4.1 Exponential Growth and Decay

A function that grows or decays by a constant percentage change over each fixed change in input is called an exponential function.

Example 1

A certain quantity has an initial value of 50 and grows at a rate of 9% each month. Fill in the table below.

$9\% \xrightarrow[\div 100]{} 0.09$ ($\times 100$ back)

Month	Amt. of Increase	Value
0	-	50
1	4.5	54.5
2	4.91	59.41
3	5.35	64.76

$50(0.09) = 4.5$

$54.5(0.09) = 4.91$

$59.41(0.09) = 5.35$

Example 2

A certain radioactive substance decays at a rate of 11.4% per hour. Suppose initially there are 100mg of this substance. Complete the table below.

$11.4\% \rightarrow .114$

Hour	Amt. of Decrease	Value
0	-	100
1	11.4	88.6
2	10.10	78.5
3	8.95	69.55

$100(.114) = 11.4$

$88.6(.114) = 10.10$

$78.5(.114) = 8.95$

Growth/Decay Factor

For an exponential function that has a constant percent change r (in decimal form), the corresponding growth/decay factor for the function is:

$a = 1 + r$ (for growth) or $a = 1 - r$ (for decay)

Example 3

Give the growth or decay factor for the exponential functions from example 1 and example 2.

Ex 1 Growth $r = .09$

$a = 1 + r = 1 + .09 = \boxed{1.09}$

Ex 2 Decay $r = .114$

$a = 1 - r = 1 - .114 = \boxed{.886}$

Exponential Function Formula

An exponential function that has initial value P and growth factor a has formula:

$\boxed{f(x) = Pa^x}$ where x has units matching the time units of the percent change

Example 4

Give the formula of the exponential functions from example 1 and example 2.

Ex 1 $a = 1.09$ $P = 50$

$f(x) = 50(1.09)^x$

Ex 2 $a = .886$ $P = 100$

$f(x) = 100(.886)^x$

Example 5

The world's population in 1900 was about 1.6 billion and has been growing by about 1.4% per year since. Give a formula for the world population x years after 1900. Use function notation to represent the world population in 1995.

x	Year	Population
0	1900	1.6
95	1995	

Growth factor $= 1 + .014 = 1.014$

$P = 1.6$

$f(x) = 1.6(1.014)^x$

$\boxed{f(95) = 1.6(1.014)^{95}}$

Example 6

Iodine-131 is a radioactive form of iodine. It decays at a rate of 8.3% per day. Suppose, in a certain experiment, 80mg of Iodine-131 is present. Give a formula for the amount of Iodine-131 present after x days. How much is remaining 36 hours into the experiment?

Decay factor $= 1 - .083 = 0.917 = a$

$P = 80$

$$\boxed{f(x) = 80(0.917)^x}$$

$$\frac{36 \text{ hours}}{24 \text{ hours}} = 1.5$$

36 hours = 1.5 days

$$f(1.5) = 80(.917)^{1.5} = \boxed{70.25\text{mg}}$$

Unit Conversion for Growth Factors

If the growth or decay factor for 1 period of time is a, then the growth or decay factor for k periods is a^k.

GROWTH FACTOR FOR ONE PERIOD OF TIME → Raise to k^{th} Power → GROWTH FACTOR FOR k PERIODS OF TIME

GROWTH FACTOR FOR k PERIODS OF TIME → Raise to $(1/k)^{th}$ Power → GROWTH FACTOR FOR ONE PERIOD OF TIME

So if a quantity is growing by 4% each hour, it has an hourly growth factor of a = 1.04.

$4\% \rightarrow .04$

$a = 1 + .04 = 1.04$

Then the daily growth factor is $1.04^{24} = 2.5633$,

and thus is growing by 156.33% per day.

$2.5633 - 1$
$= 1.5633$

1.5633×100
$= 156.33\%$

The minute growth factor is $1.04^{1/60} = 1.00065$,

and thus is growing by 0.065% per minute.

$1.00065 - 1 = 0.00065$

$0.00065 \times 100 = 0.065\%$

Example 7

Suppose a certain quantity grows by 7.4% each year.

(a) Give the yearly growth factor.

(b) Give the decade growth factor. What is the constant percent change per decade?

(c) Give the monthly growth factor. What is the constant percent change per month?

(a) $7.4\% \rightarrow .074$

Yearly Growth Factor $= 1 + .074 = \boxed{1.074}$

(b) Decade $=$ 10 Years

$DGF = (YGF)^{10}$ $\quad DGF = (1.074)^{10} = \boxed{2.04}$

Constant % Change $= (2.04 - 1) \cdot 100$

$= (1.04) \cdot 100 = \boxed{104\%}$

(c) Month $= \frac{1}{12}$ Year

$MGF = (YGF)^{1/12}$ $\quad MGF = (1.074)^{1/12} = \boxed{1.006}$

Const. % Change $= (1.006 - 1) \cdot 100$

$= 0.006 \cdot 100 = \boxed{0.6\%}$