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A NEW BOUND FOR THE EFFECTIVE MATSUSAKA BIG THEOREM

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Dedicated to Professor Shiing-shen Chern on his 90th Birthday

Abstract. Matsusaka's Big Theorem gives the very ampleness of mL for an ample line bundle L over an n-dimensional compact complex manifold X when m is no less than a number m_0 depending on L^n and $L^{n-1}K_X$. The dependence of m_0 on L^n and $L^{n-1}K_X$ is not effective. An earlier result of the author gives an effective m_0 containing the factor $\left(L^{n-1}\left((n+2)L+K_X\right)\right)^q$ with q of order 4^n. Demailly improved the order of q to 3^n by reducing the twisting required for the existence of nontrivial global holomorphic sections for anticanonical sheaves of subvarieties which occur in the verification of the numerical effectiveness of $pL-K_X$ for some effective p. Twisted sections of anticanonical sheaves are needed to offset the addition of the canonical sheaf in vanishing theorems. We introduce here a technique to get a new bound with q of order 2^n. The technique avoids the use of sections of twisted anticanonical sheaves of subvarieties by transferring the use of vanishing theorems on subvarieties to X and is more in line with techniques for Fujita conjecture type results.

0. Introduction

Let L be an ample line bundle over a compact complex manifold X of complex dimension n. Ampleness means that the holomorphic line bundle admits a smooth Hermitian metric whose curvature form is positive definite everywhere. Matsusaka's Big Theorem deals with the problem of finding conditions on the integers m so that mL is very ample. A holomorphic line bundle E over X is said to be very ample if global holomorphic sections of E over X separate any pair of distinct points of X and can give local homogeneous coordinates at any point of

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X. In other words, E is very ample over X if any basis $s_0, \cdots, s_N$ of $\Gamma(X, E)$ defines an embedding of X into $\mathbf{P}_N$.

Let $P(m)$ be a polynomial whose coefficients are rational numbers and whose values are integers at integral values of m. Matsusaka's Big Theorem [14][15] states that there is a positive integer m_0 depending on $P(m)$ such that, for every compact projective algebraic manifold X of complex dimension n and every ample line bundle L over X with

$$\sum_{\nu=0}^{n}(-1)^{\nu} \dim_{\mathbf{C}} H^{\nu}(X, mL) = P(m)$$

for every m, the line bundle mL is very ample for $m \geq m_0$. The result of Kollár and Matsusaka on Riemann-Roch type inequalities [13] improves Matsusaka's Big Theorem and shows that the positive integer m_0 can be made to depend only on the coefficients of m^n and m^{n-1} in the polynomial $P(m)$ of degree n, which means dependence only on $L^{n-1}K_X$ and L^n, where K_X denotes the canonical line bundle of X.

The proofs of Matsusaka's Big Theorem in [14][15][13] depend on the boundedness of numbers calculated for some varieties and divisors in a bounded family and thus the positive integer m_0 from such proofs cannot be effectively computed from $P(m)$.

In [20] the following effective version of Matsusaka's Big Theorem was proved.

Theorem 0.1. *Let X be a compact complex manifold of complex dimension n. Let L be an ample line bundle over X. Then mL is very ample for m no less than*

$$\frac{\left(2^{3^{n-1}}5n\right)^{4^{n-1}}\left(3\left(3n-2\right)^n L^n + L^{n-1}K_X\right)^{4^{n-1}3n}}{\left(6\left(3n-2\right)^n - 2n - 2\right)^{4^{n-1}n-\frac{2}{3}}\left(L^n\right)^{4^{n-1}3(n-1)}}.$$

Moreover, if B is a numerically effective line bundle, then $mL - B$ is very ample for m no less than

$$\frac{\left(n\left(H^n\right)^2\left(H^{n-1}B + \frac{5}{2}H^n\right)\right)^{4^{n-1}}}{\left(6(3n-2)^n - 2n - 2\right)^{4^{n-1}n-\frac{2}{3}}},$$

where $H = 2\left(K_X + 3(3n-2)^n L\right)$.

The numerical effectiveness of a line bundle means that the Chern class of its restriction to any irreducible compact complex curve is nonnegative. The proof of Theorem 0.1 hinges on the following three components (for meanings of terms see §1 below).

(1) Use the techniques of effective global generation and effective very ampleness to produce, for the adjoint line bundle of an effective multiple of an ample line bundle, a strictly positively curved singular Hermitian metric whose non-integrability set contains a prescribed point as an isolated point and whose multiplier ideal sheaf is contained in the square of the maximum ideal at the prescribed point. Then use the vanishing theorem of Nadel [17] or Kawamata-Viehweg [11][24] to produce from the metric a global holomorphic section with a prescribed 1-jet at the prescribed point.

(2) Use the Theorem of Riemann-Roch to produce a non identically zero global multi-valued holomorphic section of the difference $F - G$ of two numerically effective holomorphic line bundles F, G on a compact complex projective algebraic manifold X of complex dimension n when $F^n > nF^{n-1}G$.

(3) For any d-dimensional irreducible subvariety Y of X and any very ample line bundle H over X, use an analytic cover $Y \to \mathbf{P}_d$ defined by elements of $\Gamma(X, H)$ and Cramer's rule to produce a non identically zero section over Y of the homomorphism sheaf from the sheaf of holomorphic d-forms of Y to $(3\lambda(\lambda-1)/2 - d - 1)\, H|Y$ for $\lambda \geq H^d \cdot Y$.

Demailly [2] used a generic one-one holomorphic map from Y onto a complex-analytic hypersurface of $\mathbf{P}_{d+1}$ and the extension theorem of Ohsawa-Takegoshi [18] and Manivel [16] to improve the number in Component (3) to produce, for any d-dimensional irreducible subvariety Y of X and any very ample line bundle H over X, a non identically zero section over Y of the homomorphism sheaf from the sheaf of holomorphic d-forms of Y to $(\lambda - d - 2)\, H|Y$ for $\lambda \geq H^d \cdot Y$. His improvement in the number in Component (3) enabled him to get the following sharper bound for the effective Matsusaka Big Theorem.

Theorem 0.2. *Let L be an ample line bundle and B be a numerically effective line bundle over a compact complex manifold X of complex dimension n. Let $H = \lambda_n (K_X + (n+2)L)$ with $\lambda_2 = 1$ and $\lambda_n = \binom{3n+1}{n} - 2n$ for $n \geq 3$. Then $mL - B$ is very ample for m no less than*

$$(2n)^{\frac{3^{n-1}-1}{2}} \frac{\left(L^{n-1} \cdot (B+H)\right)^{\frac{3^{n-1}+1}{2}} \left(L^{n-1} \cdot H\right)^{3^{n-2}\left(\frac{n}{2}-\frac{3}{4}\right)-\frac{1}{4}}}{\left(L^n\right)^{3^{n-2}\left(\frac{n}{2}-\frac{1}{4}\right)+\frac{1}{4}}}.$$

In particular, mL is very ample for m no less than

$$(2n)^{\frac{3^{n-1}-1}{2}}(\lambda_n)^{3^{n-2}\left(\frac{n}{2}+\frac{3}{4}\right)+\frac{1}{4}}(L^n)^{3^{n-2}}\left(n+2+\frac{L^{n-1}K_X}{L^n}\right)^{3^{n-2}\left(\frac{n}{2}+\frac{3}{4}\right)+\frac{1}{4}}.$$

For the special case $n=2$ where X is a surface, Fernández del Busto [5] obtained the sharper result of the very ampleness of mL for

$$m>\frac{1}{2}\left(\frac{(L\cdot(K_X+4L)+1)^2}{L^2}+3\right).$$

The idea of the argument of [20] for the case of a trivial B is as follows. We sketch it in a way that facilitates the comparison with the argument in this paper. Let m_n be a positive integer such that $m_nL+E+2K_X$ is very ample for any numerically effective holomorphic line bundle E over X. Let $\tilde{K}_X=m_nL+2K_X$. It suffices to verify that $pL-\tilde{K}_X$ is numerical effective for some effective positive integers p. To do the verification, we construct by descending induction on k sequences of nested irreducible subvarieties

$$Y_k\subset Y_{k+1}\subset\cdots\subset Y_{n-1}\subset Y_n=X$$

with Y_ν of complex dimension ν. To get to the next induction step, we get inductively an effective bound of $\left(L^{k-1}\tilde{K}_X\right)\cdot Y_k$ and construct by Component (2) a non identically zero multi-valued holomorphic section ξ_k of $p_kL-\tilde{K}_X$ on Y_k for some effective positive integer p_k. From a power of the metric defined by ξ_k and by Component (1) we construct, for any prescribed effective positive number ℓ, a non identically zero holomorphic section t_k of $p'_kL-\ell\tilde{K}_X+K_k$ on Y_k for some effective positive integer p'_k, where K_k is an appropriately defined canonical sheaf of Y_k. By using Component (3) we get from t_k a non identically zero holomorphic section s_k of $p''_kL-\tilde{K}_X$ over Y_k for some effective positive integer p''_k. For each branch Y_{k-1} of the zero-set of s_k we have one new sequence of nested irreducible subvarieties

$$Y_{k-1}\subset Y_k\subset Y_{k+1}\subset\cdots\subset Y_{n-1}\subset Y_n=X.$$

When the first subvariety of each nested sequence of subvarieties so constructed by induction is of dimension zero, we get the numerical effectiveness of $pL-\tilde{K}_X$ for some effective positive integer p. From the definition of $\tilde{K}_X$ we get the very ampleness of pL. In this argument Component (3) is needed, because when Component (1) is applied to go from a non identically zero multi-valued holomorphic section to a non identically zero holomorphic section, one has to add the canonical sheaf and we need some non identically zero holomorphic section of the homomorphism sheaf from the canonical sheaf of Y_k to an effective multiple of $\tilde{K}_X$ to offset the added canonical sheaf.

For the effective bound the most essential part is the power needed for the term $L^{n-1}\left((n+2)L+K_X\right)$. We use $L^{n-1}\left((n+2)L+K_X\right)$ instead of $L^{n-1}K_X$ for the term, because K_X in general is not ample, but $(n+2)L+K_X$ is always ample according to [6]. In [20] the order of the power of $L^{n-1}\left((n+2)L+K_X\right)$ is 4^n. In [2] the order of the power of $L^{n-1}\left((n+2)L+K_X\right)$ is reduced to 3^n. The reason for the difference is the reduction, in Component (3), of the multiple of order λ^2 for $H|Y$ in [20] to the the multiple of order λ for $H|Y$ in [2].

In this paper, to give a new bound for the effective Matsusaka Big Theorem, we introduce a new technique to replace Component (3) so that in our bound the order of the power for $L^{n-1}\left((n+2)L+K_X\right)$ is further reduced to 2^n. The technique is to apply the theorem of Nadel [17] or Kawamata-Viehweg [11][24] only to X instead to the subvariety Y_k so that the canonical sheaf for Y_k is not used.

Here, to explain this technique, we use the same notations as in the sketch for the argument of [20] given above unless specified to the contrary. Now when we construct ξ_k by Component (2), we construct ξ_k as a non identically zero multi-valued holomorphic section of $p_kL-2\tilde{K}_X$ (instead of $p_kL-\tilde{K}_X$) on Y_k for some effective positive integer p_k. Choose a holomorphic section σ_k of $2\tilde{K}_X$ over X which does not vanish identically on certain subvarieties of Y_k so that its germs are not zero-divisors of the quotient of the sheaf of weakly holomorphic function germs by the structure sheaf of Y_k. Extend the multi-valued holomorphic section $\sigma_k\xi_k$ of p_kL of Y_k to a multi-valued holomorphic section η_k of p_kL over X.

Let h_n be the trivial metric of the trivial line bundle over X. Inductively we define a strictly positively curved singular metric h_j of $\left(\sum_{\lambda=j+1}^n 2^{\lambda-j-1}p_j\right)L$ over X whose non-integrability set contains Y_j as a branch. By using the inductively defined singular metric h_k of $\left(\sum_{\lambda=k+1}^n 2^{\lambda-k-1}p_\lambda\right)L$ over X from the preceding induction step and using η_k, we construct a singular metric θ_k of $\left(p_k+\sum_{\lambda=k+1}^n 2^{\lambda-k-1}p_\lambda\right)L$ over X so that the intersection with Y_k of the non-integrability set of θ_k is a proper subvariety of Y_k containing the zero-set of σ_k. We apply the theorem of Nadel [17] or Kawamata-Viehweg [24][24] to get a holomorphic section ζ_k of $\left(p_k+\sum_{\lambda=k+1}^n 2^{\lambda-k-1}p_\lambda\right)L+\tilde{K}_X$ over X whose quotient s_k by σ_k is a non identically zero holomorphic section of $\left(p_k+\sum_{\lambda=k+1}^n 2^{\lambda-k-1}p_\lambda\right)L-\tilde{K}_X$ over Y_k. Each branch of the zero-set of s_k is a new Y_{k-1}. From s_k and h_k we construct h_{k-1}, finishing the induction process. When we get to Y_0 in our construction, we get the numerical effectiveness of $\left(p_1 + \sum_{\lambda=2}^{n} 2^{\lambda-2} p_\lambda\right) L - \tilde{K}_X$ over X, from which we get the very ampleness of $\left(p_1 + \sum_{\lambda=2}^{n} 2^{\lambda-2} p_\lambda\right) L$. The technique given here is more in line with the techniques developed for the the effective freeness problem in the Fujita conjecture [1][3][4][6][7][8][9][10][12][21][22][23].

The statement of our new bound for the effective Matsusaka Big Theorem is given in Main Theorem 0.3 below. The rest of the paper is devoted to its proof. In the proof, in order to simplify the final expression of the bound, the estimate used in each step has not been optimized. There is still room to make improvements of an arithmetic nature in our argument at the expense of a bound given in a more unwieldy form.

Main Theorem 0.3. *Let X be a compact complex manifold of complex dimension n. Let L be an ample line bundle over X and B be a numerically effective line bundle over X. Then $mL - B$ is very ample for m no less than*

$$C_n \left(L^{n-1} \tilde{K}_X\right)^{2^{\max(n-2,0)}} \left(1 + \frac{L^{n-1} \tilde{K}_X}{L^n}\right)^{2^{\max(n-2,0)}},$$

where

$$C_n = 2^{n-1+2^{n-1}} \left(\prod_{k=1}^{n} \left(k 2^{\frac{(n-k-1)(n-k)}{2}} \right)^{2^{\max(k-2,0)}} \right)$$

and $\tilde{K}_X = \left(2n\binom{3n-1}{n} + 2n + 1\right) L + B + 2K_X$.

1. Notations and Terminology

Hermitian metrics $e^{-\varphi}$ of a holomorphic line bundle E over a complex manifold X are allowed to be singular, but we deal only with singular Hermitian metrics $e^{-\varphi}$ where the singularity of φ is locally at worst the sum of a plurisubharmonic function and a smooth function. Smoothness means C^∞. The curvature current of $e^{-\varphi}$ is

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi.$$

We say that the singular metric $e^{-\varphi}$ is strictly positively curved (or the curvature current of the singular metric $e^{-\varphi}$ is strictly positive on X) if, for some smooth strictly positive $(1,1)$-form ω on X,

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi \geq \omega$$

on X in the sense of currents.

The non-integrability set of the singular metric $e^{-\varphi}$ is the set of points of X where $e^{-\varphi}$ as a local function is not locally integrable. When φ is plurisubharmonic, the non-integrability set of $e^{-\varphi}$ is a complex-analytic subvariety.

The sheaf of germs of holomorphic functions F with $|F|^2 e^{-\varphi}$ locally integrable is called the multiplier ideal sheaf of the singular metric $e^{-\varphi}$ and is denoted by $\mathcal{I}_\varphi$. If a point P of X is an isolated point of the non-integrability set of a strictly positively curved singular metric $e^{-\varphi}$ of a holomorphic line bundle E on a complex manifold X, then from the vanishing of $H^1\left(X, \mathcal{I}_\varphi\left(E+K_X\right)\right)$ due to the theorem of Nadel [17] or Kawamata-Viehweg [11][24] and the cohomology long exact sequence of the short exact sequence

$$0 \to \mathcal{I}_\varphi\left(E+K_X\right) \to \mathcal{O}_X\left(E+K_X\right) \to \left(\mathcal{O}_X/\mathcal{I}_\varphi\right)\left(E+K_X\right) \to 0$$

it follows that there exists an element of $\Gamma\left(X, E+K_X\right)$ which is nonzero at P. If for any pair of distinct points P, Q of X there exists a strictly positively curved singular metric of E over X whose non-integrability set contains P and Q as isolated points and whose multiplier ideal sheaf is contained in the square of the maximum ideal at P, then the same argument using the theorem of Nadel [17] or Kawamata-Viehweg [11][24] shows that E is very ample over X.

A multi-valued holomorphic section s of a holomorphic line bundle E over a complex manifold X means that $s^q \in \Gamma\left(X, qE\right)$ for some positive integer q. A Hermitian metric $e^{-\varphi}$ of E is said to be defined by multi-valued holomorphic sections of E if there exist finitely many multi-valued holomorphic sections $s_1, \cdots, s_N$ of E over X such that

$$e^{-\varphi} = \frac{1}{\sum_{j=1}^N \left|s_j\right|^2}.$$

Let $\ell_0 \geq 2$ be an integer. We will eventually set $\ell_0 = 2$. Let m_n be a positive integer with the property that, for any ample line bundle L and any numerically effective line bundle E over a compact complex manifold X of complex dimension n and any pairs of distinct points P, Q of X and for $m \geq m_n$, the line bundle $mL+E+K_X$ admits a singular metric whose non-integrability set contains P and Q as isolated points and whose multiplier ideal sheaf is contained in the square of the maximum ideal at P. For example one can set $m_n = 2n\binom{3n-1}{n} + 2n + 1$ [21, Corollary(0.2)]. Let $\tilde{K}_X = m_n L + B + 2K_X$ which agrees with the notation given in Main Theorem 0.3 when m_n is chosen to be $2n\binom{3n-1}{n} + 2n + 1$. Define by descending induction

$$\begin{cases} p_n = n\ell_0 \left(1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor\right), \\ p_k = \ell_0 k 2^{\frac{(n-k-1)(n-k)}{2}} p_{k+1}p_{k+2}\cdots p_n \left(L^{n-1}\tilde{K}_X\right) \text{ for } 1 \le k < n. \end{cases}$$

Here $\lfloor u \rfloor$ means the largest integer not exceeding u. From the definition of p_k it is clear that $p_k > p_{k+1}$ for $1 \le k < n$. Let $X^{(n)} = X$ and $h^{(n)}$ be the trivial metric for the trivial line bundle over $X^{(n)}$.

2. Induction Statement

We are going to construct, by descending induction on $1 \le k \le n$,

(i) a non identically zero holomorphic section

$$s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \in \Gamma\left(X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}, \left(p_k + \sum_{\lambda=k+1}^{n} 2^{\lambda-k-1}p_\lambda\right) L - (\ell_0 - 1)\,\tilde{K}_X\right)$$

for

$$\begin{cases} 1 \le \nu_{k+1} \le I^{(k+1)}_{\nu_{k+2},\nu_{k+3},\cdots,\nu_n}, \\ 1 \le \nu_{k+2} \le I^{(k+2)}_{\nu_{k+3},\nu_{k+4},\cdots,\nu_n}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots \\ 1 \le \nu_{n-1} \le I^{(n-1)}_{\nu_n}, \\ 1 \le \nu_n \le I^{(n)}, \end{cases}$$

whose zero-set on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ has branches

$$X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n} \quad (1 \le \nu_k \le I^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}),$$

and

(ii) a strictly positively curved singular metric $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ of $\left(\sum_{\lambda=k}^n 2^{\lambda-k}p_\lambda\right) L$ defined by multi-valued holomorphic sections whose set of non-integrability has $X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ as a branch.

Let h_L be a strictly positively curved smooth metric of L. By replacing the metric $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ by

$$(h_L)^{\delta\left(\sum_{\lambda=k}^n 2^{\lambda-k}p_\lambda\right)} \left(h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}\right)^{1-\delta}$$

for some suitable $0 \leq \delta < 1$, we can assume without loss of generality that the non-integrability set of $\left(h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}\right)^{1-\epsilon}$ does not contain $X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ for any $0 < \epsilon < 1$.

For the construction by descending induction on $1 \leq k \leq n$ we are going to do at the same time the initial step $k = n$ and the process of going from the $(k+1)$-th to the k-th step. For the initial step $k = n$ we need to use the trivial metric $h^{(n)}$ of the trivial line bundle.

3. Degree Bound

Lemma 3.1. *Let $1 \leq k \leq n$. Suppose inductively we have a sequence of nested subvarieties*

$$Y_k \subset Y_{k+1} \subset \cdots \subset Y_{n-1} \subset Y_n = X,$$

where Y_k is an irreducible subvariety of complex dimension k. For $k < \lambda \leq n$, let t_λ be a non identically zero global holomorphic section of $\left(q_\lambda L - \ell_0 \tilde{K}_X\right)|Y_\lambda$ over Y_λ for some positive integer q_λ so that $Y_{\lambda-1}$ is an irreducible branch of the zero-set of t_λ. Then the intersection number $\left(L^{k-1}\tilde{K}_X\right) \cdot Y_k$ is no more than $\left(\prod_{\nu=k+1}^n q_\nu\right) L^{n-1}\tilde{K}_X$.

Proof. We use descending induction on $1 \leq k \leq n$. The case $k = n$ is clear when we use the fact that by the usual convention the product $\prod_{\lambda=n+1}^n q_\lambda$ of an empty set is 1. Suppose the statement holds when k is replaced by $k+1$. Then

$$\left(L^k \tilde{K}_X\right) \cdot Y_{k+1} \leq \left(\prod_{\lambda=k+2}^{n} q_\lambda\right) L^{n-1}\tilde{K}_X.$$

We write the divisor of t_{k+1} on Y_{k+1} as

$$\operatorname{div} t_{k+1} = \sum_{j=1}^{J} a_j V_j,$$

where V_j is an irreducible complex-analytic hypersurface in Y_{k+1} $(1 \leq j \leq J)$ with $V_1 = Y_k$ and $a_j \geq 1$. Then

$$\left(L^{k-1}\tilde{K}_X\right) \cdot Y_k = \left(L^{k-1}\tilde{K}_X\right) \cdot V_1$$

$$\leq \sum_{j=1}^{J} a_j \left(L^{k-1}\tilde{K}_X\right) \cdot V_j = \left(L^{k-1}\tilde{K}_X\right)\left(q_{k+1}L - \ell_0\tilde{K}_X\right) \cdot Y_{k+1}$$

$$\leq \left(L^{k-1}\tilde{K}_X\right) q_{k+1} L \cdot Y_{k+1} \leq \left(\prod_{\lambda=k+1}^{n} q_\lambda\right) L^{n-1}\tilde{K}_X.$$

□

4. Construction of Multi-valued Holomorphic Sections

Lemma 4.1. *If Y is an irreducible subvariety of complex dimension k in X and if*

$$p > \ell_0 k \left(L^{k-1}\tilde{K}_X\right) \cdot Y,$$

then there exists a non identically zero multi-valued holomorphic section of $pL - \ell_0\tilde{K}_X$ over Y.

Proof. Since $L^k \cdot Y \geq 1$, it follows from the assumption that

$$(pL)^k \cdot Y > k \left((pL)^{k-1}\tilde{K}_X\right) \cdot Y.$$

Let $\tilde{Y}$ be a desingularization of Y. Let E be the pullback of $pL - \ell_0\tilde{K}_X$ to $\tilde{Y}$. Since both $L|_Y$ and $\tilde{K}_X|_Y$ are numerically effective holomorphic line bundles over Y, by the arguments in [20] following [20, Corollary(1.2)], the complex dimension of $\Gamma\left(\tilde{Y}, qE\right)$ is at least cq^k for some positive number c when q is sufficiently large. Let Z be the subset of points where $\tilde{Y}$ is not locally biholomorphic to Y under the desingularization map. There exists some positive number ℓ such that any holomorphic section of qE over $\tilde{Y}$ vanishing to order at least ℓ at points of Z comes from some holomorphic section of $q\left(pL - \ell_0\tilde{K}_X\right)$ over Y. Since the complex dimension of the space of all holomorphic sections of qE over $\tilde{Y}$ vanishing to order at least ℓ at points of Z is no more than Cq^{k-1} for some positive number C when q is sufficiently large, it follows that there exists some non identically zero holomorphic section of qE over $\tilde{Y}$ which comes from a holomorphic section of $q\left(pL - \ell_0\tilde{K}_X\right)$ over Y. Thus there exists a non identically zero multi-valued holomorphic section of $pL - \ell_0\tilde{K}_X$ over Y. □

From the definition of p_n we have

$$p_n L^n > n\ell_0 \left(L^{n-1}\tilde{K}_X\right).$$

We now apply Lemma 3.1 to

$$\begin{cases} Y_\lambda = X^{(j)}_{\nu_{j+1},\nu_{j+2},\cdots,\nu_n} \text{ for } k \leq j \leq n, \\ \\ t_j = s^{(j)}_{\nu_{j+1},\nu_{j+2},\cdots,\nu_n} \text{ for } k < \lambda \leq n, \\ \\ q_j = p_j + \sum_{\lambda=j+1}^n 2^{\lambda-j-1} p_\lambda \text{ for } k < \lambda \leq n, \end{cases}$$

and use the strict decreasing property of the sequence p_j $(1 \leq j \leq n)$ to conclude that

$$\begin{aligned} \left(L^{k-1}\tilde{K}_X\right) \cdot X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n} &\leq \left(\prod_{j=k+1}^n \left(p_j + \sum_{\lambda=j+1}^n 2^{\lambda-j-1} p_\lambda\right)\right)\left(L^{n-1}\tilde{K}_X\right) \\ &< 2^{\frac{(n-k-1)(n-k)}{2}} p_{k+1}p_{k+2}\cdots p_n \left(L^{n-1}\tilde{K}_X\right) = \frac{p_k}{k\ell_0}, \end{aligned}$$

because

$$1 + \sum_{\lambda=j+1}^n 2^{\lambda-j-1} = 2^{n-j}$$

and

$$\sum_{j=k+1}^n (n-j) = \sum_{j=1}^{n-k-1} j = \frac{(n-k-1)(n-k)}{2}.$$

By Lemma 4.1 there exists a non identically zero multi-valued holomorphic section $\xi^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $p_k L - \ell_0 \tilde{K}_X$ over $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

5. Use of Primary Decomposition of Sheaves

Let $\tilde{\mathcal{O}}_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}$ be the sheaf of germs of weakly holomorphic functions on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$. Here a weakly holomorphic function on an open subset Ω of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ means a function which is holomorphic at the regular points of Ω and locally bounded on Ω. We use the usual notation $\mathcal{O}_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}$ to denote the sheaf of germs of holomorphic functions on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$. There exist a finite number of irreducible subvarieties

$$Z^{(k)}_{\lambda,\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \quad (1 \leq \lambda \leq N^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n})$$

in the singular set of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ with the property that at any singular point P of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ the collection of local branch germs of

$$Z^{(k)}_{\lambda,\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \quad (1 \leq \lambda \leq N^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n})$$

at P is precisely the collection of all subvariety germs of the associated prime ideals in the primary decomposition of the module

$$\tilde{\mathcal{O}}_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}} \Big/ \mathcal{O}_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}$$

over $\mathcal{O}_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}$. See, for example, [19] where primary decompositions are referred to as Noether-Lasker decompositions.

We have the following conclusion from the definition of primary decompositions. If f is a weakly holomorphic function germ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ at some point P of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ and if σ is a holomorphic function germ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ at P which is not identically zero on any of the local branch germs of

$$Z^{(k)}_{\lambda,\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \quad (1 \leq \lambda \leq N^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n})$$

at P such that $f\sigma$ is a holomorphic function germ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ at P, then f is a holomorphic function germ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ at P.

Since $\tilde{K}_X$ is very ample over X, there exists an element

$$\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \in \Gamma\left(X, \ell_0 \tilde{K}_X\right)$$

with the following three properties.

(i) The section $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ does not vanish identically on any

$$Z^{(k)}_{\lambda,\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \quad (1 \leq \lambda \leq N^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}).$$

(ii) The section $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ does not vanish identically on the singular set of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

(iii) The divisor of $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ in the regular part of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ is a nonempty irreducible complex-analytic hypersurface of multiplicity 1.

Since $p_k L$ is ample over X, the multi-valued holomorphic section

$$\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n} \xi^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$$

of $p_k L$ over $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ extends to some multi-valued holomorphic section $\eta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $p_k L$ over all of X.

6. Non-Integrability Sets of Multiplier Ideal Sheaves

Lemma 6.1. *Let (z, w) be the coordinates of $\mathbf{C}^2$ and $0 < a < 1$ and b, c are positive numbers. Then*

$$\int_{0<|z|<1,0<|w|<1} \frac{\left(\sqrt{-1}dz \wedge d\bar{z}\right) \wedge \left(\sqrt{-1}dw \wedge d\bar{w}\right)}{|z|^{2a}\left(|z|^{2b} + |w|^{2c}\right)}$$

is infinite if and only if $a + b\left(1 - \frac{1}{c}\right) \geq 1$.

Proof. By the substitution $x = |z|^2$ and $y = |w|^2$, the integral is infinite if and only if

$$\int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b + y^c)}$$

is infinite, which is the case if and only if

$$\int_{0<x<1,0<y<1} \frac{dxdy}{x^a\left(x^{\frac{b}{c}} + y\right)^c}$$

is infinite. The Lemma follows by integration first with respect to y. $\square$

Lemma 6.2. *Let $(z, w_1, \cdots, w_n)$ be the coordinates of $\mathbf{C} \times \mathbf{C}^n$. Let D (respectively Ω) be an open neighborhood of the origin in $\mathbf{C}$ (respectively $\mathbf{C}^n$). Let D' (respectively Ω') be a relatively compact open neighborhood of the origin in D (respectively Ω). Let $f(z, w)$ be a holomorphic function on $0 \times \Omega = \{z = 0\} \cap (D \times \Omega)$ and $g_j(z, w)$ $(1 \leq j \leq J < \infty)$ be a multi-valued holomorphic function on $D \times \Omega$ in the sense that for some positive integer N the function g_j^N is a well-defined single-valued holomorphic function on $D \times \Omega$ for $1 \leq j \leq J$. Assume that the common zero-set of the restrictions of $g_1, \cdots, g_J$ to $0 \times \Omega$ is equal to the zero-set of f and that $\frac{g_j}{f}\big|_{0\times\Omega}$ is a multi-valued holomorphic function on $0 \times \Omega$ for $1 \leq j \leq J$. Then there exist a positive number δ_0 and a positive-valued function $\epsilon_0(\delta)$ for $0 < \delta \leq \delta_0$ such that for $0 < \delta \leq \delta_0$ and $0 < \epsilon \leq \epsilon_0(\delta)$ the non-integrability set Z of*

$$\frac{1}{\left(\sum_{j=1}^{J} |g_j(z, w)|^2\right)^{1+\delta} |z|^{2(1-\epsilon)}}$$

in $D' \times \Omega'$ is equal to the zero-set of f in $0 \times \Omega' = \{z = 0\} \cap (D' \times \Omega')$.

Proof. Since $\epsilon > 0$ and since the common zero-set of the restrictions of $g_1, \cdots, g_J$ to $0 \times \Omega$ is equal to the zero-set of f, it follows that Z is contained in the zero-set of f in $0 \times \Omega'$. For the other inclusion it suffices to show that Z contains the regular points P of the zero-set of f in $0 \times \Omega'$. In some open neighborhood U of P in $D' \times \Omega'$ we have

$$\sum_{j=1}^{J} |g_j(z,w)|^2 \leq C\left(|h(z,w)|^2 + |z|^\eta\right)$$

for some positive numbers C, η and for some holomorphic function $h(z,w)$ on U such that dz, dh are $\mathbf{C}$-linearly independent at every point of U and the zero-set of h in $(0 \times \Omega') \cap U$ agrees with the zero-set of f in $(0 \times \Omega') \cap U$. The Lemma now follows from Lemma 6.1. $\square$

Lemma 6.3. *Let X be a compact complex manifold and F and G be ample holomorphic line bundles over X and H be a holomorphic line bundle over X. Let h_F (respectively h_G) be a strictly positively curved smooth metric of F (respectively G). Let Y be an irreducible subvariety of X and h_Y be a strictly positively curved singular metric of F over X defined by multi-valued holomorphic sections of F over X such that the non-integrability set of h_Y has Y as a branch. Assume that for any $\epsilon > 0$ the set of non-integrability of $h_Y^{1-\epsilon}$ does not contain Y. Let σ be a non identically zero holomorphic section of H over X. Let τ_j ($1 \leq j \leq J < \infty$) be a multi-valued holomorphic section of G over X such that $\frac{\tau_j}{\sigma}\big|_Y$ is a non identically zero multi-valued holomorphic section of $G - H$ over Y for $1 \leq j \leq J$. Then there exist a positive number δ_0 and a positive-valued function $\epsilon_0(\delta)$ for $0 < \delta \leq \delta_0$ such that for $0 < \delta \leq \delta_0$ and $0 < \epsilon \leq \epsilon_0(\delta)$ the metric*

$$\frac{h_Y^{1-\epsilon} h_F^{\epsilon}}{\left(\sum_{j=1}^{J} |\tau_j|^2\right)^{1+\delta} h_G^{\delta}}$$

is a strictly positively curved singular metric h of $F + G$ over X such that

(a) the intersection with Y of the non-integrability set of h contains the zero-set of σ and is a proper subvariety of Y and

(b) the non-integrability set of h is contained in the union of the non-integrability set of h_Y and the common zero-set of $\tau_1, \cdots, \tau_J$.

Proof. Since the metric h_Y is strictly positively curved, there exists a smooth strictly positive $(1,1)$-form ω on X such that both the curvature current of h_Y and the curvature form of h_F dominate ω. There exists a positive number δ_1 such that the curvature form of h_G is dominated by $\frac{1}{2\delta_1}\omega$ on X. Since the metric h_Y is defined by multi-valued holomorphic sections of F over X, there exist a holomorphic map $\pi : \tilde{X} \to X$ composed of successive monoidal transformations with regular centers and a finite collection of nonsingular hypersurfaces E_r $(r \in R)$ of $\tilde{X}$ in normal crossing such that

(1) $K_{\tilde{X}} = \pi^* K_X + \sum_{r \in R} b_r E_r$ with b_r being nonnegative integers, and

(2) $f^* h_Y$ locally on $\tilde{X}$ is equal to a nowhere zero smooth function times

$$\frac{1}{\prod_{r \in R} |s_{E_r}|^{2a_r}},$$

where a_r is a nonnegative rational number and s_{E_r} is the canonical section of the line bundle defined by the divisor E_r.

Let R_0 be the subset of all $r \in R$ such that $\pi(E)$ contains Y. Since for any $\epsilon > 0$ the non-integrability set of $h_Y^{1-\epsilon}$ does not contain Y, it follows that

(i) there exist a nonempty subset R_1 of R_0 such that $a_r - b_r = 1$ for $r \in R_1$ and

(ii) $a_r - b_r < 1$ for any $r \in R_0 - R_1$.

For every $r \in R_1$ and every point $P \in E_r$ we choose an open coordinate neighborhood U_P of P in $\tilde{X}$ such that $s_{E_r} | U_P$ can be used as one of the coordinate functions of U_P. We now apply Lemma 6.2 to the coordinate function s_{E_r} and the multi-valued holomorphic functions $\pi^* \tau_j$ $(1 \leq j \leq J)$ on U_P. Since the set $\bigcup_{r \in R_1} E_r$ can be covered by a finite number of such open sets U_P, we can get from Lemma 6.2 a positive number δ_0 and a positive-valued function $\epsilon_0(\delta)$ for $0 < \delta \leq \delta_0$ such that the metric h satisfies (a) and (b) in the statement of the Lemma. To make sure that the metric h has a strict positive curvature current, we impose the additional condition that $\delta_0 < \delta_1$. □

7. Key Step of Quotients of Sections

By applying Lemma 6.3 to the singular metric $h^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of the line bundle $\left(\sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L$ over X, to the holomorphic section $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\ell_0 \tilde{K}_X$ over X, and to the multi-valued holomorphic section $\eta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $p_k L$ over X, we obtain a strictly positively curved singular metric $\theta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of the line bundle $\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L$ over X such that the intersection with $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of the non-integrability set of $\theta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ is a proper subvariety of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ containing the zero-set of the restriction of $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ to $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

Pick a point P in $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ outside the non-integrability set of the singular metric $\theta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$. Let h_P be a strictly positively curved singular metric of $m_n L + B + K_X$ over X whose non-integrability set has P as an isolated point. Applying the theorem of Nadel [17] or Kawamata-Viehweg [11][24] to the metric $h_P \theta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\left(m_n + p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L + B + K_X$, we obtain a holomorphic section $\zeta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L + \tilde{K}_X$ over X whose zero-set in $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ is a proper subvariety of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ containing the zero-set of $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

Let

$$s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n} = \left.\frac{\zeta^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}{\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}\right|_{X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}}.$$

From the conditions imposed on the choice of $\sigma^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ in §5 we conclude that $s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ is a non identically zero holomorphic section of

$$\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L - (\ell_0 - 1)\, \tilde{K}_X$$

over $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$. We then have the branches

$$X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n} \quad \left(1 \leq \nu_k \leq I^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}\right)$$

of the zero-set of $s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$. Choose

$$\rho_1, \cdots, \rho_q \in \Gamma\left(X, (\ell_0 - 1)\, \tilde{K}_X\right)$$

without any common zero in X.

For $1 \leq j \leq q$ the multi-valued holomorphic section $\rho_j s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L$ over $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ can be extended to a multi-valued holomorphic section $\tau^{(k)}_{j,\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right) L$ over X. By Lemma 6.3, for appropriate $0 < \delta < 1$ and $0 < \epsilon < 1$ the metric

$$\tilde{h} := \frac{\left(h^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}\right)^{1-\epsilon} h_L^{\left(\sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right)\epsilon}}{\left(\sum_{j=1}^J \left|\tau^{(k)}_{j,\nu_{k+1},\nu_{k+2},\cdots,\nu_n}\right|^2\right)^{1+\delta} h_L^{\left(p_k + \sum_{\lambda=k+1}^n 2^{\lambda-k-1} p_\lambda\right)\delta}}$$

of $\left(\sum_{\lambda=k}^n 2^{\lambda-k} p_\lambda\right) L$ has strictly positive curvature current and satisfies the following two conditions.

(i) The non-integrability set of $\tilde{h}$ is contained in the non-integrability set of $h^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

(ii) The intersection with $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of the non-integrability set of $\tilde{h}$ contains the zero-set of $s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ and is a proper subvariety of $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$.

Define the metric $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ of $\left(\sum_{\lambda=k}^n 2^{\lambda-k}p_\lambda\right)L$ over X by $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n} = \tilde{h}$. Then $X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ is a branch of the non-integrability set of $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ and the curvature current of $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ is strictly positive. Note that in this construction the metric $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ is at this point independent of ν_k. However, in the induction argument when for some suitable $0 \le \alpha < 1$ we replace $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ by

$$(h_L)^{\alpha\left(\sum_{\lambda=k}^n 2^{\lambda-k}p_\lambda\right)}\left(h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}\right)^{1-\alpha}$$

so that the non-integrability set of the singular metric $\left(h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}\right)^{1-\beta}$ does not contain $X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ for any $0 < \beta < 1$, the number α depends on ν_k, resulting in the dependence of $h^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ on ν_k. This finishes the induction process.

8. Counting of Factors in the Bound

We introduce the following lemma to track the number of occurrence of a factor in the inductive definition of p_k. The lemma deals only with counting the number of symbols in a symbolic manipulation involving an inductive definition. It is in the context of symbolic manipulation and not in the context of number theory and commutative algebra. In the counting we regard each symbol as an element in a unique factorization domain which contains the symbols in question as elements.

Lemma 8.1. *Let D, A_k, C_k $(0 \le k \le N)$ be elements in a unique factorization domain. Assume that D is prime. Let D be a factor in A_0 occurring only with exponent 1. Assume that inductively*

$$A_k = C_k \cdot A_0 \cdot A_1 \cdots A_{k-1}$$

for $1 \le k \le N$ and that C_k does not contain D as a factor. Then the factor D occurs with exponent $2^{\max(k-1,0)}$ in A_k for $0 \le k \le N$.

Proof. Let a_k be the exponent of D in A_k. Then $a_0 = 1$ and $a_1 = 1$ and

$$a_k = a_0 + a_1 + \cdots + a_{k-1}.$$

Inductively, if $a_j = 2^{j-1}$ for $1 \leq j \leq k-1$,then

$$a_k = a_0 + \sum_{j=1}^{k-1} a_j = 1 + \sum_{j=1}^{k-1} 2^{j-1} = 1 + \frac{2^{k-1}-1}{2-1} = 2^{k-1}.$$

□

We now apply Lemma 8.1 to count the number of times the factor

$$1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor$$

occurs in p_1. For our counting we let

$$\begin{cases} D = 1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor, \\ A_k = p_{n-k} \text{ for } 0 \leq k \leq n-1 \end{cases}$$

By applying Lemma 8.1 to the case

$$\begin{cases} C_0 = n\ell_0, \\ C_k = \ell_0(n-k)2^{\frac{(k-1)k}{2}} \left(L^{n-1}\tilde{K}_X\right) \text{ for } 1 \leq k \leq n-1, \end{cases}$$

we conclude that the factor

$$1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor$$

occurs $2^{\max(n-2,0)}$ times in p_1.

We now apply Lemma 8.1 to count the number of times the factor $L^{n-1}\tilde{K}_X$ occurs in p_1. For our counting we let

$$\begin{cases} D = L^{n-1}\tilde{K}_X, \\ A_0 = L^{n-1}\tilde{K}_X, \\ A_k = p_{n-k} \text{ for } 1 \leq k \leq n-1 \end{cases}$$

By applying Lemma 8.1 to the case $C_0 = 1$ and $C_k = \ell_0(n-k)2^{\frac{(k-1)k}{2}}p_n$ for $1 \leq k \leq n-1$, we conclude that the factor $L^{n-1}\tilde{K}_X$ occurs $2^{\max(n-2,0)}$ times in p_1.

We now apply Lemma 8.1 to count the number of times the factor ℓ_0 occurs in p_1. For our counting we let

$$\begin{cases} D = \ell_0, \\ A_0 = \ell_0, \\ A_k = p_{n-k+1} \text{ for } 1 \leq k \leq n \end{cases}$$

By applying Lemma 8.1 to the case

$$\begin{cases} C_0 = 1, \\ C_1 = n\left(1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor\right), \\ C_k = (n-k-1)2^{\frac{k(k+1)}{2}}\left(L^{n-1}\tilde{K}_X\right) \text{ for } 2 \leq k \leq n, \end{cases}$$

we conclude that the factor ℓ_0 occurs 2^{n-1} times in p_1.

Fix $1 \leq k_0 \leq n$. We now count the number of times the factor $k_0 2^{\frac{(n-k_0-1)(n-k_0)}{2}}$ occurs in p_1. For our counting we let

$$\begin{cases} D = k_0 2^{\frac{(n-k_0-1)(n-k_0)}{2}}, \\ A_k = p_{k_0-k} \text{ for } 0 \leq k \leq k_0 - 1 \end{cases}$$

By applying Lemma 8.1 to the case

$$\begin{cases} C_0 = \ell_0\left(1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor\right) \text{ for } k_0 = n, \\ C_0 = \ell_0\left(\prod_{\lambda=k_0+1}^n p_\lambda\right)\left(L^{n-1}\tilde{K}_X\right) \text{ for } k_0 < n, \\ C_k = \ell_0(k_0-k)2^{\frac{(n-k_0+k-1)(n-k_0+k)}{2}}\left(\prod_{\lambda=k_0+1}^n p_\lambda\right)\left(L^{n-1}\tilde{K}_X\right) \text{ for } 1 \leq k < k_0, \end{cases}$$

we conclude that the factor $k_0 2^{\frac{(n-k_0-1)(n-k_0)}{2}}$ occurs $2^{\max(k_0-2,0)}$ times in p_1.

We thus conclude that p_1 is equal to the product of

$$\ell_0^{2^{n-1}}\left(\prod_{k=1}^n \left(k 2^{\frac{(n-k-1)(n-k)}{2}}\right)^{2^{\max(k-2,0)}}\right)$$

and

$$\left(L^{n-1}\tilde{K}_X\right)^{2^{\max(n-2,0)}}\left(1 + \left\lfloor \frac{L^{n-1}\tilde{K}_X}{L^n} \right\rfloor\right)^{2^{\max(n-2,0)}}.$$

9. Verification of Numerical Effectiveness

Let

$$\tilde{L} = \left(p_1 + \sum_{j=2}^n 2^{j-2}p_j\right)L - (\ell_0 - 1)\tilde{K}_X.$$

We now verify that the line bundle $\tilde{L}$ is numerically effective. Suppose the contrary. Then there exists an irreducible subvariety Y of positive dimension in X such that there does not exist any non identically zero holomorphic section of $q\tilde{L}$ on Y for any positive integer q. Let k be the smallest positive integer such that $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ contains Y for some choices of $\nu_{k+1},\nu_{k+2},\cdots,\nu_n$. The irreducible branches of the zero-set of the holomorphic section $s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ of $\left(p_k+\sum_{j=k+1}^n 2^{j-k-1}p_j\right)L-(\ell_0-1)\,\tilde{K}_X$ on $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ are the subvarieties

$$X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}\quad(1\le\nu_j\le I^{(k)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}).$$

By the choice of k the subvariety Y is not contained in any of $X^{(k-1)}_{\nu_k,\nu_{k+1},\cdots,\nu_n}$ and hence $s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ is not identically zero on Y.

Since L is ample on X, there exists some holomorphic section t of qL for some positive integer q which is not identically zero on Y. Then

$$t^{p_1+\sum_{j=2}^n 2^{j-2}p_j-\left(p_k+\sum_{j=k+1}^n 2^{j-k-1}p_j\right)}\left(s^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}\right)^q$$

is a holomorphic section of $q\tilde{L}$ over $X^{(k)}_{\nu_{k+1},\nu_{k+2},\cdots,\nu_n}$ which is not identically zero on Y. This contradicts the choice of Y with the property that there does not exist any non identically zero holomorphic section of $q\tilde{L}$ on Y for any positive integer q. Thus we conclude that $\tilde{L}$ is numerically effective on X.

We have the case $\ell_0=2$. The choice of m gives $m\ge 2^{n-1}p_1$. For any pair of distinct points P, Q of X there exists a Hermitian metric of m_nL+K_X whose non-integrability set contains P and Q as isolated points and whose multiplier ideal sheaf is contained in the square of the maximum ideal at P. It follows from the numerically effectiveness of $\left(p_1+\sum_{j=2}^n 2^{j-2}p_j\right)L-\tilde{K}_X$ and L that $mL-B$, which is equal to $\tilde{L}+(m_nL+K_X)+K_X$, is very ample on X. This finishes the proof of Main Theorem 0.3.

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