**Math 1314**
**Lesson 11: Exponential Functions as Mathematical Models**

Functions whose equations contain a variable in the exponent are called **exponential functions.**

**Exponential models** can be written in two forms: $f(x)=a\cdot e^x$ or $g(x)=a\cdot b^x$, with $e\approx 2.72$ and $0<b<1$.

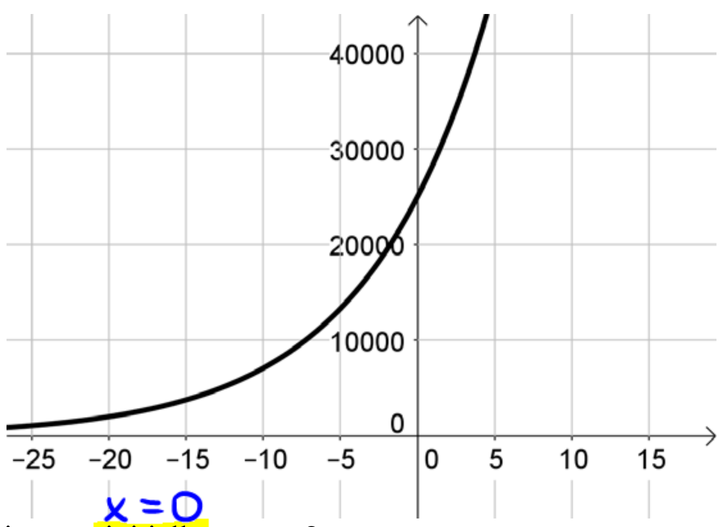
Exponential functions can be either increasing or decreasing.

- The function $f(x)=a\cdot e^{bx}$ is an **exponential growth function** and it's increasing.
- The function $f(x)=a\cdot e^{-bx}$ is an **exponential decay function** and it's decreasing.

In both equations the variable ***a*** is called the **initial amount** and ***b*** is called the **growth or decay constant**, depending on the type of function.

For a function of the form $g(x)=a\cdot b^x$, the function is an exponential growth function if $b>1$ and is an exponential decay function if $0<b<1$.

Example 1: The function $f(x)=25000e^{0.1263x}$ describes the bacteria population in a culture. *Enter the function into GGB.*




x = 0

a. How many bacteria were initially present?
Command: Answer:

f(0)

25000

$x = 10$

b. How many bacteria would be present in the culture 10 hours after beginning the experiment?
Command: Answer:

$f(10)$ 88 400 bacteria

c. How quickly is the bacteria growing 20 hours after the start of the experiment?
Command: Answer:

$f'(20)$ 39 479 bacteria / hour

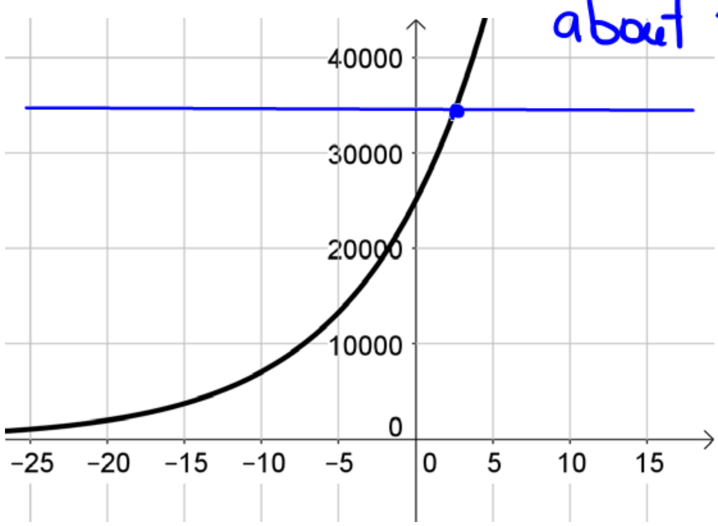
d. How long after the start of the experiment did the bacteria population reach 35,000?

Command: Answer:

intersect (f, 35000)

time # of bacteria
(2.6641, 35000)

about 2.7 hours



**Uninhibited Exponential Growth**

Some common exponential applications model **uninhibited exponential growth**. This means that there is no "upper limit" on the value of the function. It can simply keep growing and growing. Problems of this type include population growth problems and growth of investment assets.

Example 2: The sales from company ABC for the years 1998 – 2003 are given below.

x: 0 1 2 3 4 5 10

2008

list1 <

| Year | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 |
|---|---|---|---|---|---|---|
| profits in millions of dollars | 51.4 | 53.2 | 55.8 | 56.1 | 58.1 | 59.0 |

Rescale the data so that $x = 0$ corresponds to 1998. Begin by making a list.

a. Find an exponential regression model for the data.
Command: fitexp[list1]
Answer: $f(x) = 51.8595\, e^{0.0274x}$

b. Assuming the trend continues, predict the company's profit in 2008.
Command: f(10)
Answer: \$68.2099 million

**Exponential Decay**

Example 3: At the beginning of a study, there are 50 grams of a substance present. After 17 days, there are 38.7 grams remaining. Assume the substance decays exponentially.

a. State the two points given in the problem.

time weight

(0, 50) (17, 38.7)

list1 <

| | |
|---|---|
| 0 | 50 |
| 17 | 38.7 |

*Enter the two points in the spreadsheet and make a list.*

b. Find an exponential regression model.
Command: fitexp[list1]
Answer: $f(x) = 50\, e^{-0.0151x}$

c. What will be the rate of decay on day 40 of the study?

derivative x=40

Command: f'(40)
Answer: -0.4124 grams / day



On day 40 the substance is decaying at the rate 0.4124 grams / day

Exponential decay problems frequently involve the half-life of a substance. The **half-life** of a substance is the time it takes to reduce the amount of the substance by one-half.

Example 4: A certain drug has a half-life of 4 hours. Suppose you take a dose of 1000 milligrams of the drug.

a. State the two points given in the problem.

(0, 1000) (4, 500)

list1 <

| | |
|---|---|
| 0 | 1000 |
| 4 | 500 |

*Enter the two points in the spreadsheet and make a list.*

b. Find an exponential regression model.

Command: fitexp[list1]

Answer: $f(x) = 1000\,e^{-0.1733x}$

c. How much of it is left in your bloodstream 28 hours later?

Command: f(28)

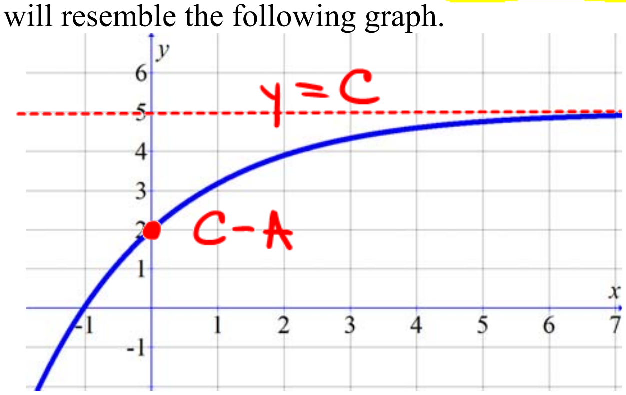
Answer: 7.8125 milligrams

**Limited Growth Models**

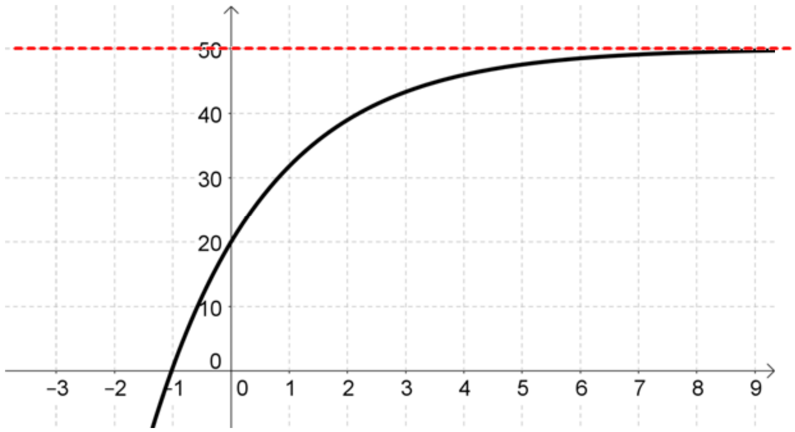
A worker on an assembly line performs the same task repeatedly throughout the workday. With experience, the worker will perform at or near an optimal level. However, when first learning to do the task, the worker's productivity will be much lower. During these early experiences, the worker's productivity will increase dramatically. Then, once the worker is thoroughly familiar with the task, there will be little change to his/her productivity.

The function that models this situation will have the form $Q(t) = C - Ae^{-kt}$ and their graphs called **learning curves** will resemble the following graph.

y-intercept at C-A
horizontal asymp. at y = C



Example 5: Suppose your company's HR department determines that an employee will be able to assemble $Q(t) = 50 - 30e^{-0.5t}$ products per day, $t$ months after the employee starts working on the assembly line. *Enter the function in GGB. The graph is pictured below.*




$t=0$

a. How many units can a new employee assemble as s/he starts the first day at work?
Command: Answer:

$Q(0)$ 20 units

$t=2$

b. How many units should an employee be able to assemble after two months at work?
Command: Answer:

$Q(2)$ 39 units

c. How many units should an experienced worker be able to assemble?

50 units Asymptote

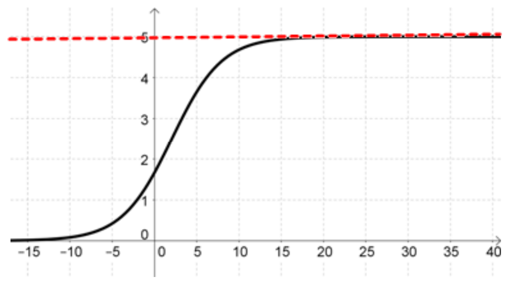
derivative $t=4$

d. At what rate is an employee's productivity changing 4 months after starting to work?
Command: Answer:

$Q'(4) = 2.03$ units/month

## Logistic Functions

The last growth model that we will consider involves the logistic function. The general form of the equation is $\boxed{Q(t) = \dfrac{A}{1+Be^{-kt}}}$ and the graph looks something like this:




Logistic functions have some of the features of both exponential models and limited growth models.

Note that the graph above has a limiting value at $y = 5$. In the context of a logistic function, this asymptote is called the **carrying capacity** (maximum number). In general the carrying capacity is $A$ from the formula above: $Q(t) = \dfrac{A}{1+Be^{-kt}}$.

Logistic curves are used to model various types of phenomena and other physical situations such as population management. Suppose a number of animals are introduced into a protected game reserve, with the expectation that the population will grow. Various factors will work together to keep the population from growing exponentially (in an uninhibited manner). The natural resources (food, water, protection) may not exist to support a population that gets larger without bound. Often such populations grow according to a logistic model.

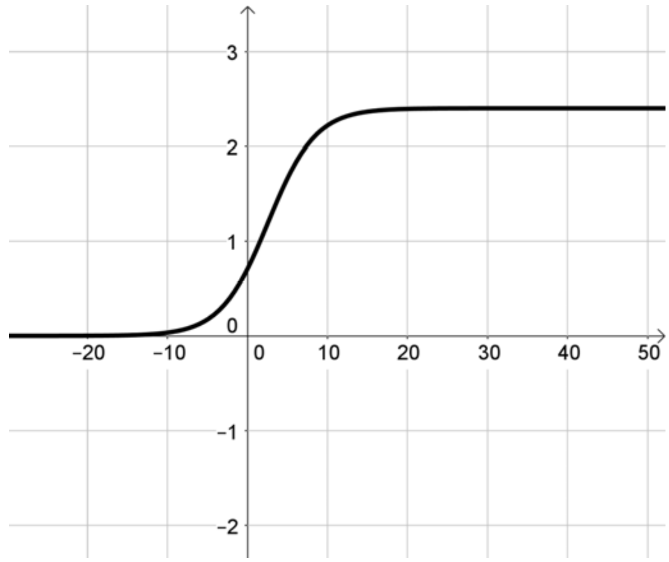
Example 6: A population study was commissioned to determine the growth rate of the fish population in a certain area of the Pacific Northwest. The function given below models the population where $t$ is measured in years and $N$ is measured in millions of tons. *Enter the function in GGB.*

$$N(t) = \frac{2.4}{(1+2.39e^{(-0.338t)})}$$

a. What was the initial number of fish in the population?
Command: Answer:

N(0)

0.708 millions of ton of fish




b. What is the maximum number of fish present, using this model?
Command: Answer:

$y = 2.4$ million of ton of fish

c. What is the fish population after 3 years?
Command: Answer:

$N(3) = 1.2855$ million ton of fish

d. How fast is the fish population changing after 2 years?
Command: Answer:

$N'(2) = 0.2009$ million tons of fish per year.

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