HOUSTON JOURNAL OF MATHEMATICS
Volume 17, No. 4, 1991

ON COMMON NONCYCLIC VECTORS FOR FAMILIES OF OPERATORS

BERNARD CHEVREAU
CARL PEARCY

This paper is dedicated to the memory of our good friend and colleague, Domingo Herrero.

1. Introduction and preliminaries. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If $\{T_\alpha\}_{\alpha\in A}$ is a family of operators in $\mathcal{L}(\mathcal{H})$ and x is a nonzero vector in $\mathcal{H}$ such that for each $\alpha \in A$,

$$\mathcal{M}_\alpha = \bigvee_{n=0}^{\infty} T_\alpha^n x \neq \mathcal{H},$$

then x is said to be a *common noncyclic vector* for the family $\{T_\alpha\}_{\alpha\in A}$. (Of course, since the subspaces $\mathcal{M}_\alpha, \alpha \in A$, need not be the same, the existence of a common noncyclic vector for a family of operators does not imply that the family has a *common* nontrivial invariant subspace.) The purpose of this paper is to use some techniques from the theory of dual algebras to obtain some (perhaps surprisingly strong) theorems concerning the existence of common noncyclic vectors for certain countable families $\{T_n\}_{n=1}^{\infty}$ of (in general) *noncommuting* operators.

We shall suppose that the reader is familiar with the theory of dual algebras, as presented in [**2**], and, in particular, the notation and terminology employed below are taken from [**2**]. If $T \in \mathcal{L}(\mathcal{H})$, we write $\mathcal{A}_T$ for the dual algebra generated by T and Q_T for the predual $\mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}_T$ of $\mathcal{A}_T$, where, as usual, $\mathcal{C}_1(\mathcal{H})$ denotes the Banach space and ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$. The elements of Q_T will be written as cosets $[L]$, where $L \in \mathcal{C}_1(\mathcal{H})$, or, when there is more than one dual algebra under consideration, as $[L]_T$. As usual, we write $\mathbb{N}$ for the set of positive integers, $\mathbb{C}$ for the complex field, $\mathbb{D}$ for the open unit disk in $\mathbb{C}$, and $\mathbb{T}$ for $\partial\mathbb{D}$. Furthermore, we shall write $H^\infty = H^\infty(\mathbb{T})$ for the Banach algebra of all bounded holomorphic functions on $\mathbb{D}$, and identify this algebra routinely with the corresponding algebra of boundary functions. Recall that a subset $\Omega \subset \mathbb{D}$ is said to be *dominating* for $\mathbb{T}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Ω. For an operator T in $\mathcal{L}(\mathcal{H})$, we write $\sigma(T)$ for the spectrum of T and $\sigma_e(T)$ for the essential (Calkin) spectrum of T. We shall employ the customary notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequence $\{T^*\}^n$ converges to zero in the strong operator topology, and write, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$ and $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

A typical corollary of our main theorem (Theorem 2.1) is as follows.

Proposition 1.1. *Let $\{T_n\}_{n=1}^\infty$ be an arbitrary (not necessarily commuting) family in $C_{\cdot 0}(\mathcal{H})$ such that $\sigma_e(T_n) \cap \mathbb{D}$ is dominating for $\mathbb{T}$ for every $n \in \mathbb{N}$. Then the family $\{T_n\}_{n=1}^\infty$ has a dense set of common noncyclic vectors.*

This proposition is pertinent to the invariant subspace problem for contraction operators with spectral radius one, as is evidenced by the following corollary.

Corollary 1.2. *Suppose T is a contraction in $\mathcal{L}(\mathcal{H})$ with spectral radius one. Then either T has a nontrivial hyperinvariant subspace, or there exists a contraction S in $\mathcal{L}(\mathcal{H})$ satisfying Lat $S =$ Lat T or Lat $S =$ Lat T^* such that the family of operators $\{S^2, S^3, S^4, \dots\}$ has a dense set of common noncyclic vectors. Moreover, in the latter case, if any pair $\{S^p, S^q\}$, where $2 \le p, q < \infty$ and p and q are relatively prime positive integers, has a common nontrivial invariant subspace, then T has a nontrivial invariant subspace.*

Proof of Corollary 1.2: If the unitary part of T is a scalar λ of modulus one, then either $T = \lambda$, in which case we set $S = \lambda$ and the theorem is proved, or T has a nontrivial eigenspace. On the other hand, if T has a (nonzero) unitary part which is not a scalar, then T has a nontrivial hyperinvariant subspace [4]. Thus we may suppose that T is completely nonunitary, and another well-known argument enables us to conclude that either $T \in C_{\cdot 0} \cup C_{0\cdot}$ or T has a nontrivial hyperinvariant subspace. If $T \in C_{\cdot 0}$, then we may apply [5, Theorem 2.1 and the discussion preceding Prop.

1.1] to conclude that either T has a nontrivial hyperinvariant subspace, or there exists a contraction S of the form $S = f(f(T))$, where f is some appropriate conformal map of $\mathbb{D}$ into $\mathbb{D}$, such that Lat $S =$ Lat T and $\sigma_{\ell e}(S) \cap \mathbb{D}$ is dominating on a subarc of $\mathbf{T}$ of length greater than $3\pi/2$. Furthermore it follows from [**3**, Theorem 7.2] that $S \in C_{\cdot 0}$, and it is easy to see that the family $\{S^n\}_{n=2}^{\infty}$ satisfies the hypotheses of Proposition 1.1, so this family has a dense set of common noncyclic vectors. In case $T \in C_{0\cdot}$, the above argument is applied to the operator T^*, and once again the desired conclusion follows, with Lat $S =$ Lat T^*. To conclude the proof, suppose now that for some $2 \leq p \leq q < \infty$ with p and q relatively prime, S^p and S^q have a common nontrivial invariant subspace $\mathcal{M}$, and let x_0 be a nonzero vector in $\mathcal{M}$. It is an exercise, using the fact that there exist integers s and t with $ps + qt = 1$, to see that the cyclic subspace

$$\mathcal{N} = \bigvee_{k=0}^{\infty} S^{p\cdot q+k} x_0$$

for S is contained in $\mathcal{M}$, so either $\mathcal{N} = (0)$, in which case S has zero as an eigenvalue, or $\mathcal{N} \neq (0)$, in which case $\mathcal{N} \in$ Lat $S =$ Lat T. In either case, Lat $T \neq \{(0), \mathcal{H}\}$, and the proof is complete.

Remark 1.3. In connection with Proposition 1.1, it is worth pointing out that two arbitrary operators may not have any common noncyclic vector. Recall (cf., for example, [**7**, Prop. 3.8 and Corollaries]) that there exist unicellular weighted backward unilateral shifts. Let T and T' be two of those defined (by their weight sequences) with respect to the orthonormal bases $\{e_n\}_{n\in\mathbb{N}}$ and $\{e'_n\}_{n\in\mathbb{N}}$. Then the only proper invariant subspaces for T (resp. T') are exactly the subspaces $\mathcal{E}_n = \vee_{k\leq n} e_k$ (resp. $\mathcal{E}'_n = \vee_{k\leq n} e'_k$), $n \in \mathbb{N}$. Thus any common noncyclic vector x for T and T' will belong to a subspace $\mathcal{E}_n \cap \mathcal{E}'_m$ for some $n, m \in \mathbb{N}$. If the orthonormal bases are chosen so that $\mathcal{E}_n \cap \mathcal{E}'_m = \{0\}, n, m \in \mathbb{N}$, then the pair $\{T, T'\}$ will not have any (nonzero) common noncyclic vector. To achieve this, consider the Hilbert space $L^2(0, 2\pi)$ (with respect to Lebesgue measure divided by π) with, on one hand, the orthonormal basis $(e_n)_{n\geq 0}$ defined by

$$\begin{cases} e_0(t) & = \frac{\sqrt{2}}{2}, \\ e_{2k-1}(t) & = \sin\,(kt), k \geq 1, \\ e_{2k}(t) & = \cos\,(kt), k \geq 1, \end{cases}$$

and, on the other hand, the orthonormal basis $(e'_n)_{n\geq 0}$ obtained from the sequence of polynomials $\{t \to t^n\}_{n\geq 0}$ via the Gram-Schmidt procedure. Take now $\mathcal{H}$ to be the orthocomplement of e_0 in $L^2(0, 2\pi)$. Then the orthonormal bases $(e_n)_{n\geq 1}$ and $(e'_n)_{n\geq 1}$ of $\mathcal{H}$ do satisfy the required conditions. (To see that, observe, for instance, that any element in $\mathcal{E}'_m$ has an identically vanishing derivative of order $(m+1)$ while nonzero elements of $\mathcal{E}_n$ have nonconstant derivatives of any order.)

2. The main theorem. In this section we provide the details which constitute a proof of our principal theorem. We recall that an absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ is said to belong to the class $\mathbb{A}(\mathcal{H})$ if its Nagy-Foias H^∞-functional calculus $\phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, such a T is said to belong to the class $\mathbb{A}_{\aleph_0}(\mathcal{H})$ if $T \in \mathbb{A}(\mathcal{H})$ and every system of equations $[x_i \otimes y_j]_T = [L_{ij}]_T, 0 \leq i, j < \infty$, in the predual Q_T, where the $L_{ij}, 0 \leq i, j < \infty$, are arbitrary trace-class operators, has a solution $(\{x_i\}_{i=0}^\infty, \{y_j\}_{j=0}^\infty)$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For more information about the class $\mathbb{A}_{\aleph_0}(\mathcal{H})$, see [**2**, Chapter V]. Our principal theorem is the following.

Theorem 2.1. *Suppose $\{T_n\}_{n=1}^\infty$ is any sequence of operators contained in the class $\mathbb{A}_{\aleph_0}(\mathcal{H}) \cap C_{\cdot 0}(\mathcal{H}), \{[L_n]_{T_n}\}_{n=1}^\infty$ is an arbitrary sequence (where $[L_n]_{T_n} \in Q_{T_n}$), and $\{\pi_n\}_{n=1}^\infty$ is any sequence of positive numbers. Then there exists a dense set $\mathcal{D} \subset \mathcal{H}$ such that for every x in $\mathcal{D}$, there exists a sequence $\{y_n^x\}_{n=1}^\infty \subset \mathcal{H}$ satisfying*

$$[x \otimes y_n^x]_{T_n} = [L_n]_{T_n}, \quad n \in \mathbb{N}, \tag{1}$$

and

$$\|y_n^x\| > \pi_n, \; n \in \mathbb{N}. \tag{2}$$

Proof of Proposition 1.1: We now show how Proposition 1.1 is a corollary of Theorem 2.1. If $\{T_n\}_{n=1}^\infty$ satisfies the hypotheses of Proposition 1.1, it follows immediately from [**2**, Theorem 6.8] that each $T_n \in \mathbb{A}_{\aleph_0}$. Thus the sequence $\{T_n\}$ satisfies the hypotheses of Theorem 2.1, and we set $[L_n]_{T_n} = 0$ and $\pi_n = 1$ for each $n \in \mathbb{N}$. Thus there exists a dense set $\mathcal{D}$ in $\mathcal{H}$ and for each x in $\mathcal{D}$ a sequence $\{y_n^x\}_{n=1}^\infty$ from $\mathcal{H}$ satisfying

$$[x \otimes y_n^x]_{T_n} = 0, \; n \in \mathbb{N},$$

and

$$\|y_n^x\| \geq 1, \quad n \in \mathbb{N}.$$

With such a nonzero x in $\mathcal{D}$ fixed, define for each $n \in \mathbb{N}, \mathcal{M}_n = \bigvee_{k=0}^{\infty}(T_n)^k x$. A standard computation (cf. [**2**, Prop. 4.8]) shows that y_n^x is orthogonal to $\mathcal{M}_n$ for each $n \in \mathbb{N}$, and hence that x is noncyclic for each $T_n, n \in \mathbb{N}$. This completes the proof of Proposition 1.1.

The proof of Theorem 2.1 is based on some preparatory lemmas which are of independent interest.

Lemma 2.2. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a commutative dual algebra with the property that for every weak* closed ideal $\mathcal{J} \subset \mathcal{A}$, the quotient $\mathcal{A}/\mathcal{J}$ is either finite dimensional or nonseparable. Then, for every x in $\mathcal{H}$, there is an orthonormal sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ satisfying*

$$\text{(3)} \qquad \|[t_n \otimes x]\| \to 0 [\textit{resp.}\ \|[x \otimes t_n]\| \to 0].$$

Proof: Since the hypothesis remains valid if $\mathcal{A}$ is replaced by $\mathcal{A}^* = \{T^* : T \in \mathcal{A}\}$, and one has $\|[t \otimes u]_{Q_{\mathcal{A}}}\| = \|[u \otimes t]_{Q_{\mathcal{A}^*}}\|$ for all $t, u \in \mathcal{H}$, it clearly suffices to prove the first statement in (3). Also, of course, we may take $x \neq 0$. Define $\psi : \mathcal{H} \to Q_{\mathcal{A}}$ by $\psi(t) = [t \otimes x], t \in \mathcal{H}$. Clearly ψ is linear and bounded. Using the Open Mapping Theorem it is easy to see that if (3) does not hold, then ker ψ is finite dimensional and $\psi|(\ker \psi)^{\perp}$ is bounded below. (Thus we must rule out this possibility.) Supposing this to be the case, the range of ψ is closed, and therefore $\psi^* : \mathcal{A} \to \mathcal{H}^*$ (where $\mathcal{H}^*$ is the Banach space dual of $\mathcal{H}$) has closed range $\mathcal{M}^*$ which is infinite dimensional (being the annihilator of ker ψ). Therefore, there exists an invertible, weak* continuous (thus bounded), linear mapping $\hat{\psi}^*$ of $\mathcal{A}/(\ker \psi^*)$ onto $\mathcal{M}^*$, and, in view of the hypothesis, to complete the argument, it suffices to show that ker ψ^* is a weak* closed ideal in $\mathcal{A}$. Thus, let $A \in$ ker ψ^* and let $B \in \mathcal{A}$. Then, for every $t \in \mathcal{H}$,

$$\text{(4)} \qquad 0 = \langle \psi^*(A), t \rangle_{\mathcal{H}^* \times \mathcal{H}} = \langle A, [t \otimes x] \rangle = (At, x)_{\mathcal{H}} = (t, A^* x),$$

and hence $A^*x = 0$. Thus $B^*A^*x = A^*B^*x = 0$, and another computation like (4) shows that $AB = BA \in$ ker ψ^*. This, together with the weak* continuity of ψ^*, shows that ker ψ^* is a weak* closed ideal in $\mathcal{A}$, which completes the proof.

In order to adapt Lemma 2.2 to our needs, we need the following proposition, whose proof was provided to us by Allen Shields.

Proposition 2.3. *If ϕ is an inner function in H^∞, then the quotient space $H^\infty/\phi H^\infty$ is separable if and only if ϕ is a finite Blaschke product, in which case this quotient space is finite dimensional.*

Proof: If ϕ is a finite Blaschke product, then $H^\infty/\phi H^\infty$ is obviously finite dimensional, so we may suppose that ϕ is not a finite Blaschke product. We consider first the case in which ϕ is divisible by a singular inner function S_μ, where μ is a singular Borel measure on $\mathbf{T}$. Then we can write $\phi = BS_\mu$, where either $B = 1$ or B is a Blaschke product. For $t > 0$, let $S_{t\mu}$ denote the singular inner function associated with the measure $t\mu$. We assert that if $0 < a < b < 1$, then

$$\|[S_{a\mu}] - [S_{b\mu}]\|_{H^\infty/\phi H^\infty} \geq 1,$$

and hence the quotient space is not separable in this case. To establish this inequality, let $f \in H^\infty$, recall that $S_{t\mu} = (S_\mu)^t$, and compute:

$$\begin{aligned}\|S_{a\mu} - S_{b\mu} - \phi f\|_{H^\infty} &= \|(S_\mu)^a - (S_\mu)^b - S_\mu B f\|_{H^\infty} \\ &= \|(S_\mu)^a[1 - (S_\mu)^{b-a} - (S_\mu)^{1-a} B f]\|_{H^\infty} \\ &= \|1 - (S_\mu)^{b-a} - (S_\mu)^{1-a} B f\|_{H^\infty}.\end{aligned}$$

But associated with every inner function ψ which is not a finite Blaschke product is a sequence $\{z_n\} \subset \mathcal{D}$ such that $|z_n| \to 1$ and $\psi(z_n) \to 0$. Choosing such a sequence $\{z_n\}$ for S_μ and evaluating along this sequence gives the desired inequality.

Finally, we consider the case in which ϕ is divisible by an infinite Blaschke product B, so $\phi = BS$, where either $S = 1$ or S is a singular inner function. By virtue of the case already done, we may suppose $S = 1$. Choose an interpolating sequence $\{z_n\}$ from among the zeros of B with the property that all the terms in $\{z_n\}$ are distinct, and let B_1 be the corresponding Blaschke product, so $\phi = B_1 B_2$ where B_2 is also a Blaschke product.

To see that $H^\infty/\phi H^\infty$ is not separable in this case, we define, for each $n \in \mathbb{N}$,

$$b_n(z) = \frac{B_1(z)(1 - \overline{z}_n z)|z_n|}{(z_n - z)\overline{z}_n}, \quad z \in \mathbb{D},$$

and recall from [**6**, p. 196] that there exists $\delta > 0$ such that

$$|b_n(z_n)| \geq \delta, \quad n \in \mathbb{N}. \tag{4}$$

Next we arrange $\mathbb{Q}$, the set of rational real numbers, in a sequence $\{q_n\}_{n=1}^{\infty}$, and for each real number x we choose a subsequence $\{q_{n_k}\}_{n_k \in N_x}$ of $\{q_n\}$ that converges to x. Observe that if $x \neq y$, then $N_x \cap N_y$ is a finite set, and for each real x, write B_x for the Blaschke product whose zeros are the sequence $\{z_{n_k}\}_{n_k \in N_x}$. We assert that

$$\|[B_x] - [B_y]\|_{H^\infty/\phi H^\infty} \geq \delta, \quad x, y \in \mathbb{R}, \quad x \neq y, \tag{5}$$

which is, of course, sufficient to prove the nonseparability of $H^\infty/\phi H^\infty$. To establish this inequality, fix $x, y \in \mathbb{R}$ with $x \neq y$, and let $f \in H^\infty$. Write $B_{(x,y)}$ for the Blaschke product that is the greatest common divisor of B_x and B_y, and note that since $N_x \cap N_y$ is finite, $B_{(x,y)}$ is a finite Blaschke product such that $B_x/B_{(x,y)}$ and $B_y/B_{(x,y)}$ are (infinite) Blaschke products with no zeros in common. Thus

$$\begin{aligned}\|B_x - B_y - \phi f\|_{H^\infty} &= \|B_x - B_y - B_1 B_2 f\| \\ &= \|B_{(x,y)}(B_x/B_{(x,y)} - B_y/B_{(x,y)} - (B_1/B_{(x,y)})B_2 f)\| \\ &= \|B_x/B_{(x,y)} - B_y/B_{(x,y)} - (B_1/B_{(x,y)})B_2 f\|,\end{aligned}$$

and if we evaluate this expression at any zero w of $B_y/B_{(x,y)}$, then we obtain

$$\|B_x - B_y - \phi f\|_{H^\infty} \geq \|(B_x/B_{(x,y)})(w)\|_{H^\infty} \geq \frac{|B_1(w)|\,|1-\overline{w}z|}{|w-z|} \geq \delta,$$

which establishes (5). (The last inequality is from (4), and the preceding one is obvious from the definitions of B_1, B_x, and $B_{(x,y)}$.)

Corollary 2.4. *Suppose $T \in \mathbb{A}(\mathcal{H})$. Then, for every x in $\mathcal{H}$, there is an orthonormal sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ satisfying*

$$\|[t_n \otimes x]\| \to 0 [\textit{resp.}, \|[x \otimes t_n]\| \to 0].$$

Proof: By definition of the class $\mathbb{A}$ one knows that the H^∞ functional calculus $\phi_T : H^\infty \to \mathcal{A}_T$ is a weak* continuous, surjective, isometric, algebra isomorphism. To see that Lemma 2.2 can be applied to $\mathcal{A}_T$, we observe that if $\mathcal{J}$ is any weak* closed ideal of $\mathcal{A}_T$, then there exists an inner function ψ such that $\phi_T(\psi H^\infty) = \mathcal{J}$, so $\mathcal{A}_T/\mathcal{J}$ is isometrically isomorphic to $H^\infty/\phi H^\infty$, and we apply Proposition 2.3 to see that $\mathcal{A}_T/\mathcal{J}$ cannot be separable and infinite dimensional.

The following lemma constitutes the essential step of an induction argument which is at the heart of the proof of Theorem 2.1.

Lemma 2.5. *Under the hypotheses of Theorem 2.1, suppose $N \in \mathbb{N}$, the numbers $\epsilon, \epsilon_N, \epsilon_{N+1}$ are positive, and there exist vectors x and $z_1, \ldots, z_N$ in $\mathcal{H}$ such that*

$$\|[x \otimes z_i]_{T_i} - [L_i]_{T_i}\| < \epsilon_N, \quad i = 1, \ldots, N. \tag{6}$$

Then there exist vectors x' and $z'_1, \ldots, z'_{N+1}$ in $\mathcal{H}$ such that

$$\|[x' \otimes z'_i]_{T_i} - [L_i]_{T_i}\| < \epsilon_{N+1}, \quad i = 1, \ldots, N+1, \tag{7}$$

$$\|x - x'\| \leq (N+1)\epsilon_N^{\frac{1}{2}}, \tag{8}$$

$$\|z_i - z'_i\| \leq \epsilon_N^{\frac{1}{2}}, \quad i = 1, \ldots, N, \tag{9}$$

and

$$\|z'_{N+1}\| > \pi_{N+1} + \epsilon. \tag{10}$$

Proof: Since $T_1 \in \mathbb{A}_{\aleph_0}, \mathcal{A}_{T_1}$ has property $X_{0,1}$ (cf. [**2**, Def. 2.7 and Prop. 6.1]), and thus there exist sequences $\{r_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ in the unit ball of $\mathcal{H}$ such that

$$\left\| \left(\frac{[x \otimes z_1]_{T_1} - [L_1]_{T_1}}{\epsilon_N} \right) - [r_n \otimes s_n]_{T_1} \right\| \to 0, \tag{11}$$

$$\|[r_n \otimes z]_{T_1}\| \to 0, \quad z \in \mathcal{H}, \tag{12}$$

and

$$\|[w \otimes s_n]_{T_1}\| \to 0, \quad w \in \mathcal{H}. \tag{13}$$

It is easy to see, using (6) (for $i = 1$), (11), (12), and (13) that if we define

$$x'_1 = x + \epsilon_N^{\frac{1}{2}} r_n, \quad z'_1 = z_1 + \epsilon_N^{\frac{1}{2}} s_n \tag{14}$$

for some n sufficiently large, then it results that

$$\|[x_1' \otimes z_1']_{T_1} - [L_1]_{T_1}\| < \epsilon_{N+1}.$$

Moreover, since $T_2, \dots, T_N \in C_{\cdot 0}(\mathcal{H})$,

$$\|[r_n \otimes z_i]_{T_i}\| \xrightarrow{n} 0, \quad i = 2, \dots, N, \tag{15}$$

and hence, using (6), we see that n may also be chosen sufficiently large to ensure that

$$\|[x_1' \otimes z_i]_{T_i} - [L_i]_{T_i}\| < \epsilon_N, \quad i = 2, \dots, N.$$

Moreover, from (14) it is automatic that

$$\|x_1' - x_1\| \le \epsilon_N^{\frac{1}{2}}, \ \ \|z_1' - z_1\| \le \epsilon_N^{\frac{1}{2}}.$$

Next, using the fact that $\mathcal{A}_{T_2}$ has property $X_{0,1}$, we obtain sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ from the unit ball of $\mathcal{H}$ such that

$$\left\| \frac{[x_1' \otimes z_2]_{T_2} - [L_2]_{T_2}}{\epsilon_N} - [u_n \otimes v_n]_{T_2} \right\| \to 0,$$
$$\|[u_n \otimes z]_{T_2}\| \to 0, \quad z \in \mathcal{H},$$
$$\|[w \otimes v_n]_{T_2}\| \to 0, \quad w \in \mathcal{H}.$$

Thus, analogous to what was done before, if we define

$$x_2' = x_1' + \epsilon_N^{\frac{1}{2}} u_n, \quad z_2' = z_2 + \epsilon_N^{\frac{1}{2}} v_n,$$

for some n sufficiently large, then we can arrange that

$$\|[x_2' \otimes z_2']_{T_2} - [L_2]_{T_2}\| < \epsilon_{N+1},$$
$$\|[x_2' \otimes z_1]_{T_1} - [L_1]_{T_1}\| < \epsilon_{n+1},$$
$$\|[x_2' \otimes z_i]_{T_i} - [L_i]_{T_i}\| < \epsilon_N, i = 3, \dots, N,$$

and

$$\|x_2' - x_1'\| \le \epsilon_N^{\frac{1}{2}}, \ \ \|z_2' - z_2\| \le \epsilon_N^{\frac{1}{2}}.$$

After repeating this argument $N-1$ times, we obtain, successively, vectors $x'_1, x'_2, \ldots, x'_N$ and vectors $z'_1, \ldots, z'_N$ satisfying

$$\|[x'_N \otimes z'_i]_{T_i} - [L_i]_{T_i}\| < \epsilon_{N+1}, \quad i = 1, \ldots, N,$$
$$\|x - x'_N\| \leq \|x - x'_1\| + \cdots + \|x'_{N-1} - x'_N\| \leq N\epsilon_N^{\frac{1}{2}},$$

and

$$\|z_i - z'_i\| \leq \epsilon_N^{\frac{1}{2}}, \quad i = 1, \ldots, N.$$

In order to obtain inequality (7) for $i = N+1$, we recall that, by **[1]**, the set

$$\mathcal{S} = \{s \in \mathcal{H} : \exists\ \omega \text{ with } [s \otimes w]_{T_{N+1}} = [L_{N+1}]_{T_{N+1}}\}$$

is dense in $\mathcal{H}$, and therefore we may choose a sequence $\{s_n\}_{n=1}^{\infty}$ from $\mathcal{S}$ satisfying

$$\|s_n - x'_N\| \to 0$$

and a sequence $\{w_n\}$ from $\mathcal{H}$ such that

$$[s_n \otimes w_n]_{T_{N+1}} = [L_{N+1}]_{T_{N+1}}, n \in \mathbb{N}.$$

Thus we may set $x' = s_n$ and $z''_{N+1} = w_n$ for n sufficiently large, and thereby arrange that (7) and (8) are satisfied. Unfortunately, (10) may not be satisfied, so the final step in the proof is to modify the definition of z''_{N+1} so that (10) will be satisfied and (7) (for $i = N+1$) will be maintained. To this end, let $\{t_k\}$ be an orthonormal sequence given by Corollary 2.4 (applied to T_{N+1}) such that

$$\|[x' \otimes t_k]_{T_{N+1}}\| \xrightarrow{k} 0,$$

and define $z'_{N+1} = z''_{N+1} + (2\pi_{N+1} + \epsilon)t_k$ for some k sufficiently large that

$$\|z'_{N+1}\| > \pi_{N+1} + \epsilon$$

and (7) for $i = N+1$ remains valid. Thus the proof of the lemma is complete.

We are now prepared to prove Theorem 2.1.

Proof of Theorem 2.1: Let x_0 in $\mathcal{H}$ be fixed, and let $\epsilon > 0$ be given. We will show that there exist a vector x and a sequence $\{y_n^x\}_{n=1}^\infty$ from $\mathcal{H}$ satisfying (1), (2), and $\|x - x_0\| < \epsilon$, which will complete the proof. The vectors x and $\{y_n^x\}_{n=1}^\infty$ are constructed by an inductive process. We begin by choosing a sequence $\{\epsilon_n\}_{n=1}^\infty$ of positive numbers such that

$$\sum_{n=1}^{\infty}(n+1)\epsilon_n^{\frac{1}{2}} < \epsilon/2. \tag{16}$$

The first step in the induction process is to consider the equation

$$[x_1 \otimes g_1]_{T_1} = [L_1]_{T_1}, \tag{17}$$

where x_1 and g_1 are unknowns. Since $T_1 \in \mathbb{A}_{\aleph_0}$, we know from [**1**] that the set of vectors x_1 in $\mathcal{H}$ for which there exists some $g_1 = g_1(x_1)$ satisfying (17) is dense in $\mathcal{H}$, and hence, in particular, we may choose such x_1 and g_1 satisfying (17) and $\|x_0 - x_1\| < 2\epsilon_1^{\frac{1}{2}}$. Since we do not know that $\|g_1\| > \pi_1+\epsilon$, we must now replace g_1 by a vector y_1 for which (17) is almost satisfied and $\|y_1\| > \pi_1 + \epsilon$. To accomplish this, we obtain from Corollary 2.4 an orthonormal sequence $\{t_n\}_{n=1}^\infty$ for which $\|[x_1 \otimes t_n]_{T_1}\| \to 0$. Therefore, by choosing n large enough and setting $y_1 = g_1 + (2\pi_2 + \epsilon)t_n$ we obtain the inequalities

$$\|[x_1 \otimes y_1]_{T_1} - [L_1]_{T_1}\| < \epsilon_1,$$
$$\|x_0 - x_1\| < 2\epsilon_1^{\frac{1}{2}},$$

and

$$\|y_1\| > \pi_1 + \epsilon.$$

Suppose now that we have constructed vectors $x_1, \ldots, x_N$, and associated with each such x_i a sequence $\{y_k^i\}_{k=1}^i$, such that the following inequalities are satisfied:

$$\|x_{i+1} - x_i\| < (i+1)\epsilon_i^{\frac{1}{2}}, \quad i = 2, \ldots, N-1,$$
$$\|[x_i \otimes y_k^i]_{T_k} - [L_k]_{T_k}\| < \epsilon_i, 1 \le k \le i, \quad i = 1, \ldots, N,$$
$$\|y_k^{i+1} - y_k^i\| < \epsilon_i^{\frac{1}{2}}, 1 \le k \le i, \quad i = 1, \ldots, N-1,$$
$$\|y_i^i\| > \pi_i + \epsilon, \quad i = 1, \ldots, N.$$

We wish to construct now a vector x_{N+1} and an associated sequence $\{y_1^{N+1}, \ldots, y_{N+1}^{N+1}\}$ such that

$$\|x_{N+1} - x_N\| < (N+1)\epsilon_N^{\frac{1}{2}},$$
$$\|[x_{N+1} \otimes y_k^{N+1}]_{T_k} - [L_k]_{T_k}\| < \epsilon_{N+1}, 1 \le k \le N+1,$$
$$\|y_k^{N+1} - y_k^N\| < \epsilon_N^{\frac{1}{2}}, 1 \le k \le N,$$

and

$$\|y_{N+1}^{N+1}\| > \pi_{N+1} + \epsilon.$$

But things have been arranged so that we may apply Lemma 2.5 to accomplish this by setting $x = x_N$ and $z_i = y_i^N, 1 \le i \le N$, and then defining $x_{N+1} = x'$ and $y_i^{N+1} = z_i', 1 \le i \le N+1$. Thus, by induction, we may construct a sequence $\{x_N\}_{N=1}^\infty$ and for each $i \in \mathbb{N}$, a sequence $\{y_i^N\}_{N=i}^\infty$ such that

$$\|x_{N+1} - x_N\| < (N+1)\epsilon_N^{\frac{1}{2}}, \ \ N \in \mathbb{N},$$
$$\|y_i^{N+1} - y_i^N\| < \epsilon_N^{\frac{1}{2}}, \ \ i \in \mathbb{N}, \ \ N = i, i+1, \ldots,$$
$$\|[x_N \otimes y_i^N]_{T_i} - [L_i]_{T_i}\| < \epsilon_n, N \in \mathbb{N}, \ \ i = 1, 2, \ldots, N,$$

and

$$\|y_i^i\| > \pi_i + \epsilon, \ \ i \in \mathbb{N}.$$

Since $\sum_{N=1}^\infty (N+1)\epsilon_N^{\frac{1}{2}} < \epsilon/2$, it follows easily from these inequalities that the sequences $\{x_N\}_{N=1}^\infty$ and $\{y_i^N\}_{N=i}^\infty, i \in \mathbb{N}$, are Cauchy, and thus converge. Let us write $x = \lim x_N$ and $y_i^x = \lim_N y_i^N, i \in \mathbb{N}$. Then it is clear from these inequalities and continuity that (1) and (2) are satisfied, and thus the proof of Theorem 2.1 is complete.

3. Some questions and examples. Theorem 2.1 and its corollaries raise some interesting questions. A first question is whether we might expect that every countable family of operators $\{T_n\}_{n=1}^\infty$ in the class $\mathbb{A}_{\aleph_0}(\mathcal{H}) \cap C_{\cdot 0}$ has a common nontrivial invariant subspace. The following example shows that this is not the case.

Example 3.1. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$, and consider the operator $B \in \mathcal{L}(\mathcal{H})$ defined by $Be_n = (n/(n+1))^{\frac{1}{2}} e_{n+1}, n \in \mathbb{N}$.

This operator B is a *Bergman shift* operator, and it is well known (cf. [**2**, Chapter IX]) that $B \in \mathbb{A}_{\aleph_0} \cap C_{00}$ and hence (cf. [**2**, Remark 5.7]) that $B^* \in \mathbb{A}_{\aleph_0} \cap C_{00}$ also. But if $\mathcal{M}$ were a nontrivial common invariant subspace for the pair B, B^*, then $\mathcal{M}$ would be reducing for B, and it is well known (cf. [**7**]) that unilateral weighted shifts with nonzero weight sequences have no nontrivial reducing subspaces. Thus the pair $\{B, B^*\}$ has no nontrivial common invariant subspace.

This example leaves open, however, the following questions.

Problem 3.2. Suppose that $\{T_1, T_2\}$ is a pair of *commuting* operators in the class $\mathbb{A}_{\aleph_0} \cap C_{\cdot 0}$. Does the pair necessarily have a nontrivial common invariant subspace?

We remark, once again, that if the answer to Problem 3.2 is affirmative, then every operator T in $\mathcal{L}(\mathcal{H})$ whose norm and spectral radius coincide has a nontrivial invariant subspace (Corollary 1.2).

Another interesting question related to Theorem 2.1 and Problem 3.2 to which we do not know the answer is the following.

Problem 3.3. Suppose that $\{T_1, T_2\} \subset \mathcal{L}(\mathcal{H})$ is a pair of operators in the class $\mathbb{A}_{\aleph_0} \cap C_{\cdot 0}$. Is it necessarily the case that there exist nonzero vectors x and y in $\mathcal{H}$ such that $[x \otimes y]_{T_1} = 0$ and $[x \otimes y]_{T_2} = 0$? Note that Theorem 2.1 provides nonzero vectors x_1, y_1, and y_2 such that $[x \otimes y_i]_{T_i} = 0, i = 1, 2$ and that, if T_1, T_2 are arbitrary operators, even this weaker property need not hold (cf. Remark 1.3 above).

One cannot hope to improve Theorem 2.1 to the extent that all of the vectors $\{y_n^x\}_{n=1}^\infty$ given in the conclusion of that theorem can be taken to be the same without placing some restrictions on the sequence $\{[L_n]_{T_n}\}$, as the following example shows.

Example 3.4. Suppose $T \in \mathbb{A}_{\aleph_0}(\mathcal{H}) \cap C_{\cdot 0}(\mathcal{H})$. It follows easily from the known structure theory of this class that $\{T^2, T^3\} \subset \mathbb{A}_{\aleph_0} \cap C_{\cdot 0}$, and for any operator $A \in \mathbb{A}$ and $\lambda \in \mathbb{D}$, let $[C_\lambda]_A$ denote that element of Q_A such that $\langle \Phi_A(f), [C_\lambda] \rangle = f(\lambda)$ for every f in H^∞. Suppose now that, for some $\lambda \neq 0$, we could solve the simultaneous system of equations

$$[x \otimes y]_{T^2} = [C_\lambda]_{T^2},$$
$$[x \otimes y]_{T^3} = [C_\lambda]_{T^3}.$$

Then it would follow trivially that

$$\lambda^2 = ((T^3)^2 x, y) = ((T^2)^3 x, y) = \lambda^3,$$

and obvious absurdity.

4. Remarks. 1. Both authors were supported by a grant from the National Science Foundation during the period when this paper was written.

2. Our friend and colleague Constantin Apostol, upon being told a weaker version of Theorem 2.1 several years ago, told the authors that he had proved a similar result (unpublished).

References

1. H. Bercovici, B. Chevreau, C. Foias, and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. II*, Math. Z. **187** (1984), 97-103.
2. H. Bercovici, C. Foias, and C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Mathematics **56** (1985), Amer. Math. Soc., Providence.
3. B. Chevreau, C. Pearcy, and A. L. Shields, *Finitely connected domains G, representations of $H^\infty(G)$ and invariant subspaces*, J. Operator Theory **6** (1981), 375-405.
4. R. G. Douglas and C. Pearcy, *Hyperinvariant subspaces and transitive algebras*, Michigan Math. J. **19** (1972), 1-12.
5. C. Foias and C. Pearcy, *(BCP)-operators and enrichment of invariant subspace lattices*, J. Operator Theory **9** (1983), 187-202.
6. H. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J. (1962).
7. A. L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys No. 13, Amer. Math. Soc., Providence RI (1974), 49-128.

UFR de Mathématiques et Informatique
Université de Bordeaux I
351, Cours de la Libération
405 Talence Cédex France

Department of Mathematics
Texas A& M University
College Station, Texas 77843

Received September 10, 1991