Section 3.5
The Inverse of a Square Matrix

In this section we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations.

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that $\underbrace{\boxed{AA^{-1}}}_{1} = \underbrace{\boxed{A^{-1}A}}_{2} = I_n$ is called the **inverse of A**.

Note: Not every square matrix has an inverse. A matrix with no inverse is called **singular**.

Example 1: Determine whether the following pairs of matrices are inverses of each other.

a. $\underset{A}{\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}}$ and $\underset{B}{\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}}$

Check if:

$$\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix} = I_2$$

$$\begin{bmatrix} 5(1)+4(-1) & 5(-4)+4(5) \\ 1(1)+1(-1) & 1(-4)+1(5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

If so, then must also check if:

$$\underset{B}{\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}}\underset{A}{\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}} = I_2$$

$$\begin{bmatrix} 1(5)-4(1) & 1(4)-4(1) \\ -1(5)+5(1) & -1(4)+5(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Inverses? YES

b. $\underset{A}{\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}}$ and $\underset{B}{\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}}$

Check if:

$$\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix} = I_2$$

$$\begin{bmatrix} \overbrace{-2\left(-\tfrac{4}{5}\right)-3\left(\tfrac{1}{5}\right)}^{8/5-3/5} & \overbrace{-2\left(-\tfrac{6}{5}\right)-3\left(\tfrac{4}{5}\right)}^{\frac{12}{5}-\frac{12}{5}} \\ 1\left(-\tfrac{4}{5}\right)+4\left(\tfrac{1}{5}\right) & \overbrace{1\left(-\tfrac{6}{5}\right)+4\left(\tfrac{4}{5}\right)}^{-\frac{6}{5}+\frac{16}{5}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \neq I_2$$

If so, then must also check if:

$$\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix} = I_2$$

No need to check!

Inverses? NO

How to Find the Inverse of a Square Matrix

1. Adjoin the square matrix A with the identity matrix of the same size, $[A \mid I]$.
2. Use the Gauss-Jordan elimination method to reduce $[A \mid I]$ to the form $[I \mid B]$, if possible.

The matrix B is the inverse of the matrix A. It may be verified by checking $AB = BA = I_n$.

If in the process of reducing $[A \mid I]$ to $[I \mid B]$, there is a row of zeros to the left of the vertical line then the matrix does not have an inverse.

Example 2: Find the inverse of each matrix, if possible.

a. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \quad -3R1 + R2 \to R2$$

$$\begin{array}{rrrrr} & -3 & -6 & -3 & 0 \\ + & 3 & 4 & 0 & 1 \\ \hline & 0 & -2 & -3 & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad -\tfrac{1}{2}R2 \to R2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \quad -2R2 + R1 \to R1$$

$$\begin{array}{rrrrr} & 0 & -2 & -3 & 1 \\ + & 1 & 2 & 1 & 0 \\ \hline & 1 & 0 & -2 & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

b. $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad -R1 - R2 + R3$$

$+$

$$\begin{array}{ccc|ccc} -1 & -2 & -3 & -1 & 0 & 0 \\ -2 & -1 & -2 & 0 & -1 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & -1 & 1 \end{array}$$

B is singular

c. $C = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 1 & 0 & 0 \end{array}\right] \quad R1 + R2 \to R2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & -1 & 1 & 1 & 0 & 0 \end{array}\right]$$

0 0

$-3R1 + R3 \to R3$

$$\begin{array}{cccccc} -3 & 0 & -3 & 0 & 0 & -3 \\ +\;3 & -1 & 1 & 1 & 0 & 0 \\ \hline 0 & -1 & -2 & 1 & 0 & -3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 & 0 & -3 \end{array}\right] \quad R2 + R3 \to R3$$

$+$

0

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -2 \end{array}\right]$$

$R1 + R3 \to R1$

$R2 + R3 \to R2$

$-1R3 \to R3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$C^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Formula for the Inverse of a 2X2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $D = ad - bc$ is not equal to zero. Then. A^{-1} exists and is given by $\boxed{A^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Example 3: Find the inverse of the following matrix.

a. $E = \begin{pmatrix} -5 & 10 \\ -2 & 7 \end{pmatrix}$

b. $F = \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix}$

$D = -5(7) - 10(-2)$
$= -35 + 20 = -15$

$E^{-1} = -\frac{1}{15}\begin{pmatrix} 7 & -10 \\ 2 & -5 \end{pmatrix} = \begin{bmatrix} -7/15 & 2/3 \\ -2/15 & 1/3 \end{bmatrix}$

$D = 2(-6) - (4)(-3)$
$= -12 - (-12) = 0$

singular
(no inverse)

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations AX = B, involving the same coefficient matrix, A, and different matrices of constants, B.

The steps are:
1. Write the system of equations in matrix form, i.e., AX= B. *A is the coefficient matrix, X is the variable matrix and B is the constant matrix.*
2. Find the inverse of the coefficient matrix, A.
3. Multiply both sides by A^{-1}.
4. State the answer.

$A^{-1}AX = A^{-1}B$
$X = A^{-1}B$

Example 4: Write each system of equations as a matrix equation, then solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 2y &= 12 \\ -3x + 5y &= -8 \end{aligned}$$

$$\begin{aligned} 2x - 2y &= 0 \\ -3x + 5y &= 4 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 12 \\ -8 \end{pmatrix}}_{B_1}$$

$$\underbrace{\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 \\ 4 \end{pmatrix}}_{B_2}$$

$$\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}$$

Now find the inverse of the coefficient matrix.

D = ad – bc = $2(5) - (-2)(-3) = 10 - 6 = 4$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

$AX = B_1$

$X = A^{-1}B_1$

$$X = \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{\cancel{4}}(\overset{3}{\cancel{12}}) + \frac{1}{\cancel{2}}(-\overset{4}{\cancel{8}}) \\ \frac{3}{\cancel{4}}(\overset{3}{\cancel{12}}) + \frac{1}{\cancel{2}}(-\overset{4}{\cancel{8}}) \end{bmatrix}$$

$$\begin{bmatrix} 15-4 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}\begin{matrix} = x \\ = y \end{matrix}$$

$$\boxed{\begin{matrix} x = 11 \\ y = 5 \end{matrix}}$$

$AX = B_2$

$X = A^{-1}B_2$

$$X = \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(4) \\ \frac{1}{2}(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{matrix} = x \\ = y \end{matrix}$$

$$\boxed{\begin{matrix} x = 2 \\ y = 2 \end{matrix}}$$