Section 6.5
Conditional Probability

Example 1: Two cards are drawn without replacement in succession from a well-shuffled deck of 52 playing cards. What is the probability that the second card drawn is an ace, given that the first card drawn was an ace?

$P(\text{First Ace}) = 4/52$

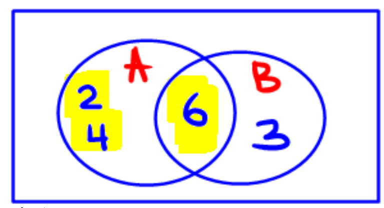
$P(\text{Second Ace}) = 3/51 = \frac{1}{17} \approx 0.0588$

The previous example is an example of **conditional probability**.

Conditional Probability of an Event

If A and B are events in an experiment and $P(A) \neq 0$, then the conditional probability that the event B will occur given that the event A has already occurred is

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

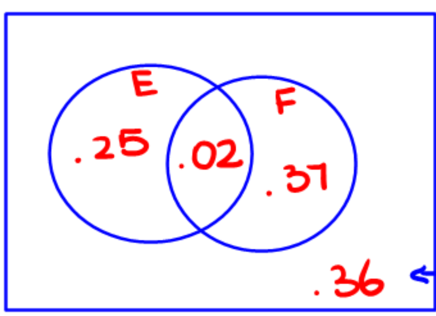


Example 2: Given P(E) = 0.26, P(F) = 0.58, and $P(E \cap F) = 0.02$. Find P(E|F).

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.02}{0.58} = .0348$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.02}{.26} = 0.0769$$

Example 3: Given $P(E^c \cap F) = 0.37$, $P(F^c) = 0.61$ and P(E) = 0.27. Find $P(F^c|E^c)$.



$P(F) = 1 - P(F^c) = 1 - .61 = .39$

$.36 \leftarrow 1 - (.25 + .02 + .37)$

$$P(F^c|E^c) = \frac{P(F^c \cap E^c)}{P(E^c)} = \frac{P((F \cup E)^c)}{P(E^c)}$$

$$= \frac{.36}{1 - .27} = \frac{.36}{.73} = .4932$$

Example 4: A pair of fair dice is cast. What is the probability that the sum of the numbers falling uppermost is 6, given that exactly one of the numbers is a 2?

E = sum is 6

F = one number is 2

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{2/36}{10/36} = \frac{2}{10} = .2$$

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
⚁	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
⚂	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
⚃	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
⚄	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
⚅	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Example 5: A pair of fair dice is cast. What is the probability that at least one of the numbers falling uppermost is a 5, if it is known that the two numbers are different?

E = at least one 5

F = two numbers are diff.

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{10/36}{30/36} = \frac{10}{30} = .3333$$

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	~~(1,1)~~	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
⚁	(1,2)	~~(2,2)~~	(3,2)	(4,2)	(5,2)	(6,2)
⚂	(1,3)	(2,3)	~~(3,3)~~	(4,3)	(5,3)	(6,3)
⚃	(1,4)	(2,4)	(3,4)	~~(4,4)~~	(5,4)	(6,4)
⚄	(1,5)	(2,5)	(3,5)	(4,5)	~~(5,5)~~	(6,5)
⚅	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	~~(6,6)~~

The Product Rule

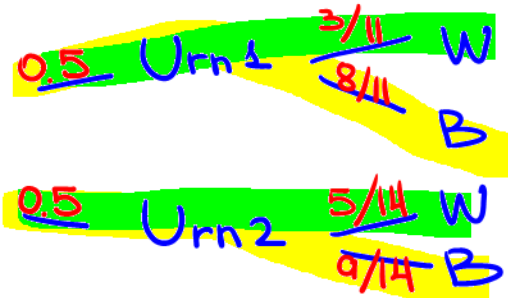
Suppose we know the conditional probability and we are interested in finding $P(A \cap B)$.

Then if P(A) $\neq 0$ then $P(A \cap B) = P(A)P(B|A)$.

In Chapter 5 we used tree diagrams to help us list all outcomes of an experiment. In this Section tree diagrams will provide a systematic way to analyze probability experiments that have two or more trials. For example, say we choose one card at random from a well-shuffled deck of 52 playing cards and then go back in for another card. The first trial would be the first draw. The second trial would be the second draw.

11 14

Example 6: Urn 1 contains 3 white and 8 blue marbles. Urn 2 contains 5 white and 9 blue marbles. One of the two urns is chosen at random with one as likely to be chosen as the other. An urn is selected at random and then a marble is drawn from the chosen urn.



a. What is the probability that Urn 2 was chosen?

0.5

b. What is the probability that a white marble was chosen, given that Urn 2 was chosen?

$P(W|U_2) = \frac{5}{14}$

c. What is the probability that Urn 1 was chosen and that a blue marble was chosen?

$P(U_1 \cap B) = P(U_1)P(B|U_1)$

$= 0.5\,(8/11) = .3636$

d. What is the probability that a blue marble was chosen?

$0.5\,(8/11) + 0.5\,(9/14) = .6851$

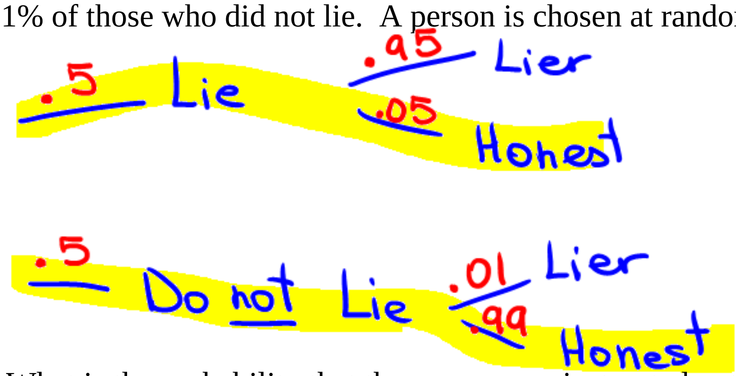
e. What is the probability that the marble drawn was white?

$0.5(3/11) + 0.5(5/14) = .3149$

OR

$1 - P(B) = 1 - .6851 = .3149$

Example 7: A new lie-detector test has been devised and must be tested before it is put into use. One hundred people are selected at random and each person draws and keeps a card from a box of 100 cards. Half the cards instruct the person to lie and the other half instruct the person to tell the truth. The test indicates lying in 95% of those who lied and in 1% of those who did not lie. A person is chosen at random.



a. What is the probability that the person was instructed not to lie and the test did not indicate lying?

$\cap$

$$.5(.99) = .495$$

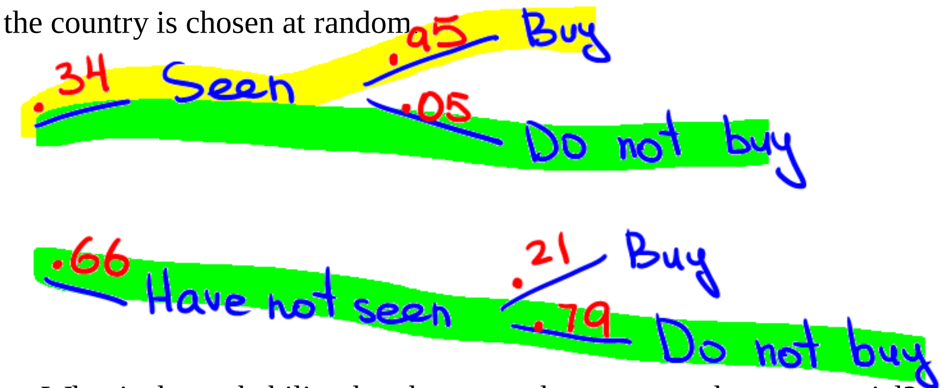
b. What is the probability that the test did not indicate lying?

$$.5(.05) + .5(.99) = .52$$

c. What is the probability that the test did not indicate lying, given that the person was instructed to lie?

$$.05$$

Example 8: Beauty Girl, a cosmetics company, estimates that 34% of the country has seen its commercial and if a person sees its commercial, there is a 5% chance that the person will not buy its product. The company also claims that if a person does not see its commercial, there is a 21% chance that the person will buy its product. A person from the country is chosen at random.



a. What is the probability that the person has not seen the commercial?

$$.66$$

b. What is the probability that the person has seen the commercial $\overset{\cap}{\text{and}}$ bought the product?

$$.34(.95) = .323$$

c. What is the probability that the person did not buy the product?

$$(.34)(.05) + (.66)(.79) = .5384$$

Example 9: A survey of 378 homeowners of new and old homes was conducted and the respondents were asked whether or not their home needed repairs.

	No Repairs Needed	Repairs Needed
Newer Home	56	14
Older Home	62	246

a. Given that no repairs are needed, find the probability that the home is a newer home.

$56/(56+62) = \boxed{.47}$

b. Given that repairs are needed, find the probability that the home is an older home.

$246/(14+246) = \boxed{.95}$

Independent Events

Two events A and B are **independent** if the outcome of one does not affect the outcome of the other.

Test for the Independence of Two Events

Two events, A and B, are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$. *This formula can be extended to a finite number of events.*

Independent and mutually exclusive does not mean the same thing.

For example, if P(A) $\neq$ 0 and P(B) $\neq$ 0 and A and B are mutually exclusive then
$P(A)*P(B) \neq 0$
$P(A \cap B) = 0$
so $P(A)P(B) \neq P(A \cap B)$

Example 10: The Ace Copy Store has four photocopying machines, A, B, C, and D. The probability that a given machine will break down on a given day is:

$P(A) = \frac{1}{40}$, $P(B) = \frac{1}{75}$, $P(C) = \frac{1}{60}$ and $P(D) = \frac{1}{50}$

Assuming independence, what is the probability that all of the machines will break down?

$P(A \cap B \cap C \cap D) = P(A) \cdot P(B) \cdot P(C) \cdot P(D)$

$= \frac{1}{40} \cdot \frac{1}{75} \cdot \frac{1}{60} \cdot \frac{1}{50} = 1.11 \times 10^{-7}$