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EXTREMAL METRICS AND GEOMETRIC STABILITY

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Abstract. This paper grew out of my lectures at Nankai Institute as well as a few other conferences in the last few years. The purpose of this paper is to describe some of my works on extremal Kähler metrics in the last fifteen years in a more unified way.

In [Ti4], [Ti2], the author developed a method of relating certain stability of underlying manifolds to Kähler-Einstein metrics. A necessary and new condition was derived in terms of the stability for a Kähler manifold to admit Kähler-Einstein metrics with positive scalar curvature. It was clear then that similar results should also hold for general extremal Kähler metrics. Extremal Kähler metrics were introduced by Calabi [Ca]. Extremal Kähler metrics are critical points of the K-energy introduced by T. Mabuchi. Most extremal metrics are Kähler metrics of constant scalar curvature. It was conjectured by the author before that the existence of Kähler metrics with constant scalar curvature is equivalent to the properness of the K-energy. This has been verified for the case of Kähler-Einstein metrics ([Ti2]).

We will explain how extremal metrics are related to the stability of the underlying manifolds and compare it with the standard picture from symplectic geometry. We will outline the proof of the Calabi's conjecture for complex surfaces. We will also list a few problems and indicate the difficulties in solving them.

1. Symplectic Quotients in Finite Dimensions

Let (M, ω) be a symplectic manifold and K be a compact Lie group acting on M by symplectic diffeomorphisms of ω. Denote by k^* the dual of the Lie algebra k of K, then K acts on k^* by the co-adjoint representation. A moment map is a K-equivariant map $\mu : M \mapsto k^*$ such that

$$d\langle \mu, a\rangle(v) = \omega(X_a, v), \ a \in k, \ v \in TM,$$

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where $\langle \cdot, \cdot \rangle$ is the natural pairing between k^* and k and X_a is the vector field on M induced by the action of the one-parameter group $\{exp(ta)\} \subset K$.

If $K = U(1)$, then the existence of the moment map simply means that K is generated by the integral curve of a Hamiltonian field. In general, the existence of the moment implies that K acts on M by Hamiltonian diffeomorphisms. In particular, if M is simply-connected, a moment map always exists. In the following, I will collect a few standard facts about moment maps (cf. [At], [GS], [Ki]).

Let μ be a moment map of the above (M, ω) with the action of K. If 0 is the regular value of μ, then the Marsden-Weinstein reduction theorem implies that the quotient $N = \mu^{-1}(0)/K$ is a symplectic manifold equipped with a natural symplectic form $\pi_* \omega|_{\mu^{-1}(0)}$, where $\pi : \mu^{-1}(0) \mapsto N$ is the projection. If M is further a Kähler manifold, so is the quotient N. Usually, N is called the symplectic quotient of M.

Suppose now that (M, ω) is a Kähler manifold and there exists a complexication $G = K^{\mathbb{C}}$ of K which acts on M by holomorphic transformations. We further assume that the natural inclusion $\iota : K \mapsto G$ induces a surjection $\iota_* : \pi_1(K) \mapsto \pi_1(G)$. Then we can construct a functional $F : M \times G \mapsto \mathbb{R}$ as follows: For any $x \in M$, we will first construct $F_x : G \mapsto \mathbb{R}$.

Lemma 1.1. *Let $\sigma : [0,1] \mapsto G$ be any loop with $\sigma(0) = I$, then*

$$\int_0^1 \langle \mu(\sigma(t)x), -\sqrt{-1}\pi(\sigma'(t)) \rangle \, dt = 0, \tag{1.1}$$

where $\pi : T_{\sigma(t)x} G \cong k \oplus \sqrt{-1}k \mapsto \sqrt{-1}k$.

Now we can define F_x by

$$F_x(\tau) = \int_0^1 \langle \mu(\sigma(t)x), -\sqrt{-1}\pi(\sigma'(t)) \rangle \, dt, \ \ \tau \in G, \tag{1.2}$$

where $\sigma : [0,1] \mapsto G$ is a path from $\sigma(0) = I$ to $\sigma(1) = \tau$.

Then we simply put $F(x, \tau) = F_x(\tau)$. It follows from the above lemma that F is well-defined. Furthermore, F has the following properties:

(1) If $\tau, \sigma \in G$, then $F(x, \tau) + F(\tau(x), \sigma) = F(x, \sigma \cdot \tau)$;
(2) For any $\kappa \in K$ and $\tau \in G$, $F(x, \kappa \cdot \tau) = F(x, \tau)$ and $F(x, 1) = 0$;
(3) For any $\kappa \in K$ and $\tau \in G$, $F(\kappa(x), \tau) = F(x, \kappa^{-1} \cdot \tau \cdot \kappa)$;
(4) $\tau \in G$ is a critical point of F_x if and only if $\mu(\tau(x)) = 0$;
(5) For any $x \in M$ and $a \in k$, $\frac{\partial^2 F}{\partial t^2}(x, e^{\sqrt{-1}ta}) \geq 0$ and the equality holds if and only if $v_a(e^{\sqrt{-1}ta}x) = 0$.

It follows from (4) and (5) that there is a unique $\tau \in G$ such that $\mu(\tau(x)) = 0$ if and only if the functional F_x is proper on G. By (1), if F_x is proper, so is $F_{\tau(x)}$. The properness of F_x can be thought as an analytic stability of x.

More generally, there is a $\tau \in G$ with $\mu(\tau(x)) = 0$ if and only if F_x is proper on G/G_x, where G_x is the isotropy group $\{\tau \in G | \tau(x) = x\}$. If $\mu(\tau(x)) = 0$ for $\tau = \tau_1, \tau_2$, then $\tau_1 = \tau_2\sigma$ for some $\sigma \in G_x$.

If $M = \mathbb{C}P^n$ with the Fubini-Study metric ω and $K = SU(n+1)$, then its complexication is $G = \mathbf{SL}(n+1, \mathbb{C})$ and its moment map is defined by

$$\langle \mu([z_0 : \cdots : z_n]), A\rangle = \frac{\sqrt{-1} z^* A z}{z^* z},$$

where $A \in su(n+1)$ is a skew-Hermitian matrix with vanishing trace and $z = (z_0, \cdots, z_n)^T$ and z^* is the conjugate transpose of z.

Now let M be a complex submanifold in $\mathbb{C}P^N$ with ω being the restriction of the Fubini-Study metric to M, let G be a subgroup of $\mathbf{SL}(N+1, \mathbb{C})$ acting on $\mathbb{C}P^N$ in the usual way and K be its maximal compact subgroup. Then the moment map is simply

$$\langle \mu([z_0 : \cdots : z_N]), A\rangle = \frac{\sqrt{-1} z^* A z}{z^* z},$$

where $A \in k \subset su(N+1)$ and $z = (z_0, \cdots, z_N)^T$ with $[z_0 : \cdots : z_N] \in M$.

If $G_0 = \{\sigma(s) | s \in \mathbb{C}^*\}$ is a one-parameter algebraic subgroup of G, then by changing coordinates if necessary, we may assume that

$$\sigma(s)(z) = (s^{\alpha_0} z_0, \cdots, s^{\alpha_N} z_N)^T,$$

where $x = [z_0 : \cdots : z_N] \in M$ and $\alpha_i \in \mathbb{Z}$ with $\sum_{i=0}^N \alpha_i = 0$. Put $\lambda(G_0) = \max\{\alpha_i | z_i \neq 0\}$, i.e., the maximal weight of G_0. Then a direct computation shows that

$$\lambda(G_0) = \lim_{t \to \infty} F_x(\sigma(e^t)).$$

It follows that if F_x is proper on G, then $\lambda(G_0) > 0$ for every such a G_0. Its converse is also true.

On the other hand, the Hilbert numerical criterion for stability states that $x \in M$ is a stable point of G-action if and only if $\lambda(G_0) > 0$ for every one-dimensional algebraic subgroup G_0 of G. Here z being stable means that its lifting $z \in \mathbb{C}^{N+1}$ satisfies: The orbit of z under the action of $G = \mathbf{SL}(N+1, \mathbb{C})$ on $\mathbb{C}^{N+1}$ is closed and the stabilizer of z is finite.

Therefore, the existence of zeroes for the moment μ in case of an algebraic manifold M is equivalent to either the analytic stability or algebraic stability of x in M.

2. Extremal Kähler Metrics

Let M be a compact Kähler manifold of dimension n with a Kähler metric. A Kähler metric can be given by specifying its Kähler form ω, in local coordinates $z_1, \cdots, z_n$. It is of the form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive Hermitian matrix-valued function such that $d\omega = 0$. We will simply use ω to denote both a metric and its Kähler form.

Recall that the Kähler class of ω is the cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$ represented by ω. It follows from the Hodge theory that if ω' is another Kähler metric with $[\omega'] = [\omega]$, then there is a smooth function φ on M such that

$$\omega' = \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\varphi.$$

We will often denote the right side by ω_φ. Thus, the space $\mathcal{K}_{[\omega]}$ of Kähler metrics with the same Kähler class $[\omega]$ can be identified with

$$\{\varphi \in C^\infty(M, \mathbb{R}) \mid \int_M \varphi\omega^n = 0,\ \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi > 0\}.$$

Consider the functional on $\mathcal{K}_{[\omega]}$,

$$\mathrm{Ca}(\omega) = \int_M s(\omega)^2 \omega^n.$$

E.Calabi proved in [Ca] that ω is a critical metric of Ca in $\mathcal{K}_{[\omega]}$ if and only if there is a holomorphic vector field v on M such that $i_v\omega = \bar{\partial} s(\omega)$. Following Calabi, we call such a critical metric extremal metric.

Clearly, a Kähler metric ω is an extremal metric if its scalar curvature $s(\omega)$ is constant. If ω has constant scalar curvature, then

$$s(\omega) = \frac{n c_1(M) \cup [\omega]^{n-1}([M])}{[\omega]^n([M])},$$

where $c_1(M)$ is the first Chern class and $[M]$ denotes the fundamental class of M.

If ω has constant scalar curvature and $c_1(M) = \lambda[\omega]$ for some constant λ, then ω is a Kähler-Einstein metric, that is, $\mathrm{Ric}(\omega) = \lambda\omega$, where $\mathrm{Ric}(\omega)$ is the Ricci curvature form, in local coordinates $z_1, \cdots, z_n$,

$$\mathrm{Ric}(\omega) = -\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log\det(g_{i\bar{j}}).$$

Basic problems include the uniqueness and existence of extremal metrics as well as their moduli spaces and applications to the geometry of Kähler manifolds.

Let $\eta(M)$ be the space of holomorphic vector fields on M and ω be any fixed Kähler metric with Kähler class $\Omega \in H^2(M, \mathbb{R})$. Then one can define the Calabi-Futaki invariant as

$$f_\Omega(v) = \int_M v(h_\omega)\omega^n, \ v \in \eta(M),$$

where h_ω is determined by the equations

$$s(\omega) - \underline{s}_\omega = \Delta h_\omega, \ \int_M \left(e^{h_\omega} - 1\right)\omega^n = 0,$$

where $\underline{s}_\omega$ denotes the average of $s(\omega)$. It was proved ([Fut]) that $f_\Omega(v)$ is actually independent of the choice of ω. Therefore, it is an invariant. Consequently, if M admits an extremal metric with the Kähler class Ω and constant scalar curvature, then $f_\Omega \equiv 0$. On the other hand , there are examples of Kähler manifolds with nonvanishing Calabi-Futaki invariant for certain Kähler classes, so there do not exist metrics of constant scalar curvature for those Kähler classes. Also, extremal metrics with the Kähler class Ω have constant scalar curvature if and only if the Calabi-Futaki invariant f_Ω vanishes identically.

In 1986, T. Mabuchi introduced the K-energy which can be defined as follows: For any $\varphi \in \mathcal{K}_{[\omega]}$, let φ_t be any path joining 0 to φ, then

$$\mathbf{T}_\omega(\varphi) = -\int_M \dot{\varphi}_t(s(\omega_{\varphi_t}) - \underline{s}_\omega)\omega^n_{\varphi_t}.$$

It can be proved that $\mathbf{T}_\omega(\varphi)$ is independent of the choice of the path. Its critical points are clearly Kähler metrics of constant scalar curvature. A straightforward computation shows

$$\mathbf{T}_\omega(\varphi) + \mathbf{T}_{\omega_\varphi}(\psi) = \mathbf{T}_\omega(\varphi + \psi).$$

In fact, one can show that the K-energy is certain Donaldson functional restricted to the space of Kähler metrics. The Donaldson functionals were first introduced in [Do] in terms of the Bott-Chern classes [BC].

Here is a brief summary of known results on extremal metrics. The celebrated solution of Yau [Ya] for the Calabi conjecture established the existence of a Ricci-flat metric, now named as Calabi-Yau metric, in each Kähler class on a compact Kähler manifold M with $c_1(M) = 0$. If $c_1(M) < 0$, the existence of Kähler-Einstein metrics was proved by Aubin [Au] and Yau [Ya], independently. Not every M with $c_1(M) > 0$ admits a Kähler-Einstein metric and additional geometric conditions are needed to assure the existence. If M is a complex surface with $c_1(M) > 0$, then it admits a Kähler-Einstein metric if and only if the

Lie algebra of holomorphic vector fields is reductive[Ti3]. For general M with $c_1(M) > 0$, the existence of Kähler-Einstein metrics is equivalent to certain analytic stability which is related to certain geometric stability[Ti2]. The existence of extremal metrics other than Kähler-Einstein ones is basically open, but C. Lebrun constructed scalar-flat Kähler metrics on blow-ups of ruled surfaces by using the twistor method. Also, examples were constructed on manifolds with many symmetries.

The uniqueness problem of extremal metrics is also highly nontrivial. The uniqueness for Kähler-Einstein metrics of nonpositive scalar curvature was already known to Calabi in the 50's. In 1986, Bando and Mabuchi [BM] proved the uniqueness for Kähler-Einstein metrics with positive scalar curvature. Their proof is very nice and highly nontrivial. So far, the best uniqueness result on general extremal metrics was due to X.X. Chen [Ch]. He proved the uniqueness of extremal metrics in any Kähler class on a Kähler manifold with nonpositive first Chern class.

Compactness theorems on Kähler-Einstein metrics were established in [Ti2] (also see [An]) for complex dimension two and in [CCT] for general dimensions. There are still many open questions to be solved, especially in higher dimensions. For general extremal metrics, very little is known.

In the following, we will only concentrate on the existence problem and discuss its relation to the geometric stability of underlying manifolds. We will not follow the chronicle order and present our results in contrast with those properties discussed in the first section.

3. Scalar Curvature As Moment Map

Our discussions in this section follow [Ma], [Fuj] and [Do]. We will show that the scalar curvature in Kähler geometry can be thought as an infinite dimensional moment map in a formal sense as we discussed in Section 1. Then in the next two sections, we explain how my previous theorems on Kähler-Einstein metrics fit into the picture for the finite dimensional moment maps described in Section 1.

Let (X, ω) be a compact Kähler manifold. An ω-compatible almost complex structure is an endomorphism $J : TX \mapsto TX$ such that $J^2 = -Id$ and $g_J(u, v) = \omega(u, Jv)$ $(u \in TX)$ defines a Hermitian metric with respect to J. It is integrable if it induces a complex structure on X. We put $\mathcal{J}_\omega$ to be the set of all ω-compatible, integrable almost complex structures. For simplicity, we will often abbreviate it as $\mathcal{J}$. Let K be the group of all Hamiltonian diffeomorphisms of (X, ω). Then K

acts naturally on $\mathcal{J}_\omega$: For any $\phi \in K$ and $J \in \mathcal{J}_\omega$, $\phi_*(J) = d\phi^{-1} \cdot J \cdot d\phi$. The Lie algebra $\mathbf{g}$ of K is made of all Hamiltonian vector fields v_h defined by

$$\omega(v_h, u) = u(h),\ u \in TX,$$

where h is any smooth function. Therefore, the Lie algebra of K is naturally identified with $C^\infty(M, \mathbb{R})_0$ of all smooth functions h satisfying

$$\int_X h\omega^n.$$

For any $J \in \mathcal{J}$, the tangent space $T_J\mathcal{J}$ consists of all endomorphisms $\mu : TX \mapsto TX$ which anti-commutes with J and is symmetric with respect to g_J. Clearly, $J\mu \in T_J\mathcal{J}$ whenever μ is. So we can define a complex structure $\jmath$ on $\mathcal{J}$ by $\jmath(\mu) = J\mu$ for any $\mu \in T_J\mathcal{J}$. There is also a metric $\langle \cdot, \cdot \rangle$ on $\mathcal{J}$: For each J,

$$\langle \mu, \mu' \rangle_J = \int_X g_J(\mu, \mu')\omega^n,$$

where we also denote by g_J the induced metric on $TX \otimes T^*X$. This metric is $\jmath$-invariant and induces naturally a two form Ω on $\mathcal{J}$ which is actually closed, at least in a formal sense. So $(\mathcal{J}, \Omega)$ becomes an infinite dimensional symplectic manifold. Moreover, the above action of K on $\mathcal{J}$ is made of symplectomorphisms of $(\mathcal{J}, \Omega)$.

For any $J \in \mathcal{J}$, let $K(J)$ be the orbit of K passing through J, its tangent space at J is

$$\{L_{v_h} J \mid h \in C^\infty(M, \mathbb{R})_0\}.$$

Thus, any $h \in C^\infty(X, \mathbb{R})_0$ induces a vector field V_h on $\mathcal{J}$ defined by

$$V_h(J) = L_{v_h} J \in T_J K(J) \subset T_J\mathcal{J}.$$

It can be shown in [Ma], [Fuj] and [Do] (also see [Ti5]) that

$$\frac{d}{dt}\int_X s(g_{J_t}) h\, \omega^n \Big|_{t=0} = \Omega(V_h, \mu)|_J,\ h \in C^\infty(M\mathbb{R})_0,\ J \in \mathcal{J},$$

where $s(g_J)$ is the scalar curvature of the metric g_J, $\{J_t\}$ is any path in $\mathcal{J}$ with $J_0 = J$ and $\frac{dJ_t}{dt}|_0 = \mu$. Through the L^2 inner product on the left side of the above, we may regard $C^\infty(M, \mathbb{R})_0$ as the dual $\mathbf{g}^*$ of the Lie algebra $\mathbf{g}$. Also any g_J is a Kähler metric, so the average $\underline{s}$ of its scalar curvature is determined by the first Chern class and ω. Thus, we have a map $m : \mathcal{J} \mapsto \mathbf{g}^*$ defined by $m(J) = s(g_J) - \underline{s}$. The above identity simply means that m is a moment map of the action by K with respect to the symplectic structure Ω (in a formal sense).

Now let us see how this fits into the problem of extremal Kähler metrics and why the discussions in Section 1 are relevant.

Define a distribution $\mathcal{D} \subset T\mathcal{J}$:

$$\mathcal{D}_J = \{L_{v_h}J, L_{Jv_h}J \mid h \in C^\infty(M,\mathbb{R})\}.$$

One can show that $\mathcal{D}$ is involutive.[1] Let us construct maximal integral manifolds of $\mathcal{D}$ in $\mathcal{J}$. Any $J \in \mathcal{J}$ gives rise to a complex structure on X. We may assume that ω is a Kähler form. Any other Kähler metric with the Kähler class $[\omega]$ is of the form ω_φ for some $\varphi \in C^\infty(X,\mathbb{R})_0$. By Moser's theorem on deformation of symplectic forms, we can get a smooth map $F : C^\infty(X,\mathbb{R})_0 \times X \mapsto X$ such that for each φ, $F_\varphi = F(\varphi,\cdot)$ defines a diffeomorphism of X satisfying: $F_\varphi^*\omega_\varphi = \omega$. Define

$$\iota : C^\infty(X,\mathbb{R})_0 \times K \mapsto \mathcal{J},\ \iota(\varphi,\psi) = \psi^* \cdot F_\varphi^* J.$$

One can prove that its image $\mathcal{M}$ is a maximal integral manifold of $\mathcal{D}$ through J. For this, we suffice to prove that $T\mathcal{M}$ is contained in $\mathcal{D}$. It is also preserved by K. Therefore, we can regard the space of Kähler metrics with the Kähler class $[\omega]$ as the quotient $\mathcal{M}/K$. It follows from these discussions

Proposition 3.1. *The moduli space of Kähler metrics with constant scalar curvature and Kähler class being $[\omega]$ is the symplectic quotient $m^{-1}(0)/K$.*

Furthermore, we observe that the K-energy $\mathbf{T}_\omega$ in the last section is just the infinite dimensional version of the function in (1.1). This makes us consider two problems analogous to those at the end of Section 1, namely, the existence of constant scalar curvature Kähler metrics versus the properness of the K-energy and geometric stability. For example, is there a notion of the Hilbert criterion for extremal Kähler metrics? These problems will be studied in the following two sections.

As in the case of finite dimensional moment maps, the K-energy $\mathbf{T}_\omega$ is convex on the space of Kähler metrics with a fixed Kähler class. Hence, many profound understanding towards extremal Kähler metrics can be studied by using this convexity ([Do]). However, since the space of Kähler metrics is of infinite dimension, it is unclear if this space is complete with the L^2-metric on Kähler potentials. For example, is there a geodesic joining any two given metrics? In [Do] (also [Se]), the existence of geodesics is reduced to solving a degenerate complex Monge-Ampere equation. Though it seems very hard to solve this equation, X. Chen [Ch] proved existence of its weak solutions. Following a suggestion of S. Donaldson, he used them to prove the uniqueness of extremal metrics under certain negativity conditions.

[1]The Hamiltonian group K does not have any complexification, but it admits a complexication on the level of Lie algebra through the distribution $\mathcal{D}$.

4. Analytic Stability

In Section 1, we have seen that on a finite dimensional symplectic manifold (M,ω) with an action by a compact group K, the associated moment map μ has a zero if and only if the function defined in (1.1) is proper. In this section, we study its analogue for constant scalar curvature Kähler metrics. First, we need to introduce the notion of the analytic stability, which is equivalent to certain properness of $\mathbf{T}_\omega$.

Because of infinite dimension, the properness of $\mathbf{T}_\omega$ depends on the norms we use. The space of Kähler metrics with a fixed Kähler class $[\omega]$ is

$$\mathcal{K}_{[\omega]} = P(M,\omega)/\sim,\ P(M,\omega) = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi > 0\}.$$

Here $\varphi \sim \varphi'$ means $\varphi = \varphi' + c$ for some constant c. Note that $\mathbf{T}_\omega$ can be lifted to $P(M,\omega)$.

Define

$$\mathbf{I}_\omega(\varphi) = \frac{1}{V}\int_M \varphi(\omega^n - (\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi)^n), \tag{4.1}$$

where $V = \int_M \omega^n$. Also define

$$\mathbf{J}_\omega(\varphi) = \int_0^1 \frac{\mathbf{I}_\omega(t\varphi)}{t}dt = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\frac{\sqrt{-1}}{2}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i\wedge\omega_\varphi^{n-i-1}. \tag{4.2}$$

Then

$$\mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{n-i}{n+1}\frac{\sqrt{-1}}{2}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-i-1}.$$

Notice that $\mathbf{I}_\omega(\varphi) \geq 0$ whenever $\varphi \in P(M,\omega)$. We will denote by (M,Ω), where $\Omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$, a compact Kähler manifold polarized by a Kähler class Ω. By an automorphism of (M,Ω), we mean a biholomorphism σ of M such that $\sigma^*\Omega = \Omega$. Denote by $\mathrm{Aut}(M,\Omega)$ the group of all such automorphisms.

Recall

Definition 1. *We say that $(M,[\omega])$ is analytically stable, or equivalently, the K-energy $\mathbf{T}_\omega$ is proper, if there is an increasing function $a(t) \geq 0$ ($t \in (-\infty,\infty)$) such that $\lim_{t\to\infty} a(t) = \infty$ and for any $\varphi \in P(M,\omega)$,*

$$\mathbf{T}_\omega(\varphi) \geq a(\mathbf{I}_\omega(\varphi)). \tag{4.3}$$

We say that $(M,[\omega])$ is analytically semi-stable if $\mathbf{T}_\omega$ is bounded from below.

If G is a compact Lie group acting on M by automorphisms of $(M,[\omega])$, then we say that $(M,[\omega])$ is analytically G-stable if (4.3) holds for all G-invariant φ in $P(M,\omega)$ for some G-invariant metric ω in $[\omega]$.

Since the K-energy satisfies the cocycle condition

$$\mathbf{T}_\omega(\varphi) = \mathbf{T}_\omega(\psi) + \mathbf{T}_{\omega_\psi}(\varphi - \psi),$$

the analytic stability is independent of the choice of ω representing the Kähler class $[\omega]$.

Conjecture 4.1. *(cf.[Ti5]) A compact Kähler manifold M admits an Kähler metric of constant scalar curvature and with the Kähler class Ω if and only if the polarized manifold (M,Ω) is analytically G-stable for some maximal compact subgroup G of $\mathrm{Aut}(M,\Omega)$.*

This conjecture has a counterpart in finite dimensional case, which is true as shown in Section 1. This conjecture was solved in the case of Kähler-Einstein metrics (cf. [Ti5]). Precisely, we have

Theorem 4.2. *Let (M,Ω) be a compact polarized Kähler manifold such that $c_1(M) = \lambda\Omega$ for some constant λ. Then M admits a Kähler-Einstein metric within the Kähler class Ω if and only if (M,Ω) is analytically G-stable for some maximal compact subgroup G of $\mathrm{Aut}(M,\Omega)$.*

Let us outline its proof here by using a result in [Ti2]. A simple computation shows that when $c_1(M) = \lambda\Omega$ and ω is a metric with $[\omega] = \Omega$,

$$\mathbf{T}_\omega(\varphi) = \int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n - \int_M h_\omega(\omega^n - \omega_\varphi^n) - \lambda\, V\, \left(\mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi)\right).$$

Following [Ti1], we introduce the invariant $\alpha(M,\Omega)$ for any polarized compact Kähler manifold (M,Ω): Let G be a maximal compact subgroup of $\mathrm{Aut}(M,\Omega)$. Choose a G-invariant Kähler metric ω with $[\omega] = \Omega$. Define

$$\alpha(M,\Omega) = \sup\{\alpha \mid \exists C_\alpha, \mathbf{s.t.} \int_M e^{-\alpha(\varphi - \sup_M \varphi)}\omega^n \le C_\alpha,\ \forall\varphi\in P_G(M,\omega)\}, \tag{4.4}$$

where $P_G(M,\omega)$ consists of all G-invariant functions in $P(M,\omega)$. It can be shown that it is independent of the choice of ω and it is always positive (cf. [Ti1]). It follows that for any $\alpha < \alpha(M,\Omega)$ and $\varphi \in P_G(M,\omega)$,

$$\frac{1}{V}\int_M e^{-\log\frac{\omega_\varphi^n}{\omega^n} - \alpha(\varphi - \sup_M\varphi)}\omega_\varphi^n = \frac{1}{V}\int_M e^{-\alpha(\varphi - \sup_M\varphi)}\omega^n \le C_\alpha. \tag{4.5}$$

Taking the logarithm on both sides and using its concavity, we get

$$\frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \geq \frac{\alpha}{V}\int_M (\sup_M \varphi - \varphi)\omega_\varphi^n - \log C_\alpha.$$

Therefore, $\mathbf{T}_\omega$ is proper if $\lambda \leq 0$.[2] On the other hand, by [Ya], M admits a unique Kähler-Einstein metric with the Kähler class Ω whenever $\lambda \leq 0$.

It remains to show the theorem when $\lambda > 0$, say $\lambda = 1$. Recall

$$\mathbf{F}_\omega(\varphi) = \mathbf{J}_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega}\omega^n\right).$$

Similarly, one can define the properness of $\mathbf{F}_\omega$. The following theorem was proved in [Ti2].

Theorem 4.3. *Let $(M, c_1(M))$ be a polarized compact Kähler manifold with $c_1(M)$ positive. Then M admits a Kähler-Einstein metric if and only if $\mathbf{F}_\omega$ is G-proper for a maximal compact subgroup of* $\mathrm{Aut}(M, c_1(M))$*, where ω is any fixed G-invariant Kähler metric in $c_1(M)$.*

We will deduce Theorem 4.3 from this theorem when $\lambda = 1$. A direct computation shows

$$\mathbf{F}_\omega(\varphi) = \frac{1}{V}\mathbf{T}_\omega(\varphi) + \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^n - \frac{1}{V}\int_M h_\omega\omega^n.$$

It follows that $\mathbf{F}_\omega \leq \mathbf{T}_\omega + c$ for some constant c. So the necessary part of Theorem 4.3 follows. On the other hand, in order to prove the existence, it is sufficient to establish the properness of $\mathbf{F}_\omega$ along solutions of (cf. [Ti1])

$$\left(\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi\right)^n = e^{h_\omega - t\varphi}\omega^n,\ t \in [0,1]. \tag{4.6}$$

If φ solves (4.4) for t, then $h_{\omega_\varphi} = -(1-t)\varphi$, however, Bando-Mabuchi showed (cf. [BM]) that $(1-t)\varphi$ is bounded if the K-energy $\mathbf{T}_\omega$ is bounded from below. So Theorem 4.3 follows.

Remark. The necessary part of Conjecture 4.2 would follow if one could prove that $\mathcal{K}_{[\omega]}$ is geodesically complete with respect to the L^2-metric as pointed out by S. Donaldson.

Theorem 4.4 can be used to construct many Kähler-Einstein manifolds with positive scalar curvature (cf. [Ti5]).

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[2] In fact, $\mathbf{T}_\omega$ is proper and so M has a Kähler-Einstein metric whenever $\alpha(M,\Omega) > \frac{n\lambda}{n+1}$. So Theorem 4.3 implies the main result in [Ti1].

5. Kähler-Einstein Metrics On Complex Surfaces

The existence problem for Kähler-Einstein metrics on a complex surface has been completely solved due to Yau [Ya] in the case of vanishing first Chern class, Aubin-Yau [Au], [Ya] in the case of negative first Chern class and myself [Ti3] in the case of positive first Chern class. More precisely, if (M, Ω) is a polarized Kähler surface with $c_1(M) = \lambda\Omega$ for some constant λ and vanishing Futaki invariant, then M admits a unique Kähler-Einstein metric with Kähler class Ω. Of course, the condition on the Futaki invariant is void if $c_1(M)$ is not positive. Now we recall the main result of [Ti3].

Theorem 5.1. *Let M be a compact complex surface with positive first Chern class. Then M admits a Kähler-Einstein metric if and only if it has vanishing Futaki invariant.*

In rest of this section, we will outline the proof of this theorem following arguments in [Ti3]. Hopefully, it will make the readers easier to understand the proof which was very long in [Ti3]. What we really did in [Ti3] is to get the properness of $\mathbf{F}_\omega$ on certain finite dimensional space of "algebraically" defined Kähler metrics. This properness is equivalent to certain stability condition as we will see in the next section.

Without loss of generality, we may assume that $c_1(M) = \Omega$. By the classification theory of complex surfaces, M is either $\mathbb{C}P^2$ or $S^2 \times S^2$ or the blow-up Σ_m of $\mathbb{C}P^2$ at generic m points $(1 \leq m \leq 8)$. Here the genericity means that none of three points are collinear, none of six points are on a common quadratic curve and, if $m = 8$, not all eight points lie on a cubic curve with a cusp which is one of the blow-up points. The first two surfaces are homogeneous and so have canonical Kähler-Einstein metrics. It was proved by Futaki that M has a Kähler-Einstein metric only if its associated Futaki invariant vanishes. It was also known that the Futaki invariant of $M = \Sigma_m$ is nonzero if and only if $m = 1$ or 2. Therefore, we suffice to establish the existence of Kähler-Einstein metrics on Σ_m for $3 \leq m \leq 8$. Denote by $\mathcal{M}_m$ the moduli spaces of Σ_m. It consists of all possible m points in $\mathbb{C}P^2$ in general position. Clearly, it is connected.

The proof in [Ti3] consists of three steps. In the first step, we proved that there is at least one complex surface M in $\mathcal{M}_m$ which admits a Kähler-Einstein metric. It was actually done in [TY] as an application of the main theorem in [Ti1]. In [Ti1], it was proved that if the invariant $\alpha(M)$, the abbreviation of $\alpha(M, c_1(M))$ in last section, is greater than $n/n+1$, where $n = \dim_{\mathbb{C}} M$, then M admits a Kähler-Einstein metric. Here is a brief summary of estimates on the invariant $\alpha(M)$ in [TY]: (1) There is only one complex surface $M \in \mathcal{M}_3$ whose $\alpha(M) \geq 1$; (2) The only one surface $M \in \mathcal{M}_4$ has $\alpha(M) \geq 3/4$; (3) Every surface $M \in \mathcal{M}_5$ is a complete intersection in $\mathbb{C}P^4$ defined by two quadratic polynomials $\sum_{i=0}^4 z_i^2 = 0$ and $\sum_{i=0}^4 \lambda_i z_i^2 = 0$, then $\alpha(M) \geq 1$; (4) Every surface in $\mathcal{M}_6$ is a cubic surface in $\mathbb{C}P^3$. If M is the Fermat surface, then $\alpha(M) \geq 1$; (5) Every surface M in $\mathcal{M}_7$ is a double branch covering of $\mathbb{C}P^2$ along a quartic curve Q. Then $\alpha(M) \geq 3/4$ when Q is a quartic curve with certain finite symmetries [3]; (8) Certain M in $\mathcal{M}_8$ with finite symmetries has $\alpha(M) \geq 5/6$ (cf. [TY] for details). These surfaces described above admit a Kähler-Einstein metric.

Next we used the continuity method: For each $m \geq 5$, let $\mathcal{E}_m$ be the subset of all $M \in \mathcal{M}_m$ which admits a Kähler-Einstein metric. It follows from the first step that $\mathcal{E}_m$ is nonempty. Choose a smooth family of Kähler metrics ω_τ on $M_\tau \in \mathcal{M}_m$ with Kähler class $c_1(M_\tau)$. Then M_τ admits a Kähler-Einstein metric if and only if the following Monge-Ampere equation is solvable

$$(\omega_\tau + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi)^2 = e^{h_\tau - \varphi}\omega_\tau^2, \ \omega_\tau + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi > 0 \text{ on } M_\tau, \tag{5.1}$$

where h_τ is determined by

$$\mathrm{Ric}(\omega_\tau) - \omega_\tau = \frac{\sqrt{-1}}{2}\partial\overline{\partial}h_\tau, \ \int_{M_\tau}(e^{h_\tau} - 1)\omega_\tau^2 = 0.$$

Since $m \geq 5$, any surface M_τ does not have any nontrivial holomorphic vector fields, it follows that if (5.1) is solvable on M_τ, so is every $M_{\tau'}$ sufficiently close to M_τ. This is a simple application of the Implicit Function Theorem. So $\mathcal{E}_m$ is open. It remains to show that $\mathcal{E}_m$ is closed in $\mathcal{M}_m$. In order to deduce this closedness, we need an a prior C^3-estimate on solutions of (5.1). As we explained in [Ti1], this C^3-estimate follows from an a prior C^0-estimate. In general, there does not exist such an estimate for an equation of the type like (5.1). The idea of [Ti2] is to get a partial C^0-estimate and then use geometric information of underlying manifolds to check if the required C^0-estimate holds. Now let us recall the partial C^0-estimate.

Theorem 5.2. *There are two constants $c > 0$ and $l_0 > 0$ such that for any Kähler-Einstein surface (M, ω) with* $\mathrm{Ric}(\omega) = \omega$*, there is some $l \in [l_0, 2l_0]$ such that*

$$c \geq \frac{1}{l}\log\left(\sum_{i=0}^N ||S_i||^2\right) \geq -c, \tag{5.2}$$

[3]As an application of the main theorem in [Ti2], it was shown in [Ti5] that every surface M in $\mathcal{M}_7$ has proper $\mathbf{F}_\omega$, and consequently has a Kähler-Einstein metric.

where $\{S_i\}_{0\le i\le N}$ *is any orthonormal basis of* $H^0(M, K_M^{-l})$ *with respect to the inner product induced by* ω. [4]

Let us first explain why it implies a partial C^0-estimate. Let φ be a solution of (5.1). Write ω for $\omega_\tau + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi$. Choose a Hermitian metric $||\cdot||_\tau$ on K_M^{-1} such that its curvature form is ω_τ. This Hermitian metric and ω_τ induce an inner product on $H^0(M, K_M^{-l})$. We may choose $\{S_i\}$ in (5.2) such that $\{\mu_i S_i\}$ is an orthonormal basis of this inner product associated to ω_τ for some positive constants μ_i $(i = 0, \cdots, N)$. It follows from the Maximum principle

$$\varphi - \frac{1}{l}\log\left(\frac{\sum_{i=0}^N ||S_i||_\tau^2}{\sum_{i=0}^N ||S_i||^2}\right) = c', \tag{5.3}$$

where c' is some constant. Write $\sigma_i = \mu_i S_i$. We may arrange $\mu_0 \ge \mu_1 \ge \cdots \ge \mu_N$ and put $\lambda_i = \mu_N/\mu_i$. Then $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_N = 1$ and

$$||\varphi - \sup_M \varphi - \frac{1}{l}\log(\sum_{i=0}^N \lambda_i^2 ||\sigma_i||_\tau^2)||_{C^0} \le C. \tag{5.4}$$

Note that C always denotes a uniform constant. In particular, $\varphi - \sup_M \varphi$ is bounded away from the zero locus of σ_N. So we have a partial C^0-estimate of $\varphi - \sup_M \varphi$.

Now let us say a few words about the proof of Theorem 5.2. It follows from a compactness theorem on Kähler-Einstein metrics. Fix m, let $\mathcal{M}_{m,KE}$ be the set of $M \in \mathcal{M}_m$ which admits a Kähler-Einstein metric. If (M, ω) is any Kähler-Einstein surface in $\mathcal{M}_{m,KE}$ with $\mathrm{Ric}(\omega) = \omega$, it has uniformly bounded diameter, fixed volume and by the Gauss-Bonnet formula, the L^2-norm of its curvature is uniformly bounded. It was then proved in [Ti3] that $\mathcal{M}_{m,KE}$ can be compactified in the Cheeger-Gromov topology by adding Kähler-Einstein orbifolds with isolated singularities. Those singularities are of the form U/Γ, where $U \subset \mathbb{C}^2$ and Γ is a finite group of $U(2)$ with uniformly bounded order. Next, using the L^2-estimate for holomorphic sections, we proved in [Ti3] that the function in (5.2) is continuous with respect to the Cheeger-Gromov topology. Then one can deduce (5.2) by choosing $l = ak$, where a is the product of all orders of possible Γ for the singularities.

The third step of the proof in [Ti3] is to prove

$$\mathbf{F}_{\omega_\tau}(\varphi) \ge \epsilon I_{\omega_\tau}(\varphi) - C, \tag{5.5}$$

[4]In [Ti3], (5.2) is actually proved for any $l = 6k > 0$. It was also conjectured that (5.2) holds for any l sufficiently large.

where ϵ is a fixed positive constant and φ solves (5.1).[5] Note that we always use C to denote a uniform constant.

Now let us explain why (5.5) implies Theorem 5.1. Since φ solves (5.1), we have

$$\begin{aligned} \mathbf{F}_{\omega_\tau}(\varphi) &= \tfrac{\sqrt{-1}}{3V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\tau + \tfrac{\sqrt{-1}}{6V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega - \tfrac{1}{V}\int_M \varphi\omega_\tau^2 \\ &= -\tfrac{1}{3V}\int_M \varphi\omega^2 - \tfrac{2}{3V}\int_M \varphi\omega_\tau^2 + \tfrac{\sqrt{-1}}{6V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\tau. \end{aligned} \tag{5.6}$$

Recall the following facts from [Ti1] (also see [Ti5]):

$$\mathbf{F}_{\omega_\tau}(\varphi) \leq 0; \tag{5.7}$$

$$\sup_M \varphi \leq \frac{1}{V}\int_M \varphi\omega_\tau^2 + C; \tag{5.8}$$

$$-\inf_M \varphi \leq C\left(1 - \frac{1}{V}\int_M \varphi\omega^2\right). \tag{5.9}$$

It follows from the above that

$$||\varphi||_{C^0} \leq C(1 + I_{\omega_\tau}(\varphi)), \tag{5.10}$$

$$-\inf_M \varphi \leq 2\sup_M \varphi + C. \tag{5.11}$$

Hence, Theorem 5.1 follows from (5.5).

Lemma 5.3. *For any $\alpha < 2/3$, there is a uniform constant C_α such that*[6]

$$\int_M e^{-\alpha(\varphi - \sup_M \varphi)}\omega_\tau^2 \leq C_\alpha. \tag{5.12}$$

This lemma can be proved by using geometry of M and L^2-estimate.

Using the concavity of logarithm and (5.1), we get

$$\begin{aligned} &\alpha \sup_M \varphi + \frac{1-\alpha}{V}\int_M \varphi\omega^2 \\ \leq\ & \log\left(\frac{1}{V}\int_M e^{\alpha sup_M \varphi + (1-\alpha)\varphi} e^{h_\tau - \varphi}\omega_\tau^2\right) - \frac{1}{V}\int_M h_\tau\omega^2 \\ \leq\ & \log C_\alpha - \frac{1}{V}\int_M h_\tau\omega^2. \end{aligned}$$

For any small $\epsilon > 0$, choose $\alpha = 2/3 - \epsilon$, then

$$\sup_M \varphi \leq -\frac{1+\epsilon}{2V}\int_M \varphi\omega^2 + C. \tag{5.13}$$

[5] This implies that $\mathbf{F}_{\omega_\tau}$ is proper along the solutions of (5.1).

[6] In fact, by using the L^2-estimate for $\overline{\partial}$-operator, one can show that $\alpha(M) \geq 2/3$ for any M in $\mathcal{M}_m$ ($3 \leq m \leq 8$).

Hence, by (5.4), (5.6), (5.7), (5.8) and (5.13), (5.5) follows from the following

$$\frac{\sqrt{-1}}{2V}\int_M \partial\psi\wedge\overline{\partial}\psi\wedge\omega_\tau \geq -10\epsilon\inf_M\varphi - C, \tag{5.14}$$

where $\psi = \frac{1}{l}\log(\sum_{i=0}^N \lambda_i^2||\sigma_i||_\tau^2)$.

Lemma 5.4. *Define $D_k = \cap_{i\geq k}\sigma_i^{-1}(0)$ and a by requiring that D_{a+1} contains a divisor while D_a is isolated. Then for any δ, there is a uniform C_δ such that for any $k \geq a$,*

$$\frac{\sqrt{-1}}{2V}\int_M \partial\psi\wedge\overline{\partial}\psi\wedge\omega_\tau \geq \frac{2\pi(1-\delta)(-\log\lambda_k)}{l^2\,V}\int_{D_k}\omega_\tau + \frac{4}{l}\log\lambda_{k+1} - C_\delta. \tag{5.15}$$

Proof. Put

$$\psi_k = \frac{1}{l}\log(\sum_{i\leq k}\lambda_i^2||\sigma_i||_\tau^2 + \sum_{i>k}||\sigma_i||_\tau^2).$$

Then $\psi_k \geq \psi \geq \psi_k + \frac{2}{l}\log\lambda_{k+1} - \log 2$, so

$$\frac{\sqrt{-1}}{2V}\int_M \partial\psi\wedge\overline{\partial}\psi\wedge\omega_\tau \geq \frac{\sqrt{-1}}{2V}\int_M \partial\psi_k\wedge\overline{\partial}\psi_k\wedge\omega_\tau + \frac{4}{l}\log\lambda_{k+1} - 2\log 2.$$

Then (5.15) follows from estimating the integral on the right of the above inequality. □

The following can be proved by using L^2-estimate of $\overline{\partial}$-operator.

Lemma 5.5. *Let a be defined in last lemma. Then for η sufficiently small, we have*

$$\int_M \frac{1}{(\sum_{i\geq a}||\sigma_i||_\tau^2)^{\frac{1}{l}(\frac{2}{3}+\eta)}}\omega_\tau^2 \leq C. \tag{5.16}$$

It follows from (5.16) and the concavity of logarithm that

$$-\eta\inf_M\varphi \leq -\frac{2}{l}\log\lambda_a + C. \tag{5.17}$$

Then (5.14) follows from (5.15) and (5.17) by choosing $\epsilon \ll \eta$. Theorem 5.1 is proved.

Remark. It will be an interesting problem when there is a Kähler-Einstein orbifold metric on complex surfaces with isolated quotient singularities. Part of the proof of Theorem 5.1 can be extended for this problem, but there are substantial new difficulties due to presence of singularities.

Remark. In the course of proving Theorem 5.2, we proved that the moduli space of Kähler-Einstein surfaces with positive scalar curvature can be compactified by adding Kähler-Einstein orbifolds with isolated singularities. Furthermore, certain constraints were shown on the singularities (cf. [Ti3]). It was conjectured that the moduli space can be compactified by adding only Kähler-Einstein orbifolds with rational double points.

6. The K-stability

In Section 1, we have seen that a moment map on an algebraic manifold associated to a linear group action has a unique zero along an orbit if and only if this orbit is stable in Geometry Invariant Theory. Further, this algebraic stability can be checked through the Hilbert criterion. We will discuss its analogue in the case of extremal Kähler metrics in this section. Our results here were already presented in [Ti2] for Kähler-Einstein metrics. Their extension to general extremal Kähler metrics is straightforward.

First let us introduce the K-stability (cf. [Ti2]). Let $(M, c_1(L))$ be a compact Kähler manifold polarized by a line bundle L.[7] By the Kodaira embedding theorem, for m sufficiently large, a basis of $H^0(M, L^m)$ gives an embedding $\phi_m : M \mapsto \mathbb{C}P^{N_m}$, where $N_m + 1 = \dim_{\mathbb{C}} H^0(M, L^m)$. Any other basis gives an embedding of the form $\sigma \cdot \phi_m$, where $\sigma \in G = \mathbf{SL}(N_m + 1, \mathbb{C})$. Let $|| \cdot ||$ be a Hermitian metric on L such that its curvature form ω is a Kähler metric. Then for any $\sigma \in G$, there is a unique function φ_σ such that

$$\phi_m^* \sigma^* (|| \cdot ||_{FS}^{\frac{2}{m}}) = e^{-\varphi_\sigma} || \cdot ||^2, \tag{6.1}$$

where $|| \cdot ||_{FS}$ is a Hermitian metric on the hyperplane bundle over $\mathbb{C}P^{N_m}$ whose curvature form is the Fubini-Study metric.

Lemma 6.1. *Let $G_0 = \{\sigma_t\}_{t \in \mathbb{C}^*}$ be any one-parameter algebraic subgroup of $\mathbf{SL}(N_m + 1, \mathbb{C})$. Then*

$$- \lim_{t \to -\infty} \frac{d}{dt} \mathbf{T}_\omega (\varphi_{\sigma(e^t)}) \tag{6.2}$$

exists. We will denote it by $\mathbf{w}(M, L, G_0)$, called the weight of G_0 associated to (M, L) and often abbreviated as $\mathbf{w}(G_0)$.

[7] It is also possible to define the K-stability for general polarized Kähler manifolds, for instance, through geodesic rays in the space of Kähler metrics. It was explained in [AT] that geodesic rays correspond to special degenerations used in the definition of the K-stability for algebraic manifolds.

This lemma was proved in [DT] under some further conditions on G_0, but it actually holds for general G_0 and can be proved with slightly more efforts. It was also proved in [DT] that $\mathbf{w}(G_0)$ is equal to the generalized Futaki invariant of $\mathrm{Re}(\sigma'(1))$ on $\lim_{t\to 0}\sigma(t)(\phi_m(M))$ if the limit is a normal variety. Furthermore, if the limit has only quotient singularities [DT] or M is a hypersurface [Lu], then the generalized Futaki invariant can be computed in terms of localization.

Let G_0 be any one-parameter subgroup of $\mathbf{SL}(N+1)$. Here for simplicity, we write $N = N_m$. Then there is a coordinate system $z_0, \cdots, z_N$ in which $\sigma(t) \in G_0$ is represented by

$$\mathbf{diag}(t^{\alpha_0}, \cdots, t^{\alpha_N}),\ t \in \mathbb{C}^*,$$

where $\alpha_0 \leq \cdots \leq \alpha_N$ are integers. Define a height $\mathbf{h}(M, G_0)$ or simply $\mathbf{h}(G_0)$ to be the smallest $\alpha_N - \alpha_i$ such that $\phi_m^* z_i, \cdots, \phi_m^* z_N$ have no common zero on M. It is easy to show that $\varphi_{\sigma(t)} - \sup_M \varphi_{\sigma(t)}$ is unbounded on $t \in \mathbb{C}^*$ if and only if $\mathbf{h}(G_0) > 0$.

Definition 2. *We say that M is K-semistable with respect to L^m if $\mathbf{w}(G_0) \geq 0$ for any one-parameter algebraic subgroup $G_0 \in \mathbf{SL}(N_m+1)$. We say that M is K-stable with respect to L^m if it is K-semistable and $\mathbf{w}(G_0) > 0$ whenever $\mathbf{h}(G_0) > 0$.*[8]

Definition 3. *Let (M, L) be a compact Kähler manifold polarized by L. We say that (M, L) is asymptotically K-stable if M is K-stable with respect to L^m for sufficiently large m.*

Let $\mathrm{Aut}(M, L)$ be the group of all automorphisms which can be lifted to L.

Conjecture 6.2. *Let (M, L) be a compact Kähler manifold polarized by a line bundle L. For simplicity, assume that $\mathrm{Aut}(M, L)$ is finite. Then M admits a Kähler metric with constant scalar curvature and Kähler class $c_1(L)$ if and only if (M, L) is asymptotically K-stable.*[9]

Let us give some evidence for this conjecture. The same arguments in [Ti2] show that for any m, $\mathbf{T}_\omega(\varphi_\sigma)$ is proper over $\sigma \in \mathbf{SL}(N_m+1)$ if and only if M is K-stable with respect to L^m. On the other hand, we have [Ti2]

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[8] The K-stability was first introduced in [Ti2] for Fano manifolds in a slightly weaker sense. But I believe that two definitions should be equivalent

[9] In general, (M, L) admits a constant scalar curvature metric if and only if M is asymptotically weakly K-stable. The weak K-stability means that M is semistable and $\mathbf{w}(G_0) > 0$ for any G_0 with $\mathbf{h}(G_0) > 0$ and transverse to identity component of $\mathrm{Aut}(M, L)$ in a suitable sense.

Theorem 6.3. *Let (M,ω) be a Kähler-Einstein manifold with $c_1(M) = \lambda[\omega] = \lambda c_1(L)$ for some constant λ. Assume that the Lie algebra of holomorphic vector fields is trivial. Then (M,L) is asymptotically K-stable.*

If $\lambda \leq 0$, then the converse is automatically true because of the main theorem in [Ya]. If $\lambda > 0$, the converse can be derived from the following partial C^0-estimate proposed in [Ti6]. To see this, we assume that (M,ω) is a n-dimensional compact Kähler manifold with $[\omega] = c_1(M)$. Then there is a unique h_ω such that

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial} h_\omega, \ \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

The existence of Kähler-Einstein metrics is equivalent to the solvability of the following complex Monge-Ampere equation at $t = 1$,

$$(\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \ \varphi \in P(M,\omega), \tag{6.3}$$

where $t \in [0,1]$. It can be proved that the set of $t \in [0,1]$ contains 0 and is open. The more difficult part is to prove the closedness. As we said in the last section (cf. [Ti1]), the closedness follows from an a prior C^0-estimate for solutions of (6.3). In general, such an estimate does not exist. In our case, we need to use the stability to get this C^0-estimate. The stability condition can be used only after we can approximate a solution of (6.3) by φ_σ for some $\sigma \in \mathbf{SL}(N_m + 1)$ for m sufficiently large. This is exactly what the partial C^0-estimate gives. Here is the general conjecture on the partial C^0-estimate [Ti6].

Conjecture 6.4. *For any $\epsilon > 0$, there are constants m_0, δ_m $(m \geq 1)$ and an integer A, depending only n and ϵ, such that if (M,ω) satisfies $\mathrm{Ric}(\omega) \geq \epsilon\omega$ and $c_1(M) = [\omega]$, then for $m = Al \geq m_0$,*

$$\frac{1}{m}\log(\sum_{i=0}^{N_m} ||S_i||^2) \geq \delta_m, \tag{6.4}$$

where $\{S_i\}$ is an orthonormal basis of $H^0(M, K_M^{-m})$ with respect to any inner product induced by ω and a Hermitian metric $||\cdot||$ on K_M^{-1} with ω as its curvature form. Note that $||\cdot||$ induces a Hermitian metric on K_M^{-m}, still denoted by $||\cdot||$.

Remark. It will be an interesting question to find the smallest A. Is A possible to be one?

If φ is a solution of (6.3) at t, then the equation implies that ω_φ has its Ricci curvature no less than $t > 0$. If the above conjecture is true, then for some m

sufficiently large, $\varphi - \varphi_\sigma$ is bounded for some $\sigma \in \mathbf{SL}(N_m + 1)$. The stability implies that F_ω is proper along these φ_σ and we get the required estimate.

Remark. The existence of Kähler metrics is also related to the stability of underlying manifolds in Geometric Invariant Theory. We will discuss it in detail elsewhere. We refer the readers to [Ti2] for discussions in the case of Kähler-Einstein metrics.

Theorem 6.3 can be used to prove that certain Fano 3-folds without holomorphic vector fields do not admit any Kähler-Einstein metrics (cf.[Ti2]). In fact, these manifolds do not have any extremal Kähler metrics.

7. Extremal Metrics vs Complex Structures

Not every Kähler manifold admits an extremal metric. However, I still believe that essentially all Kähler manifolds admit extremal Kähler metrics. Let us explain this in the following. Let $\mathcal{K}_n$ be the set of all polarized Kähler manifolds (M, Ω) of dimension n. First let us introduce a partial ordering on $\mathcal{K}_n$.

Definition 4. *Let (M, Ω) and (M', Ω') be two polarized Kähler manifolds. We say $(M', \Omega') \succ (M, \Omega)$ if there is a fibration $\pi : Y \mapsto \Delta \subset \mathbb{C}$ from a polarized manifold (Y, Φ) onto the unit disk Δ such that $(\pi^{-1}(0), \Phi|_{\pi^{-1}(0)}) = (M, \Omega)$ and there is a fiber-preserving biholomorphic map ψ from $Y \backslash \pi^{-1}(0)$ onto $M' \times \Delta^*$ such that $\psi^*\Omega' = \Phi$.*

For any (M, Ω), we denote by $[M, \Omega]$ the subset of all polarized manifolds $(M'\Omega') \succ (M, \Omega)$ of $\mathcal{K}_n$. Clearly, if $(M', \Omega') \succ (M, \Omega)$, then $[M', \Omega'] \subset [M, \Omega]$. In this way, we can decompose $\mathcal{K}_n$ into a disjoint union of subsets. Further, one should be able to show that each subset is of the form $[M, \Omega]$ for some polarized Kähler manifold (M, Ω). Such a (M, Ω) will be called a minimal polarized Kähler manifold. We conjecture that *each minimal polarized Kähler manifold admits an extremal Kähler metric within the given Kähler class.*[10] One may have the following uniqueness theorem in strong form: *Each* $[M, \Omega]$ *contains at most one polarized manifold with an extremal metric.*

A similar conjecture was proposed before for Kähler manifolds polarized by the first Chern class. It is believed that every minimal polarized $(M, c_1(M))$ admits a unique Kähler-Ricci soliton, that is a Kähler metric ω such that $\mathrm{Ric}(\omega) = \omega + L_X\omega$ for some holomorphic vector field X on M. The uniqueness part was proved in [TZ].

[10]In this conjecture, one may need to allow minimal polarized (M, Ω) to be a Kähler varieties with only mild singularities, such as singularities of quotient type.

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