnan]).

3. The geometric meaning of Abel's DE's

Abel's DE's may be interpreted as expressing an isomorphism

$$TCH^2(X) \cong TZ^2(X)/TZ^2_{\text{rat}}(X).$$

Here the left hand side is the expression given by Bloch [Bloch] for the formal tangent space to the Chow groups for 0-cycles on an algebraic surface. The terms $TZ^2(X)$ and $TZ^2_{\text{rat}}(X)$ are geometric in character and are defined in the paper [Green-Griffiths I]. The above isomorphism is established using an extension of Grothendieck's duality theory. We shall briefly discuss one definition of $TZ^n(X)$ for 0-cycles on an n-dimensional smooth variety when $n = 1, 2$, and then use this to lead into a geometric interpretation of Abel's DE's. [3]

For X an algebraic curve, $\underline{\underline{T}}Z^1(X)$ may be defined in several equivalent ways (loc. cit.) and there turns out to be an isomorphism

$$\underline{\underline{T}}Z^1(X) \cong \bigoplus_{p\in X} \operatorname{Hom}^c(\Omega^1_{X/\mathbb{C},p}, \mathbb{C}),$$

[3] The formal definition of the sheaf $\underline{\underline{T}}Z^n(X)$ obtained by localization of $TZ^n(X)$ is

$$\underline{\underline{T}}Z^n(X) \stackrel{\text{def}}{=} \bigoplus_{p\in X} \lim_{k\to\infty} \operatorname{Ext}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\mathfrak{m}_p^k, \Omega^{n-1}_{X/\mathbb{Q},p}).$$

The justification of this definition on both formal and geometric grounds - especially as regards the appearance of absolute differentials - is given in the paper referred to above. Below we shall discuss some of the geometric motivation for it.

where in the right-hand side only finitely many terms appear in any tangent vector and $\mathrm{Hom}^c(\Omega^1_{X/\mathbb{C},p},\mathbb{C}) \stackrel{\mathrm{def}}{=} \lim_{k\to\infty} \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},p}\otimes\mathcal{O}_X/\mathfrak{m}_p^k,\mathbb{C})$ and where $\Omega^1_{X/\mathbb{C},p}$ is the stalk at p of $\Omega^1_{X/\mathbb{C}}$. The map goes as follows: Given an arc

$$z(t) = \sum_i n_i p_i(t), \qquad z(0) = z$$

in $Z^1(X)$ with

$$\lim_{t\to 0}\{\text{support } z(t)\} = p,$$

and given $\varphi \in \Omega^1_{X/\mathbb{C},p}$, the action of the tangent vector $z' \in T_z Z^1(X)$ on φ is given by

$$z'(\varphi) = \frac{d}{dt}\left(\sum_i n_i \int_p^{p_i(t)} \varphi\right)_{t=0}.$$

Essentially we are taking the infinitesimal form of abelian sums. That the equations

$$z'(\varphi) = 0 \qquad \text{for all } \varphi \in H^0(\Omega^1_{X/\mathbb{C}})$$

define the subspace

$$TZ^1_{\mathrm{rat}}(X) \subset TZ^1(X)$$

is a consequence of the Riemann-Roch theorem together with duality on the curve.

Turning to an algebraic surface X, the tangent space $TZ^2(X)$ may be defined (loc. cit.), and for the corresponding sheaf $\underline{\underline{T}}Z^2(X)$ it turns out that there is an exact sequence

$$0 \to A \to \underline{\underline{T}}Z^2(X) \to B \to 0$$

where

$$B \cong \bigoplus_{p\in X} \mathrm{Hom}^c(\Omega^1_{X/\mathbb{C},p},\mathbb{C})$$

and

$$A \cong \bigoplus_{p\in X} \mathrm{Hom}^c(\Omega^2_{X/\mathbb{C}.p},\Omega^1_{\mathbb{C}/\mathbb{Q}}).$$

The B-term is essentially the same as for curves; the new geometry comes in the A-term, both as regards the appearance of 2-forms and (see below) of differentials over $\mathbb{Q}$. There are essentially three ingredients in understanding the A-term.

(i) The infinitesimal structure of $X^{(\infty)} = \lim_{d\to\infty} X^{(d)}$ is reflected by collections of differential forms φ_d on $X^{(d)}$ with the hereditary property

$$(*) \qquad \varphi_{d+1}\big|_{X^{(d)}} = \varphi_d.$$

Among such forms are those given by traces $\operatorname{Tr}\varphi$ for $\varphi \in \Omega^q_{X/\mathbb{C}}$. Especially important is the structure of $X^{(d)}$ near the diagonals, which geometrically measures the behavior of 0-cycles when points come together. An interesting fact, reflecting the property that the $X^{(d)}$ are singular along the diagonals when $\dim X = n \geqq 2$, is:

Fact. *The forms satisfying* $(*)$ *are generated as an algebra over* $\mathcal{O}_{X^{(d)}}$ *by*

$$\operatorname{Tr}\varphi \quad \textit{for} \quad \varphi \in \Omega^q_{X/\mathbb{C}}, \quad 1 \leqq q \leqq n.$$

New generators are added for each q in the above range.

Thus, on an algebraic surface the 2-forms as well as the 1-forms should enter into the definition of $TZ^2(X)$. From the above exact sequence we see that for a tangent vector $\tau \in TZ^2(X)$ the value $\tau(\omega)$ on a 2-form ω is only well-defined modulo the values of τ on 1-forms, the new information provided by $\Omega^2_{X/\mathbb{C},p}$ should only be considered modulo that provided by $\Omega^1_{X/\mathbb{C},p}$. (All of this is quite clear, including what is meant by working modulo the information coming from the 1-forms, when things are computed out in local coordinates – cf. §§3-5 in the paper by the authors referred to above.)

(ii) Formally, we proceed as follows: There is an evaluation map

$$\Omega^1_{X/\mathbb{Q},p} \to \Omega^1_{\mathbb{C}/\mathbb{Q}}$$

defined for $f, g \in \mathcal{O}_{X,p}$ by

$$f dg \to f(p) dg(p) \qquad\qquad (d = d_{\mathbb{C}/\mathbb{Q}}).$$

Using exterior algebra this map induces a pairing

$$\Omega^2_{X/\mathbb{Q},p} \otimes T_p X \to \Omega^1_{\mathbb{C}/\mathbb{Q}}.$$

Working globally, for X regular and defined over $\mathbb{Q}$, we have by base change

$$H^0(\Omega^2_{X(\mathbb{Q})/\mathbb{Q}}) \otimes \mathbb{C} \cong H^0(\Omega^2_{X(\mathbb{C})/\mathbb{C}}),$$

from which we infer a well-defined pairing

$$H^0(\Omega^2_{X/\mathbb{C}}) \otimes TZ^2(X) \to \Omega^1_{\mathbb{C}/\mathbb{Q}}.$$

Theorem. *The right kernel of this pairing is*

$$TZ^2_{\mathrm{rat}}(X) \subset TZ^2(X).$$

For X a smooth surface in $\mathbb{P}^3$ we recover Abel's DE's as stated above.

When X is not defined over $\mathbb{Q}$ and/or is not regular, there are correction terms that must be added.

(iii) Now one may say: *The above is fine formally, but geometrically why should absolute differentials turn up in the definition of* $TZ^2(X)$*?* [4]

There are a number of responses, at different levels, to this question.

Differentials over $\mathbb{Q}$ have among other things to do with whether complex numbers are algebraic or not. Using the assumption that the tangent map

$$\left\{\begin{matrix}\text{arcs in}\\ Z^2(X)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{complex vector}\\ \text{space } TZ^2(X)\end{matrix}\right\}$$

should be a homomorphism forces arithmetic considerations to enter. For example, in $\mathbb{C}^2$ for $\alpha \neq 0$ let

$$z_\alpha(t) = 0\text{-cycle with ideal } (x^2+\alpha y^2, xy-t).$$

Then it can be shown that

$$z'_\alpha(0) = z'_1(0) \Leftrightarrow \alpha \text{ is algebraic.}$$

As a special case, suppose $\alpha^m = 1$. Then $mz_\alpha(t)$ is defined by the ideal

$$((x^2+\alpha y^2)^m, xy-t).$$

In this ideal, $x^2y^2 = t^2$ and using $\alpha^m = 1$ we have

$$\begin{aligned}(x^2+\alpha y^2)^m &\equiv x^{2m} + y^{2m} \operatorname{mod} t^2\\ &\equiv (x^2+y^2)^m \operatorname{mod} t^2\\ \Longrightarrow mz'_\alpha(0) &= mz'_1(0)\\ \Longrightarrow z'_\alpha(0) &= z'_1(0).\end{aligned}$$

Similar considerations show that for any reasonable geometric definition of 1st-order equivalence of arcs in $Z^2(X)$, the axioms for $\operatorname{Der}(\mathbb{C}/\mathbb{Q})$ enter.

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[4] By definition, for any sub-field $k \subset \mathbb{C}$ the differentials $\Omega^1_{X/k,p}$ are the Kähler differentials of $\mathcal{O}_{X,p}$ over k. For $k = \mathbb{Q}$ we obtain the absolute differentials. In the algebraic case, but *not* in the analytic case, for $k = \mathbb{C}$ we have

$$\Omega^1_{X/\mathbb{C},p} \cong \mathcal{O}_{X,p}(T^*X).$$

In other words, only in *algebraic* geometry do absolute differentials have geometric meaning.

From a different and perhaps more fundamental level we have the concept of a *spread*, which we shall now describe informally. Let $Z \subset \mathbb{P}^N$ be given by homogeneous equations

$$F_\lambda(x) = \sum_I a_{\lambda I} x^I = 0.$$

Let

$$G_\alpha(\ldots, a_{\lambda I}, \ldots) = 0$$

generate the relations defined over $\mathbb{Q}$ satisfied by the coefficients of the F_λ, and let S be the variety given by

$$G_\alpha(\ldots, y_{\lambda I}, \ldots) = 0.$$

For each $s \in S$ we define Z_s by the equations

$$F_{\lambda,s}(x) = \sum_I y_{\lambda I} x^I = 0$$

where $s = (\ldots, y_{\lambda I}, \ldots)$. Then $Z_{s_0} = Z$ for the generic point $s_0 = (\ldots, a_{\lambda I}, \ldots) \in S$. The *spread of* Z is the family

$$\begin{array}{c} \mathcal{Z} \\ \big\downarrow{\scriptstyle \pi} \\ S \end{array}$$

where $\pi^{-1}(s) = Z_s$. Spreads have the following properties:

(i) For a general $s \in S$ the algebraic properties of the Z_s are all "the same" as those for Z_{s_0} – in particular, the algebraic properties of the ideal of Z – such as the character of its minimal free resolution, the ideals containing it and therefore the configurations of its subvarieties – are "constant" in the family $\{Z_s\}$.

(ii) $\mathcal{Z}$ and S are defined over $\mathbb{Q}$ and

$$\mathbb{Q}(S) =: k \cong \mathbb{Q}(\ldots, a_{\lambda I}, \ldots)$$

is a field of definition of Z. If we think of Z as an abstract variety defined over a field k, then speaking roughly the spread arises from the different embeddings $k \hookrightarrow \mathbb{C}$.

(iii) Absolute differentials are interpreted geometrically via a natural isomorphism

$$\Omega^2_{Z(k)/\mathbb{Q},z} \cong \Omega^2_{\mathcal{Z}(\mathbb{Q})/\mathbb{Q},z}$$

for $z \in Z(k)$. Again roughly speaking, both sides are generated by the quantities $da_{\lambda I}$, dx_i subject to the relations

$$\begin{cases} dF_{\lambda,s_0}(x) = 0 \\ dG_\alpha(\ldots, a_{\lambda I}, \ldots) = 0 \end{cases}$$

where $dF_{\lambda,s_0}(x) = \sum_I da_{\lambda I} x^I + \sum_I a_{\lambda I} dx^I$.

(iv) Infinitesimally, spreads are characterized among all families $\mathcal{Z} \to S$ by the property that the Kodaira-Spencer class

$$\rho \in H^1(\mathrm{Hom}(\Omega^1_{Z(k)/k}, T^*_{s_0} S))$$

is identified with the extension class of

$$0 \to \Omega^1_{k/\mathbb{Q}} \otimes \mathcal{O}_{Z(k)} \to \Omega^1_{Z(k)/\mathbb{Q}} \to \Omega^1_{Z(k)/k} \to 0$$

via the evaluation map

$$T^*_{s_0} S \to \Omega^1_{k/\mathbb{Q}}$$

given by $da_{\lambda I} \to d_{\mathbb{C}/\mathbb{Q}} a_{\lambda I}$, where $da_{\lambda I}$ is the usual differential of the function $y_{\lambda I}$ considered in $T^*_{s_0} S_0$.

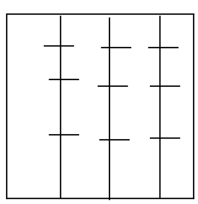
(v) The evaluation mappings

$$\Omega^1_{Z(k)/\mathbb{Q},z} \to \Omega^1_{k/\mathbb{Q}}$$

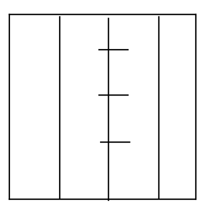
may, via the above identification $\Omega^1_{Z(k)/\mathbb{Q},z} \cong \Omega^1_{\mathcal{Z}(\mathbb{Q})/\mathbb{Q},z}$, be thought of as inducing maps

$$T^*_z \mathcal{Z} \to T^*_{s_0} S$$

that give a k-linear, but not an $\mathcal{O}_{Z(k)}$ linear, connection along Z_{s_0} in the family $\mathcal{Z} \to S$.



usual picture of a connection



connection only along central fiber as for spreads

Now let X be a regular surface defined over $\mathbb{Q}$, and let $z \in Z^2(X(k))$ where $k \cong \mathbb{Q}(S)$. We may apply the spread construction to $Z = (X, z)$ to obtain

$$\begin{array}{ccc} \mathcal{Z} & \subset & X \times S \\ \big\downarrow & & \big\downarrow \\ S & = & S \end{array}$$

where $\mathcal{Z} \cdot X \times \{s\} = z_s$. Note that a k-rational equivalence has a spread over $S \times \mathbb{P}^1$. If

$$\tau = \sum_i (p_i, v_i) \in T_z Z^2(X(k))$$

where the $p_i \in X(k)$ and $v_i \in T_{p_i} X(k)$, then for $\omega \in H^0(\Omega^2_{X(k)/\mathbb{Q}})$ we may consider

$$\omega \rfloor \tau \in T_z^* \mathcal{Z},$$

where in fact $\omega \rfloor \tau \in T^*_{s_0} S \hookrightarrow T^*_z \mathcal{Z}$ is "horizontal" relative to the connection along Z_{s_0} described above. This gives the geometric interpretation of how (absolute) 2-forms may be evaluated on geometric tangent vectors to 0-cycles; in summary:

The value of $\omega \in H^0(\Omega^2_{X/\mathbb{Q}})$ on $\tau \in T_z Z^2(X(k))$ is interpreted geometrically as a 1-form on the spread $\mathcal{Z}$ at z.

4. Integration of Abel's DE's; caveat

By "integration" of Abel's DE's we mean constructing *Hodge-theoretically* a space $\mathcal{H}$ and map

$$\psi\colon\ Z^n(X) \to \mathcal{H}$$

such that the fibers of ψ have tangent spaces given by (A). Actually, what will be meant is to have a sequence $(\mathcal{H}_i, \psi_i)$ for $i = 0, 1, \dots, n$ such that ψ_i is defined on

$$Z^n(X)_{i-1} =: \ker\, \psi_{i-1}$$

where $Z^n(X)_{-1} = Z^n(X)$ and the tangent spaces for $\ker\, \psi_i$ are given by

$$(A_i) = 0$$

where (A_i) is the i^{th} level of Abel's DE's. We shall now illustrate this for curves and surfaces.

For $n = 1$, we have

- $\mathcal{H}_0 = \mathrm{Hom}(H^0(\Omega^0_{X/\mathbb{C}}), \mathbb{C})$ and
$$\psi_0\colon\ Z^2(X) \to \mathbb{C}$$
is given by
$$\psi_0(z)(1) = \int_z 1 = \deg\, z, \qquad\qquad 1 \in H^0(\Omega^0_{X/\mathbb{C}})$$
- $\mathcal{H}_1 = \mathrm{Hom}(H^0(\Omega^1_{X/\mathbb{C}}), \mathbb{C})/\text{periods} = \mathrm{Jac}(X)$ and
$$\psi_1\colon\ Z^2(X)_0 \to \mathcal{H}_1$$
is given by
$$\psi_1(z)(\varphi) = \int_\gamma \varphi \quad \text{mod periods}, \qquad\qquad \varphi \in H^0(\Omega^1_{X/\mathbb{C}})$$

where γ is a 1-chain satisfying
$$\partial\gamma = z,$$
which may be constructed using the assumption that $\psi_0(z) = 0$. As has been observed earlier, the fibers of ψ_1 have tangent spaces defined by Abel's DE's in the case of algebraic curves.

For $n = 2$ we may define $(\mathcal{H}_0, \psi_0)$ and $(\mathcal{H}_1, \psi_1)$ as in the curve case, where now $\mathcal{H}_1 = \mathrm{Alb}(X)$. For the same reasons as in the curve case, the tangent spaces to the fibers of ψ_1 are given by
$$\varphi = 0, \qquad\qquad \varphi \in H^0(\Omega^1_{X/\mathbb{C}}).$$
The interesting question is to define ψ_2. Thinking of ψ_0 and ψ_1 as given by
$$\int_z 1, \qquad\qquad 1 \in H^0(\Omega^0_{X/\mathbb{C}})$$
$$\int_\gamma \varphi, \qquad\qquad \varphi \in H^0(\Omega^1_{X/\mathbb{C}})$$
one way of looking at the issue is the question of how to define
$$\int_\Gamma \omega, \qquad\qquad \omega \in H^0(\Omega^2_{X/\mathbb{C}}).$$
Here, Γ is to be a 2-chain that may only be constructed using the assumption $\psi_0(z) = \psi_1(z) = 0$.

In fact, the summary at the end of the previous section suggests one way to proceed. Namely, assume that X is defined over $\mathbb{Q}$ and let $z \in Z^2(X)_1$ satisfy

$$\psi_0(z) = \psi_1(z) = 0.$$

Based on the discussion above we should consider the field k over which z is defined. Let then $k \cong \mathbb{Q}(S)$ and $\{z_s\}_{x \in S}$ be the spread of z. We observe that

$$\psi_0(z_s) = \psi_1(z_s) = 0$$

for all $s \in S$. According to the geometric interpretation of Abel's DE's, in the picture

$$\begin{array}{ccc} \mathcal{Z} & \subset & X \times S \\ \big\downarrow & & \big\downarrow \\ S & = & S \end{array}$$

at $z \in Z_{s_0}$ we should think of $\omega(z) \in \Lambda^2 T_z^* \mathcal{Z}$ as having one "vertical" component corresponding to a variation of z in $Z^2(X(k))_1$ and one "horizontal" component corresponding to the variation of z in the spread directions. The vertical variation of z is given by the tangent $z' \in T_z Z^2(X(k))$ at z. For the horizontal variation we choose a closed curve $\gamma \in S$ based at s_0 and take the tangent γ' to γ at s_0. Abel's DE's may then be expressed as

$$\langle \omega, z' \wedge \gamma' \rangle = 0.$$

Thus we are led to the following construction: By the assumption $\psi_1(z_s) = 0$, we have that the induced map

$$\mathcal{Z}_* \colon\ H_1(S, \mathbb{Q}) \to H_1(X, \mathbb{Q})$$

is zero. Setting $\mathcal{Z}_\gamma = \pi^{-1}(\gamma) \cap \mathcal{Z}$, we see that the cycle $\mathcal{Z}_\gamma$ represents $\mathcal{Z}_*(\gamma)$ and working modulo torsion we have that $\mathcal{Z}_\gamma = \partial \Gamma$ for 2-chain Γ. For $\omega \in H^0(\Omega^2_{X/\mathbb{Q}})$ we define

$$(**) \qquad \boxed{\psi_2(z)(\omega) = \int_\Gamma \omega \qquad \text{mod periods}}$$

A different choice of Γ changes $(**)$ by a period. If $\gamma = \partial \Delta$ for some 2-chain $\Delta \subset S$, by Stokes' theorem

$$\psi_2(z)(\omega) = \int_\Delta \mathrm{Tr}_{\mathcal{Z}} \omega$$

where $\mathrm{Tr}_{\mathcal{Z}} \omega$ is induced by the trace map; specifically,

$$\mathcal{Z}_* \colon\ H^0(\Omega^2_{X/\mathbb{C}}) \to H^0(\Omega^2_{S/\mathbb{C}})$$

is the map induced by the component

$$[\mathcal{Z}]^{(0,2),(2,0)} \in H^{0,2}(X) \otimes H^{2,0}(S) \cong \mathrm{Hom}(H^0(\Omega^2_{X/\mathbb{C}}), H^0(\Omega^2_{S/\mathbb{C}}))$$

of the fundamental class $[\mathcal{Z}]$. Here we are writing

$$[\mathcal{Z}] = [\mathcal{Z}]^{(2,0),(0,2)} + [\mathcal{Z}]^{(1,1),(1,1)} + [\mathcal{Z}]^{(0,2),(2,0)}$$

under the Künneth-Hodge decomposition

$$\begin{aligned} H^{2,2}(X \times S) &= \big(H^{(2,0)}(X) \otimes H^{0,2}(S)\big) \oplus \big(H^{1,1}(X) \otimes H^{1,1}(S)\big) \\ &\quad \oplus \big(H^{(0,2)}(X) \otimes H^{(2,0)}(S)\big). \end{aligned}$$

It follows that $\psi_2(z)(\omega)$ is a *differential character* on S.[5]

Moreover, it follows from the above discussion that

$$\psi_2(z)(\omega) \equiv 0 \text{ mod periods}$$

if z is k-rationally equivalent to zero. Indeed, we first argue heuristically as follows: Set

$$\psi_2(z)(\omega, \gamma) = \int_\Gamma \omega$$

in $(**)$ above. Thinking of γ as parametrized by $0 \leqq s \leqq 2\pi$ and setting $\Gamma \cdot X_s = \lambda_s$ we may iterate the integral to have

$$\psi_2(z)(\omega, \gamma) = \int_0^{2\pi} \left(\int_{\lambda_s} \omega \rfloor \partial/\partial s \right) ds.$$

Now suppose we have a k-rational equivalence

$$z_1 \equiv_{\mathrm{rat}} z_2.$$

This is given by

$$\mathcal{Z} \subset X \times S \times \mathbb{P}^1$$

where

$$\begin{cases} \mathcal{Z} \cdot X \times S \times \{0\} = k\text{-spread of } z_1 \\ \mathcal{Z} \cdot X \times S \times \{\infty\} = k\text{-spread of } z_2. \end{cases}$$

[5]Invariants associated to general families of 0-cycles have been constructed by Schoen in [Schoen] (based in part on a suggestion by Nori) and Voisin in [Voisin]. For families arising from spreads, the above construction differs in an interesting way from that of Voisin – c.f. the appendix to section 9(iii) in [Green-Griffiths I] for a discussion of this point.

Letting $\mathcal{Z}_t = \mathcal{Z} \cdot X \times S \times \{t\}$ be the k-spread of z_t, and setting $z = z_0$, and $z' =$ tangent to z_t at $t = 0$, we have

$$\begin{aligned}\frac{d}{dt}(\psi_2(z_t)(\omega,\gamma))_{t=0} &= \int_0^{2\pi} \frac{d}{dt}\Bigg(\int\limits_{\lambda_{s,t}} \omega\rfloor\partial/\partial s\Bigg)_{t=0} ds \\ &= \int_0^{2\pi} \langle \omega, z' \wedge \gamma'\rangle\, ds\end{aligned}$$

where $\gamma' = \partial/\partial s$. Then

$$\langle \omega, z' \wedge \gamma' \rangle = 0 \tag{$***$}$$

is exactly Abel's DE as expressed above.

Similar heuristic reason gives that the fiber of ker ψ_2 through $z \in Z^2(X(k))$ has tangent space equal to $T_z Z^2_{\mathrm{rat},z}(X(k))$, where $Z^2_{\mathrm{rat},z}(X(k))$ are the cycles in $Z^2(k)$ that are k-rationally equivalent to z. Indeed, if z_t is an arc in $Z^2(X(k))$ with $z_0 = z$ and tangent z' at $t = 0$, then if

$$\frac{d}{dt}(\psi_2(z_t)(\omega,\gamma))_{t=0} = 0$$

for all ω and γ, we may reverse the above calculation and conclude that $(***)$ holds for all ω and γ, which by the theorem stated above means that

$$z' \in T_z Z^2_{\mathrm{rat},z}(X(k)).$$

The reason that the above calculations are "heuristic" is this: The geometric interpretation of Abel's DE's via spreads requires that we restrict our attention to a field k that is finitely generated over $\mathbb{Q}$ – i.e., we consider $X(k)$, $Z^2(X(k))$, $Z^2_{\mathrm{rat}}(X(k))$ etc. On the other hand the above calculations using loops in S and paths in X as if we were in the setting of complex manifolds seems, at least on the face of it, to require that we be working with $X(\mathbb{C})$, $Z^2(X(\mathbb{C})), Z^2_{\mathrm{rat}}(X(\mathbb{C}))$ etc. However, the geometric situation

$$\begin{array}{ccc}\mathcal{Z}(\mathbb{C}) & \subset & X(\mathbb{C}) \times S(\mathbb{C}) \\ \big\downarrow & & \big\downarrow \\ S(\mathbb{C}) & = & S(\mathbb{C}),\end{array}$$

and the same situation crossed with $\mathbb{P}^1(\mathbb{C})$, are the complex points of these same situations with k replacing $\mathbb{C}$. Moreover, Abel's DE's are *algebraic* and *defined* *over* k and are satisfied in the above situation over the field k. Therefore they also hold in the above situation over the field $\mathbb{C}$, thereby justifying the heuristic calculations.[6]

Caveat. Although we have described how to integrate Abel's DE's in the sense of differential equations, we have *not* asserted what would amount to integrating Abel's DE's in the sense of geometry; namely, that modulo torsion

$$\ker \psi_2 = Z^2_{\mathrm{rat}}(X).$$

We do not know that the above Hodge-theoretic construction captures rational equivalence.

One reason is that, due to the fact that $Z^2(X)$ and $CH^2(X)$ are not varieties in anything like the usual sense, the rules of calculus "break down". Thus, e.g., one has the phenomenon of *null-curves*, which are curves B in $Z^2(X)$ such that the induced map

$$\mathrm{Jac}\, B \to CH^2(X)$$

is non-constant but whose differential vanishes identically. For example, suppose that X is a regular surface defined over $\mathbb{Q}$ and $Y \subset X$ is an algebraic curve also defined over $\mathbb{Q}$. Then

$$J(Y) \to CH^2(X)$$

is certainly non-constant in general, while since the spread of any 0-cycle contained in Y is supported on Y and

$$\omega\big|_Y = 0$$

for any $\omega \in H^0(\Omega^2_{X/\mathbb{Q}})$, we infer that the differential of the above map vanishes identically. One might say that *in the world of $Z^2(X)$ and $CH^2(X)$ the usual uniqueness theorems from DE's fail.* This failure does seem to occur for arithmetic reasons – the above construction is conjecturally the only way null curves can arise. We refer to [Green-Griffiths I] for further discussion of this, and the related failure of the usual existence theorems in DE's in the setting of algebraic cycles.

[6] The above is a special case of a general construction given in [Green-Griffiths II] of Hodge - theoretic invariants of algebraic cycles. These invariants are constructed using the fundamental class and Abel-Jacobi map applied to spreads; assuming the generalized Hodge conjecture and conjecture of Bloch-Beilinson they give a complete set of invariants of the rational equivalence class (modulo torsion) of algebraic cycles.

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(Mark L. Green) Department of Mathematics, UCLA, Los Angeles, CA 90095
E-mail address: mlg@math.ucla.edu

(Phillip A. Griffiths) Institute for Advanced Study Einstein Drive Princeton, NJ 08540
E-mail address: pg@ias.edu