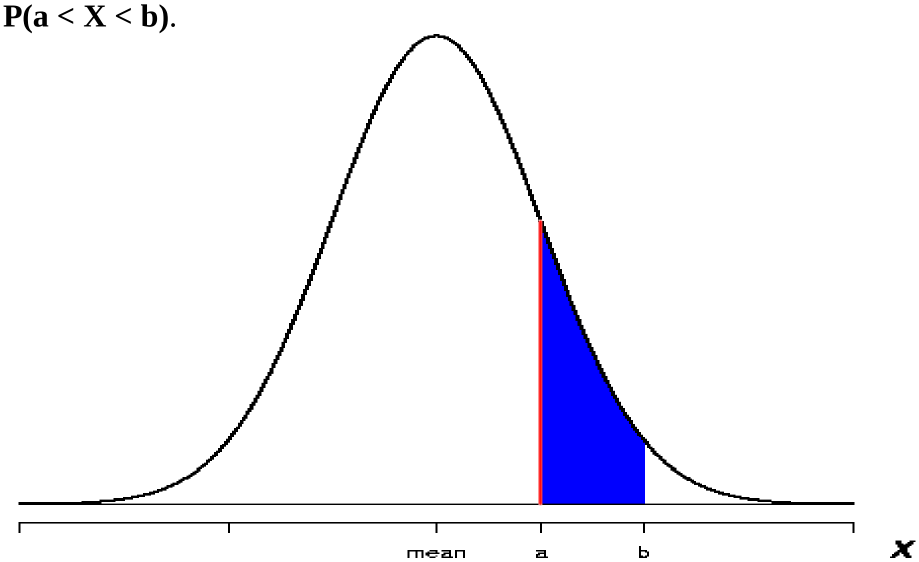
Section 7.5 – The Normal Distribution
Section 7.6 – Application of the Normal Distribution

A random variable that may take on infinitely many values is called a **continuous random variable**.

A continuous probability distribution is defined by a function *f* called the **probability density function**.

The probability that the random variable X associated with a given probability density function assumes a value in an interval $a < x < b$ is given by the area of the region between the graph of f and the x-axis from $x = a$ to $x = b$.

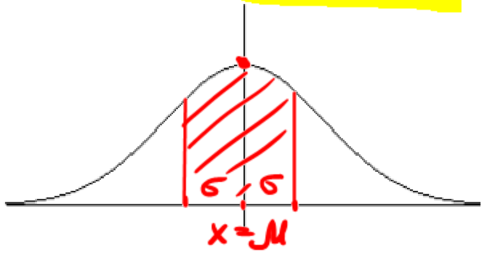
The following graph is a picture of a normal curve and the shaded region is **P(a < X < b)**.



Note: $P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$, since the area under one point is 0. The **area of the region under the standard normal curve to the left** of some value z, i.e. $P(Z < z)$ or $P(Z \leq z)$, is calculated for us in the **Standard Normal Cumulative Probability Table** found in Chapter 7 of the online book.

Normal distributions have the following characteristics:

1. The graph is a bell-shaped curve.



The curve always lies above the *x*-axis but approaches the *x*-axis as *x* extends indefinitely in either direction.

2. The curve has peak at $x = \mu$. The mean, μ, determines where the center of the curve is located.

3. The curve is symmetric with respect to the vertical line $x = \mu$.

4. The area under the curve is 1.

5. σ determines the sharpness or the flatness of the curve.

6. For any normal curve, 68.27% of the area under the curve lies within 1 standard deviation of the mean (i.e. between $\mu - \sigma$ and $\mu + \sigma$), 95.45% of the area lies within 2 standard deviations of the mean, and 99.73% of the area lies within 3 standard deviations of the mean.

The Standard Normal Variable will commonly be denoted *Z*. The **Standard Normal Curve** has $\mu = 0$ and $\sigma = 1$.

Example 1: Let *Z* be the standard normal variable. Find the values of:

a. P(*Z* < -1.91) = .0281

Table

b. $P(Z > 0.5)$ $= 1 - P(Z < 0.5) = 1 - .6915$

Table

$= .3085$

OR $P(Z < -0.5) = .3085$

c. $P(-1.65 < Z < 2.02)$ $= P(Z < 2.02) - P(Z < -1.65)$

Table

$= .9783 - .0495 = \boxed{.9288}$

Example 2: Let Z be the standard normal variable. Find the value of z if z satisfies:

a. $P(Z < -z) = 0.9495$ ← table

$-z = 1.64$ $\quad z = -1.64$

b. $P(Z > z) = 0.9115$

$P(Z < z) = 1 - P(Z > z) = 1 - .9115 = .0885$

Table

$z = -1.35$

c. $P(-z < Z < z) = 0.8444$

Formula: $P(Z < z) = \frac{1}{2}[1 + P(-z < Z < z)]$

$P(Z < z) = \frac{1}{2}[1 + 0.8444] = .9222$ Table

$\boxed{z = 1.42}$

When given a normal distribution in which $\mu \neq 0$ and $\sigma \neq 1$, we can transform the normal curve to the standard normal curve by doing whichever of the following applies.

$$P(X < b) = P\left(Z < \frac{b-\mu}{\sigma}\right)$$

$$P(X > a) = P\left(Z > \frac{a-\mu}{\sigma}\right)$$

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

Example 3: Suppose X is a normal variable with $\mu = 7$ and $\sigma = 4$. Find P(X > -1.35).

$$P(X > -1.35) = P\left(Z > \frac{-1.35-7}{4}\right)$$

$$= P(Z > -2.08) = \underbrace{P(Z < 2.08)}_{\text{Table}} = \boxed{.9812}$$

Applications of the Normal Distribution

Example 4: The heights of a certain species of plant are normally distributed with a mean of 20 cm and standard deviation of 4 cm. What is the probability that a plant chosen at random will be between 10 and 33 cm tall?

$$P(10 < X < 33) = P\left(\frac{10-20}{4} < Z < \frac{33-20}{4}\right)$$

$$= P(-2.5 < Z < 3.25)$$

$$= P(Z < 3.25) - P(Z < -2.5)$$

Table

$$= .9994 - 0.0062 = \boxed{.9932}$$

Example 5: Reaction time is normally distributed with a mean of 0.7 second and a standard deviation of 0.1 second. Find the probability that an individual selected at random has a reaction time of less than 0.6 second.

$$P(X<0.6) = P\left(Z < \frac{0.6-0.7}{0.1}\right)$$

$$= P(Z<-1) = .1587$$

Table

Approximating the Binomial Distribution Using the Normal Distribution

Theorem

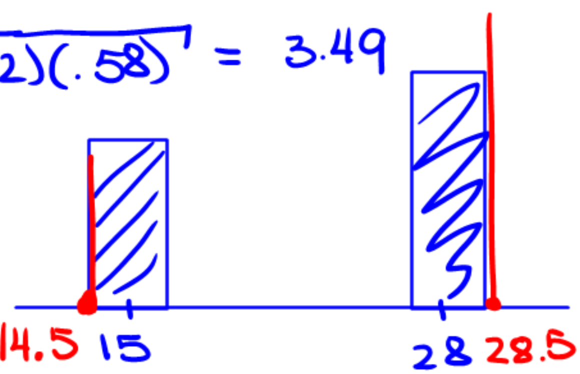
Suppose we are given a binomial distribution associated with a binomial experiment involving n trials, each with probability of success p and probability of failure q. Then if n is large and p is not close to 0 or 1, the binomial distribution may be approximated by a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$.

Example 6: Consider the following binomial experiment. Use the normal distribution to approximate the binomial distribution. A company claims that 42% of the households in a certain community use their Squeaky Clean All Purpose cleaner. What is the probability that between 15 and 28, inclusive, households out of 50 households use the cleaner?

$n=50 \quad p=.42 \quad q=1-p=.58$

$\mu = np = 50(.42) = 21$

$\sigma = \sqrt{npq} = \sqrt{50(.42)(.58)} = 3.49$



$$P(15 \le X \le 28)$$

$$= P(14.5 \le Y \le 28.5)$$

$$= P(14.5 < Y < 28.5)$$

$$= P\left(\frac{14.5-21}{3.49} < Z < \frac{28.5-21}{3.49}\right) = P(-1.86 < Z < 2.15)$$

$$= P(Z<2.15) - P(Z<-1.86)$$

Table

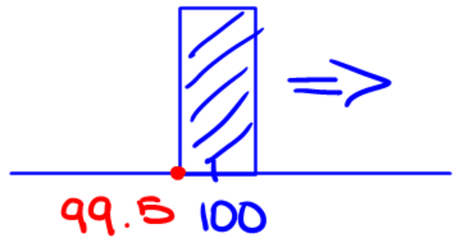
$$= .9842 - .0314 = .9528$$

$n = 120 \quad p = .75 \quad q = .25$

$\mu = np = 120(.75) = 90 \quad \sigma = \sqrt{npq} = \sqrt{120(.75)(.25)} = 4.7434$

Example 7: Use the normal distribution to approximate the binomial distribution. A basketball player has a 75% chance of making a free throw. She will make 120 attempts. What is the probability of her making:

a. 100 or more free throws?



$P(X \geq 100)$

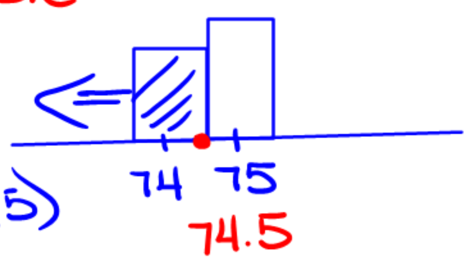
$= P(Y \geq 99.5)$

$= P(Y > 99.5) = P\left(Z > \dfrac{99.5 - 90}{4.7434}\right)$

$= P(Z > 2.00) = P(Z < -2.00) = .0228$

Table

b. fewer than 75 free throws?



$P(X < 75)$

$= P(Y \leq 74.5) = P(Y < 74.5)$

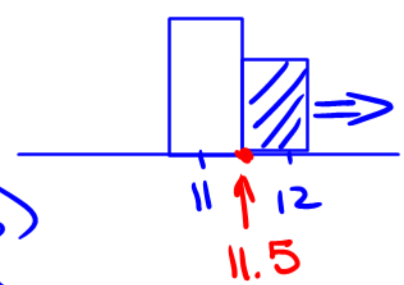
$= P\left(Z < \dfrac{74.5 - 90}{4.7434}\right) = P(Z < -3.27) = 0.0005$

Table

Example 8: Use the normal distribution to approximate the binomial distribution. A die is rolled 84 times. What is the probability that the number 2 occurs more than 11 times?

$n = 84 \quad p = 1/6 \quad q = 5/6 \quad \mu = 84\left(\frac{1}{6}\right) = 14$

$\sigma = \sqrt{84\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} - 3.4157$



$P(X > 11)$

$= P(Y \geq 11.5) = P(Y > 11.5)$

$= P\left(Z > \dfrac{11.5 - 14}{3.4157}\right) = P(Z > -.73)$

$P(Z < 0.73)$

Table $= \boxed{.7673}$