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ON SIMULTANEOUS TRIANGULARIZATION OF COLLECTIONS OF OPERATORS

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0. Introduction. A collection $\mathcal{S}$ of bounded linear operators is (simultaneously) *triangularizable* if there is a maximal chain of subspaces all of which are invariant under the operators in $\mathcal{S}$. (Note that the chain is required to be maximal as a subspace chain, not merely maximal as a chain of invariant subspaces of $\mathcal{S}$.)

There has been a lot of work on triangularizability: see the references below. In the present paper we discuss several results on triangularizability of algebras and semigroups of operators. Most arguments that are used to prove triangularizability depend on results that insure the existence of invariant subspaces; the key idea is obvious, but worth stating.

If M and N are subspaces and $N \subset M$, and $\mathcal{G}$ is the algebra of all operators leaving both M and N invariant, then there is a natural homomorphism $\Phi : \mathcal{G} \to B(M/N)$ defined by $\Phi(A)(x+N) = Ax + N$. In the Hilbert space case, we can identify M/N with $K = M \cap N^{\perp}$; under this identification the map Φ is given by $\Phi(A) = PA|K$ where P is the orthogonal projection onto K. If $\mathcal{S} \subset \mathcal{G}$, then there is a natural correspondence between the invariant subspaces of $\mathcal{S}$ that lie between M and N and the invariant subspaces of the collection $\Phi(\mathcal{S})$ of operators on K. This explains how theorems on the existence of invariant subspaces yield theorems on triangularizability.

Section 1 concerns the existence of invariant subspaces for algebras that contain a bilateral shift and an injective backward bilateral weighted shift. We show how the problem of existence of invariant subspaces for such algebras is related to the periodicity of the weights of the weighted shift. We also show how the problem is related to harmonic analysis and weak*-closed translation-invariant unital subalgebras of $\ell^\infty(\mathbb{Z})$. Such algebras must be annihilators of closed ideals in the Banach algebra $\ell^1(\mathbb{Z})$.

Section 2 deals with a stronger notion of triangularizability. A collection $\mathcal{S}$ of operators is *hypertriangularizable* if there is a maximal subspace chain of invariant subspaces for $\mathcal{S}$ whose projections generate a masa (maximal abelian self-adjoint algebra). Unlike triangularizability, hypertriangularizability is not preserved under similarity [**13**, **14**]. Theorem 2.4 states that a direct integral of weakly closed algebras is hypertriangularizable if and only if almost every one of the algebras is hypertriangularizable. The analogous result for triangularizability is false (Lemma 2.3), since every direct integral of algebras with respect to a nonatomic measure is triangularizable.

Section 3 deals with the question of the triangularizability of a Banach algebra $\mathcal{G}$ of operators for which $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, where $\mathrm{Rad}\mathcal{G}$ is the Jacobson radical of $\mathcal{G}$. If such an algebra is weakly closed and contains a masa, then it is triangularizable (Theorem 3.1); also if such an algebra is contained in a direct integral of algebras acting on finite-dimensional spaces, then it is hypertriangularizable (Theorem 3.4).

In section 4 we consider collections of nilpotent operators. We show that an algebra of nilpotents is triangularizable if the index of nilpotence is bounded (Theorem 4.1), which is implied by the algebra being uniformly closed (Corollary 4.2). We also generalize the example in [**9**] of an algebra of nilpotents whose norm closure is semisimple and transitive.

1. Algebras generated by bilateral shifts. Recall that an algebra of operators on a Hilbert space H is *transitive* if the only subspaces that are invariant under the algebra are the trivial ones $\{0\}$ and H. The *transitive algebra problem* asks if every transitive algebra must be dense in the weak (or, equivalently, strong) operator topology. The answer is known to be affirmative if the algebra includes a masa or the unilateral shift operator [**1**]; many other special cases are also known (see [**20**]). It is somewhat surprising that it is not known if the presence of the bilateral shift in a transitive algebra implies the density of the algebra. We consider the case in which the algebra contains the bilateral shift and a weighted backwards bilateral shift.

Let $\{e_n : -\infty < n < \infty\}$ be an orthonormal basis for H, and define S and A by $Se_n = e_{n+1}$ and $Ae_n = w_n e_{n-1}$, where $\{w_n\}$ is a bounded sequence of non-zero complex numbers. Then S is the bilateral shift and A is a backwards bilateral weighted shift. Let $\mathcal{G}(S, A)$ be the weakly closed unital algebra generated by S and A.

Let D be the diagonal operator defined by $De_n = w_{n+1}e_n$. Then $A = DS^*$. Since $S^*S = 1$, it is clear that $D = AS\epsilon\mathcal{G}(S, A)$.

If $H = L^2(m)$, where m is normalized Lebesgue measure on the unit circle in the complex plane, then we may take $e_n(e^{i\theta}) = e^{in\theta}$ for $-\infty < n < \infty$, and S becomes multiplication by $z = e^{i\theta}$. The invariant subspaces of S are known to have the form $\chi_E L^2(m)$ or qH^2, where E is a measurable subset of the circle, χ_E is the characteristic function of E, q is a function of unit modulus a. e., and H^2 is the closed linear span of $\{e_n : n \geq 0\}$. In the cases in which $S^*\epsilon\mathcal{G}(S, A)$, the only possible invariant subspaces of $\mathcal{G}(S, A)$ are those of the form $\chi_E L^2(m)$. It seems unlikely that many diagonal operators leave such subspaces invariant.

The following theorem shows that few diagonal operators are reduced by such subspaces. We first require a technical lemma.

Lemma 1.1. *Suppose that A^nS^n and S^nA^n, which are diagonal for all n, always agree at fixed i and $i+p$ for some $p \geq 1$; i. e., if B is any one of the above operators, then $(Be_i, e_i) = (Be_{i+p}, e_{i+p})$. Then $\{w_k\}$ is periodic of period p.*

Proof: Calculation shows

$$S^nA^ne_k = (w_kw_{k-1}\cdots w_{k+1-n})e_k$$

and

$$A^nS^ne_k = (w_{k+1}w_{k+2}\cdots w_{k+n})e_k.$$

For $n = 1$, the hypothesis implies $w_i = w_{i+p}$. By induction, with $n = k+1$, we obtain $w_{i-k} = w_{i-k+p}$ and $w_{i+k} = w_{i+k+p}$. □

Theorem 1.2. *Suppose that $\mathcal{D}$ is a self-adjoint collection of diagonal operators and E is a measurable subset of the circle with $0 < m(E) < 1$. The following are equivalent.*

(1) $\mathcal{D}$ leaves $\chi_E L^2(m)$ invariant

(2) There is a positive integer p such that

(a) *the sequence of eigenvalues of each operator in $\mathcal{D}$ is periodic of period p, and*

(b) $E = e^{2\pi i/p}E$ a. e.

Proof: (1) $\Rightarrow$ (2). Statement (1) implies that the weakly closed algebra $\mathcal{B}$ generated by $\{1, S, S^*\} \cup \mathcal{D}$ has a non-trivial invariant subspace, namely, $\chi_E L^2(m)$. Clearly, the statement (2) (a) is implied by the periodicity of the sequence of eigenvalues of each Hermitian diagonal operator in $\mathcal{B}$. Suppose that $T \in \mathcal{B}$ and, $T = T^*$, and T is diagonal. Since we can replace T by $\lambda + T$ for some scalar λ, we can assume that none of the eigenvalues of T is 0. It follows from the preceding lemma, replacing A with TS^*, that if the sequence of eigenvalues of T is not periodic, then, for each pair i, j of distinct integers, there is a Hermitian diagonal operator B in $\mathcal{B}$ such that $(Be_i, e_i) \neq (Be_j, e_j)$. Since $\mathcal{B}$ is weakly closed it contains the spectral projections of all of its Hermitian elements. Hence, for each pair i, j of distinct integers, there is a diagonal projection P in $\mathcal{B}$ such that $Pe_i = 0$ and $Pe_j = e_j$. It follows that every diagonal projection is a strong limit of projections in $\mathcal{B}$, which implies that $\mathcal{B}$ contains all diagonal operators. This contradicts the fact that $\mathcal{B}$ leaves $\chi_E L^2(m)$ invariant; whence (2) (a) is proved.

To prove (2) (b), we can assume that p is the smallest period of the sequences of eigenvalues of all the diagonal operators in $\mathcal{B}$. The preceding argument implies that $\mathcal{B}$ contains all diagonal operators whose sequence of eigenvalues has period p. In particular, $\mathcal{B}$ contains the projection P onto the subspace spanned by $\{e_{np} : n \in \mathbb{Z}\}$. If $\omega = e^{2\pi i/p}$ is a primitive p^{th} root of unity, then, for each f in $L^2(m)$, we have

$$(Pf)(z) = \frac{1}{p} \sum_{k=0}^{p-1} f(\omega^k z).$$

(This is easily checked on the basis vectors $\{e_n : n \in \mathbb{Z}\}$.) Since P and $1-P$ are both in $\mathcal{B}$ and leave $\chi_E L^2(m)$ invariant, it follows that P commutes with multiplication by χ_E. Applying this commutativity relation to the constant 1 function, we obtain $P\chi_E = \chi_E$. It follows from the above formula for P that $E = \omega E = e^{2\pi i/p} E$. $\square$

Corollary 1.3. *Suppose that $\mathcal{D}$ is a self-adjoint collection of diagonal operators and $\mathcal{B}$ is the von Neumann algebra generated by $\mathcal{D}$ and S. Then $\mathcal{B} = B(H)$ if and only if there is no positive integer p such that the sequence of eigenvalues of every element of $\mathcal{D}$ is periodic with period p.*

An operator T is *reductive* if each invariant subspace of T reduces T. Note that every Hermitian operator is reductive and every diagonal unitary operator is reductive **[22]**. In the case in which the diagonal operator D is reductive, the algebra $\mathcal{G}(S, A)$ is the von Neumann algebra generated by S, S^*, D, and D^*.

Corollary 1.4. *If D is reductive, then $\mathcal{G}(S, A)$ is the von Neumann algebra generated by $\{D, S\}$. Furthermore $\mathcal{G}(S, A) = B(H)$ if and only if the weight sequence $\{w_n\}$ is not periodic.*

Proof: Since $D \in \mathcal{G}(S, A)$, we know that $\mathcal{G}(S, A)$ contains the weakly closed algebra generated by D and 1. However, since every normal operator is reflexive **[23]**, this latter algebra equals AlgLat$D$. Since D is reductive, $D^* \in$ AlgLat$D$. Hence $D^* \in \mathcal{G}(S, A)$. The proof will be complete if we can show that $S^* \in \mathcal{G}(S, A)$. For each positive integer n, define $\varphi_n : \mathbb{C} \to \mathbb{C}$ by

$$\varphi_n(z) = \begin{cases} 1/z \text{ if } |z| \geq 1/n \\ n \text{ if } |z| < 1/n. \end{cases}$$

Since none of the eigenvalues of D is 0, $\varphi_n(D)D \to 1$ in the strong operator topology. Thus $\varphi_n(D)A = \varphi_n(D)DS^* \to S^*$ strongly. Hence $S^* \in \mathcal{G}(S, A)$. $\square$

Part of the preceding proof can be carried out, in spirit, under more general conditions than the reductivity of D. A compact subset K of the complex plane *separates* 0 *from* ∞ if there is no piecewise continuously differentiable path joining 0 to ∞ that doesn't intersect K, except possibly at 0. We let $D(a, r)$ denote the open disk in $\mathbb{C}$ with center a and radius r.

Lemma 1.5. *If K is a compact subset of the plane that does not separate 0 from ∞, then there is a sequence $\{p_n(z)\}$ of complex polynomials that are uniformly bounded on K such that $p_n(0) = 0$ for each n and $p_n(z) \to 1$ on $K \setminus \{0\}$.*

Proof: It is clear that the condition $p_n(0) = 0$ for each n can be replaced by $p_n(0) \to 0$. (Simply replace each p_n by $p_n - p_n(0)$.) It follows from the hypothesis that there is a bounded simply connected region G containing $K \setminus \{0\}$ and whose boundary is a simple rectifiable Jordan curve containing 0. There is a continuous function $\varphi : G^- \to D(-1, 1)^-$ with $\varphi(0) = 0$ whose restriction to G is the Riemann map. Mergelyan's theorem says that φ is a uniform limit of polynomials. This reduces the problem to the case $K = D(-1, 1)^-$. However, $\varphi_n(z) = \frac{nz}{nz-1}$ is 0 at 0, and converges pointwise to 1 on $\mathbb{C} \setminus \{0\}$. Also on $K, |\varphi_n(z)| = |1 + \frac{1}{nz-1}| \leq 1 + 1 = 2$. Since each φ_n is a uniform limit of polynomials on K, the lemma is proved. $\square$

The following lemma provides examples in which $S^* \in \mathcal{G}(S, A)$, but in which $\mathcal{G}(S, A)$ is possibly not self-adjoint.

Lemma 1.6. *If the spectrum of D does not separate 0 from ∞, then $\mathcal{G}(S, A)$ is the weakly closed algebra generated by $\{1, D, S, S^*\}$.*

Proof: Using the preceding lemma, we can choose a sequence $\{p_n(z)\}$ of polynomials that is uniformly bounded on $\sigma(D)$ such that $p_n(0) = 0$ for each n and $p_n(z) \to 1$ on $\sigma(T) \setminus \{0\}$. Write $p_n(z) = q_n(z)z$. Since none of the eigenvalues of D is zero, it follows that $q_n(D)D = p_n(D) \to 1$ in the strong operator topology. Thus $q_n(D)A = q_n(D)DS^* \to S^*$ in the strong operator topology. Thus $S^* \in \mathcal{G}(S, A)$. $\square$

Corollary 1.7. *If the spectrum of D does not separate 0 from ∞, then any transitive algebra containing S and A is weakly dense in $B(H)$.*

Proof: It follows from the preceding lemma that any unital weakly closed algebra containing S and D must contain S^*. Since the weakly closed algebra generated by S and S^* is a masa, it follows from Arveson's theorem **[1]** that such a transitive algebra must be $B(H)$. $\square$

The problem of the transitivity of $\mathcal{G}(S, A)$ is related to the following problem from harmonic analysis: does there exist a closed ideal of $\ell^1(\mathbb{Z})$ whose annihilator is a proper aperiodic unital subalgebra of $\ell^\infty(\mathbb{Z})$? A weak*-closed linear subspace $\mathcal{S}$ of $\ell^\infty(\mathbb{Z})$ is translation invariant (in both directions) if and only if its preannihilator in $\ell^1(\mathbb{Z})$ is an ideal. Thus the question about ideals of $\ell^1(\mathbb{Z})$ is equivalent to the question of the existence of weak*-closed unital subalgebras of $\ell^\infty(\mathbb{Z})$ that are translation-invariant. For each positive integer p, the p-periodic sequences in $\ell^\infty(\mathbb{Z})$ form a proper weak*-closed unital translation invariant subalgebra of $\ell^\infty(\mathbb{Z})$. The question of whether there are any others is related to our transitivity problem.

Proposition 1.8. *The following are equivalent.*

(1) There is a proper unital weak-closed translation invariant subalgebra of $\ell^\infty(\mathbb{Z})$ that is not periodic.*

(2) There is an aperiodic sequence $\{w_n\}$ of complex numbers such that $\sigma(D)$ does not separate 0 from ∞ and $\mathcal{G}(S, A)$ is not transitive.

Proof: (1) $\Rightarrow$ (2). It follows from (1) that we can choose $\{w_n\}$ so that $\{w_n\}$ is aperiodic and such that the weak*-closed translation invariant unital algebra generated by $\{w_n\}$ is not $\ell^\infty(\mathbb{Z})$. By adding an appropriate constant sequence to $\{w_n\}$, if necessary, we can assume that $\sigma(D)$ does not separate 0 from ∞. It follows from Lemma 1.6 that $\mathcal{G}(S,A)$ is the weakly closed algebra generated by $\{1, D, S, S^*\}$. It is easily seen that the diagonal sequence $\{(Be_n, e_n)\}$ of any polynomial B in $1, D, S, S^*$ is in the translation invariant unital algebra generated by $\{w_n\}$. Thus the set of diagonal operators in $\mathcal{G}(S,A)$ is not the set of all diagonal operators. Thus $\mathcal{G}(S,A) \neq B(H)$. It follows from Corollary 1.7 that $\mathcal{G}(S,A)$ is not transitive.

(2) $\Rightarrow$ (1). Suppose that $\{w_n\}$ is aperiodic, $\sigma(D)$ does not separate 0 from ∞, and $\mathcal{G}(S,A)$ is not transitive. Let $\mathcal{D}$ be the sequences of eigenvalues of the operators in $\mathcal{G}(S,A)$ that are diagonal with respect to the basis $\{e_n\}$. Since S and S^* are both in $\mathcal{G}(S,A)$, it is clear that $\mathcal{D}$ is translation invariant (consider S^*TS and STS^* for a diagonal operator T in $\mathcal{G}(S,A)$). Furthermore, $\mathcal{D}$ is a weak*-closed unital subalgebra that is not periodic (since $\{w_n\} \in \mathcal{D}$). If $\mathcal{D} = \ell^\infty(\mathbb{Z})$, then $\mathcal{G}(S,A)$ contains all diagonal operators. This would imply that the only invariant subspaces for $\mathcal{G}(S,A)$ would be the ranges of diagonal projections. Clearly, none of these is both nontrivial and invariant under S and S^*. Thus $\mathcal{D} \neq \ell^\infty(\mathbb{Z})$, and (1) is proved. □

2. Direct integrals and hypertriangularizability. In this section we consider hypertriangularizability. Although hypertriangularizability is defined in terms of masa-generating chains of invariant projections, the following lemma states that on a separable Hilbert space the condition that the projections form a chain is unnecessary (see [**20**, Lemma 7.11 and its proof]).

Lemma 2.1. *An algebra $\mathcal{G}$ of operators on a separable Hilbert space is hypertriangularizable if Lat$\mathcal{G}$ contains a subset that generates a masa.*

Our main results concern a weakly closed algebra $\mathcal{G}$ that is a direct integral of a measurable family $\{\mathcal{G}_\omega : \omega \in \Omega\}$ of unital algebras, i.e., $\mathcal{G} = \int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ where $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$ and μ is a σ-finite measure on Ω. The questions that interest us deal with the relationship between the triangularizability or hypertriangularizability of $\mathcal{G}$ and the corresponding property for almost all of the $\mathcal{G}_\omega$'s. We first consider the case in which the measure μ is discrete. In this case direct integrals become direct sums. The proof of the following lemma is elementary and is left to the reader.

Lemma 2.2. *A direct sum of algebras is triangularizable (hypertriangularizable) if and only if each summand is triangularizable (hypertriangularizable).*

If the measure μ is σ-finite, then every direct integral is a direct sum of direct integrals in which the measures are either nonatomic or consist of a single atom. The preceding lemma reduces both of our questions to the case in which the measure is nonatomic. Moreover, the preceding lemma and [4] imply that we can assume that there is a fixed Hilbert space M such that $H_\omega = M$ for every ω in Ω. The question of triangularizability in this case is independent of the $\mathcal{G}_\omega$'s.

Lemma 2.3. *If μ is nonatomic, then $\int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ is triangularizable.*

Proof: There is a natural embedding $\pi : L^\infty(\mu) \to B(H)$ defined by $\pi(\varphi)(f)(\omega) = \varphi(\omega)f(\omega)$ for each ω in Ω and each f in $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$. The map π is a unital *-homomorphism and is continuous with respect to the weak*-topology on $L^\infty(\mu)$ and the weak operator topology on $B(H)$. Moreover, ran (π) is the commutant of $\int_\Omega^\oplus B(M)d\mu(\omega)$. Choose a maximal chain $\mathcal{C}$ of projections in $L^\infty(\mu)$. Then $\pi(\mathcal{C}) = \{\pi(P) : P \in \mathcal{C}\}$ is a maximal chain of projections in $B(H)$ that are not only invariant for, but reduce $\int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$. $\square$

The question of hypertriangularizability of a direct integral is more difficult than that of triangularizability, but the answer is the exact analogue of the situation for direct sums.

Theorem 2.4. *A direct integral $\mathcal{G} = \int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ on the separable Hilbert space $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$ is hypertriangularizable if and only if almost every $\mathcal{G}_\omega$ is hypertriangularizable.*

The proof of the preceding theorem requires a lemma involving a topological property of the spaces of sequences of projections that generate a masa. Suppose that M is a separable space, and $\mathcal{P}(M)$ is the set of projections in $B(M)$ with the strong operator topology. Since $\mathcal{P}(M)$ is a complete separable metric space, so is a countable cartesian product of copies of $\mathcal{P}(M)$. Since the set $\mathcal{P}$ of sequences of projections in $\mathcal{B}(M)$ that are pairwise commuting is closed in the cartesian product of countably many copies of $\mathcal{P}(M)$, it follows that $\mathcal{P}$ is also a complete separable metric space. We are interested in the set $\mathcal{Q}$ of sequences in $\mathcal{P}$ that generate a masa.

Clearly, $\mathcal{Q}$ is not generally closed (except when dim $M < \infty$). However, we shall prove that $\mathcal{Q}$ is always a G_δ in $\mathcal{P}$.

Lemma 2.5. *There is an equivalent metric on $\mathcal{Q}$ that makes $\mathcal{Q}$ a complete separable metric space.*

Proof: It follows from [**2**, Thm. 3.1.2] that the lemma is equivalent to the statement that $\mathcal{Q}$ is a G_δ subset of $\mathcal{P}$. First note that there is a sequence $\{p_n\}$ of finite polynomials in infinitely many variables $x_1, x_2, x_3, \ldots$, such that, for each $P = (P_1, P_2, \ldots)$ in $\mathcal{P}$, the set $\{p_n(P) : n \geq 1\}$ is strongly dense in the unit ball of the von Neumann algebra generated by $\{P_1, P_2, \ldots\}$. To see this let F be the field of complex numbers with rational real and imaginary parts, and, for each positive integer N, let $\mathcal{F}_N$ be the linear combinations over F with coefficients of modulus less than or equal to 1 of the 2^N products $y_1 y_2 \ldots y_N$ where each y_k is x_k or $(1 - x_k)$. It is clear that, for each P in $\mathcal{P}, \{f(P) : f \in \mathcal{F}_N\}$ is dense in the unit ball of the von Neumann algebra generated by $\{P_1, P_2, \ldots, P_N\}$. Thus $\{P_1, P_2, \ldots\} = \cup_n \mathcal{F}_n$ has the required properties.

Note that, for each n, the map $P \mapsto p_n(P)$ is continuous with respect to the strong operator topology on $B(M)$. Let d be a metric on the unit ball of $B(M)$ that yields the weak operator topology. For each positive integer k, let E_k be the set of those P in $\mathcal{P}$ for which there is a contraction T commuting with $\{P_1, P_2, \ldots\}$ such that $d(T, p_n(P)) \geq 1/k$ for $n = 1, 2, \ldots$. Clearly, each E_k is closed in $\mathcal{P}$. (This follows from the fact that every sequence $\{T_n\}$ of contractions has a subsequence that converges in the weak operator topology to a contraction.) It follows from the Kaplansky density theorem that $\mathcal{Q}$ is the complement in $\mathcal{P}$ of the union of the $E_k's$. Thus $\mathcal{Q}$ is a G_δ, and the proof is complete. $\square$

Proof of Theorem 2.4: We can assume that $H = \int_\Omega^\oplus M d\mu(\omega)$ for some separable Hilbert space M. Let $\mathcal{D}$ be the corresponding algebra of diagonal operators, i. e., $\mathcal{D} = \int_\Omega^\oplus \mathbb{C} \cdot 1 d\mu(\omega)$. Since each $\mathcal{G}_\omega$ is unital, we have $\mathcal{D} \subset \mathcal{G}$. Thus a projection P in Lat$\mathcal{G}$ must commute with $\mathcal{D}$, and must therefore be a direct integral, $P = \int_\Omega^\oplus P_\omega d\mu(\omega)$.

First suppose that $\mathcal{G}$ is hypertriangularizable, and let $\{P_1, P_2, \ldots\}$ be a chain in Lat$\mathcal{G}$ that generates a masa $\mathcal{M}$. For each ω, let $\mathcal{M}_\omega$ be the von Neumann algebra generated by $\{P_1(\omega), P_2(\omega), \ldots\}$. Then $\mathcal{M} \subset \int_\Omega^\oplus \mathcal{M}_\omega d\mu(\omega)$, and since $\mathcal{M}$ is a masa, we conclude that $\mathcal{M} = \int_\Omega^\oplus \mathcal{M}_\omega d\mu(\omega)$. Since

$\mathcal{M}' = \int_{\Omega}^{\oplus} \mathcal{M}'_\omega d\mu(\omega)$ is commutative, it follows that $\mathcal{M}'_\omega$ is commutative for almost every ω. The commutativity of $\mathcal{M}'_\omega$ implies that almost every $\mathcal{M}_\omega$ is a masa generated by a chain $\{P_1(\omega), P_2(\omega), \dots\}$ in Lat$\mathcal{G}_\omega$. Hence almost every $\mathcal{G}_\omega$ is hypertriangularizable.

Conversely, suppose that almost every $\mathcal{G}_\omega$ is hypertriangularizable. Let $\mathcal{B}$ be a product of countably many copies of the unit ball of $B(M)$ with the product topology induced by the strong operator topology in each coordinate. Then $\mathcal{Q} \times \mathcal{B}$ is a complete separable metric space. Let $\mathcal{E}$ be the set of (P,T) in $\mathcal{Q} \times \mathcal{B}$ such that $(1 - P_i)T_j P_i = 0$ for all i, j. Clearly $\mathcal{E}$ is a closed subset of $\mathcal{Q} \times \mathcal{B}$ and is therefore a complete separable metric space. It follows from [**2**, Thm 3.4.3] that the projection map $\pi_2 : \mathcal{E} \to \mathcal{B}$ has an absolutely measurable cross-section ρ.

Suppose that $\{A_n\}$ is a strongly dense sequence in the unit ball of $\mathcal{G}$. Then, for almost every ω, the sequence $\{A_n(\omega)\}$ is in the range of π_2. It follows that $\{P_n(\omega)\} = \pi_1(\rho(\{A_n(\omega)\}))$ defines a commuting sequence $\{P_n\}$ in Lat $\mathcal{G}$ such that, for almost every ω, $\{P_n(\omega)\}$ generates a masa in $B(M)$. The von Neumann algebra $\mathcal{W}$ generated by$\{P_1, P_2, \dots\}$ and the projections in $\mathcal{D}$ equals the direct integral of the von Neumann algebras generated by $\{P_1(\omega), P_2(\omega), \dots\}$. Since a direct integral of masas is a masa, it follows from Lemma 2.1 that $\mathcal{G}$ is hypertriangularizable. □

3. Commutativity modulo the radical and triangularizability. In [**17**] it is shown that a Banach algebra $\mathcal{G}$ of compact operators is simultaneously triangularizable if and only if for all A, B, and C in $\mathcal{G}$, $(AB - BA)C$ is quasinilpotent. G. Murphy [**16**] pointed out that this condition is equivalent to $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, where Rad$\mathcal{G}$ is the Jacobson radical of $\mathcal{G}$. To understand Murphy's observation, recall that Rad$\mathcal{G}$ is the ideal $\{A\epsilon\mathcal{G} : 1 + AX$ is invertible for every Xin $\mathcal{G}\}$. Thus, if $\mathcal{G}$ is a Banach algebra, then Rad$\mathcal{G} = \{A\epsilon G : AX$ is quasinilpotent for every X in $\mathcal{G}\}$.

In [**19**] it was shown that the aforementioned result implies that a trace condition is equivalent to triangularizability for a multiplicative semigroup of trace-class operators. Namely, a semigroup $\mathcal{S}$ of trace-class operators is triangularizable if and only if

$$trABC = trBAC \qquad \text{for every } A, B, C \text{ in } \mathcal{S}.$$

In [**8**] it was shown that if $\mathcal{G}$ is a Banach algebra of polynomially compact operators and $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, then $\mathcal{G}$ is triangularizable.

It was shown in [**18**] that, in this case, the commutativity of $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is not necessary for the triangularizability of $\mathcal{G}$. A condition that is related to the commutativity of $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is $\mathrm{Rad}\mathcal{G} = \{T \epsilon G : T \text{ is quasinilpotent}\}$. This latter condition holds whenever $\mathcal{G}$ is commutative. In fact, if $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, then $\mathrm{Rad}(\mathcal{G}/\mathrm{Rad}\mathcal{G}) = \{0\}$ implies $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ contains no nonzero quasinilpotents; whence, $\mathrm{Rad}\mathcal{G}$ contains all of the quasinilpotents in $\mathcal{G}$. A detailed list of conditions equivalent to the commutativity of $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is given in [**3**].

Theorem 3.1. *Suppose that $\mathcal{G}$ is a weakly closed algebra of operators that contains a maximal abelian self-adjoint algebra. If $\mathrm{Rad}\mathcal{G} = \{T \epsilon \mathcal{G} : T \text{ is quasinilpotent}\}$, then $\mathcal{G}$ is triangularizable.*

Proof: Let $\mathcal{G}$ be a weakly closed algebra of operators containing a masa $\mathcal{G}_0$; assume that $\mathrm{Rad}\mathcal{G}$ contains all quasinilpotents in $\mathcal{G}$. Suppose that $\mathcal{G}$ is not simultaneously triangularizable. Then there is a pair of $\mathcal{G}$-invariant subspaces M and N such that $N \subset M$ and such that $\dim(N \cap M^{\perp}) > 1$ and there are no $\mathcal{G}$-invariant subspaces strictly between M and N. Let $K = N \cap M^{\perp}$, and let P be the projection onto K. Since $\mathcal{G}_0$ is a masa, the projections onto M and N are in $\mathcal{G}_0$. Hence $P \epsilon \mathcal{G}_0$, and $P\mathcal{G}P$ is weakly closed. Define $\Phi : \mathcal{G} \to B(K)$ by $\Phi(A) = PA|K$. Since K is semi-invariant for $\mathcal{G}$ (i. e., $K = N \cap M^{\perp}$ with $M, N \epsilon \mathrm{Lat}\mathcal{G}$), Φ is a homomorphism. Also the range $\mathcal{B}$ of Φ is weakly closed since $P\mathcal{G}P$ is weakly closed.

We have that $\mathcal{B}$ is a weakly closed transitive algebra that contains the masa $\Phi(\mathcal{G}_0)$; thus $\mathcal{B} = B(K)$ by Arveson's theorem [**1**]. Thus $\mathrm{Rad}\mathcal{B} = 0$. Since $\dim K > 1$, there is a nonzero nilpotent B in $\mathcal{B}$. If $A \epsilon \mathcal{G}$ and $B = \Phi(A)$, then PAP is nilpotent and $\Phi(PAP) = B$. However, $\Phi(\mathrm{Rad}\mathcal{G}) \subset \mathrm{Rad}\Phi(\mathcal{G})$ implies that $\mathrm{Rad}\mathcal{B} \neq 0$. This contradiction implies that $\mathcal{G}$ is indeed triangularizable. □

Corollary 3.2. *Suppose that $\mathcal{G}$ is a weakly closed algebra of operators that contains a maximal abelian self-adjoint algebra. If $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, then $\mathcal{G}$ is triangularizable.*

Remarks 1. A crucial step in the above proof involved showing that the compression of $\mathcal{G}$ to K is closed in the weak operator topology. This was done using the fact that the projections in $\mathrm{Lat}\mathcal{G}$ are contained in $\mathcal{G}$. Note that if P and Q are projections, then $i(PQ - QP)$ is Hermitian; thus $PQ - QP$ is quasinilpotent if and only if $PQ = QP$. Thus if $\mathcal{G}$ is an algebra that contains all of the projections in Lat$\mathcal{G}$, and if $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, then the projections in Lat$\mathcal{G}$ commute with each other. It is possible, however, for the compression of $\mathcal{G}$ to K to be weakly closed without $\mathcal{G}$ containing the projection onto K.

2. If, in the above proof, it could not be shown that $\mathcal{B}$ is weakly closed, then one would hope that the commutativity of $\mathcal{B}/\mathrm{Rad}\mathcal{B}$ would imply the same for the weak closure of $\mathcal{B}$. However, it was shown in [**9**] that this is not the case. (A more general construction is given in the next section of this paper.) $\square$

For direct integrals of algebras on Hilbert spaces of bounded dimension, we can prove results related to [**19**], [**8**], [**16**], [**17**]. We let $\mathcal{M}_{n\times k}$ denote the complex $n \times k$ matrices, and we let $\mathcal{M}_n = \mathcal{M}_{n\times n}$.

We first prove a lemma that may be of independent interest. It is the analogue for direct integrals of the fact that every unital finite-dimensional complex algebra is the linear span of its nilpotents and idempotents.

Lemma 3.3. *Suppose that $\mathcal{G} = \int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ is an algebra on a separable Hilbert space $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$ and for some positive integer n, dim $H_\omega \leq n$ a. e. . Let $\mathcal{S}$ be the linear span of $\{T\epsilon\mathcal{G} : T^2 = T \text{ or } T^n = 0\}$. Then every element of $\mathcal{G}$ is the limit in the strong operator topology of a sequence in $\mathcal{S}$.*

Proof: By considering finite direct sums of algebras, we can assume [**4**] that $\dim H_\omega = n$ for every ω, and we can assume that $H_\omega = \mathbf{C}^n$ and $\mathcal{G}_\omega \subset \mathcal{M}_n$ for every ω. It follows from the Jordan canonical form that the unital algebra generated by a single $n \times n$ matrix is spanned by n (not necessarily distinct) idempotents $\{P_1, \dots P_n\}$ and n (not necessarily distinct) nilpotents $\{Q_1, \dots, Q_n\}$. That is, each P_i and each Q_j is a linearcombination of $\{T^0, T^1, \dots, T^{n-1}\}$, and each $T^k (1 \leq k \leq n-1)$ is a linear combination of $\{P_1, \dots, P_n, Q_1 \dots, Q_n\}$. (In the next paragraph we express these linear combinations as formal matrix products.)

Let X be the product of $\mathcal{M}_n$ and $\mathcal{M}_{2n\times n}$ and $\mathcal{M}_{n\times 2n}$ with $2n$ copies of $\mathcal{M}_n$ with the product topology. Let E be the set of those $(T, A, B, P_1, \dots, P_n, Q_1, \dots, Q_n)$ in X such that $P_i^2 = P_i$ and $Q_i^n = 0$ for each i, and such that the formal products hold:

$$A\begin{pmatrix} T^0 \\ T^1 \\ \vdots \\ T^n \end{pmatrix} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \\ Q_1 \\ \vdots \\ Q_n \end{pmatrix} \text{ and } B\begin{pmatrix} P_1 \\ \vdots \\ P_n \\ Q_1 \\ \vdots \\ Q_n \end{pmatrix} = \begin{pmatrix} T^0 \\ T^1 \\ \vdots \\ T^n \end{pmatrix}.$$

Clearly E is closed and is thus a complete separable metric space. Therefore, by [**2**, Thm. 3.4.3] the projection map π_1 into the first coordinate has an absolutely measurable cross-section α.

It follows from the previous remarks about the Jordan canonical form that the range of π_1 is all of $\mathcal{M}_n$. Hence $\alpha : \mathcal{M}_n \to E$ is absolutely measurable. It follows that there are absolutely measurable maps $p_k, q_k : \mathcal{M}_n \to \mathcal{M}_n$ for $1 \leq k \leq n$, and absolutely measurable maps $\alpha_{ij}, \beta_{ji} : \mathcal{M}_n \to \mathbb{C}$ for $1 \leq i \leq 2n$ and $0 \leq j \leq n-1$ such that, for each T in $\mathcal{M}_n$,

$$\alpha(T) = (T, (\alpha_{ij}(T)), (\beta_{ji}(T)), p_1(T), \ldots, p_n(T), q_1(T), \ldots, q_n(T)).$$

Now suppose that $T \epsilon \mathcal{G}$ and $T = \int_\Omega^\oplus T_\omega d\mu(\omega)$. Define measurable maps $P_k, Q_k : \Omega \to \mathcal{M}_n$ for $1 \leq k \leq n$, and absolutely measurable maps $c_i, d_i : \Omega \to \mathbb{C}$ for $1 \leq i \leq n$ by $P_k(\omega) = p_k(T_\omega), Q_k(\omega) = q_k(T_\omega)$, and $c_i(\omega) = \beta_{1i}(T_\omega)$, and $d_i(\omega) = \beta_{1(n+i)}$. We can write Ω as a disjoint union of measurable sets $\Omega_1, \Omega_2, \ldots$ such that all of the $P_k's, Q_k's, c_i's$, and $d_i's$ are bounded on each Ω_m.

We will show that T is a strong limit of a sequence in $\mathcal{S}$ by showing that, for each m, $\chi_{\Omega_m} T$ is in the norm closure of $\chi_{\Omega_m}\mathcal{S}$. Hence we can assume that all of the $P_k{}'s, Q_k{}'s, c_i{}'s$ and $d_i{}'s$ are bounded on Ω. It follows that $P_k = \int_\Omega^\oplus P_k(\omega)d\mu(\omega)$ is an idempotent operator in $\mathcal{G}$ and $Q_k = \int_\Omega^\oplus Q_k(\omega)d\mu(\omega)$ is a nilpotent operator in $\mathcal{G}$. Moreover, for each ω, we have $T_\omega = \sum_{k=1}^n c_k(\omega)P_k(\omega) + \sum_{k=1}^n d_k(\omega)Q_k(\omega)$. Suppose $\epsilon > 0$. We can approximate the $c_k{}'s$ and $d_k{}'s$ by simple functions $e_k{}'s$ and $f_k{}'s$ so that, for each ω in Ω we have $\| \sum_{k=1}^n e_k(\omega)P_k(\omega) + \sum_{k=1}^n f_k(\omega)Q_k(\omega) - T_\omega \| \leq \epsilon$.

If F is a measurable subset of Ω on which each of the $e_k{}'s$ and the $f_k{}'s$ are constant, then $\chi_F P_k$ is an idempotent in $\mathcal{G}$ and $\chi_F Q_k$ is a nilpotent in $\mathcal{G}$ and $\chi_F \left(\sum_{k=1}^n e_k(\omega)P_k(\omega) + \sum_{k=1}^n f_k(\omega)Q_k(\omega)\right)$ is therefore in $\mathcal{S}$.

Hence, by adding over finitely many $F's$, we see that $\sum_{k=1}^{n} e_k(\omega)P_k(\omega) + \sum_{k=1}^{n} f_k(\omega)Q_k(\omega) \in \mathcal{S}$. Thus T is in the norm closure of $\mathcal{S}$. This completes the proof. □

Under the hypothesis $\dim(H_\omega) \leq n$ a. e., we can, by **[4]** assume that there is a measurable integer-valued function $n(\omega)$ such that $H_\omega = \mathbb{C}^{n(\omega)}$ a. e. . Hence the following theorem is more general than it might first appear.

Theorem 3.4. *Suppose $\mathcal{G} = \int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ acts on the separable Hilbert space $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$. Suppose also that there is a bounded measurable integer-valued function $n(\omega)$ such that $H_\omega = \mathbb{C}^{n(\omega)}$ a. e. . The following are equivalent:*

(1) *$\mathcal{G}$ is hypertriangularizable,*
(2) *$\mathcal{G}/Rad\mathcal{G}$ is commutative,*
(3) *There is a multiplicative semigroup $\mathcal{S} \subset \mathcal{G}$ such that each element of $\mathcal{G}$ is a limit in the weak operator topology of a sequence in $\mathcal{S}$, and such that for each $A, B, C \in \mathcal{S}$, we have $trA_\omega B_\omega C_\omega = trB_\omega A_\omega C_\omega$ a.e.,*
(4) *$Rad\mathcal{G} = \{T \in \mathcal{G} : T \text{ is nilpotent }\}$,*
(5) *There is a unitary operator $U = \int_\Omega^\oplus U_{(\omega)d}\mu(\omega)$ such that $U(\omega)^*\mathcal{G}_\omega U(\omega)$ is upper triangular in $\mathcal{M}_{n(\omega)}$ a. e. .*

Proof: We can write $\mathcal{G}$ as a finite direct sum of algebras for which $\dim H_\omega$ is constant. Thus we can assume that $\dim H_\omega = n$ for all ω, and hence that $H_\omega = \mathbb{C}^n$ for each ω.
(2) $\Rightarrow$ (3). First note that if $T = \int_\Omega^\oplus T_\omega d\mu(\omega)$ is quasinilpotent, then so is almost every T_ω, whence, $(T_\omega)^n = 0$ a. e. . It follows from (2), for all A, B, C in $\mathcal{G}$, that $(AB - BA)C \in \mathrm{Rad}\mathcal{G}$. Thus $(A_\omega B_\omega - B_\omega A_\omega)C_\omega$ is nilpotent a. e., and therefore has trace 0. This proves (3) with $\mathcal{S} = \mathcal{G}$.
(3) $\Rightarrow$ (1). Suppose (3) holds. It follows that almost every $\mathcal{G}_\omega$ is the closed linear span of the multiplicative semigroup $\{S_\omega : S \in \mathcal{S}\}$. It follows from **[19]** that almost every $\mathcal{G}_\omega$ is hypertriangularizable. Thus, by Theorem 2.4, $\mathcal{G}$ must be hypertriangularizable.
(1) $\Rightarrow$ (5). Suppose (1) holds. It follows from Theorem 2.4 that almost every $\mathcal{G}_\omega$ is hypertriangularizable. Let X be a countable product of copies of $\mathcal{M}_n$ with the product topology, and let E be the set of $(U, T_1, T_2, \dots)$ in X such that U is unitary and U^*T_kU is upper triangular for $k \geq 1$. Clearly, E is a closed subset of X and is therefore a complete separable

metric space. Hence, by [**2**, Thm. 3.4.3], the map that sends $(U, T_1, T_2, \dots)$ to $(T_1, T_2, \dots)$ has an absolutely measurable cross-section α. Let $\beta = \pi_1 \circ \alpha$, where π_1 is the projection onto the first coordinate. Choose $\{A_1, A_2, \dots\}$ to be sequentially dense in $\mathcal{G}$ with respect to the weak operator topology, write each $A_k = \int_\Omega^\oplus A_k(\omega) d\mu(\omega)$ and let $U(\omega) = \beta(A_1(\omega), A_2(\omega), \dots)$ for each ω. Then $U(\omega)^* \mathcal{G}_\omega U(\omega)$ is upper triangular a. e. . This proves (5).
(5) $\Rightarrow$ (4). The beginning of the proof of (2) $\Rightarrow$ (3) shows that Rad$\mathcal{G} \subset \{T \in \mathcal{G} : T^n = 0\}$. It follows from (2) that we can assume that $\mathcal{G}_\omega$ is upper triangular a.e. . Thus Rad$\mathcal{G} \subset \{T \in \mathcal{G} : T_\omega$ is strictly upper triangular a.e.$\}$. However, the latter set is an ideal of nilpotents in $\mathcal{G}$, and is therefore contained in Rad$\mathcal{G}$ [**11**, Thm. II.8].
(4) $\Rightarrow$ (2). It follows from (4) that Rad$\mathcal{G}$ is closed under limits of sequences in the strong operator topology. It follows from the preceding lemma and (4) that it suffices to show that $PQ - QP \in$ Rad$\mathcal{G}$ whenever P and Q are idempotents. However, if $P^2 = P$, then $PQ - QP = PQ(1-P) - (1-P)QP$. Since each of the operators $PQ(1-P)$ and $(1-P)QP$ is nilpotent, it follows from (4) that $PQ - QP \in$ Rad$\mathcal{G}$. □

Theorem 3.5. *Suppose that $\mathcal{B}$ is a (not necessarily closed) unital subalgebra of the direct integral $\mathcal{G} = \int_\Omega^\oplus \mathcal{G}_\omega d\mu(\omega)$ on the separable Hilbert space $H = \int_\Omega^\oplus H_\omega d\mu(\omega)$ with dim $H_\omega < \infty$ a.e. . If either*

(1) *$\mathcal{B}/Rad(\mathcal{B})$ is commutative, or*
(2) *For each $A, B, C \in \mathcal{B}$, we have $trA_\omega B_\omega C_\omega = trB_\omega A_\omega C_\omega$ a.e.,*

then $\mathcal{B}$ is hypertriangularizable.

Proof: Suppose that $\{T_n\}$ is a sequence that is closed under multiplication and dense in the weak operator topology on $\mathcal{B}$. For each n, write $T_n = \int_\Omega^\oplus T_n(\omega) d\mu(\omega)$. For each ω let $\mathcal{B}_\omega$ be the algebra generated by $\{T_n(\omega) : n \geq 1\}$.

If (2) above holds, then it follows from [**19**] that each $\mathcal{B}_\omega$ is hypertriangularizable. On the other hand, condition (1) above implies that $(AB - BA)C \in$ Rad$(\mathcal{B})$ for all A, B, C in $\mathcal{B}$. This implies that $(AB - BA)C$ is quasinilpotent, which in turn implies that $(A_\omega B_\omega - B_\omega A_\omega)C_\omega$ is nilpotent a.e. . Hence condition (1) above implies condition (2) since every nilpotent matrix has trace 0.

It follows from the preceding paragraph that almost every $\mathcal{B}_\omega$ is hypertriangularizable. Thus, by Theorem 2.4, $\int_\Omega^\oplus \mathcal{B}_\omega d\mu(\omega)$ is a hypertriangularizable algebra that contains $\mathcal{B}$. Hence $\mathcal{B}$ is hypertriangularizable. $\square$

The following example shows that the converse of the preceding theorem is false.

Example. Suppose that Ω is the unit circle in the plane and μ is normalized linear Lebesgue (i. e., Haar) measure on Ω. For each A in $\mathcal{M}_2$ and each positive integer k, let $A_k = \int_\Omega^\oplus z^k A d\mu(z)$ acting on the Hilbert space $H = \int_\Omega^\oplus \mathbb{C}^2 d\mu(\omega)$. It is clear that $A_k B_j = (AB)_{k+j}$. Thus $\{A_k : A \in \mathcal{M}_2, k \geq 1\}$ is a multiplicative semigroup. Let $\mathcal{G}$ be the weakly closed unital algebra generated by $\mathcal{S}$. It is clear that condition (2) in Theorem 3.5 does not hold; whence neither does condition (1) (in fact, $\mathcal{G}$ is semisimple).

Suppose that $\{e_1, e_2\}$ is the standard basis for $\mathbb{C}^2$. We know that $\{z^n : n = 0, \pm 1, \pm 2, \dots\}$ is an orthonormal basis for $L^2(\mu)$. For each integer n and each j in $\{0, 1\}$, let $e_{nj} = \int_\Omega^\oplus z^n e_j d\mu(z)$. It is easy to show that the e_{nj}'s form an orthonormal basis for H. Furthermore, if we define $M_{nj} = \overline{sp}\{e_{mi} : m \geq n, i \leq j\}$, we see that the M_{nj}'s form a (discrete) masa-generating chain. It is also clear that the M_{nj}'s are invariant subspaces for the A_k's, and thus are invariant subspaces for $\mathcal{G}$. $\square$

The preceding example suggests another possible result. Suppose that $H = \int_\Omega^\oplus \mathbb{C}^n d\mu(\omega)$ and μ is a finite measure. There is a natural trace TR defined on $\int_\Omega^\oplus B(\mathbb{C}^n) d\mu(\omega) = \int_\Omega^\oplus \mathcal{M}_n d\mu(\omega)$ by

$$TR(A) = \int_\Omega^\oplus tr(A_\omega) d\mu(\omega).$$

It is reasonable to ask whether a subalgebra $\mathcal{G}$ of $\int_\Omega^\oplus \mathcal{M}_n d\mu(\omega)$ such that $TR(ABC) = TR(BAC)$ always holds must be hypertriangularizable. In the preceding example, we had $TR(A_k) = 0$, so the above trace condition held.

We suspect that the preceding question has a negative answer in general. However, it is possible to say something about algebras satisfying the trace condition $TR(ABC) = TR(BAC)$ under the additional rather stringent hypothesis that no element in the algebra has spectrum that separates 0 from ∞. We first require a lemma.

The following lemma is a generalization of **[12]** and **[19]**.

Lemma 3.6. *Suppose $A = \int_\Omega^\oplus A_\omega d\mu(\omega)$ is an operator acting on the separable Hilbert space $H = \int_\Omega^\oplus \mathbb{C}^n d\mu(\omega)$ with μ a finite measure. If $\sigma(A)$ does not separate 0 from ∞ and if $TR(A^k) = 0$ for $k = 1, 2, 3, \ldots$, then $A^n = 0$.*

Proof: It follows from part (5) of Theorem 3.4 that we can assume that each A_ω is upper triangular. Let $\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_n(\omega)$ be the diagonal entries of A_ω for each ω. By throwing away a set of measure zero, we can assume that each $\lambda_k(\omega)$ is in $\sigma(A)$.

Using Lemma 1.5, we can choose a sequence $\{p_k\}$ of polynomials uniformly bounded on $\sigma(A)$ such that $p_k(0) = 0$ for each k, and $p_k(z) \to 1$ on $\sigma(A) \setminus \{0\}$. It follows from the hypothesis that $0 = TR(p_k(A)) = \int_\Omega p_k(\lambda_1(\omega))d\mu(\omega) + \int_\Omega p_k(\lambda_2(\omega))d\mu(\omega) + \cdots + \int_\Omega p_k(\lambda_1(\omega))d\mu(\omega)$ for each k. It follows from the dominated convergence theorem that $\lambda_1(\omega) = \cdots = \lambda_n(\omega) = 0$ a. e. . Thus almost every A_ω is nilpotent of index at most n. Hence $A^n = 0$. $\square$

Theorem 3.7. *Suppose that $\mathcal{G}$ is a (not necessarily closed) unital real subalgebra of $\int_\Omega^\oplus \mathcal{M}_n d\mu(\omega)$ acting on $\int_\Omega^\oplus \mathbb{C}^n d\mu(\omega)$ with μ finite. If $\sigma(A)$ does not separate 0 from ∞ for each A in $\mathcal{G}$, and if $TR(ABC) = TR(BAC)$ for every A, B, C in $\mathcal{G}$, then*

(1) *$Rad\mathcal{G} = \{T \in \mathcal{G} : TR(TC) = 0$ for every C in $\mathcal{G}\}$,*
(2) *$\mathcal{G}/Rad\mathcal{G}$ is commutative, and*
(3) *$\mathcal{G}$ is hypertriangularizable.*

Proof: (1). It follows from the preceding lemma that $\{T \in \mathcal{G} : TR(TC) = 0$ for every C in $\mathcal{G}\}$ is an ideal consisting of nilpotents, and is thus contained in Rad$\mathcal{G}$. On the other hand, it is clear that if $T \in \mathcal{G}$ and T is nilpotent, then $TR(T) = 0$. This proves the reverse inclusion.
(2). The trace condition $TR(ABC) = TR(BAC)$ implies, by (1) that $AB - BA \in$ Rad$\mathcal{G}$ for every A, B in $\mathcal{G}$. Thus $\mathcal{G}/$Rad$\mathcal{G}$ is commutative.
(3). This follows from Theorem 3.5. $\square$

4. Algebras of nilpotent operators.

Theorem 4.1. *Let $\mathcal{G}$ be a subalgebra of $B(H)$ such that for some fixed k, $A^k = 0$ for all A in $\mathcal{G}$. Then $\mathcal{G}$ is triangularizable.*

Proof: We can certainly assume that $k > 1$ and that there is an A in $\mathcal{G}$ with $A^{k-1} \neq 0$. For every B in $\mathcal{G}$ and every complex z we have $(A + zB)^k = 0$, implying that the coefficient of each z^m in this polynomial is zero. In particular, $m = 1$ yields the equation

$$BA^{k-1} = -A(BA^{k-2} + ABA^{k-3} + \cdots + A^{k-2}B).$$

Thus $B(A^{k-1}H) \subset AH$. Fix nonzero vectors x in $A^{k-1}H$ and y orthogonal to AH. Then the equation $(Bx, y) = 0$ holds for every B in $\mathcal{G}$. Hence either the closure of $\mathcal{G}x$ or $\{\lambda x : \lambda \in \mathbb{C}\}$ is a non-trivial invariant subspace for $\mathcal{G}$ depending on whether $\mathcal{G}x \neq 0$ or $\mathcal{G}x = 0$. Thus we have the existence of a non-trivial invariant subspace whenever the underlying subspace has dimension greater than 1.

By Zorn's lemma there exists a maximal chain $\mathcal{C}$ of invariant subspaces for $\mathcal{G}$. We must show that $\mathcal{C}$ is a maximal subspace chain. This follows from the observation that if M and N are invariant subspaces for $\mathcal{G}$ with $M \subset N$, then the compression of $\mathcal{G}$ to $N \ominus M$ consists of nilpotent operators of index at most k. Thus if $N \ominus M$ has dimension greater than one, this compression will have a non-trivial invariant subspace by above argument. The span of this subspace and M is clearly an invariant subspace for $\mathcal{G}$ lying strictly between M and N. It follows that any gaps in $\mathcal{C}$ are one-dimensional and $\mathcal{G}$ is hypertriangularizable. ☐

Corollary 4.2. *If $\mathcal{G}$ is a norm closed algebra of nilpotent operators, then $\mathcal{G}$ is triangularizable.*

Proof: It was shown by S. Grabiner [7] that if $\mathcal{G}$ is a Banach algebra of nilpotents, then there is a fixed k with $A^k = 0$ for every A in $\mathcal{G}$. ☐

The above theorem is not true if "algebra" is replaced with "semigroup" as examples given below show. It is also not true if "algebra" is replaced with "linear manifold" as the following simple example of 3×3 matrices shows: the linear manifold

$$\left\{ \begin{bmatrix} 0 & \beta & 0 \\ \alpha & 0 & -\beta \\ 0 & \alpha & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

consists of nilpotents but has no nontrivial invariant subspaces.

However, there is one easy special case where linear manifolds are triangularizable.

Theorem 4.3. *If $\mathcal{L}$ is a linear manifold of operators on H with $A^2 = 0$ for all A in $\mathcal{L}$, then $\mathcal{L}$ is triangularizable.*

Proof: As before, we must only show that $\mathcal{L}$ has a non-trivial invariant subspace. Assuming, with no loss of generality, that there is a non-zero A in $\mathcal{L}$, we observe that $(A+B)^2 = 0$ implies $BA = -AB$, so that $B(AH) \subset AH$ for all B in $\mathcal{L}$. Thus the (non-trivial) closure of AH is invariant for $\mathcal{L}$. □

In [**9**] we constructed an algebra of nilpotent operators on Hilbert space with a semisimple uniform closure. Here we make a more general construction that includes the example in [**9**] as a special case.

Suppose $\{n_k\}$ is a sequence of integers greater than 1. There is a natural embedding of $\mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2} \otimes \cdots \otimes \mathcal{M}_{n_k}$ into $\mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2} \otimes \cdots \otimes \mathcal{M}_{n_k} \otimes \mathcal{M}_{n_{k+1}}$ that sends $A_1 \otimes A_2 \otimes \cdots \otimes A_k$ to $A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes 1$. The resulting C^*-algebraic direct limit is the *Glimm algebra* $\mathcal{J}$ associated with the sequence $\{n_k\}$.

Note that $\mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2} \otimes \cdots \otimes \mathcal{M}_{n_k}$ is naturally isomorphic to $\mathcal{M}_{m_k}$ where $m_k = n_1 n_2 \cdots n_k$. Furthermore, the natural embedding described above induces the embedding of $\mathcal{M}_{m_k}$ into $\mathcal{M}_{m_{k+1}}$ that sends a matrix A to the matrix $\begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A \end{bmatrix}$.

Using the latter view, we can easily represent $\mathcal{J}$ on a separable Hilbert space H with an orthonormal basis $\{e_1, e_2, \dots\}$. If $T = (t_{ij})$ is an $n \times n$ matrix, let $\hat{T}$ be the operator on H whose matrix is defined as follows for integers $m, k \geq 0$ and integers $1 \leq i, j \leq n$:

$$(\hat{T}e_{j+kn}, e_{i+mn}) = \begin{cases} t_{ij} \text{ if } k = m \\ 0 \text{ if } k \neq m \end{cases}.$$

Thus $\hat{T}$ is just a direct sum of infinitely many copies of T. It is clear that if $\hat{M}_{m_k} = \{\hat{T} : T \in \mathcal{M}_{m_k}\}$, then $\hat{M}_{m_k} \subset \hat{M}_{m_{k+1}}$, and that the inclusion map coincides with the above embedding of $\mathcal{M}_{m_k}$ into $\mathcal{M}_{m_{k+1}}$. Thus $\mathcal{J}$ is represented as the norm closure $\hat{\mathcal{J}}$ of the union of the $\hat{M}'_{m_k}s$.

Our general construction proceeds as follows. Suppose, for each $k \geq 1$, that $\mathcal{L}_k$ is a non-zero subalgebra of $\mathcal{M}_{n_k}$, and let $\mathcal{S}_k = \mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2} \otimes \cdots \otimes \mathcal{M}_{n_{k-1}} \otimes \mathcal{L}_k$. It is clear that each $\mathcal{S}_k$ is a closed subalgebra of $\mathcal{J}$ and that $\mathcal{S}_k\mathcal{S}_m \subset \mathcal{S}_{\max(k,m)}$ for all $k, m \geq 1$. Let $\mathcal{G}_k = \mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_k$ for $k \geq 1$. It follows that $\mathcal{G}_k^2 = \mathcal{G}_k\mathcal{G}_k \subset \mathcal{S}_1^2 + \mathcal{S}_2 + \cdots + \mathcal{S}_k, \mathcal{G}_k^3 = \mathcal{G}_k\mathcal{G}_k\mathcal{G}_k \subset$

$\mathcal{S}_1^3 + \mathcal{S}_2 + \mathcal{S}_3 + \cdots + \mathcal{S}_k, \ldots$. Thus $\mathcal{G}_k$ is an algebra for each k, and if $\mathcal{S}_k$ consists of nilpotents, then $\mathcal{G}_k$ consists of nilpotents (i. e., if $\mathcal{S}_i^{m_i} = 0$, then $\mathcal{G}_k^{m_1} \subset \mathcal{S}_2 + \cdots + \mathcal{S}_k$, $\mathcal{G}_k^{m_1 m_2} \subset \mathcal{S}_3 + \cdots + \mathcal{S}_k, \ldots, \mathcal{G}_k^{m_1 m_2 \cdots m_k} = 0$). Finally, we let $\mathcal{S}_\infty = \cup_n \mathcal{S}_n$, and $\mathcal{G}_\infty = \cup_n \mathcal{G}_n$, and let $\mathcal{G}$ be the norm closure of $\mathcal{G}_\infty$. It is clear that if $\mathcal{L}_k$ contains the identity matrix for infinitely many $k's$, then $\mathcal{S}_\infty$ is the union over k of the sets $\mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2} \otimes \cdots \otimes \mathcal{M}_{n_k}$, which makes $\mathcal{G} = \mathcal{J}$. The example constructed in **[9]** is the special case of the above construction where each $n_k = 2$, and each $\mathcal{L}_k = \left\{ \begin{bmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{bmatrix} : \alpha \in \mathbb{C} \right\}$.

Theorem 4.4. *If $\mathcal{G}$ is constructed as above, then*

(1) *$\mathcal{G}$ is semisimple;*
(2) *$\hat{\mathcal{G}}$ is a strictly dense subalgebra of $B(H)$;*
(3) *if each $\mathcal{L}_k$ is an algebra of nilpotents, then so is $\mathcal{G}_\infty$.*

Proof: (3) follows from the remarks preceding the theorem, and (1) follows from (2). Hence, we need to prove (2). Since $\mathcal{L}_k \neq 0$, we can choose a nonzero operator S_k in $\mathcal{L}_k$ for each positive integer k. Since the numerical radius of an operator is at least one-half its norm, we can assume that $\|S_k\| \leq 2$ and that the numerical range of S_k contains the number 1 for each $k \geq 1$. Since applying a unitary automorphism on each $\mathcal{M}_{n_k}$ corresponds to a single unitary automorphism on $\hat{\mathcal{G}}$, we can assume that the $(1,1)$ entry of the matrix S_k is 1 for each $k \geq 1$.

For each positive integer k, let P_k be the orthogonal projection of H onto the span of $\{e_1, e_2, \ldots, e_{m_k}\}$. Suppose $T \in B(H)$ and $\|T\| \leq 1$. The upper $m_k \times m_k$ left hand corner of the matrix for T is a matrix in $\mathcal{M}_{m_k}$ of norm at most 1, and corresponds to a tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_k$ with $\|A_i\| \leq 1$, and $A_i \in \mathcal{M}_{n_i}$ for each i. Let $T_k = A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes S_{k+1}$. Then $T_k \in \mathcal{S}_{k+1}$, and $\|T_k\| \leq 2$, and $P_k \hat{T}_k P_k = P_k T P_k$. Thus $\hat{T}_k \to T$ in the strong operator topology. Hence twice the unit ball of $\hat{G}$ is strongly dense in the unit ball of $B(H)$. It now follows as in **[9]** that $\hat{G}$ is strictly dense in $B(H)$. $\square$

5. Some unsolved problems.

5.1 Does Theorem 1.2 hold if $\mathcal{D}$ is not assumed to be self-adjoint?

5.2 If $\mathcal{G}$ is uniformly closed and $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, must $\mathcal{G}$ be triangularizable?

5.3 If $\mathcal{G}$ is weakly closed and $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ is commutative, must $\mathcal{G}$ be triangularizable? What if $\mathcal{G}$ contains the unilateral shift?

5.4 If $\mathcal{G}$ is triangularizable, what are necessary and sufficient conditions that $\mathcal{G}/\mathrm{Rad}\mathcal{G}$ be commutative (see [**3**])?

5.5 What other sufficient conditions are there that insure that a collection of operators is hypertriangularizable?

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