IN Variant MEAN CHARACTERIZATIONS
OF AMENABLE C*-ALGEBRAS

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Abstract. It is shown that unital amenable and strongly amenable C*-algebras can be characterized by the existence of a right invariant mean on a certain subspace of \( \ell_\infty(H) \), where \( H \) is the unitary group. A fixed-point theorem for amenable C*-algebras is obtained.

1. Introduction. The following result of Haagerup ([11, Theorem 2.1]) is the main motivation for this paper. Let \( R \) be a von Neumann algebra with isometry semigroup \( S \). Let \( \text{Bil}^\sigma(R) \) be the space of bounded bilinear forms on \( R \) which are separately, \( \sigma \)-weakly continuous on \( R \). Then \( R \) is injective if and only if there exists a mean \( m \) on \( S \) such that for all \( V \in \text{Bil}^\sigma(R) \) and all \( a \in R \), we have

\[
\int_S V(au^*, v)dm(v) = \int_S V(u^*, av)dm(v).
\]

Haagerup uses (1) in his proof that nuclear C*-algebras are amenable. (Another proof which avoids (1) and the use of approximate finite dimensionality has been given by Effros ([7, 8].)

Since injectivity and amenability are equivalent for \( R \), it is natural to ask if (1) can be interpreted as asserting the existence of a suitably invariant mean on a subspace of \( \ell_\infty(S) \) associated with \( \text{Bil}^\sigma(R) \). A corresponding question, of course, can be asked for amenable unital C*-algebras with the unitary group \( H \) in place of \( S \). In both cases, the answer is positive, and this opens the way to interpreting operator algebra amenability in terms of a classical right invariant mean (RIM), replacing the more complex notion of virtual diagonal by the more accessible and better understood notion of invariant mean.

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In this paper, we examine the $C^*$-case; the author plans to discuss the von Neumann case in another paper.

Let $A$ be a unital $C^*$-algebra, and $Bil(A)$ be the Banach space of bounded bilinear forms on $A$. Let $Bil_{22}(A)$ be the subspace of completely bounded bilinear forms in $Bil(A)$. Recall that a $C^*$-algebra $A$ is called amenable if there exists a virtual diagonal for the definition, see (8) below, for $A$. This notion was introduced by Johnson ([14]); in his memoir [15], Johnson introduced the notion of a strongly amenable $C^*$-algebra, and Haagerup ([11]) has observed that this notion is characterized by the existence of a special kind of virtual diagonal. (See Proposition 4.) Our results can be interpreted as asserting that such virtual diagonals can be taken as arising from a RIM on spaces of functions on $H$.

We start by showing that amenability for $A$ is associated with $Bil_{22}(A)$. We show in Proposition 2 that $A$ is amenable if and only if there exists a virtual diagonal on $Bil_{22}(A)$. This is the analogue of the result of Effros ([7, 8]) that a von Neumann algebra $R$ is amenable if and only if there exists a virtual diagonal on the subspace of completely bounded elements of $\ell_\infty(R)$.

We then turn to the subspaces of $\ell_\infty(H)$ which support a RIM when $A$ is amenable or strongly amenable. These spaces are quite simple to define. We define a map $\Delta : Bil(A) \rightarrow \ell_\infty(H)$ by

$$\Delta(V)(u) = V(u^*, u) \quad (u \in H).$$

Let $B(A)$ be $\Delta(Bil(A)) \subset \ell_\infty(H)$. The subspace $B_{22}(A)$ of $B(A)$ is defined: $B_{22}(A) = \Delta(Bil_{22}(A))$. Both $B_{22}(A), B(A)$ are invariant and contain 1. The main result of this paper is the following (Theorem 1, Theorem 2):

(a) $A$ is strongly amenable if and only if there exist a RIM on $B(A)$
(b) $A$ is amenable if and only if there exists a RIM on $B_{22}(A)$

In the final part of the paper, we prove a fixed-point theorem for amenable $C^*$-algebras. One would expect such a theorem to exist in view of the well known fact in the theory of amenable groups that such theorems are associated with invariant means on subspaces of $\ell_\infty(G)$. Bunce ([2, 3]) proved such a theorem for strongly amenable $C^*$-algebras, and this easily follows by amenable group techniques using the invariant mean result (a) above. We prove a fixed-point theorem associated with (b) above, using
the notion of weakly completely bounded $A$-modules. Here, a locally convex space $E$ which is a unital $A$-module is called weakly completely bounded if, for every $F \in E^*$ and every $x \in E$, the bilinear map $F_x$, where

$$F_x(a,b) = F(axb)$$

is a completely bounded bilinear form on $A$. This result emphasizes a theme of the paper that amenability for $C^*$-algebras is a completely bounded phenomenon. (An elegant account of the theory of completely bounded maps is given in [21].)

2. Amenable $C^*$-algebras and invariant means. Let $A$ be a unital $C^*$-algebra. Then $Bil(A) = (A \hat{\otimes} A)^*$ is the Banach space of bounded bilinear forms on $A \times A$. The norm on $Bil(A)$ can also be given by:

$$\|V\| = \sup\{|V(a,b)| : a, b \in A, \|a\| = \|b\| = 1\}.$$ 

Let $Bil_{22}(A)$ be the subspace of completely bounded elements of $Bil(A)$. So a bilinear form $V$ on $A \in Bil_{22}(A)$ if it is completely bounded as a bilinear map $V : A \times A \to \mathbb{C}$. (See, for example, [4].) For our purposes, such forms can be conveniently specified as follows. Let $(a, \xi) \to a\xi$ be the universal representation of $A$ on its Hilbert space $\mathcal{H}$. Then (cf. [8]) $V \in Bil(A)$ is completely bounded if and only if there exist $\xi, \eta$ in $\mathcal{H}$ and $T \in B(\mathcal{H})$ such that for all $a, b \in A$,

$$V(a,b) = aTb\xi \cdot \eta.$$ 

We note that such a representation of $V$ has been extended to the non-scalar case by Christensen and Sinclair ([4])—an elegant account of this is given in [22]. We also note that there are subspaces $Bil_{ij}(A)$ for $i, j \in \{1, 2\}$ which arise naturally and are discussed in [16]. These do not play a role in the present paper but are significant in the von Neumann case.

We will require another characterization of completely bounded bilinear forms in the proof of Proposition 5. For $u \in A \otimes A$, define $\|u\|_{22} \geq 0$ as follows:

$$\|u\|_{22} = \inf \left\{ \| \sum a_j a_j^* \|^{\frac{1}{2}} \| \sum b_j b_j^* \|^{\frac{1}{2}} : u = \sum a_j \otimes b_j \right\}.$$ 

In [10, 8], the map $\|\cdot\|_{22}$ is shown to be a norm on $A \otimes A$ and is called the Haagerup norm. It is also shown that a bilinear form on $A \times A$ is completely bounded if and only if it is bounded on $A \otimes A$ for the Haagerup norm. Recent accounts of the Haagerup norm and other operator space norms are given in [1, 9].

Let $R = A^{**}$ be the enveloping von Neumann algebra of $A$ realised on $H$. It follows from [13, Theorem 2.3] that each $V \in Bil(A)$ extends uniquely, without change of norm to an element, also denoted $V$, of $Bil^\sigma(R)$. (The latter space is defined in the Introduction.) So we can identify $Bil(A)$ with $Bil^\sigma(R)$ and can identify $Bil_{22}(A)$ with the appropriate subspace of $Bil^\sigma(R)$. This subspace is denoted by $Bil_{22}(R)$. The elements $V$ of $Bil_{22}(R)$ are also given by the formula (3) with $a, b$ allowed to lie in $R$.

We recall that $Bil(A)$ is a dual Banach $A$-module with actions

\[ xV(a, b) = V(a, bx) \quad Vx(a, b) = V(xa, b). \]

Direct checking in (3) shows that $Bil_{22}(A)$ is an invariant subspace of $Bil(A)$. There is another useful module action $\circ$ which we postpone till later ((21)).

The next result seems to be well known, but for convenience we give the simple proof.

**Proposition 1.** Let $V \in Bil_{22}(R)$. Then the maps $x \to Vx^*$, $x \to xV$ are strong operator-norm continuous from $R$ into $Bil^\sigma(R)$.

**Proof:** If $V$ is as in (3), then

\[ \|Vx^* - Vy^*\| \leq \|\xi\| \|T\| \|x\eta - y\eta\|, \tag{6} \]
\[ \|xV - yV\| \leq \|\eta\| \|T\| \|x\xi - y\xi\|. \tag{7} \]

The result now follows. \qed

We now discuss amenability for $A$. This involves the notion of a virtual diagonal for $A$. Let $\pi : A \hat{\otimes} A \to A$ be the multiplication map. An element $M$ of $(A \hat{\otimes} A)^{**}$ is called a **virtual diagonal** if, for all $a \in A$:

\[ aM = Ma \quad (a \in A) \quad \pi^{**}(M) = 1. \tag{8} \]

The algebra $A$ is called **amenable** if there exists a virtual diagonal for $A$. 
The subspace $\pi^*(A^*)$ can easily be identified with $A^*$ by associating $\pi^*(\phi) = V_\phi$ with $\phi$, where

$$V_\phi(a, b) = \phi(ab).$$

It is simple to check that the natural $A$-module structure of $A^*$ coincides with the submodule structure that it inherits as a subspace of $Bil(A)$, and that $A^* \subset Bil_{22}(A)$. Further, regarding $A \subset A^{**}$, the second equality of (8) becomes:

$$M|A^* = 1.$$  

The first equality of (8) can be reformulated:

$$(9) \quad v^*Mv = M \quad (v \in H).$$

Indeed, (9) is equivalent to $Mv = vM$ for all $v \in H$, which in turn is equivalent to $aM = Ma$ for all $a \in A$ since $H$ spans $A$.

Virtual diagonals for submodules of $Bil(A)$ containing $A^*$ are defined in the obvious way.

There is a natural $H$-action on $Bil(A)$ associated with the module actions of (5) and (21). We define:

$$(10) \quad v.V(a, b) = V(v^*a, bv) \quad V.v(a, b) = V(av^*, vb).$$

Clearly, $Bil(A)$ is a Banach $H$-module. Using (5), we have

$$(11) \quad v.V = vVv^*.$$  

Since $Bil_{22}(A)$ is an $A$-submodule of $Bil(A)$, it follows that it is also an $H$-submodule.

Note also that for $\phi \in A^*$, we have $v.V_\phi = V_{v\phi v^*}$, and since $v^*v = 1$, we also have $V_\phi.v = V_\phi$. In particular, $A^*$ is an $H$-submodule of $Bil(A)$. In the dual $H$-module action on $A^{**}$, where we regard $A \subset A^{**}$, we have $v.1 = 1 = 1.v$ for all $v \in H$.

The actions of (10) of course dualise to give an $H$-module action on $(Bil(A))^*$. These actions will be denoted by:

$$(v, M) \rightarrow v.M \quad (M, v) \rightarrow M.v.$$  

Note that, using (11):

$$(12) \quad M.v = v^*Mv.$$  

The following proposition shows that for amenability for $A$, we require a virtual diagonal only on $Bil_{22}(A)$.  


Proposition 2. The $C^*$-algebra $A$ is amenable if and only if there exists a virtual diagonal on $\text{Bil}_{22}(A)$.

Proof: Suppose that there exists a virtual diagonal $M$ on the space $\text{Bil}_{22}(A)$. Let $G$ be the unitary group of $R$. Let $V \in \text{Bil}_{22}(R)$. Let $v \in G$ and $\{u_\alpha\}$ be a net in $H$ such that $u_\alpha \to v$ strongly in $R$ ([23, Theorem 2.3.3]). Now since the strong and weak operator topologies coincide on $G$ ([26, p. 84]), it follows that the map $u \to u^*$ is strong operator continuous, and using Proposition 1 and the triangular inequality, we have $\|u_\alpha^* V u_\alpha - v^* V v\| \to 0$. Hence

$$v M v^*(V) = \lim u_\alpha M u_\alpha^*(V) = M$$

and so identifying $\text{Bil}(A)$ with $\text{Bil}_{22}(R)$, we see that $M$ is a virtual diagonal on $\text{Bil}_{22}(R)$. By a result of [7, 8], $R$ is amenable and so injective. So $A$ is amenable (=nuclear) by the well-known result (due to Connes and Choi-Effros): $A$ is nuclear if and only if $A^{**}$ is injective.

The rest of the proof is trivial.

We now discuss invariant means on groups. Let $G$ be a group. Convolution on $\ell_1(G)$ dualises to give a $G$-action on $\ell_\infty(G)$:

$$(f s_0)(s) = f(s_0 s) \quad (s_0 f)(s) = f(s_0 s)$$

for all $s_0, s \in G$ and all $f \in \ell_\infty(G)$. A right invariant mean (RIM) on $\ell_\infty(G)$ is a mean (=state) on $\ell_\infty(G)$ which is right invariant under the right dual $G$-action on $(\ell_\infty(G))^*$. So a mean $m$ on $G$ is a RIM if and only if

$$m(s f) = m(f)$$

for all $f \in \ell_\infty(G)$ and all $s \in G$. The group $G$ is called right amenable if there exists a RIM on $\ell_\infty(G)$. Left amenability and two-sided amenability for $G$ are defined in the obvious ways. Recent accounts of amenability theory are given in [19, 24, 25].

A subspace $X$ of $\ell_\infty(G)$ is called left invariant if $sf \in X$ for all $f \in X$ and all $s \in G$. If $X$ is left invariant and contains 1, then a RIM on $X$ is an element $m \in X^*$ satisfying $m(1) = 1 = \|m\|$ and $m(s f) = m(f)$ for all $f \in X$ and all $s \in G$. Similarly we can define left invariant means (LIM's) for right invariant unital subspaces of $\ell_\infty(G)$. We will be concerned with
invariant means on subspaces of $\ell_\infty(H)$. Since $H$ is so large and (usually) highly non-commutative, it is rarely going to be amenable, and we are interested in the existence of invariant means on certain smaller, though significant, subspaces of $\ell_\infty(H)$.

The subspaces $B_{22}(A)$ and $B(A)$ that will concern us are associated with the following map $\Delta : \text{Bil}^\sigma(A) \rightarrow \ell_\infty(S)$:

\[ \Delta(V)(v) = V(v^*, v). \]

We define the following subspaces of $\ell_\infty(G)$:

\[ B(A) = \Delta(\text{Bil}(A)) \quad B_{22}(A) = \Delta(\text{Bil}_{22}(A)). \]

We give $H$ the relative $\sigma(A, A^*)$ (i.e. the weak) topology. Then ([20]) $H$ is a topological group. The invariant, unital $C^*$-algebra $\text{LUC}(H)$ (resp $\text{RUC}(H)$) is the set of functions $f \in \ell_\infty(H)$ such that the map $s \rightarrow sf$ (resp $s \rightarrow fs$) is norm continuous. Since $1 \in H$, each $f \in \text{LUC}(H)$ is continuous.

We now collect some simple facts relating to the spaces $B(A)$ and $B_{22}(A)$.

**Proposition 3.** (a) The map $\Delta$ is an $H$-equivariant, norm decreasing, linear map onto $B(A)$. Further, the spaces $B(A), B_{22}(A)$ are invariant subspaces of $\ell_\infty(H)$, and $\Delta(A^*) = C1$.

(b) $\Delta^*(m)$ is a virtual diagonal for every RIM $m$ on $B(A)$.

(c) Both subspaces $B(A)$ and $B_{22}(A)$ are closed under the complex conjugation map $f \rightarrow \overline{f}$.

(d) $1 \in B_{22}(A) \subset \text{LUC}(H)$.

**Proof:** (a) For $V \in \text{Bil}(A)$, $u, v \in H$, we have

\[ \Delta(V.v)(u) = V.u(u^*, u) = V(u^*v^*, vu) = \Delta(V)(vu) = \Delta(V)v(u) \]
\[ \Delta(v.V)(u) = v.V(u^*, u) = V(v^*u^*, uv) = \Delta(V)(uv) = v\Delta(V)(u) \]

so that $\Delta$ is $H$-equivariant. Obviously, $\Delta$ is norm-decreasing and linear. Since $\Delta$ is equivariant and the spaces $\text{Bil}(A), \text{Bil}_{22}(A)$ are $H$-modules, it follows that $B(A)$ and $B_{22}(A)$ are invariant. Finally, if $\phi \in A^*$, then

\[ \Delta(V_\phi)(v) = V_\phi(v^*, v) = \phi(v^*v) = \phi(1) \]
so that $\Delta(V_\phi) = \phi(1)1$.

(b) If $m$ is a RIM on $B(A)$, then, for $v \in H$, $\phi \in A^*$, using (a), (12) and (15):

$$v^*\Delta^*(m)v = \Delta^*(m)v = \Delta^*(mv) = \Delta^*(m)$$

$$\Delta^*(m)(V_\phi) = m(\Delta(V_\phi)) = \phi(1) = 1(V_\phi).$$

So using (9), $\Delta^*(m)$ is a virtual diagonal.

(c) For $V \in Bil(A)$, define $V^* \in Bil(A)$ by:

$$V^*(a,b) = \overline{V(b^*,a^*)}.$$ 

Then $\Delta(V) = \Delta(V^*)$, and $B(A)$ is closed under complex-conjugation. The same property holds for $B_{22}(A)$: we observe that the conjugate $\overline{f}$ of $f \in B_{22}(A)$ is obtained by replacing the $T$ in (3) by its adjoint and interchanging $\xi$ and $\eta$.

(d) Since $A^* \subset Bil_{22}(A)$, it follows from (a) that $1 \in B_{22}(A)$. If $V \in Bil_{22}(A)$, then for $u, v \in H$, 

$$\|u\Delta(V) - v\Delta(V)\| \leq \|Vu^* - Vv^*\| + \|uV - vV\|.$$ 

Now $u_\delta \rightarrow u$ weakly in $A$ if and only if $u_\delta \rightarrow u$ in the strong operator topology of $R = A^{**}$. It follows from (16) and Proposition 1 that $\Delta(V) \in LUC(H)$. 

The next result gives an invariant mean characterization of amenable $C^*$-algebras.

**Theorem 1.** The following statements are equivalent:

(a) $A$ is amenable 
(b) there exists a RIM on $B_{22}(A)$ 
(c) there exist a RIM on $LUC(H)$

**Proof:** The equivalence of (a) and (c) follows by [20]. Since $B_{22}(A) \subset LUC(H)$ by (d) of Proposition 3, we have that (c) implies (b). Now suppose that (b) holds and let $R = A^{**}$. Let $m$ be a RIM on $B_{22}(A)$. By (3), each $f \in B_{22}(A)$ is of the form $f_{T\xi\eta}$, where:

$$f_{T\xi\eta}(u) = u^*Tu\xi.\eta.$$
For \( g \in B_{22}(A) \), define \( g^* \in \ell_\infty(H) \) by setting \( g^*(u) = f(u^{-1}) \), and let \( Y = \{ g^* : g \in B_{22}(A) \} \). Then \( m^* \), where \( m^*(g^*) = m(g) \), is a left invariant mean (LIM) on \( Y \). Now \( H \) is strongly dense in the unitary group \( G \) of \( R \), and \( G \) is a topological group in the strong operator topology ([12]). From (17), each \( g = f_{T, \xi, \eta} \) extends uniquely by continuity to a continuous function \( g' \) on \( G \)-just allow \( u \) in (17) to belong to \( G \). Let \( Y' = \{ g' : g \in Y \} \). Then \( Y' \subset RUC(G) \): this is easily checked as in Proposition 1. (See also [12].) As in [20, Proposition 1], there exists an LIM on \( Y' \), and a result of de la Harpe (cf [19, p. 78]) gives that \( R \) is injective. Hence \( A \) is nuclear and so amenable. So (b) implies (a). 

We will show in Theorem 2 below that strong amenability for \( A \) is equivalent to the existence of a RIM on \( B(A) \). For convenience, we write \( \overline{co} S \) for the weak* closure of the convex hull of a subset \( S \) of a Banach space dual \( X^* \), and for any Banach space \( X \), will regard \( X \subset X^{**} \).

Recall that ([15]) the algebra \( A \) is called strongly amenable if, whenever \( X \) is a unital Banach \( A \)-module and \( D : A \to X^* \) is a derivation, then there exists \( \alpha_0 \) in \( \overline{co}\{u^*D(u) : u \in H\} \) such that \( D(a) = a\alpha_0 - \alpha_0 a \) for all \( a \in A \). Haagerup ([11, Lemma 3.4 seq]) remarks that the following characterization of strong amenability holds.

**Proposition 4.** The C*-algebra \( A \) is strongly amenable if and only if there exists a virtual diagonal \( M \) in \( \overline{co}\{u^* \otimes u : u \in H\} \).

**Theorem 2.** The C*-algebra \( A \) is strongly amenable if and only if there exists a RIM on \( B(A) \).

**Proof:** Suppose that \( m \) is a RIM on \( B(A) \). From (b) of Proposition 3, \( \Delta^*(m) \) is a virtual diagonal for \( A \). For \( u \in G \), let \( \hat{u} \in \ell_\infty(G)^* \) be given by: \( \hat{u}(\phi) = \phi(u) \). It is easily checked that \( \Delta^*(\hat{u}) = u^* \otimes u \). Since \( m \) is in \( \overline{co}\{\hat{u} : u \in G\} \), it follows that \( \Delta^*(m) \) is in \( \overline{co}\{u^* \otimes u : u \in G\} \) in \((A^{\otimes}A)^*\). By Proposition 4, \( A \) is strongly amenable.

Conversely, suppose that \( A \) is strongly amenable, and let \( M \) be as in Proposition 4. Then there exists a net \( \{f_\delta\} \) in \( P(G) \) such that in the weak* topology

\[
\left( \sum_{u \in G} f_\delta(u)(u^* \otimes u) \right) \to M.
\]
In particular, if $V \in Bil(A)$, then

$$(18) \quad (\sum f_{\delta}(u)\hat{u})(\Delta(V)) = (\sum f_{\delta}(u)(u^* \otimes u))(V) \to M(V).$$

(19) Define $m(\Delta(V)) = M(V)$. Then $m$ is well-defined and is a mean on $B(A)$. Let $v \in G$. By (9) and (11), $M(v.V) = M(V)$. Further, by (a) of Proposition 3, $\Delta(v.V) = v\Delta(V)$.

It follows that $m$ is a RIM. \[ \square \]

We conclude by discussing how some characterizations of amenable and strongly amenable $C^*$-algebras can be interpreted as fixed-point or extension theorems of classical amenability type. In particular, using modules with a certain completely bounded property, we will prove a fixed-point theorem for amenable $C^*$-algebras which fills a gap in the literature.

We begin with strongly amenable $C^*$-algebras for which the literature is more complete. In [2, 3], Bunce gives six characterizations of strongly amenable $C^*$-algebras. An account of the results of Bunce is given in [25, Chapter 2]. Three of these can be interpreted as fixed-point theorems for the unitary group $H$ analogous to the classic fixed-point theorem of Day. A fourth can be interpreted as a stronger version of a result in [17] which is valid for amenable Banach algebras. (See [6] for an elegant proof.) The remaining two give invariant extension characterizations. All of these characterizations can be readily proved using Theorem 2 and the approach of the fixed-point theorems for amenable groups ([19, (2.16) ff.]). We are particularly interested in the following fixed-point theorem of Bunce.

**Theorem 3.** The $C^*$-algebra $A$ is strongly amenable if and only if whenever $X$ is a unital Banach $A$-module and $S$ is weak*-closed convex subset of $X^*$ such that $v^*Sv = S$ for all $v \in H$, then there exists $g \in S$ such that $v^*gv = g$ for all $v \in H$.

Bunce gives two characterizations of amenable $C^*$-algebras. As in the strongly amenable case, one of these is of the Khelemskii-type and the other is an invariant extension result. Both have Banach algebra versions, the extension version appearing in [18]. We now discuss a fixed-point theorem for amenable $C^*$-algebras corresponding to Theorem 3.

Let $E$ be a locally convex space which is a unital $A$-module. The module $E$ is called weakly completely bounded if, for every $F \in E^*$ and
every $x \in E$, the bilinear map $F_x$, where
\begin{equation}
F_x(a, b) = F(axb),
\end{equation}
is a completely bounded bilinear form on $A$.

If $X$ is a completely bounded normed $A$-module in the sense of ([5]),
then $X$ is weakly completely bounded.

We can make the Banach space $A \hat{\otimes} A$ into a unital Banach $A$-module
with the actions $\circ$:
\begin{equation}
a \circ (b \otimes c) = b \otimes ac \quad (b \otimes c) \circ a = ba \otimes c.
\end{equation}
These actions are discussed in [2, 3].

**Proposition 5.** Let $E = Bil_{22}(A)$ with the relative weak*-topology which
it inherits as a subspace of $(A \hat{\otimes} A)^*$. Then $E$ is a weakly completely bounded
$A$-submodule of $(A \hat{\otimes} A)^*$ under the dual actions $\circ$ for (21).

**Proof:** The fact that $a \circ V \circ a' \in E$ for $a, a' \in A$ and $V \in E$
follows by expressing $V$ in the form of (3) and checking that
\begin{equation}
a \circ V \circ a'(b \otimes c) = b(aTa')c\xi \eta
\end{equation}
which is also of the form of (3). So $E$ is an $A$-module. Now the dual of $E$
is just $(A \hat{\otimes} A)/E^\perp$. If $F$ is the restriction of $(b \otimes c)$ to $E$, then using (22),
\begin{equation}
F_V(a \otimes a') = aTa'(c\xi \eta)\eta
\end{equation}
which is also of the form of (3). Now let $F$ be a general element of $A \hat{\otimes} A$.
We can write
\[ F = \sum b_i \otimes c_i \left( \sum ||b_i|| \ ||c_i|| < \infty \right). \]
Let $u = \sum a_j \otimes a_j' \in A \otimes A$. Then using (23),
\[ |F_V(u)| = \left| \sum_{i,j} a_j T a_j' c_i \xi \eta b_i^* \eta \right| \]
\[ \leq ||T|| \sum_i \sum_j ||a_j' c_i \xi || \ ||a_j^* b_i^* \eta || \]
\[ \leq ||T|| \sum_i \left( \sum_j ||a_j' c_i \xi ||^2 \right)^{1/2} \left( \sum_j ||a_j^* b_i^* \eta ||^2 \right)^{1/2} \]
\[ = ||T|| \sum_i \left( \left( \sum_j a_j^* a_j' \right) c_i \xi \eta \cdot \left( \sum_j a_j a_j^* \right) b_i \eta b_i^* \eta \right)^{1/2} \]
\[ \leq ||T|| \sum_i \left( \sum_j a_j^* a_j' \right)^{1/2} ||c_i \xi || \ ||\sum_j a_j a_j^* \right)^{1/2} ||b_i \eta ||. \]
It follows that

$$|F_V(u)| \leq \|T\| \left( \sum_i \|b_i\| \|c_i\| \|\xi\| \|\eta\| \|u\|_2 \right).$$

Hence $V$ is bounded for the Haagerup norm (4) and so is completely bounded.

\[ \square \]

**Theorem 4.** The $C^*$-algebra $A$ is amenable if and only if $H$ has the fixed-point property: whenever $X$ is a weakly completely bounded $A$-module and $S$ is a non-empty, compact, convex subset of $X$ such that $v^*Sv = S$ for all $v \in H$, then there exists $h \in S$ such that $v^*hv = h$ for all $v \in H$.

**Proof:** Suppose that $A$ is amenable. Let $X$ be a weakly completely bounded $A$-module and $S$ be a non-empty compact, convex, invariant subset of $X$. Let $\alpha \in X^*$ and $g \in S$. Then in the notation of (20), $\alpha_g \in B_{l2}(A)$. By Theorem 1, there exists a RIM $m$ on $B_{l2}(A)$. Since $\Delta(\alpha_g) \in B_{l2}(A)$, we can define $h : X^* \to C$ by:

$$h(\alpha) = \int_H \Delta(\alpha_g)dm = \int_H \alpha(u^*gu)dm(u).$$

Clearly, $h$ is linear, and by approximating $m$ by convex combinations of point masses, we can, using the invariance of $S$ and regarding the elements of $S$ as functionals on $X^*$, find a net $\{g,\}$ in $S$ such that $g, \to h$ pointwise on $X^*$. Since $S$ is weakly compact, it follows that $h \in S$. Now for $v \in H$,

$$v\Delta(\alpha_g)(u) = \Delta(\alpha_g)(uv) = (v\alpha u^*)(u^*gu)$$

so that

$$h(v\alpha u^*) = m(v\Delta(\alpha_g)) = m(\Delta(\alpha_g)) = h(\alpha).$$

Hence $v^*hv = h$ for all $v \in H$.

Conversely suppose that $A$ has the fixed-point property of the theorem. The amenability of $A$ will follow from Theorem 1 once we have shown that $B_{l2}(A)$ has a RIM. For this purpose, we will use [19, Theorem (2.13)]. The latter asserts the existence of a RIM provided we can show that $B_{l2}(A)$ is right introverted (defined below) and that for each $\phi \in B_{l2}(A)$, there exists a constant function in the pointwise closure of the set

$$C_\phi = \text{co}\{\phi v : v \in H\}.$$
We will establish these two facts in turn.

Let $m$ be a mean on $H$. Let $V \in Bil_{22}(A)$. We wish to define an element $V * m \in Bil_{22}(A)$ such that for $v \in H$, we have $V * \delta_v = V.v$ (as in (10)). Indeed, for $a, b \in A$, we have $b V a \in Bil_{22}(R)$, and can thus define

$$V * (a, b) = \int_H V(au^*, ub)dm(u).$$

(24)

It is obvious that $V * m \in Bil(A)$ and that $V * \delta_v = V.v$. In fact, by approximating $m$ weak* by convex combinations of elements $\delta_v$, we see that if $V$ satisfies (3), then $V * m$ satisfies (3) with $T$ replaced by some ultraweak cluster point of the set $\text{co}\{v^*Tv : v \in H\}$ in $B(H)$. So $V * m \in Bil_{22}(R)$.

A left invariant subspace $Y$ of $\ell_\infty(H)$ is called right introverted ([19, (2.6)]) if for each $F \in \ell_\infty(H)$ and $b \in Y$, we have $F \in Y$, where $c_F(v) = F(v\phi)$. We claim that $Y = B_{22}(A)$ is right introverted. Indeed, if $m$ and $V$ are as above, then

$$A(V)m(v) = m(vA(V)) = \int_v A(V)(u)dm(u) = \int_v V(v^* u^*, uv)dm(u) = V * (v^*, v) = A(V * m)(v).$$

Since $\ell_\infty(H)^*$ is spanned by means, it follows that $B_{22}(A)$ is right introverted.

We now turn to the second fact to be established. By Proposition 5, $Bil_{22}(A)$ is a weakly completely bounded $A$-module with the weak*-topology and the action dual to that in (21). Note that as in (11), $V.v = v^* o V o v$. Let $V \in Bil_{22}(A)$. Let $S = \overline{\text{co}}\{v^* o V o v : v \in H\}$ in $(A \widehat{\otimes} A)^*$. As in the preceding paragraph, $S$ is a weak*-compact convex subset of $Bil_{22}(A)$. Of course, $v^* o S o v = S$. By hypothesis, there exists $W \in S$ such that $W.v = W$ for all $v \in H$. Further there exists a net $\{g_\delta\}$ in $P(H)$ such that $V.g_\delta \to W$. Then

$$\Delta(V)g_\delta(u) = V.g_\delta(u^* \otimes u) \to W(u^*, u)$$

so that $\Delta(V)g_\delta \to \Delta(W)$ pointwise on $H$. Since

$$\Delta(W)(u) = W(u^*, u) = W.u(1, 1) = W(1, 1)$$
it follows that $\Delta(W)$ is a constant function.

This completes the proof. $\square$

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REFERENCES


