HOUSTON JOURNAL OF MATHEMATICS, Volume 1, No. 1, 1975.

A NOTE ON THE CARTAN INTEGERS

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With each pair of roots of a Lie algebra there is defined a normalized inner product which turns out to be an integer. These are called the Cartan Integers. In this note we show how these integers may be defined for a compact, connected, semisimple Lie group without recourse to Lie algebra. They turn out to be intersection numbers of certain subgroups or, alternatively, they are the degrees of certain maps of the circle S^1 to itself. Along the way we get the roots and coroots of a group without Lie algebras or even the adjoint representation.

Our main motivation for doing without Lie algebras is to be able to define and study these things for H-spaces.

1. Coroots and roots. Let G be a compact connected Lie group of dimension n and rank r. Let T be a maximal torus in G, N its normalizer and W = N/T the Weyl group. Thus we have

$$0 \longrightarrow T \longrightarrow N \xrightarrow{p} W \longrightarrow 1,$$

and, since T is abelian, W acts on T (by conjugation; i.e., if $w \in W$ and $t \in T$, then w sends t to ntn^{-1} where $n \in p^{-1}(w)$).

Let $w \in W$ be a reflection. Precisely this means that if $w: T \to T$ is lifted to the covering space

$$\widetilde{w}: \mathbf{R}^r \to \mathbf{R}^r,$$

then $\widetilde{w}$ is reflection in an (r-1)-hyperplane H in $\mathbf{R}^r$. Let $T' = p^{-1}(w)$. Since $w^2 = 1 = p(T)$ we see that if $x \in T'$ then $x^2 \in T$.

DEFINITION 1: For the reflection $w \in W$ we let V be the set of fixed points of the action of w on T.

Clearly V is a closed subgroup of T, and since a neighborhood of 1 in V is the image of a neighborhood of 0 in H under the covering map, we see that dim V = r-1. The identity component V^o of V is thus an (r-1)-torus.

Let ϕ: G → G be the squaring map

$$\phi(x) = x^2.$$

Then $\phi(T') \subset V$. For if $x \in T'$ we have $x^2 \in T$ and

$$x(x^2)x^{-1} = x^2.$$

DEFINITION 2: $Q = \{t \in T \mid xtx^{-1} = t^{-1} \text{ for } x \in T'\}$.

Q is a closed subgroup of T. Clearly every element of $Q \cap V$ is its own inverse; i.e., it is a square root of 1 in T. Thus $Q \cap V$ is finite and since dim V = r-1 we see that dim Q = 1. Thus Q is topologically a disjoint finite set of circles.

LEMMA 1: *$\phi(T')$ is a component of V.*

PROOF: We look at the fibers of ϕ. If $\phi(x) = \phi(y)$; i.e., $x^2 = y^2$, we see that

$$x(xy^{-1})x^{-1} = x^2y^{-1}x^{-1} = y^2y^{-1}x^{-1} = yx^{-1} = (xy^{-1})^{-1}$$

so $xy^{-1} \in Q$. Thus the fibers of this (smooth) map ϕ have dimension one so $\phi(T')$ is a closed connected (r-1)-manifold in V and is thus a component of V.

Let Q^o be the identity component of Q, so that Q^o is a circle group. Let $\alpha \in Q^o$ be the square root of 1 ($\neq 1$) in Q^o. Obviously $\alpha \in V$.

LEMMA 2: *V has at most two components.*

PROOF: If V had three or more components Q^o and V would have at least three points in common, but $Q^o \cap V = \{1, \alpha\}$ since each point of this intersection must be a square root of 1 in Q^o.

We have noted that the identity component V^o of V is an (r-1)-torus. By taking an appropriate line segment in $\mathbf{R}^r$ and projecting to T by the covering map we can get a circle subgroup C of T such that $C \cap V^o = \{1\}$. Then C and V^o together are a "coordinate system" for T. In particular, any homomorphism of T to another group is determined by specifying its restriction to C and V^o. We are going to define a homomorphism $\theta: T \to S^1$ (a *root*) corresponding to the reflection $w \in W$.

Let $U = V^o \cup \phi(T')$. This is going to be the kernel of the homomorphism θ. There are two cases:

(i) $\alpha \in U$ and

(ii) $\alpha \notin U$.

(Case (i) occurs when the simisimple group associated to the root is S^3; case (ii) occurs when it is SO(3) .) In case (i) the homomorphism is to be such that the combined map $Q^o \xrightarrow{\text{incl.}} T \xrightarrow{\theta} S^1$ has degree 2. In case (ii) this combined map is to have degree 1. To accomplish these things we specify the degree on C. In case (i) we may have

$\alpha \in V^o$ in which case the degree on C is to be 1 (since Q^o already "goes around twice"). If $\phi(T') \neq V^o$ then α must be in $\phi(T')$ and then θ is to have degree 2 on C (since Q^o goes around only once). Now case (ii) can occur only if $\phi(T') = V^o$ (and $\alpha \notin V^o$), so in this case the degree on C is to be 1.

DEFINITION 3: The homomorphism $\theta: T \to S^1$ just defined is the *root* associated with the reflection w. The homomorphic inclusion $Q^o \to T$ is the *coroot* associated with the reflection w.

Since U is the kernel of θ it is a subgroup of T. Let P be the centralizer of U; i.e.,

$$P = \{ x \in G | \, xu = ux \text{ for all } u \in U \}.$$

LEMMA 3: *P is connected.*

PROOF: If U is connected it is a torus and the centralizer of a torus is connected ([1], Prop. 4.25). If Z( ) denotes "centralizer of", then we have

$$Z^o(U) \subseteq Z(U) \subseteq Z(U^o).$$

The end terms are connected and all terms are closed manifolds. Furthermore they all have dimension r+2 because $Z^o(U)/U$ and $Z(U^o)/U^o$ both have rank 1 and hence dimension 3 (recall dim U = r-1). Thus all three terms are equal and Z(U) is connected.

LEMMA 4: *Let $j: T \to P$ be the inclusion and let $\beta: S^1 \to T$ be the inclusion of the coroot Q^o. Then in $\pi_1(P)$ we have*

$$2j_{\#}[\beta] = 0.$$

PROOF: Let ρ be a path in P from 1 to a point x in T′ (P is connected and, clearly, $T' \subset P$). Then conjugation by $\rho(t)$ gives a homotopy from the identity to conjugation by x which is the inverse map on Q^o. This proves the lemma.

2. Cartan Integers. Let $w_1, w_2, \ldots, w_k$ be all of the reflections in W. These generate W, and, if G is semisimple some r of them generate W. For each w_i we have defined in section 1 a coroot $\beta_i: S^1 \to T$ and a root $\theta_i: T \to S^1$. Thus for each pair w_i, w_j we have a map

$$S^1 \xrightarrow{\beta_i} T \xrightarrow{\theta_j} S^1$$

and we denote by d_{ij} the degree of $\theta_j \beta_i$.

Also for each w_i we have defined the degree d_i of $[\beta_i]$ in $\pi_1(G)$. By Lemma 4 d_i is either 1 or 2.

DEFINITION 4: For each pair w_i, w_j of reflections we have an integer $C_{ji} = d_i d_{ij}$. These are called the Cartan Integers of G.

THEOREM: *If G is semisimple and* $w_1, w_2, \ldots, w_r$ *generate W, then the Cartan Integers defined above agree with the usual Cartan Integers.*

PROOF: Let $\langle,\rangle$ be an inner product on $\mathcal{L}(T)^*$ (the dual of the Lie algebra $\mathcal{L}(T)$ of T) which is invariant under the action of the Weyl group ([1], page 116). The Cartan Integers are defined by

$$C_{ji} = 2\,\frac{\langle \theta_i, \theta_j \rangle}{\langle \theta_i, \theta_i \rangle}$$

or, equivalently,

$$C_{ji} = \theta_j(\tau_i)$$

where τ_i is the i^{th} basic translate ([2], page 41).

First we note that the degree d_{ij} defined above is the same as the intersection number of Q_i^o and U_j which we denote by

$$Q_i^o \cap U_j.$$

Now it suffices to prove the theorem in case G is simple. We first prove the theorem when G is also simply connected. Then we want to show that $C_{ji} = d_{ij} = Q_i \cap U_j$. In $\mathcal{L}(T)$ we have the line $\mathcal{L}(Q_i)$ and the hyperplane $\mathcal{L}(U_j)$, $\mathcal{L}(U_j)$ being in a family of equally spaced parallel hyperplanes; viz. $d\theta_j^{-1}(Z)$, with $\mathcal{L}(U_j) = d\theta_j^{-1}(0)$.

Let $\bar{\mathcal{L}}(Q_i)$ be the segment of $\mathcal{L}(Q_i)$ from 0 out just far enough to cover Q_i once under the exponential map. We see that $Q_i \cap U_j$ (at least in absolute value) is just the number of the hyperplanes in $d\theta_j^{-1}(Z)$ crossed by $\bar{\mathcal{L}}(Q_i)$ (including one end of $\bar{\mathcal{L}}(Q_i)$). (Note that if θ_i and θ_j are perpendicular roots then $Q_i \subset U_j$ and the intersection number is zero.)

Since we are assuming $\pi_1(G) = 0$ we have that the basic translates $\{\tau_k\}$ generate the integer lattice I. So in the simply connected case $\bar{\mathcal{L}}(Q_i) = \tau_i$. Thus we have

$$|Q_i \cap U_j| = |\theta_j(\tau_i)| = |C_{ji}|.$$

To get the signs we need to consider orientations. We orient the Q_i's and U_i's so that each $Q_i \cap U_i$ is positive. Then $Q_i \cap U_j$ and $Q_j \cap U_i$ will have the same sign (since Q_i is perpendicular to U_i). So we orient the Q_i's and U_i's by starting at any end of the Dynkin diagram and orient each new Q_j and U_j to give the right sign. Since the Dynkin diagram contains no cycles this is always possible.

Now suppose $\pi_1(G)$ is not necessarily zero. Then the number of points of the integer lattice lying on τ_i (counting one end) is just the order of Q_i in $\pi_1(G)$, and we must multiply the intersection number by this to get $\theta_j(\tau_i) = C_{ji}$.

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Received May 2, 1975