Section 5.4
Permutations and Combinations

Definition: n-Factorial
For any natural number n, $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.
$0! = 1$

A **combination** of a set is arranging the elements of the set without regard to order.

Example: The marinade for my steak contains soy sauce, Worchester sauce and a secret seasoning.

Formula: $C(n,r) = \dfrac{n!}{r!(n-r)!}$, $r \le n$, where n is the number of distinct objects and r is the number of distinct objects taken r at a time.

$$C(\overset{n}{5},\overset{r}{2}) = \frac{5!}{2!\;3!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\overset{2}{\cancel{4}}\cdot 5}{\cancel{1}\cdot\cancel{2}\cdot 1\cdot\cancel{2}\cdot\cancel{3}} = 10$$

A **permutation** of a set is arranging the elements of the set with regard to order.

Example: My previous pin number was 2468, now it's 8642.

$$P(10,4) = 5040$$

Formula: $P(n,r) = \dfrac{n!}{(n-r)!}$, $r \le n$, where n is the number of distinct objects and r is the number of distinct objects taken r at a time.

$$P(\overset{n}{5},\overset{r}{2}) = \frac{5!}{3!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}} = 20$$

The deck of 52 playing cards is a good set to use with some of these problems, so let's make some notes:

52 total cards, no jokers--26 red and 26 black
Suits are Hearts, Diamonds, Clubs, and Spades.
Each suit has 13 cards, one of each 2 – 10, Jack, Queen, King, Ace
Face cards are J, Q, K only = 12 face cards

2 ♡	3 ♡	4 ♡	5 ♡	6 ♡	7 ♡	8 ♡	9 ♡	10 ♡	J ♡	Q ♡	K ♡	A ♡
2 ♢	3 ♢	4 ♢	5 ♢	6 ♢	7 ♢	8 ♢	9 ♢	10 ♢	J ♢	Q ♢	K ♢	A ♢
2 ♣	3 ♣	4 ♣	5 ♣	6 ♣	7 ♣	8 ♣	9 ♣	10 ♣	J ♣	Q ♣	K ♣	A ♣
2 ♠	3 ♠	4 ♠	5 ♠	6 ♠	7 ♠	8 ♠	9 ♠	10 ♠	J ♠	Q ♠	K ♠	A ♠

Example 1: In how many ways can 7 cards be drawn from a well-shuffled deck of 52 playing cards?
Combination or Permutation

$$C(52,7) = 133{,}784{,}560$$

Example 2: An organization has 30 members. In how many ways can the positions of president, vice-president, secretary, treasurer, and historian be filled if not one person can fill more than one position?
Combination or Permutation

P(30, 5) = 17,100,720

Example 3: In how many ways can 10 people be assigned to 5 seats?
Combination or Permuation

P(10, 5) = 30,240

Example 4: An organization needs to make up a social committee. If the organization has 25 members, in how many ways can a 10 person committee be made?
Combination or Permuation

C(25, 10) = 3,268,760

Example 5: Seven people arrive at a ticket counter at the same time to buy concert tickets. In how many ways can they line up to purchase their tickets?
Combination or Permuation

P(7, 7) = 5040

Formula: Permutations of n objects, not all distinct

Given a set of n objects in which n_1 objects are alike and of one kind, n_2 objects are alike and of another kind,…, and, finally, n_r objects are alike and of yet another kind so that

$$n_1 + n_2 + ... + n_r = n$$

then the number of permutations of these *n* objects taken *n* at a time is given by

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Example: All arrangements that can be made using all of the letters in the word COMMITTEE.

Example 6: REENNER, a small software company would like to make letter codes using all of the letters in the word REENNER . How many codes can be made from all the letters in this word?
Combination or Permuation

n = 7
R = 2
E = 3
N = 2

$$\frac{7!}{2!\,3!\,2!} = 210$$

Example 7: A coin is tossed 5 times.
a. How many outcomes are possible?

$$(2)(2)(2)(2)(2) = 2^5 = 32$$

b. In how many outcomes do exactly 3 heads occur? $C(5,3) = 10$

{ ($H_1H_2H_3TT$), ($H_1H_2T\,H_4T$), ($H_1H_2TT\,H_5$), ($H_1TH_3T\,H_5$), ($H_1T\,TH_4H_5$),
($H_1T\,H_3H_4T$), ($TH_2H_3H_4T$), ($T\,H_2H_3T\,H_4$), ($TH_2TH_4H_5$), ($T\,TH_3H_4H_5$) }

c. In how many outcomes do exactly 2 tails occur?

$$C(5,2) = 10 \qquad C(n,k) = C(n, n-k)$$

Example 8: A coin is tossed 18 times.
a. How many outcomes are possible?

$$2^{18} = 262{,}144$$

$$C(n,0) = 1$$
$$C(n,1) = n$$
$$C(n,n) = 1$$
$$C(n,n-1) = n$$

b. In how many outcomes do exactly 7 tails occur?

$$C(18,7) = 31{,}824$$

c. In how many outcomes do at most 2 tails occur?

0T or 1T or 2T

$$\underbrace{\cancel{C(18,0)}}_{1} + \underbrace{\cancel{C(18,1)}}_{18} + C(18,2) = 1+18+153 = \boxed{172}$$

d. In how many outcomes do at least 17 tails occur?

17T or 18T

$$\underbrace{\cancel{C(18,17)}}_{18} + \underbrace{\cancel{C(18,18)}}_{1} = 18+1 = 19$$

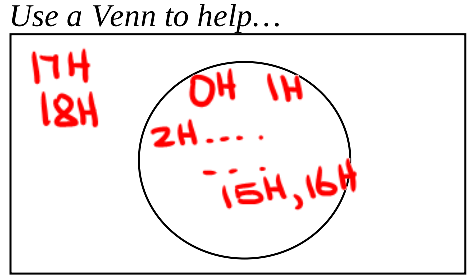
e. In how many outcomes do at most 16 heads occur?

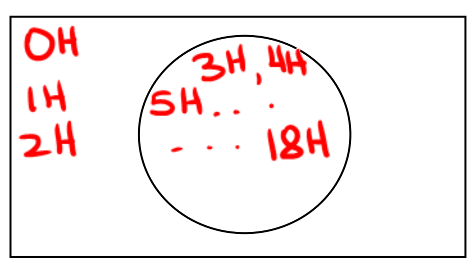
$$2^{18} - [C(18,17) + C(18,18)] = 2^{18} - 19$$

f. In how many outcomes do at least 3 heads occur?

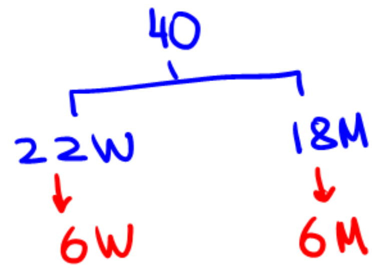
$$2^{18} - [C(18,0) + C(18,1) + C(18,2)]$$
$$= 2^{18} - 172 = 261{,}972$$

Use a Venn to help…





Example 9: A judge has a jury pool of 40 people that contains 22 women and 18 men. She needs a jury of 12 people.



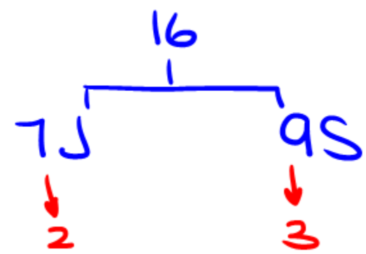
a. How many juries can be made?

$C(40,12)$

b. How many juries contain 6 women and 6 men?

$C(22,6)\,C(18,6)$

Example 10: A club of 16 students, 7 juniors and 9 seniors, is forming a 5 member subcommittee.

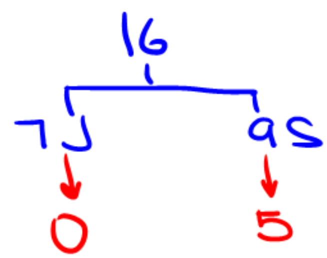


a. How many subcommittees can be made?

$C(16,5)$

b. How many subcommittees contain 2 juniors and 3 seniors?

$C(7,2)\,C(9,3)$

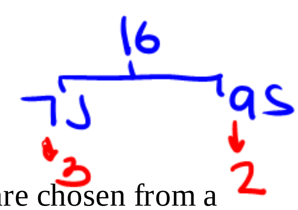


c. How many subcommittees contain all seniors?

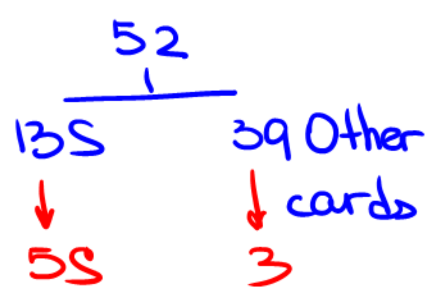
$\cancel{C(7,0)}^{1}\,C(9,5) = C(9,5)$

d. How many subcommittees contain 3 juniors?

$C(7,3)\,C(9,2)$



Example 11: In how many ways can 5 spades be chosen if 8 cards are chosen from a well-shuffled deck of 52 playing cards?



$C(13,5)\,C(39,3)$

Example 12: A customer at a fruit stand picks a sample of 7 oranges at random from a crate containing 35 oranges of which 5 are rotten.

35

30 Good → 3   5 Rotten → 4

a. How many selections can be made?

$C(35,7)$

b. How many selections contain 4 rotten?

$C(30,3)\,C(5,4)$

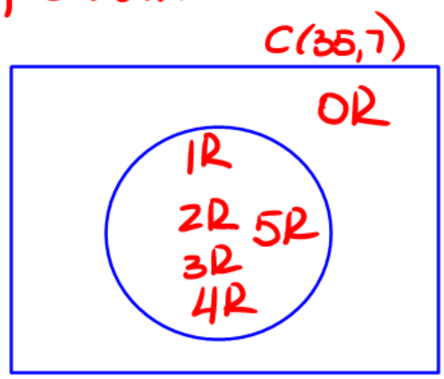
c. How many selections contain at least 4 rotten? 3G 4R, 2G 5R

$C(30,3)\,C(5,4) + C(30,2)\,C(5,5)$

d. How many selections contain at most 4 rotten? 0R, 1R, 2R, 3R, 4R

complement 5R

$C(35,7) - C(30,2)\,C(5,5)$



e. How many selections contain at least 1 rotten?

$C(35,7) - C(30,7)\,C\cancel{(5,0)}^{1}$

f. How many selections contain at least 3 rotten? 4G 3R, 3G 4R, 2G 5R   ~~0R~~, ~~1R~~, ~~2R~~

$C(30,4)\,C(5,3)$

$+ C(30,3)\,C(5,4)$

$+ C(30,2)\,C(5,5)$

Example 13: A store receives a shipment of 35 calculators including 8 that are defective. A sample of 6 calculators is chosen at random. How many selections contain at least 5 defective calculators?

5D or 6D

35

27G → 1, 0   8D → 5, 6

$C(27,1)\,C(8,5)$

$+ C\cancel{(27,0)}^{1}\,C(8,6)$