Section 5.1: Logistic Functions

Logistic Growth Curve

The logistic growth curve has the following properties:

- Initially the growth is rapid, nearly exponential
- The inflection point represents the location of most rapid growth
- After the inflection point, the growth rate declines. The function has a limiting value, known as the carrying capacity.
- The point of inflection occurs at an output of half of the carrying capacity. This is the level of maximum growth. This level is often called the optimum yield level.

Logistic Model Formula

$$f(x) = \frac{K}{1 + be^{-rx}}$$

- The constant K is the carrying capacity. It is the limiting value of f.
- The point of inflection, or optimum yield level, occurs at an output of $\frac{K}{2}$.
- The constant b is determined by the formula $b = \frac{K}{f(0)} - 1$.
- The r-value is the intrinsic exponential growth rate. In the absence of limiting factors, growth would be exponential according to the formula $y = f(0)e^{rx}$. The corresponding growth factor would be $a = e^{r}$.

For example:

The function $S(x)=\dfrac{2.4}{1+239e^{-0.338x}}$ models the Pacific sardine population (measure in million tons of fish) x years from now.

- In the absence of limiting factors, the annual growth factor for the Pacific sardine would be $a=e^{0.338}\approx 1.402$.
    - So the annual growth rate would be 40.2% per year.
- The carrying capacity is 2.4 million tons of fish.
- The optimum yield level is $\frac{2.4}{2}=1.2$ million tons of fish.
- Using the crossing graphs method, we can find the input x = 16.203 years corresponds to the optimum yield level.

Example 1

We begin selling a new magazine in a small town. Initial sales are 250 magazines per month. We believe that in the absence of limiting factors, our sales will increase by 6% per month, but the size of the town limits our total sales to 1000 magazines per month.

(a) Construct a logistic model for our magazine sales under these conditions.

(b) When can we expect sales to reach 750 magazines per month?

Example 2

Compact fluorescent light bulbs save energy when compared to traditional incandescent bulbs. Our green energy campaign includes efforts to get local residents to exchange their incandescent bulbs for fluorescent ones. Initially 200 households make the change. Market studies suggest that his would increase by 25% per month in the absence of limiting factors. In one target area, there are 250000 households.

(a) Make a logistic model for the number of households converting to fluorescent bulbs after x months.

(b) What is the optimal yield level of this function, and when is it achieved?

Example 3

In a city of half a million people, there are initially 800 cases of a particularly virulent strain of flu. The C.D.C. in Atlanta claims that the cumulative number of infections will increase by 40% per week if there are no limiting factors.

(a) Make a logistic model for the cumulative number of people infected?

(b) When is this quantity increasing most rapidly?

Logistic Regression

When a scatterplot of data shows the behavior of logistic growth (initially exponential looking but then leveling off after an inflection point), then the data may be best modeled with a logistic regression. Following the same steps we have done in the TI for other regressions, we can get a logistic regression model.

Example 4

The table gives data showing the number C, in millions, of basic cable subscribers in various years. Let x represent the number of years since 1970.

Year	C
1975	9.8
1980	17.5
1985	35.4
1990	50.5
1995	60.6
2000	66.3

(a) Find a logistic regression model for this data.

(b) What is the limiting value (aka carrying capacity) of your model?