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CONTROL SYNTHESIS IN HYBRID SYSTEMS WITH FINSLER DYNAMICS

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Abstract. This paper is concerned with a symbolic-based synthesis of feedback control policies for hybrid and continuous dynamic systems. A key step in our synthesis procedure is a new method to solve the following dynamic programming problem:

$$\frac{\partial V}{\partial t}(z,t) = \min_v \frac{\partial V}{\partial z}(z,t)\rho_v(z,v)v$$

$$\dot{z} = \rho(z,v), \quad t \in [0,T] \tag{1}$$

$$V(z,T) = \Psi(z,T) \tag{2}$$

Here $V(z,t)$ is the cost-to-go function associated with a certain type of homogeneous calculus of variations problem on a Finsler manifold and (z,v) is a positive homogeneous function of degree one in v. This optimization problem is at the core of the control synthesis procedure for many hybrid control problems [1], [2].

1. Dedication

We dedicate this paper to Professor Chern with gratitude, and for good reason. Nerode learned his differential geometry from Chern's classes at the University of Chicago in 1951. As a lifelong mathematical logician, Nerode never expected to use this knowledge, especially knowledge of Finsler Geometry, which he had later acquired from his friend and fellow student of Chern's, Louis Auslander. When developing feedback policies for optimal control systems, we discovered that these can be modeled as Finsler geodesic fields and connections. One never knows in advance what mathematics one may need, and it is nice to be in a position to recognize what is needed. A great virtue of the University of Chicago program, of which Chern was one of its founders, was that general knowledge of broad fields was an explicit aim. Kohn learned differential geometry at MIT and found its tremendous potential in control applications. He has been an avid student of Professor Chern's work. Vladimir Brayman who is completing his PhD in Electrical Engineering is working on the application of the differential geometric methods to enterprise control problems.

2. Introduction

The discipline of hybrid systems emerged in the decade 1990-2000 as an important science at the interface of control theory, electrical engineering, and computer science. Hybrid systems are systems that incorporate (discrete) logical control programs that interact with continuous physical plants in a changing environment. In this context, a plant is thought of as an evolving vector field of plant states. Before the Hybrid System approach, the standard models for such systems tended to ignore either the continuous or the discrete aspects of the system. There are many cases where non-hybrid approaches are not adequate to develop optimal control laws. An enterprise control system is an example. The discipline of hybrid systems attempts to build and analyze models in which both the continuous and discrete aspects of the control problem are taken into account in a combined continuous field encoding both the discrete (event driven) and continuous dynamics of the system, with a transformation to the discrete domain when needed. A good place to see examples of these systems, is in the four volumes of hybrid systems which have appeared in the Lecture Notes in Computer Science series by Springer-Verlag, see references [1], [2]. There have been many conferences on this subject worldwide since the publication of these seminal volumes.

We design logical control laws for hybrid systems to force the evolution of the plant states to satisfy a performance specification, even when subjected to disturbances in the environment, and in the presence of unmodeled plant dynamics. A control law is implemented by a real time control architecture. The way a control architecture module operates is as follows: when the plant enters a certain prescribed region of the state manifold, the event is sensed by the control architecture, which triggers the control program to change the control parameters for the plant actuators, thereby changing the plant evolution vector field to a new vector field. The plant state trajectory is thus a piecewise differentiable path. Therefore, the discontinuities in the direction of the trajectory take place at the times of the control law intervention. In other words, the hybrid control programs are event-driven finite automata, switching the plant from one vector field evolution to another for the purpose of enforcing plant performance specifications. The use of hybrid control programs extends the range of applications of conventional continuous control theory to complex non-linear non-homogeneous systems. The problem is, how does one develop such a program?

The fundamental problem of hybrid systems is to produce control algorithms and control implementation architectures that enforce the performance specification for the system, given the plant state models. In our approach, we introduce a manifold on which evolution of plant state trajectory $y(t)$ take place. The manifold is determined by the constraints. The control policies are formulated as functions $\gamma(y(t), \dot{y}(t))$ that determine the direction $\dot{y}(t) = \rho(y(t), u(t))$ at time t when the plant state is $y(t)$. In practice, the control effort should depend on the current state y and rate $\dot{y}$, and not on the current time, because the unknown disturbances and inaccuracies in the modelling parameters may lead to timing and positional inaccuracies. We introduce a suitable non-negative Lagrangian $\tilde{L}(y, u, t)$ and a goal set G on the manifold. We rephrase the performance specification so that the requirement on the control policy can be chosen such that the plant state trajectory $y(t)$ remains on the manifold. The trajectory $y(t)$ leads from current state to the goal set G, and minimizes $\int \tilde{L}(y, \gamma(y, \dot{y}), t)dt$ among all trajectories $y(t)$ arising from admissible control policies.

The discussion is confined to the case when the goal is a single point. Therefore, a control policy indicates, given the current state, the optimal direction to go in order to end at the goal point. We allow the tangent field along the optimal state trajectory to be measure-valued. This is done to ensure that mathematically optimal trajectories exist. Furthermore, this implies that we allow control policies that are generalized curves $u(t)$ in the sense of L.C. Young [7]. Measure-valued optimal control policies are generally not physically realizable, but there are close approximations that are realizable. Given a positive ϵ, we can generate algorithms that allow one to compute a piecewise constant approximation to $\gamma(y, \dot{y})$ to an optimal control policy which brings $\int \tilde{L}(y, \gamma(y, \dot{y}), t)dt$ within ϵ of its minimum among all admissible trajectories. If the states y and the directions $\dot{y}$ are then discretized, the approximate control policy can be implemented as a logical control program that is a hybrid control automaton [4]. Unlike the true optimal control policy, this approximate control policy can be implemented in such a way as to guarantee the ϵ-optimality. This automaton is easily realizable in a generic form with Horn clauses [6], [5].

The Pontryagin School of optimal control was based on necessary, not sufficient, conditions. In this approach, one solves the necessary conditions, and among the feasible solutions to the plant equations one finds candidates to an optimal solution. In our Young-based sufficiency approach, we approximate to an optimal weak solution already known to exist by the convexity properties of Finsler spaces.

Finsler manifolds enter through the Caratheodory-Cartan reformulation of a Hamiltonian variational problem as a feedback control extraction problem. Following the example of Weierstrass for ordinary differential equations and variational calculus, time is introduced as an additional explicit variable, replacing y by $x = (y, t)$. With the transformed variables and Finsler Lagrangian, the positive homogeneity condition that $L(x, \lambda\dot{x}) = \lambda L(x, \dot{x})$ for positive λ is produced. An optimal control policy yields a path from the present position x to the goal point minimizing the cost-to-go function $\int L(x, \dot{x})dt$. When $(g_{ij}(x, \dot{x})) = \left(\frac{\partial^2(L^2(x,\dot{x})}{\partial\dot{x}_i\partial\dot{x}_j}\right)$ is positive definite, the Finsler interpretation of the cost-to-go function $\int L(x, \dot{x})dx$ measures the "Finsler length" of the curve $x(t)$. The Finsler fundamental ground form $\Sigma_{ij}g_{ij}(x, \dot{x})dx_i dx_j$ represents infinitesimal length as a function of position x and velocity $\dot{x}$. Integrating this quantity along curves gives its Finsler length. Thus the optimal plant state trajectories are Finsler geodesics. One may take as admissible control policies functions $u(t) = \gamma(x, \dot{x})$, giving the velocity $\dot{x} = \rho(x, \gamma(x, \dot{x}))$ at each position x, with ρ a positive homogeneous function of degree one in the second argument. ρ is the Euler-Lagrange form of the Lagrangian L. The generalized control policies $\gamma(x, \dot{x})$ may be interpreted as having probability measures on sets of values of $\dot{x}$ and x. These generalized control policies yield a Lebesgue measurable plant state trajectory on which the expected value of $\dot{x}(t)$ is almost always the velocity of the actual plant state trajectory $x(t)$.

When the apparently artificial homogeneity in $\dot{x}$ is introduced, where no homogeneity was originally present, the Finsler manifold structure introduced allows one to compute these control policies explicitly. But one only sees the control policies as geodesic fields or connections when this transformation is carried out. Reading the introduction to Cartan's book and Finsler's thesis, the transformation of variational problems to Finsler form seems to have been the inspiration for developing Finsler geometry in the first place. It is fitting that this source of Finsler spaces is now found to be very powerful for computing optimal control policies. Another inspiration for Finsler's thesis was undoubtedly Caratheodory's famous "Golden Path" to the calculus of variations, which was an axiomatic treatment of the relation between a geodesic field and its family of wave front hypersurfaces. Finsler's work made this Caratheodory relation arise automatically from Lagrangian problems recast in Finsler form. Now much of the structure of control policies can be seen clearly through the duality between the finite dimensional tangent space unit sphere and its cotangent space.

The process for extracting digital programs to control continuous physical systems breaks into several stages. First, the control problem is reinterpreted as an optimal control problem, by the inverse problem method of the calculus of variations, and then the optimal control problems are translated to the corresponding Finsler Manifold. On the Finsler manifold, the control problem becomes one of computing a geodesic field. This amounts to finding the connection matrix in the Cartan sense. Connections or geodesic fields are the required control policies. They often exist only as weak limits of sectionally smooth geodesic fields if one uses the sufficient conditions of the calculus of variations attributed to L. C. Young. Weak limits are usually not physically realizable, but they can be approximated by digital control programs that are real-time finite automata. Thus for any ϵ, a sectionally smooth trajectory that will produce a result within ϵ of the minimum for the Lagrangian involved, and a discrete real time digital control program that issues control orders to achieve such a trajectory can be generated. The digital control program arises by decomposing the Finsler tangent manifold into a finite number of regions, and when a new region is entered, the system communicates this fact to the control automaton, which issues the appropriate control to the actuators, usually a chattering control. Digital control programs can achieve near optimal behavior when they arise from continuous models by discretization. In this paper, we do not delve into the discretization process. It is more appropriate for this volume to describe the classical differential geometry tools used to compute the needed control policies. These are in the tradition of Cartan, and we use his notation. At the Hynomics corporation, the needed algorithms have been implemented in symbolic software. The first commercialization of this software is for agent based supply chain programming. Included here are some of the tools used.

3. An Approach to Control Synthesis

Our approach for designing adaptive feedback control laws is not the usual one. We begin by describing the *desired behavior* of the intended closed-loop system. This desired behavior is a trajectory on a suitable constructed manifold, called the *carrier manifold*, generated by a variational formulation

$$\min_{\alpha} \int_0^T L(\alpha(t))dt, \text{where } L: TM \to R, \tag{3}$$

with $\alpha(t) = ((x(t), \dot{x}(t)),\ \alpha(0) = \alpha_0)$ given and $\alpha(T) \in G$. We shall call such an L a *closed-loop Lagrangian.* L is constructed from the equations of motion of the system and isoperimetric constraints. The necessary conditions for the extremal trajectory (Euler-Lagrange equations, Nöether invariants) represent the desired dynamics of the closed-loop system.

We first construct a *constraint manifold* by incorporating the geometric and logical constraints imposed on the system, and then embed the constraint manifold into a "carrier manifold". Such an embedding is often not possible without relaxing the problem. Physically reasonable changes in the model, the closed-loop Lagrangian and the carrier manifold can be made to get this embedding. In applications, such modifications are made iteratively on-line as an adaptive mechanism for altering the controller to meet performance specifications.

Think of a large building as the result of two final blueprints which agree at the end of the design process. One is the architect's blueprint the other is the structural engineer's blueprint. Blueprints are handed from one to the other until one gets a final common blueprint. This is a feedback loop. The conventional synthesis procedure is that the architect presents his current blueprint to the structural engineer, who dictates the modifications she thinks are required and returns her blueprint to the architect, who then modifies his blueprint to conform more closely, et al... The feedback proceeds till the blueprints agree.

But the feedback can be in the opposite direction. This is the paradigm we follow. In this paradigm, the structural engineer hands her blueprint to the architect, who makes an architecture blueprint which approximately matches the structural engineer's blueprint. This blueprint is then handed back to the structural engineer, and this continues till the blueprints agree. That is, the architect formulates a blueprint containing the desired livability conditions, the construction industry analogue to our choice of constraints, manifold, and Lagrangian. The structural engineer then attempts to design a physical system approximately matching the architect's blueprint. This is iterated until the blueprints correspond. In the process, the architect may well have to relaxed constraints if they cannot be met, including cost constraints.

In this paper we concentrate on the Euclidean connections of Cartan. Other connections can be equally useful. A great variety of geodesic fields appear to be useful when one is doing control by chattering between different geodesic fields in the course of the evolution.

4. Cartan Absolute Differentiation

Let X be a vector field on a section of TM in a neighborhood of a given trajectory $\alpha(t) = (x(t), \dot{x}(t))$. The absolute Cartan differential of X measures the local change in X as we move along $\alpha(t)$ from point $(x(t), \dot{x}(t))$ to an infinitesimally close point. The *Cartan matrix* [8] is the matrix of forms with (i, k)-th entry

$$\omega_k^i = \sum_{h=1}^{n} C_{kh}^i(x, \dot{x}) d\dot{x}_h + \sum_{h=1}^{n} \Gamma_{kh}^i(x, \dot{x}) dx_h. \tag{4}$$

The Cartan absolute differential of X is

$$DX = dX + \omega X. \tag{5}$$

For a vector field X to be a Cartan geodesic field, it must satisfy the following two conditions:

$$(i) \quad DX = 0 \tag{6}$$

$$(ii) \quad dL^2(x, X) = 0, \tag{7}$$

where $L^2(x, X)$ is

$$L^2(x, X) = X^T g(x, \dot{x}) X. \tag{8}$$

Condition (i) and equation (5) imply

$$dX = -\omega X. \tag{9}$$

Substitute (8) into (7) and use (9) to get

$$\begin{aligned} 0 = dL^2(x, X) &= X^T g(x, \dot{x}) dX + dX^T g(x, \dot{x}) X + X^T dg(x, \dot{x}) X \\ &= -X^T g(x, \dot{x}) \omega X - X^T \omega^T g(x, \dot{x}) X + X^T dg(x, \dot{x}) X \\ &= X^T(-g(x, \dot{x}) w - \omega^T g(x, \dot{x}) + dg(x, \dot{x})) X. \end{aligned} \tag{10}$$

Because X is arbitrary, (10) implies

$$dg(x, \dot{x}) = g(x, \dot{x}) \omega + \omega^T g(x, \dot{x}). \tag{11}$$

Thus for the (i, j)-th entry of g, the following holds

$$\begin{aligned} dg_{ij} &= \sum_{k=1}^{n} g_{ik} \omega_j^k + \sum_{k=1}^{n} g_{jk} \omega_i^k \\ &= \sum_{k=1}^{n} g_{ik} \left(\sum_{h=1}^{n} C_{jh}^k d\dot{x}_h + \sum_{h=1}^{n} \Gamma_{jh}^k dx_h \right) \\ &= \sum_{h=1}^{n} \left[\sum_{k=1}^{n} \left(g_{ik} C_{jh}^k + g_{jk} C_{ih}^k \right) \right] d\dot{x}_h + \sum_{h=1}^{n} \left[\sum_{k=1}^{n} \left(g_{ik} \Gamma_{jh}^k + g_{jk} \Gamma_{ih}^k \right) \right] dx_h. \end{aligned} \tag{12}$$

At the same time,

$$dg_{ij} = \sum_{h=1}^{n} \frac{\partial g_{ij}}{\partial \dot{x}_h} d\dot{x}_h + \sum_{h=1}^{n} \frac{\partial g_{ij}}{\partial x_h} dx_h. \tag{13}$$

We deduce from (12) and (13) that

$$\frac{\partial g_{ij}}{\partial \dot{x}} = \sum_{k=1}^{n} g_{ik} C_{jh}^{k} + \sum_{k=1}^{n} g_{jk} C_{ih}^{k} \tag{14}$$

$$\frac{\partial g_{ij}}{\partial x} = \sum_{k=1}^{n} g_{ik} \Gamma_{jh}^{k} + \sum_{k=1}^{n} g_{jk} \Gamma_{ih}^{k} \tag{15}$$

These identities are useful for computing C and Γ.

Lemma 4.1. *Let $\{e\} = \{e_1, \ldots, e_n\}$ be a frame in $TM_{\alpha(0)}$. Let U be a neighborhood of $\alpha(0)$ defined by*

$$\frac{de}{dt} = e \frac{\omega}{dt} \tag{16}$$

with the initial conditions $g(x(0), \dot{x}(0)) = e^T(x(0), \dot{x}(0)) e(x(0), \dot{x}(0))$, where g is defined in (8). Then

$$e^T(x(t), \dot{x}(t)) e(x(t), \dot{x}(t)) = g(x(t), \dot{x}(t)) \tag{17}$$

for $t \in (0, \bar{t}]$.

Proof. From (16),

$$\frac{d(e^T e)}{dt} = \frac{de^T}{dt} e + e^T \frac{de}{dt} = \frac{\omega^T}{dt} e^T e + e^T e \frac{\omega}{dt}. \tag{18}$$

By (16) e is Lipschitz continuous because w is absolutely continuous in t. Compare equations (11) and (18). Since they share the same initial conditions and the same equations, the result follows from the uniqueness theorem. □

Here is the relation between Cartan connection coefficients and Finsler geometry.

5. Finsler Connections

Connection coefficients are computed from a given metric ground form as follows. Assume the following conditions [8]:

- **A:** If the direction of a vector X coincides with the direction of its element of support $(x, \dot{x})$, then the length of X is $L(x, \dot{x})$.

B: Let X and Y are two vectors with a common element of support. The infinitesimal rotation of the element of support around its center $(x, \dot{x} + d\dot{x})$ leaves the components of X and Y fixed, and the following condition holds:

$$g(x, \dot{x}) X D Y = g(x, \dot{x}) Y D X, \tag{19}$$

where DX and DY are the absolute Cartan differentials (see equation (5)).

C: If the direction of a vector X coincides with direction of its element of support and if the latter undergoes an infinitesimal rotation about its center, then its absolute differential (5) vanishes.

D: If Γ^{*k}_{ij} are the connection coefficients when the displacement is such that the element of support is transported parallel to itself, then these coefficients are symmetric in their lower indices i, j.

Condition **A** states that $L^2(x, \dot{x}) = \dot{x}^T g(x, \dot{x}) \dot{x}$.

Condition **B,** together with the equation (14) leads to the identity

$$C_{ijh} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{x}_h}, \tag{20}$$

where $C_{ijh} = g_{jk} C^k_{ih}$.

Condition **C** implies that

$$\sum_{k=1}^{n} C_{kih} \dot{x}_k = 0 \tag{21}$$

The unit vector l in the direction of the element of support $(x, \dot{x})$ is

$$l = \frac{\dot{x}}{L(x, \dot{x})}. \tag{22}$$

The absolute differential of l is

$$\begin{aligned} Dl_i &= dl_i + \sum_{h=1}^{n} \sum_{k=1}^{n} C^i_{kh}(x, \dot{x}) l_k d\dot{x}_h + \sum_{h=1}^{n} \sum_{k=1}^{n} \Gamma^i_{kh}(x, \dot{x}) l_k dx_h \\ &= dl_i + \sum_{h=1}^{n} \sum_{k=1}^{n} \Gamma^i_{kh}(x, \dot{x}) l_k dx_h, \end{aligned} \tag{23}$$

where the last equality is implied by the equations (21) and (22). Under parallel transport of a vector, conditions (i) and (ii) are satisfied. Thus if l is displaced parallel to itself, we get from (23) the following expression for $d\dot{x}$.

$$d\dot{x}_i = -\sum_{h=1}^{n}\sum_{k=1}^{n}\Gamma^i_{kh}(x,\dot{x})l_k dx_h. \tag{24}$$

Then along a geodesic, the Cartan matrix becomes

$$\omega^i_j = \sum_{h=1}^{n}\left(\Gamma^j_{ih} - \sum_{k=1}^{n}\sum_{r=1}^{n}\dot{x}_k\Gamma^r_{kh}C^i_{kr}(x,\dot{x})\right)dx_h = \sum_{h=1}^{n}\Gamma^{*j}_{ih}dx_h, \tag{25}$$

where

$$\Gamma^{*j}_{ih} = \Gamma^j_{ih} - \sum_{k=1}^{n}\sum_{r=1}^{n}\dot{x}_k\Gamma^r_{kh}C^i_{kr}(x,\dot{x}) \tag{26}$$

Condition **D** states that

$$\Gamma^{*i}_{kj} = \Gamma^{*i}_{jk}.$$

Then from (26)

$$\Gamma^i_{kj} - \Gamma^i_{jk} = \sum_{r=1}^{n}\left(\sum_{h=1}^{n}C^i_{kh}\Gamma^h_{rj} - \sum_{h=1}^{n}C^i_{jh}\Gamma^h_{rk}\right)\dot{x}_r. \tag{27}$$

Using the identities $\Gamma_{ijh} = g_{jk}\Gamma^k_{ih}$ and $C_{ijh} = g_{jk}C^k_{ih}$, and interchanging the indices, we rewrite (27) as follows:

$$\Gamma_{ijh} - \Gamma_{hji} = \sum_{k=1}^{n}\left(\sum_{r=1}^{n}C_{ijr}\Gamma^r_{kh} - \sum_{r=1}^{n}C_{hjr}\Gamma^r_{ki}\right)\dot{x}_k \tag{28}$$

These are n^3 equations in n^3 unknowns which we could solve for Γ_{ijh} by computing the coefficients C_{ijr} from (20). But there is a better way to compute Γ_{ijh}

5.1. **Computing Connection Coefficients.** Suppose we are given a function $G(x,\dot{x})$ that is positive homogeneous in $\dot{x}$ and such that the equation for the desired geodesic curves $\alpha(t)$ is in the local coordinates

$$\frac{d^2x}{ds^2} = -2G(x,\frac{dx}{ds}), \tag{29}$$

where s is an arc length (a curvilinear coordinate of a point moving along the geodesic curve). Consider a unit vector l tangent to α assuming, of course, that its element of support $(x,\dot{x})$ is also tangent to α [8]. In this case we can write $l = \frac{dx}{ds}$. Define a differential form $\bar{\omega}$ such that

$$\frac{\bar{\omega}^i}{ds} = \frac{d^2x_i}{ds^2} + 2G^i(x,l) = \frac{d^2x_i}{ds^2} + \frac{2}{L^2(x,\dot{x})}G^i(x,\dot{x}), \tag{30}$$

where the second equality is the consequence of the homogeneity of G. Then equation (29) can be viewed as the condition for $\frac{\bar{\omega}^i}{ds}$ to vanish along a geodesic curve. From (30) we deduce that

$$\bar{\omega}^i = dl_i + \frac{1}{L}\sum_{h=1}^{n}\frac{\partial G^i}{\partial \dot{x}_h}dx_h. \tag{31}$$

Compare equations (31) and (23) to conclude that $\bar{\omega}$ is the absolute differential of l and

$$\sum_{i=1}^{n}\dot{x}_i\Gamma^h_{ij} = \frac{\partial G^h}{\partial \dot{x}_j}. \tag{32}$$

Following Cartan [8], introduce Christoffel symbols γ_{ijh} as solutions to the following equations

$$\begin{aligned} &\gamma_{ijh} = \gamma_{hji} \\ &\frac{\partial g_{ij}}{\partial x_h} = \gamma_{ijh} + \gamma_{jih}. \end{aligned} \tag{33}$$

These equations imply that

$$\gamma_{ijh} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x_h} + \frac{\partial g_{jh}}{\partial x_i} - \frac{\partial g_{ih}}{\partial x_j}\right). \tag{34}$$

Comparing equations (15) and (28) with (33), we deduce that

$$\Gamma_{ijh} = S_{ijh} + \gamma_{ijh}, \tag{35}$$

where S_{ijh} are components of an antisymmetric tensor. The latter property implies the following identity

$$S_{ijh} + S_{jhi} + S_{hij} = 0. \tag{36}$$

From (35),

$$\Gamma_{ijh} - \Gamma_{hji} = S_{ijh} - S_{hji} = S_{ihj}, \tag{37}$$

where the first equality is implied by the symmetry of γ and the second equality follows from (36) and the antisymmetry of S. Substitute (37) and (35) into equation (28) to get

$$S_{ihj} = \sum_{r=1}^{n}\sum_{k=1}^{n}C_{ijr}\dot{x}_k\Gamma^r_{kh} - \sum_{r=1}^{n}\sum_{k=1}^{n}C_{hjr}\dot{x}_k\Gamma^r_{ki}. \tag{38}$$

Using (32) in (38), we get

$$S_{ihj} = \sum_{r=1}^{n}\sum_{k=1}^{n} C_{ijr}\frac{\partial G^r}{\partial \dot{x}_h} - \sum_{r=1}^{n}\sum_{k=1}^{n} C_{hjr}\frac{\partial G^r}{\partial \dot{x}_i}. \tag{39}$$

Equations (35) and (39) give the desired result

$$\Gamma_{ihj} = \gamma_{ihj} + \sum_{r=1}^{n}\sum_{k=1}^{n} C_{ijr}\frac{\partial G^r}{\partial \dot{x}_h} - \sum_{r=1}^{n}\sum_{k=1}^{n} C_{hjr}\frac{\partial G^r}{\partial \dot{x}_i}, \tag{40}$$

where γ_{ihj} can be found from (34) and C_{ijr} can be found from (20).

6. Bellman's Blueprint In Finsler Spaces

We now show how Bellman equation is a blueprint in the Finsler context for the extraction of control policies and the implementation of control loops. A second, subtler, use for the Bellman equation is to extract feedback correction to revise control polices when additional information about the environment faced and the inadequacies of the system model become available on line during the operation of the system. This is a form of adaptation. The Bellman equation here arises from a dynamic programming formulation of the variational problem on the associated Finsler manifold. Solutions are implemented by a differential inclusion procedure. Bellman's equation plays a role here similar to the role of Hamilton-Jacobi equations in classical mechanical systems.

Below are the necessary and sufficient conditions for optimal solutions of the Finsler variational problem assuming that Bellman's Principle of Optimality holds for the problem [3].

Let L be a constrained Finsler Lagrangian over TM (see Section 4). We formulate the desired behavior of the system as solution trajectories of the following variational problem

Problem 1.

$$\textit{minimize} \quad \int_0^T L(x(t), \dot{x}(t))dt, \tag{41}$$

over a family of curves $\alpha(t)$ on an open set U of a constraint manifold M, with the prescribed boundary condition $\alpha(T) \in G$ and subject to $(\alpha(t), \dot{\alpha}(t)) = (x(t), v(t))$, $v(t) \in TM_{x(t)}$.

Define $n+1$ dimensional vector function as follows

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{n+1}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ L(x(t), v(t)) \end{bmatrix} = \rho(\tilde{x}(t), v(t)), \tag{42}$$

where $\tilde{x} = (x_1, \dots, x_{n+1})$ and $\dot{\tilde{x}}(t) = \rho(\tilde{x}(t), v(t))$.

We note that $\rho(\tilde{x}, v)$ is positive homogeneous of degree 1 in v since by construction, the first n entries of ρ are linear in v and the $n+1$-th entry is a positive homogeneous Finsler Lagrangian.

Consider now the following problem

Problem 2.

$$\textit{minimize} \quad \{x_{n+1}(T; v) \mid x(T) \in G\} \tag{43}$$

over v and subject to

$$\dot{\tilde{x}}(t) = \rho(\tilde{x}(t), v(t)), \quad x_{n+1}(0) = 0. \tag{44}$$

Under mild assumptions, we can prove the following.

Lemma 6.1. *If $L(x(t), \dot{x}(t))$ is smooth, then Problem 2 is equivalent to Problem 1.*

Lemma 6.2 (Bellman)**.** *For Problem 2, define a twice differentiable function (cost-to-go function)as follows:*

(45)
$$V(y,t) := \inf_{\substack{v(\tau) \\ t \le \tau \le T}} \{x_{n+1}(\tau, v) |\ \tilde{x} \ \textit{satisfies} \ (44),\ \tilde{x}(t) = y,\ \tilde{x}(T) \in G \times x_{n+1}(T)\}.$$

Then

$$-\frac{\partial V}{\partial t}(y,t) = \min_{v} \{ \frac{\partial V}{\partial \tilde{x}}(y,t) \left[\frac{dX}{dt}(y,v) + \frac{\omega}{dt}(y,v) X(y,v) \right] \}, \tag{46}$$

where $X(y, v) = \rho(y, v)$ is a vector field defining the direction of the infinitesimal variation of V.

Proof. By the principle of optimality,

$$V(y,t) \le 0 + V(y + dy, t + dt). \tag{47}$$

By the local version of the fundamental theorem of calculus on manifolds [9],

$$V(y,t) \le V(y,t) + dV|_y(dt) + O(dt^2). \tag{48}$$

Thus for every y,

$$dV|_y(dt) + O(dt^2) \ge 0. \tag{49}$$

Expanding dV, we get

$$0 \leq \frac{\partial V}{\partial t}(y,t)dt + \frac{\partial V}{\partial \tilde{x}}(y,t)DX + O(dt^2), \tag{50}$$

where DX is the absolute differential of X (see equation (5)). Then we can rewrite (50) as follows

$$0 \leq \frac{\partial V}{\partial t}(y,t)dt + \frac{\partial V}{\partial \tilde{x}}(y,t)\left[\frac{dX}{dt} + \frac{\omega}{dt}X\right]dt + O(dt^2) \tag{51}$$

Dividing by dt and taking the limit as $dt \to 0$,

$$-\frac{\partial V}{\partial t}(y,t) = \min_v\{\frac{\partial V}{\partial \tilde{x}}(y,t)\left[\frac{dX}{dt} + \frac{\omega}{dt}X\right]\}, \tag{52}$$

where X and ω depend on y and v. □

Equation (52) is a necessary condition for optimality in Problem 2 and hence Problem 1. Let $\alpha(t)$ be an optimal solution of Problem 1.

Theorem 6.3. *The Solution of Problem 2 is a Cartan geodesic field.*

Proof. Observe that the only term in (52) that depends on v is $\left[\frac{dX}{dt}(y,v) + \frac{\omega}{dt}(y,v)X(y,v)\right]$. Since $V(\cdot,\cdot)$ is constant along optimal solutions,

$$\frac{\partial V}{\partial t}(y,t) = 0. \tag{53}$$

Since this is true along a Cartan geodesic field (see equation (9)), the result follows. □

We showed that $V(\cdot,\cdot)$ is constant along optimal solutions and non-decreasing along solutions of (44). These properties together with the final value $V(\cdot,T) = x_{n+1}(T)$ characterize this function [10].

Later on in Section 7, we will consider a mechanism that will construct a Cartan geodesic using the mathematical machinery developed in the present section.

7. The Control Loop

7.1. **Active Geometric Constraints.** Consider a gradient of the constraint vector $K_{\dot{z}}^T$ evaluated at a certain state $(z,\dot{z})$. Let $(\tilde{v}_1,\ldots,\tilde{v}_{\tilde{m}})$ be column vectors of $K_{\dot{z}}^T$. Then we define $(v_1,\ldots,v_m)$ to be an orthonormal basis obtained from $\tilde{v}$'s using the KVD procedure. Define also $(e_1,\ldots,e_n)$ to be the canonical orthonormal basis of $\mathbf{R}^n$. Introduce a transformation $(e_1,\ldots,e_n) \to (e'_1,\ldots,e'_n)$ as follows

$$e'_j = e_j - \sum \frac{e_j^T v_i}{v_i^T v_i} v_i. \tag{54}$$

Let $E' = [e'_1, \ldots, e'_n]$. Orthogonalize E' to get E. A projection map $\pi : M \to N$ associated with the embedding of the constraint manifold N into the carrier manifold M is computed as $\pi = E^T$.

7.2. **Feedback Control Law.** We construct a control law for the process to be controlled. The dynamics of the process under control is modeled locally by

$$\ddot{z} = \tilde{P}(z, \dot{z}, u), \tag{55}$$

where $z = (z_1, \ldots, z_n)$ is a vector of local coordinates of the carrier manifold.

Embed the constraint manifold into the carrier manifold. A projection map associated with this embedding is $\pi : M \to N$, where M is the carrier manifold and N is the constraint manifold. Then in local coordinates, $y = \pi(z)$ and $z = \pi^+ y$, where π^+ is the generalized inverse of π. If π is linear, then applying π to the both sides of (55) we get

$$\begin{aligned} \pi\ddot{z} &= \pi\tilde{P}(\pi^+ y, \pi^+ \dot{y}, u), \\ \ddot{y} &= \pi\tilde{P}(\pi^+ y, \pi^+ \dot{y}, u), \\ \ddot{y} &= P(y, \dot{y}, u), \end{aligned} \tag{56}$$

where $y_1, \ldots, y_m$ are the local coordinates of the points along a curve on the constraint manifold and the definition of P is obvious.

The desired behavior of the system is described by certain trajectories generated by a variational formulation

$$\min_z \int_0^T L^{(1)}(z, \dot{z}) dt, \quad z(T) \in G, \quad \dot{z}(T) \in G_z, \tag{57}$$

where $L^{(1)}(z, \dot{z}) = \sqrt{L^{00}}(z, \dot{z})$, where $L^{00}(z, \dot{z})$ the effective Lagrangian of the system. Terminal conditions are defined by an open set G.

Using the projection map π, we define a “constraint” Lagrangian as follows

$$L^{(2)}(y, \dot{y}) = L^{(1)}(\pi^+ y, \pi^+ \dot{y}). \tag{58}$$

With terminal conditions given by

$$\begin{aligned} y(T) &= \pi G = \tilde{G}, \\ \dot{y}(T) &= \pi G_y = \tilde{G}_y. \end{aligned} \tag{59}$$

the Euler-Lagrange necessary conditions associated with the minimization of $L^{(2)}(y, \dot{y})$ are given by

$$\frac{d}{dt} L^{(2)}_{\dot{y}}(y, \dot{y}) - L^{(2)}_{y}(y, \dot{y}) = 0, \tag{60}$$

together with (59). In the expanded form (60) is

$$L^{(2)}_{\dot{y}y}(y, \dot{y})\dot{y} + L^{(2)}_{\dot{y}\dot{y}}(y, \dot{y})\ddot{y} - L^{(2)}_{y}(y, \dot{y}) = 0. \tag{61}$$

Solving for $\ddot{y}$, we get

$$\ddot{y} = -\left(L^{(2)}_{\dot{y}\dot{y}}(y, \dot{y})\right)^{-1}\left[L^{(2)}_{\dot{y}y}(y, \dot{y})\dot{y} - L^{(2)}_{y}(y, \dot{y})\right]. \tag{62}$$

In order to find a vertical section of the constraint manifold, we need to compute $\dddot{y}$ from (62). We first introduce notations. Let $D := \left(L^{(2)}_{\dot{y}\dot{y}}(y, \dot{y})\right)^{-1}$, $e := L^{(2)}_{\dot{y}y}(y, \dot{y})\dot{y}$, and $d := L^{(2)}_{y}(y, \dot{y})$. Then, for $i, j = 1, \ldots, m$,

$$D_{ij} = \left[\left(\frac{\partial^2 L^{(2)}}{\partial \dot{y} \partial \dot{y}}\right)^{-1}\right]_{ij}, \tag{63}$$

$$e_j = \sum_{k=1}^{m} \frac{\partial^2 L^{(2)}}{\partial \dot{y}_j \partial y_k} \dot{y}_k, \tag{64}$$

$$d_j = \frac{\partial L^{(2)}}{\partial y_j}. \tag{65}$$

Writing (62) in terms of the constructs introduced above, we get

$$\ddot{y}_i = \sum_{j=1}^{m} D_{ij}\left(d_j - e_j\right), \quad i = 1, \ldots, m. \tag{66}$$

Take derivatives w.r.t t in both sides of (66) to obtain

$$\dddot{y}_i = \sum_{j=1}^{m}\left[\dot{D}_{ij}(d_j - e_j) + D_{ij}(\dot{d}_j - \dot{e}_j)\right], \quad i = 1, \ldots, m, \tag{67}$$

where definitions for $\dot{D}_{ij}$, $\dot{d}_j$, and $\dot{e}_j$ are given below.

$$\dot{D}_{ij} = -\sum_{l=1}^{m}\sum_{k=1}^{m} D_{ik}\left(\frac{d}{dt}\frac{\partial^2 L^{(2)}}{\partial \dot{y}_k \partial \dot{y}_l}\right) D_{lj}, \tag{68}$$

where

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 L^{(2)}}{\partial \dot y_k \partial \dot y_l} &= \sum_{s=1}^{m}\left[\frac{\partial^3 L^{(2)}}{\partial \dot y_k \partial \dot y_l \partial y_s}\dot y_s + \frac{\partial^3 L^{(2)}}{\partial \dot y_k \partial \dot y_l \partial \dot y_s}\ddot y_s\right] \\
&= \sum_{s=1}^{m}\left[\frac{\partial^3 L^{(2)}}{\partial \dot y_k \partial \dot y_l \partial y_s}\dot y_s + \frac{\partial^3 L^{(2)}}{\partial \dot y_k \partial \dot y_l \partial \dot y_s}\left[\sum_{r=1}^{m} D_{sr}(d_r - e_r)\right]\right]. \qquad (69)
\end{aligned}
$$

To find $\dot d_j$, we take derivative w.r.t t in both sides of (65):

$$
\begin{aligned}
\dot d_j &= \sum_{s=1}^{m}\left[\frac{\partial^2 L^{(2)}}{\partial y_j \partial y_s}\dot y_s + \frac{\partial^2 L^{(2)}}{\partial y_j \partial \dot y_s}\ddot y_s\right] \\
&= \sum_{s=1}^{m}\left[\frac{\partial^2 L^{(2)}}{\partial y_j \partial y_s}\dot y_s + \frac{\partial^2 L^{(2)}}{\partial y_j \partial \dot y_s}\left[\sum_{r=1}^{m} D_{sr}(d_r - e_r)\right]\right]. \qquad (70)
\end{aligned}
$$

From (64),

$$
\begin{aligned}
\dot e_j &= \sum_{s=1}^{m}\sum_{p=1}^{m}\left[\frac{\partial^3 L^{(2)}}{\partial \dot y_j \partial y_s \partial y_p}\dot y_p \dot y_s + \frac{\partial^3 L^{(2)}}{\partial \dot y_j \partial y_s \partial \dot y_p}\ddot y_p \dot y_s + \frac{\partial^2 L^{(2)}}{\partial \dot y_j \partial y_s}\ddot y_s\right] \\
&= \sum_{s=1}^{m}\sum_{p=1}^{m}\left[\frac{\partial^3 L^{(2)}}{\partial \dot y_j \partial y_s \partial y_p}\dot y_p \dot y_s + \frac{\partial^3 L^{(2)}}{\partial \dot y_j \partial y_s \partial \dot y_p}\dot y_s\left[\sum_{r=1}^{m} D_{pr}(d_r - e_r)\right]\right. \\
&\quad \left. + \frac{\partial^2 L^{(2)}}{\partial \dot y_j \partial y_s}\left[\sum_{r=1}^{m} D_{sr}(d_r - e_r)\right]\right]. \qquad (71)
\end{aligned}
$$

Equation (67) can be written in vector form as follows

$$
\dddot y = \mathcal{F}(y, \dot y), \qquad (72)
$$

where the i-th term of $\mathcal{F}$ is given by the RHS of (67).

At the same time, from (56),

$$
\begin{aligned}
\dddot y &= \dot P(y, \dot y, u) = P_y(y, \dot y, u)\dot y + P_{\dot y}(y, \dot y, u)\ddot y + P_u(y, \dot y, u)\dot u \\
&= P_y(y, \dot y, u)\dot y + P_{\dot y}(y, \dot y, u)P(y, \dot y, u) + P_u(y, \dot y, u)\dot u. \qquad (73)
\end{aligned}
$$

Let $ysol(t)$ denote a solution to (60) and let $(ym(t), \dot ym(t)$ denote a measured state of the system at time t (we assume that all states are observed). By minimizing the distance between the vector $\dddot y\, sol(t)$ and the vector $\dddot{ym}(t)$, we force the process to adopt the vertical sector behavior of the geodesic field. This approach is similar to the ones described in [11] and [12].

Fact: (i) If the system is moving along a geodesic $\alpha(t)$ in the embedding of the constraint manifold in the carrier manifold, no correction control is needed. Thus $\gamma_*(\alpha(t)) = 0$, where $\gamma_*(y(t)) := \dot{u}(t)$ is the differential of the map $\gamma(y(t)) := u(t)$

(ii) If the invariant condition is not satisfied but the quasi-geodesic condition [13] is satisfied, then the process is following a curve $\beta(t)$ (quasi-geodesic) satisfying the following condition.

$$|\alpha(t) - \beta(\tau)| \leq \kappa |t - \tau|, \tag{74}$$

where κ, a Lipschitz-like constant, defines the range of curves close enough to the geodesic pipe so that the control law proposed below maintains a bounded distance between α and β. The control law, $u(t) = \gamma(y(t))$ must be applied to the process to achieve this condition. Then we formulate a control problem as follows. Find $\dot{u}$ such that the distance between $ysol(t)$ and $ym(t)$ is minimal in curvature, i.e.

$$\min_{\dot{u}} \left\{ (\mathcal{F} - \dot{P})^T (\mathcal{F} - \dot{P}) \right\}. \tag{75}$$

Theorem 7.1. *If $P(y, \dot{y}, u)$ and $\mathcal{F}(y, \dot{y})$ satisfy sufficient smoothness conditions and there exists a differentiable optimal feedback control law, then the rate of this control law is given by*

$$\begin{aligned}
\dot{u}(t) &= \gamma_*(ysol(t), \dot{y}sol(t), ym(t), \dot{y}m(t)) \\
&= \left(P_u^T(ym(t), \dot{y}m(t), u(t)) P_u(ym(t), \dot{y}m(t), u(t))\right)^{-1} P_u^T(ym(t), \dot{y}m(t), u(t)) \\
&\quad \Big(\mathcal{F}(ysol(t), \dot{y}sol(t)) - P_y(ym(t), \dot{y}m(t), u(t)) \dot{y}m \\
&\quad - P_{\dot{y}}(ym(t), \dot{y}m(t), u(t)) P(ym(t), \dot{y}m(t), u(t))\Big)
\end{aligned} \tag{76}$$

Proof. The existence of the optimal $\dot{u}$ is assured by the convexity of the objective function in (75). Necessary condition for optimality (without constraints) can be written as

$$\frac{d}{d\dot{u}} \left(\mathcal{F}^T \mathcal{F} - \mathcal{F}^T \dot{P} - \dot{P}^T \mathcal{F} + \dot{P}^T \dot{P} \right) = 0. \tag{77}$$

Substitute for $\dot{P}$ from (73) and simplify to get

$$\begin{aligned}\mathcal{F}(ysol(t), \dot{y}sol(t)) &- P_y(ym(t), \dot{y}m(t), u(t))\dot{y}m \\ &- P_{\dot{y}}(ym(t), \dot{y}m(t), u(t))P(ym(t), \dot{y}m(t), u(t)) \\ &= P_u(ym(t), \dot{y}m(t), u(t))\dot{u}(t).\end{aligned} \tag{78}$$

Using the right pseudo-inverse of P_u, the result follows immediately. □

Note that, according to Theorem 7.1, if two curvatures match exactly, $\dot{u}(t) = 0$ and thus no change in control is needed.

7.3. **The Inverse Variational Problem.** This subsection is devoted to the construction of a non-negative Lagrangian L on the carrier manifold of the system, the traditional "inverse variational problem" when the problem starts without a Lagrangian. Given a second order differential equation, we construct a non-negative Lagrangian such that the extremals which minimize this Lagrangian are the solutions of the original differential equation. These extremals might or might not be physically realizable. We use approximation techniques to find a physically realizable approximations to them.

Consider a vector second order differential equation

$$\ddot{x} = F(x, \dot{x}, t) \tag{79}$$

Consider also the Euler-Lagrange equations

$$\frac{d}{dt}L_{\dot{x}_i}(x, \dot{x}) - L_{x_i}(x, \dot{x}) = 0, \quad i = 1, \ldots, n \tag{80}$$

Write (80) as

$$\partial L_{\dot{x}}(x, \dot{x}) \begin{bmatrix} \dot{x} \\ \ddot{x} \\ 1 \end{bmatrix} - L_x(x, \dot{x}) = 0, \tag{81}$$

where $\partial L_{\dot{x}} = \begin{bmatrix} (L_{\dot{x}_i x_j}) & (L_{\dot{x}_i \dot{x}_j}) & L_{\dot{x}t} \end{bmatrix}$, $i, j = 1, \ldots, n$.

Substitute for $\ddot{x}$ from the equation (79):

$$\partial L_{\dot{x}}(x, \dot{x}) \begin{bmatrix} \dot{x} \\ F \\ 1 \end{bmatrix} - L_x(x, \dot{x}) = 0, \tag{82}$$

Take the derivative with respect to $\dot{x}$ from the both sides of this equation:

$$\begin{bmatrix} \partial L_{\dot{x}\dot{x}_1}(x,\dot{x}) & \partial L_{\dot{x}\dot{x}_2}(x,\dot{x}) & \dots & \partial L_{\dot{x}\dot{x}_n}(x,\dot{x}) \end{bmatrix} \begin{bmatrix} \dot{x} \\ F \\ 1 \end{bmatrix} + \partial L_{\dot{x}}(x,\dot{x}) \begin{bmatrix} I \\ F_{\dot{x}} \\ 0 \end{bmatrix}$$

(83)
$$-L_{x\dot{x}}(x,\dot{x}) = 0.$$

where I is the identity matrix and $F_{\dot{x}} = \begin{bmatrix} \frac{\partial F}{\partial \dot{x}_1} & \frac{\partial F}{\partial \dot{x}_2} & \dots & \frac{\partial F}{\partial \dot{x}_n} \end{bmatrix}$. With the assumptions that L is in C^3 and the mixed partials are continuous, the following relations hold: $L_{\dot{x}x\dot{x}} = L_{\dot{x}\dot{x}x}$, and $L_{\dot{x}t\dot{x}} = L_{\dot{x}\dot{x}t}$. Thus the equation (83) can be rewritten as

$$\frac{d}{dt}L_{\dot{x}\dot{x}}(x,\dot{x}) + L_{\dot{x}x}(x,\dot{x}) + L_{\dot{x}\dot{x}}(x,\dot{x})F_{\dot{x}}(x,\dot{x},t) - L_{x\dot{x}}(x,\dot{x}) = 0. \tag{84}$$

Take the transpose of (84) using the fact that $\left(\frac{d}{dt}L_{\dot{x}\dot{x}}\right)^T = \frac{d}{dt}L_{\dot{x}\dot{x}}$

$$\frac{d}{dt}L_{\dot{x}\dot{x}}(x,\dot{x}) + L_{x\dot{x}}(x,\dot{x}) + \left(F_{\dot{x}}\right)^T(x,\dot{x},t)L_{\dot{x}\dot{x}}(x,\dot{x}) - L_{\dot{x}x}(x,\dot{x}) = 0. \tag{85}$$

Add (84) and (85) and divide by 2 to get the Lyapunov equation

$$-\frac{d}{dt}\Psi = \frac{1}{2}F_{\dot{x}}^T\Psi + \frac{1}{2}\Psi F_{\dot{x}} \tag{86}$$

where $\Psi(x,\dot{x}) = L_{\dot{x}\dot{x}}$. The solution of this equation will give the desired $L_{\dot{x}\dot{x}}$.

7.4. **Bellman's Inverse Problem.** Here is the Bellman inverse problem [3]. Given a local control law $\dot{x} = v(y,t)$ that determines $V(y,t)$, we would like to find a function $L(x,\dot{x})$ such that

$$V(y,t) = \min_x \int_t^T L(x,\dot{x})dt, \;\; x(t) = y. \tag{87}$$

Note that the cost-to-go function $V(y,t)$ is constant along a geodesic line given by $v(y,t)$ and the boundary condition is $V(y,T) = 0$. Under the conditions for Bellman's principle of optimality,

$$V(y,t) = \min_v \left[L(y,v)\Delta + V(exp(\Delta v)y, t+\Delta)\right] + O(\Delta^2). \tag{88}$$

Expand $V(\exp(\Delta v)y, t+\Delta)$ in the Lie-Taylor series [14], page 31.

$$\begin{aligned} V(\exp(\Delta v)y, t+\Delta) &= V(y,t) + \Delta v(V)(y,t) + \Delta \frac{\partial V}{\partial t}(y,t) + O(\Delta^2) \\ &= V(y,t) + \Delta \sum_{i=1}^{n} \xi_i(y,t) \frac{\partial V}{\partial y_i}(y,t) + \Delta \frac{\partial V}{\partial t}(y,t) + O(\Delta^2), \end{aligned} \tag{89}$$

where $(y_1, \dots, y_n)$ are local coordinates of the point y and $v(y,t) = \sum_{i=1}^{n} \xi_i(y,t) \frac{\partial}{\partial y_i}$. If we denote $v(y,t) = [\xi_1(y,t), \dots, \xi_n(y,t)]^T$ and $\frac{\partial V}{\partial y} = [\frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n}]$, then (89) can be rewritten in a vector form as

$$V(\exp(\Delta v)y, t+\Delta) = V(y,t) + \Delta \frac{\partial V}{\partial y}(y,t) v(y,t) + \Delta \frac{\partial V}{\partial t}(y,t) + O(\Delta^2). \tag{90}$$

Substitute (90) into (88) and simplify, to get

$$-\Delta \frac{\partial V}{\partial t}(y,t) = \min_{v} \left[L(y,v)\Delta + \Delta \frac{\partial V}{\partial y}(y,t) v(y,t) \right] + O(\Delta^2). \tag{91}$$

Divide by Δ and let $\Delta \to 0$, to get the Hamilton-Jacobi-Bellman equation:

$$-\frac{\partial V}{\partial t}(y,t) = \min_{v} \left[L(y,v) + \frac{\partial V}{\partial y}(y,t) v(y,t) \right], \quad V(y,T) = 0. \tag{92}$$

Now assume that L is twice differentiable and convex in $\dot{x}$. Then for the given (optimal) v, $L_{\dot{x}}(y,v) + \frac{\partial V}{\partial y}(y,t) = 0$. Then (92) can be rewritten as

$$\frac{\partial V}{\partial t}(y,t) = L_{\dot{x}}(y,v)v(y,t) - L(y,v) \tag{93}$$

$$\frac{\partial V}{\partial y}(y,t) = -L_{\dot{x}}(y,v). \tag{94}$$

The differential of $V(y,t)$ is given by

$$dV = \frac{\partial V}{\partial y}(y,t)dy + \frac{\partial V}{\partial t}(y,t)dt \tag{95}$$

$$= -L_{\dot{x}}(y,v)dy + (L_{\dot{x}}(y,v)v(y,t) - L(y,v))\, dt. \tag{96}$$

Along a geodesic, V is constant and thus $dV = 0$. Then

$$L_{\dot{x}}(y,v)v = L(y,v), \tag{97}$$

$$L_{\dot{x}} = 0. \tag{98}$$

Equation (97) shows that $L(y,v)$ is positive homogeneous of degree one in $\dot{x}$.

We also can show that the optimal control can be written in the form [15]

$$\gamma(y) = \exp(\gamma_y \dot{y}) \varepsilon \gamma(0). \tag{99}$$

8. Conclusion

This paper illustrated some tools from Finsler geometry used to extract control policies, mostly by symbolic computation. These have all been implemented.

There are many other areas of science in which the Caratheodory-Finsler-Cartan translation of variational problems, and these same tools, may prove to be of equal value. The strategy that physicists have used for 25 years in designing connections, gauge fields, for a given problem is valuable here too. We are currently investigating algorithms for constructing connections to meet given system goals without using an explicit variational formulation or the inverse variational method. Much of the algebra of linear connections and gauge fields has already been used in particle physics for similar purposes. We believe this algebra is equally useful here for boutique control program design. We see useful Finsler spaces now all around us.

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